Canad. Math. Bull. Vol. **64** (2), 2021, pp. 282–291
http://dx.doi.org/10.4153/S0008439520000442

On Pisier's inequality for UMD targets

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Abstract. We prove an extension of Pisier's inequality (1986) with a dimension-independent constant for vector-valued functions whose target spaces satisfy a relaxation of the UMD property.

1 Introduction

Let $(X, \|\cdot\|_X)$ be a Banach space. For $p \in [1, \infty)$, the vector-valued L_p norm of a function $f : \Omega \to X$ defined on a measure space $(\Omega, \mathcal{F}, \mu)$ is given by $\|f\|^p_{L_p(\Omega,\mu;X)} = \int_\Omega \|f(\omega)\|^p_X \, d\mu(\omega)$. When Ω is a finite set and μ is the normalized counting measure, we will simply write $\|f\|_{L_p(\Omega;X)}$.

Let $\mathcal{C}_n = \{-1, 1\}^n$ be the discrete hypercube. For $i \in \{1, \dots, n\}$, the ith partial derivative of a function $f : \mathcal{C}_n \to X$ is defined by

$$\forall\, \varepsilon \in \mathcal{C}_n, \quad \partial_i f(\varepsilon) \stackrel{\text{def}}{=} \frac{f(\varepsilon) - f(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n)}{2}. \tag{1}$$

In [Pis86], Pisier showed that for every $n \in \mathbb{N}$ and $p \in [1, \infty)$, every $f : \mathcal{C}_n \to X$ satisfies

$$\Big\| f - \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} f(\delta) \Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{P}^n_p(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i \partial_i f \Big\|^p_{L_p(\mathcal{C}_n;X)} \Big)^{1/p}, \tag{2}$$

with $\mathfrak{P}^n_p(X) \leqslant 2e \log n$. Showing that $\mathfrak{P}^n_p(X)$ is bounded by a constant depending only on p and the geometry of the given Banach space X, is of fundamental importance in the theory of nonlinear type (see [Pis86, NS02]). The first positive and negative results in this direction were obtained by Talagrand in [Tal93], who showed that $\mathfrak{P}^n_p(\mathbb{R}) \asymp_p 1$ and $\mathfrak{P}^n_p(\ell_\infty) \asymp_p \log n$ for every $p \in [1, \infty)$.

Talagrand's dimension-independent scalar-valued inequality (2) was greatly generalized in the range $p \in (1, \infty)$ by Naor and Schechtman [NS02]. Recall that a Banach space $(X, \|\cdot\|_X)$ is called a UMD space if for every $p \in (1, \infty)$, there exists a constant $\beta_p \in (0, \infty)$ such that for every $n \in \mathbb{N}$, every probability space $(\Omega, \mathcal{F}, \mu)$ and every filtration $\{\mathcal{F}_i\}_{i=0}^n$ of sub- σ-algebras of $\mathcal{F}$, every martingale $\{\mathcal{M}_i : \Omega \to X\}_{i=0}^n$ adapted to $\{\mathcal{F}_i\}_{i=0}^n$ satisfies

Received by the editors March 12, 2020.
Published online on Cambridge Core June 15, 2020.
The author was supported by a postdoctoral fellowship of the Fondation Sciences Mathématiques de Paris.
AMS subject classification: **46B07**, 46B85, 42C10, 60G46.
Keywords: Pisier's inequality, Banach space valued martingale, UMD Banach space.

https://doi.org/10.4153/S0008439520000442 Published online by Cambridge University Press

$$\max_{\delta=(\delta_1,\ldots,\delta_n)\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i(\mathcal{M}_i-\mathcal{M}_{i-1})\Big\|_{L_p(\Omega,\mu;X)} \leqslant \beta_p\|\mathcal{M}_n-\mathcal{M}_0\|_{L_p(\Omega,\mu;X)}. \tag{3}$$

The least constant $\beta_p\in(0,\infty)$ for which (3) holds is called the UMD$_p$ constant of X and is denoted by $\beta_p(X)$. In [NS02], Naor and Schechtman proved that for every UMD Banach space X and $p\in(1,\infty)$,

$$\sup_{n\in\mathbb{N}}\mathfrak{P}_p^n(X)\leqslant\beta_p(X). \tag{4}$$

Their result was later strengthened by Hytönen and Naor [HN13] in terms of the random martingale transform inequalities of Garling; see [Gar90]. Recall that a Banach space $(X,\|\cdot\|_X)$ is a UMD$^+$ space if for every $p\in(1,\infty)$ there exists a constant $\beta_p^+\in(0,\infty)$ such that for every martingale $\{\mathcal{M}_i:\Omega\to X\}_{i=0}^n$ as before, we have

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i(\mathcal{M}_i-\mathcal{M}_{i-1})\Big\|^p_{L_p(\Omega,\mu;X)}\Big)^{1/p}\leqslant\beta_p^+\|\mathcal{M}_n-\mathcal{M}_0\|_{L_p(\Omega,\mu;X)}. \tag{5}$$

Similarly, X is a UMD$^-$ Banach space if for every $p\in(1,\infty)$ there exists a constant $\beta_p^-\in(0,\infty)$ such that for every martingale $\{\mathcal{M}_i:\Omega\to X\}_{i=0}^n$ as before, we have

$$\|\mathcal{M}_n-\mathcal{M}_0\|_{L_p(\Omega,\mu;X)}\leqslant\beta_p^-\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i(\mathcal{M}_i-\mathcal{M}_{i-1})\Big\|^p_{L_p(\Omega,\mu;X)}\Big)^{1/p}. \tag{6}$$

The least positive constants β_p^+,β_p^- for which (5) and (6) hold are respectively called the UMD$_p^+$ and UMD$_p^-$ constants of X and are denoted by $\beta_p^+(X)$ and $\beta_p^-(X)$. In [HN13], Hytönen and Naor showed that for every Banach space X whose dual X^* is a UMD$^+$ space and $p\in(1,\infty)$,

$$\sup_{n\in\mathbb{N}}\mathfrak{P}_p^n(X)\leqslant\beta^+_{p/(p-1)}(X^*). \tag{7}$$

In fact, in [HN13, Theorem 1.4], the authors proved a generalization (see (28)) of inequality (2) for a family of n functions $\{f_i:\mathcal{C}_n\to X\}_{i=1}^n$ under the assumption that the dual of X is UMD$^+$.

The main result of the present note is a different inequality of this nature with respect to a Fourier-analytic parameter of X. For a Banach space $(X,\|\cdot\|_X)$ and $p\in(1,\infty)$, let $\mathfrak{s}_p(X)\in(0,\infty]$ be the least constant $\mathfrak{s}\in(0,\infty]$ such that the following holds. For every probability space $(\Omega,\mathcal{F},\mu)$, $n\in\mathbb{N}$ and filtration $\{\mathcal{F}_i\}_{i=1}^n$ of sub- σ-algebras of $\mathcal{F}$ with corresponding vector-valued conditional expectations $\{\mathcal{E}_i\}_{i=1}^n$, every sequence of functions $\{f_i:\Omega\to X\}_{i=1}^n$ satisfies

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_if_i\Big\|^p_{L_p(\Omega,\mu;X)}\Big)^{1/p}\leqslant\mathfrak{s}\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|^p_{L_p(\Omega,\mu;X)}\Big)^{1/p}. \tag{8}$$

The square function inequality (8) originates in Stein's classical work [Ste70], where he showed that $\mathfrak{s}_p(\mathbb{R})\asymp_p 1$ for every $p\in(1,\infty)$. In the vector-valued setting which is of interest here, it has been proved by Bourgain in [Bou86] that for every UMD$^+$ Banach space and $p\in(1,\infty)$,

$$\mathfrak{s}_p(X)\leqslant\beta_p^+(X). \tag{9}$$

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For a function $f : \mathcal{C}_n \to X$ and $i \in \{0, 1, \dots, n\}$ denote by

$$\forall\, \varepsilon \in \mathcal{C}_n, \quad \mathcal{E}_i f(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{2^{n-i}} \sum_{\delta_{i+1},\dots,\delta_n \in \{-1,1\}} f(\varepsilon_1, \dots, \varepsilon_i, \delta_{i+1}, \dots, \delta_n), \tag{10}$$

so that $\mathcal{E}_n f = f$ and $\mathcal{E}_0 f = \frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} f(\delta)$. The main result of this note is the following theorem.

Theorem 1 *Fix $p \in (1, \infty)$ and let $(X, \|\cdot\|_X)$ be a Banach space with $\mathfrak{s}_p(X) < \infty$. If, additionally, X is a UMD$^-$ space, then for every $n \in \mathbb{N}$ and functions $f_1, \dots, f_n : \mathcal{C}_n \to X$, we have*

$$\Big\| \sum_{i=1}^n (\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i) \Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{s}_p(X) \beta_p^-(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i \partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)}^p \Big)^{1/p}. \tag{11}$$

Choosing $f_1 = \cdots = f_n = f$, we deduce that the constants in Pisier's inequality (2) satisfy

$$\sup_{n \in \mathbb{N}} \mathfrak{P}_p^n(X) \leqslant \mathfrak{s}_p(X) \beta_p^-(X). \tag{12}$$

Combining (12) with Bourgain's inequality (9), we deduce that $\sup_{n\in\mathbb{N}} \mathfrak{P}_p^n(X) \leqslant \beta_p^+(X)\beta_p^-(X)$, which is weaker than Naor and Schechtman's bound (4). Nevertheless, it appears to be unknown (see [Pis16, p. 197]) whether every Banach space X with $\mathfrak{s}_p(X) < \infty$ is necessarily a UMD$^+$ space. Therefore, it is conceivable that there exist Banach spaces X for which inequality (12) does not follow from the previously known results of [NS02, HN13]. We will see in Proposition 5 below that if the dual X^* of a Banach space X is UMD$^+$, then X satisfies the assumptions of Theorem 1. Therefore, Theorem 1 also contains the aforementioned result of [HN13].

Moreover, Theorem 1 implies an inequality similar to [HN13, Theorem 1.4] (see also Remark 3 below for comparison), under different assumptions. We will need some standard terminology from discrete Fourier analysis. Recall that every function $f : \mathcal{C}_n \to X$ can be expanded in a Walsh series as

$$f = \sum_{A \subseteq \{1,\dots,n\}} \widehat{f}(A) w_A, \tag{13}$$

where $\widehat{f}(A) \in X$ and the Walsh function $w_A : \mathcal{C}_n \to \{-1, 1\}$ is given by $w_A(\varepsilon) = \prod_{i\in A} \varepsilon_i$ for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}_n$ and $A \neq \varnothing$. As usual, we agree that $w_\varnothing \equiv 1$. Moreover, the fractional hypercube Laplacian of a function $f : \mathcal{C}_n \to X$ is given by

$$\forall\, \alpha \in \mathbb{R}, \quad \Delta^\alpha \Big(\sum_{A \subseteq \{1,\dots,n\}} \widehat{f}(A) w_A \Big) \stackrel{\text{def}}{=} \sum_{\substack{A \subseteq \{1,\dots,n\} \\ A \neq \varnothing}} |A|^\alpha \widehat{f}(A) w_A. \tag{14}$$

Corollary 2 *Fix $p \in (1, \infty)$ and let $(X, \|\cdot\|_X)$ be a Banach space with $\mathfrak{s}_p(X) < \infty$. If, additionally, X is a UMD$^-$ space, then for every $n \in \mathbb{N}$ and functions $f_1, \dots, f_n : \mathcal{C}_n \to X$, we have*

$$\Big\| \sum_{i=1}^n \Delta^{-1} \partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{s}_p(X) \beta_p^-(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i \partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)}^p \Big)^{1/p}. \tag{15}$$

Asymptotic notation In what follows we use the convention that for $a, b \in [0, \infty]$ the notation $a \gtrsim b$ (respectively $a \lesssim b$) means that there exists a universal constant $c \in (0, \infty)$ such that $a \geqslant cb$ (respectively $a \leqslant cb$). Moreover, $a \asymp b$ stands for $(a \lesssim b) \wedge (a \gtrsim b)$. The notations $\lesssim_\xi$, $\gtrsim_\chi$ and $\asymp_\psi$ mean that the implicit constant c depends on ξ, χ and ψ respectively.

2 Proofs

We first present the proof of Theorem 1.

Proof of Theorem 1 For a function $h : \mathcal{C}_n \to X$ and $i \in \{1, \ldots, n\}$ consider the averaging operator

(16)
$$\forall\ \varepsilon \in \mathcal{C}_n, \quad \mathsf{E}_i h(\varepsilon) \stackrel{\text{def}}{=} \frac{h(\varepsilon) + h(\varepsilon_1, \ldots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_n)}{2} = (\mathsf{id} - \partial_i)h(\varepsilon),$$

where id is the identity operator. Then, for every $i \in \{0, 1, \ldots, n\}$ we have the identities

$$\mathcal{E}_i h = \mathsf{E}_{i+1} \circ \cdots \circ \mathsf{E}_n h = \mathbb{E}[h|\mathcal{F}_i], \tag{17}$$

where $\mathcal{F}_i = \sigma(\varepsilon_1, \ldots, \varepsilon_i)$. Since for every $i \in \{1, \ldots, n\}$,

$$\mathbb{E}\big[\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i \big| \mathcal{F}_{i-1}\big] = 0, \tag{18}$$

the sequence $\{\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i\}_{i=1}^n$ is a martingale difference sequence and thus the UMD^- condition and (8) imply that

$$\begin{aligned}\Big\| \sum_{i=1}^n (\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i) \Big\|_{L_p(\mathcal{C}_n;X)} &\overset{(6)}{\leqslant} \beta_p^-(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i (\mathcal{E}_i f_i - \mathcal{E}_{i-1} f_i) \Big\|_{L_p(\mathcal{C}_n;X)}^p \Big)^{1/p} \\ &\overset{(16)}{=} \beta_p^-(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i \mathcal{E}_i \partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)}^p \Big)^{1/p} \\ &\overset{(8)}{\leqslant} \mathfrak{s}_p(X) \beta_p^-(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i \partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)}^p \Big)^{1/p}\end{aligned} \tag{19}$$

which completes the proof. ∎

We will now derive Corollary 2 from Theorem 1. The proof follows a symmetrization argument of [HN13].

Proof of Corollary 2 As noticed in (19) above, (11) can be equivalently written as

$$\Big\| \sum_{i=1}^n \mathcal{E}_i \partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{s}_p(X) \beta_p^-(X) \Big(\frac{1}{2^n} \sum_{\delta \in \mathcal{C}_n} \Big\| \sum_{i=1}^n \delta_i \partial_i f_i \Big\|_{L_p(\mathcal{C}_n;X)}^p \Big)^{1/p}. \tag{20}$$

Fix a permutation $\pi \in S_n$ and consider the filtration $\{\mathcal{F}_i^\pi\}_{i=0}^n$ given by $\mathcal{F}_i^\pi = \sigma(\varepsilon_{\pi(1)}, \ldots, \varepsilon_{\pi(i)})$ with corresponding conditional expectations $\{\mathcal{E}_i^\pi\}_{i=0}^n$. Repeating the argument of the proof of Theorem 1 for this filtration and the martingale difference sequence $\{\mathcal{E}_i^\pi f_{\pi(i)} - \mathcal{E}_{i-1}^\pi f_{\pi(i)}\}_{i=1}^n$, we see that for every $\pi \in S_n$,

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$$\Big\|\sum_{i=1}^{n}\mathcal{E}_i^{\pi}\partial_{\pi(i)}f_{\pi(i)}\Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^{n}\delta_i\partial_{\pi(i)}f_{\pi(i)}\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}$$

$$(21)\qquad = \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^{n}\delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p},$$

since $(\delta_1,\ldots,\delta_n)$ has the same distribution as $(\delta_{\pi(1)},\ldots,\delta_{\pi(n)})$. An obvious adaptation of (10) along with (13) shows that for every $h:\mathcal{C}_n\to X$,

$$(22)\qquad \mathcal{E}_i^{\pi}h = \sum_{A\subseteq\{\pi(1),\ldots,\pi(i)\}}\widehat{h}(A)w_A$$

where $\widehat{h}(A)$ are the Walsh coefficients of h. Therefore, expanding each $f_{\pi(i)}$ as a Walsh series (13) we have

$$(23)\qquad \forall\ i\in\{1,\ldots,n\},\quad \mathcal{E}_i^{\pi}\partial_{\pi(i)}f_{\pi(i)} = \sum_{\substack{A\subseteq\{1\ldots,n\}\\ \max\pi^{-1}(A)=i}}\widehat{f_{\pi(i)}}(A)w_A$$

and therefore

$$(24)\qquad \sum_{i=1}^{n}\mathcal{E}_i^{\pi}\partial_{\pi(i)}f_{\pi(i)} = \sum_{A\subseteq\{1,\ldots,n\}}\widehat{f_{\pi(\max\pi^{-1}(A))}}(A)w_A.$$

Averaging (24) over all permutations $\pi\in S_n$ and using the fact that $\pi(\max\pi^{-1}(A))$ is uniformly distributed in A, we get

$$\frac{1}{n!}\sum_{\pi\in S_n}\sum_{i=1}^{n}\mathcal{E}_i^{\pi}\partial_{\pi(i)}f_{\pi(i)} = \sum_{\substack{A\subseteq\{1,\ldots,n\}\\ A\neq\varnothing}}\frac{1}{|A|}\sum_{i\in A}\widehat{f_i}(A)w_A$$

$$= \sum_{i=1}^{n}\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ i\in A}}\frac{1}{|A|}\widehat{f_i}(A)w_A = \sum_{i=1}^{n}\Delta^{-1}\partial_i f_i.$$

Hence, by convexity we finally deduce that

$$\Big\|\sum_{i=1}^{n}\Delta^{-1}\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \frac{1}{n!}\sum_{\pi\in S_n}\Big\|\sum_{i=1}^{n}\mathcal{E}_i^{\pi}\partial_{\pi(i)}f_{\pi(i)}\Big\|_{L_p(\mathcal{C}_n;X)}$$

$$(25)\qquad \overset{(21)}{\leqslant} \mathfrak{s}_p(X)\beta_p^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^{n}\delta_i\partial_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p},$$

which completes the proof. ■

Remark 3 In [HN13], Hytönen and Naor obtained a different extension of Pisier's inequality (2) for Banach spaces whose dual is UMD$^+$. For a function $F:\mathcal{C}_n\times\mathcal{C}_n\to X$ and $i\in\{1,\ldots,n\}$, let $F_i:\mathcal{C}_n\to X$ be given by

$$(26)\qquad \forall\ \varepsilon\in\mathcal{C}_n,\quad F_i(\varepsilon)\overset{\text{def}}{=}\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\delta_i F(\varepsilon,\delta).$$

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In [HN13, Theorem 1.4], it was shown that for every $p \in (1,\infty)$ and every function $F : \mathcal{C}_n \times \mathcal{C}_n \to X$,

$$\Big\| \sum_{i=1}^{n} \Delta^{-1}\partial_i F_i \Big\|_{L_p(\mathcal{C}_n;X)} \leqslant \beta^{+}_{p/(p-1)}(X^*)\|F\|_{L_p(\mathcal{C}_n\times\mathcal{C}_n;X)}. \tag{27}$$

In fact, since every Banach space whose dual is UMD$^+$ is K-convex (see [Pis16] and Section 3 below) the validity of inequality (27) is equivalent to its validity for functions of the form $F(\varepsilon,\delta) = \sum_{i=1}^{n} \delta_i F_i(\varepsilon)$, where $F_1,\dots,F_n : \mathcal{C}_n \to X$. In other words, [HN13, Theorem 1.4] is equivalent to the fact that if X^* is UMD$^+$, then for every $F_1,\dots,F_n : \mathcal{C}_n \to X$ and $p \in (1,\infty)$,

$$\Big\| \sum_{i=1}^{n} \Delta^{-1}\partial_i F_i \Big\|_{L_p(\mathcal{C}_n;X)} \leqslant A_p(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n} \Big\|\sum_{i=1}^{n}\delta_i F_i\Big\|^p_{L_p(\mathcal{C}_n;X)}\Big)^{1/p}, \tag{28}$$

up to the value of the constant $A_p(X)$. In particular, applying (28) to $F_i = \partial_i f_i$, one recovers Corollary 2, so inequality (28) of [HN13] is formally stronger than (15) in the class of spaces whose dual is UMD$^+$.

3 Concluding Remarks

In this section we will compare our result with existing theorems in the literature. Recall that a Banach X space is K-convex if X does not contain the family $\{\ell_1^n\}_{n=1}^{\infty}$ with uniformly bounded distortion. We will need the following lemma.

Lemma 4 *If a space $(X,\|\cdot\|_X)$ satisfies $\mathfrak{s}_p(X) < \infty$ for some $p \in (1,\infty)$, then X is K-convex.*

Proof It is well known since Stein's work [Ste70] that inequality (8) does not hold for $p \in \{1,\infty\}$ even for scalar valued functions. In fact, an inspection of the argument in [Ste70, p. 105] shows that for every $n \in \mathbb{N}$ there existn functions $g_1,\dots,g_n : \mathcal{C}_n \to \{0,1\}$ such that for every $q \in (2,\infty)$,

$$\Big\|\Big(\sum_{i=1}^{n}(\mathcal{E}_i g_i)^2\Big)^{1/2}\Big\|_{L_q(\mathcal{C}_n;\mathbb{R})} \gtrsim \Big(\int_0^n y^{q/2}e^{-y}\,\mathrm{d}y\Big)^{1/q}\Big\|\Big(\sum_{i=1}^{n} g_i^2\Big)^{1/2}\Big\|_{L_q(\mathcal{C}_n;\mathbb{R})}, \tag{29}$$

where $\{\mathcal{E}_i\}_{i=0}^{n}$ are the conditional expectations (10). Using the fact that $L_\infty(\mathcal{C}_n;\mathbb{R})$ is 2-isomorphic to $L_n(\mathcal{C}_n;\mathbb{R})$, we thus deduce that

$$\begin{aligned}\Big\|\Big(\sum_{i=1}^{n}(\mathcal{E}_i g_i)^2\Big)^{1/2}\Big\|_{L_\infty(\mathcal{C}_n;\mathbb{R})} &\gtrsim \Big(\int_0^n y^{n/2}e^{-y}\,\mathrm{d}y\Big)^{1/n}\Big\|\Big(\sum_{i=1}^{n} g_i^2\Big)^{1/2}\Big\|_{L_\infty(\mathcal{C}_n;\mathbb{R})}\\ &\asymp \sqrt{n}\Big\|\Big(\sum_{i=1}^{n} g_i^2\Big)^{1/2}\Big\|_{L_\infty(\mathcal{C}_n;\mathbb{R})}\end{aligned} \tag{30}$$

Therefore, by duality in $L_\infty(\mathcal{C}_n;\ell_2^n)$ and Khintchine's inequality [Khi23], we deduce that there exist n functions $h_1,\dots,h_n : \mathcal{C}_n \to \mathbb{R}$ such that

$$\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^{n}\delta_i\mathcal{E}_i h_i\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})} \gtrsim \frac{\sqrt{n}}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^{n}\delta_i h_i\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})}. \tag{31}$$

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Suppose that a Banach space X with $\mathfrak{s}_p(X) < \infty$ is not K-convex, so that there exists a constant $K \in [1,\infty)$ such that for every $n \in \mathbb{N}$, there exists a linear operator $\mathsf{J}_n : L_1(\mathcal{C}_n;\mathbb{R}) \to X$ satisfying

$$(32) \qquad \forall\, h \in L_1(\mathcal{C}_n;\mathbb{R}), \quad \|h\|_{L_1(\mathcal{C}_n;\mathbb{R})} \leqslant \|\mathsf{J}_n h\|_X \leqslant K\|h\|_{L_1(\mathcal{C}_n;\mathbb{R})}.$$

Consider the functions $H_1,\ldots,H_n : \mathcal{C}_n \to L_1(\mathcal{C}_n;\mathbb{R})$ given by

$$(33) \qquad \forall\, \varepsilon,\varepsilon' \in \mathcal{C}_n, \quad \big[H_i(\varepsilon)\big](\varepsilon') = h_i(\varepsilon_1\varepsilon_1',\ldots,\varepsilon_n\varepsilon_n'),$$

where $h_i \in L_1(\mathcal{C}_n;\mathbb{R})$ are the functions satisfying (31). Then, for every $i \in \{1,\ldots,n\}$, we have $[\mathcal{E}_i H_i(\varepsilon)](\varepsilon') = \mathcal{E}_i h_i(\varepsilon_1\varepsilon_1',\ldots,\varepsilon_n\varepsilon_n')$ and, by translation invariance, for every $\varepsilon,\delta \in \mathcal{C}_n$ we have

$$\Big\|\sum_{i=1}^n \delta_i\mathcal{E}_i H_i(\varepsilon)\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})} = \Big\|\sum_{i=1}^n \delta_i\mathcal{E}_i h_i\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})} \quad \text{and}$$
$$\Big\|\sum_{i=1}^n \delta_i H_i(\varepsilon)\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})} = \Big\|\sum_{i=1}^n \delta_i h_i\Big\|_{L_1(\mathcal{C}_n;\mathbb{R})}$$

Therefore, considering the mappings $f_1,\ldots,f_n : \mathcal{C}_n \to X$ given by $f_i = \mathsf{J}_n \circ H_i$, we see that

$$(34) \quad \Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\mathcal{E}_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p} \gtrsim K^{-1}\sqrt{n}\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p},$$

thus showing that $\mathfrak{s}_p(X) \gtrsim K^{-1}\sqrt{n}$, which is a contradiction. ∎

Recall that the X-valued Rademacher projection is defined to be

$$(35) \qquad \mathsf{Rad}\Big(\sum_{A\subseteq\{1,\ldots,n\}} \widehat{f}(A)w_A\Big) \stackrel{\text{def}}{=} \sum_{i=1}^n \widehat{f}(\{i\})w_{\{i\}}.$$

A deep theorem of Pisier [Pis82] asserts that a Banach space is K-convex if and only if

$$(36) \qquad \forall\, r \in (1,\infty), \quad \mathsf{K}_r(X) \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} \|\mathsf{Rad}\|_{L_r(\mathcal{C}_n;X)\to L_r(\mathcal{C}_n;X)} < \infty.$$

In particular, it follows from Lemma 4 that $\mathfrak{s}_p(X) < \infty$ for some $p \in (1,\infty)$ implies that $\mathsf{K}_r(X) < \infty$ for every $r \in (1,\infty)$. We proceed by showing that Banach spaces belonging to the class considered in [HN13, Theorem 1.4] satisfy the assumptions of Theorem 1.

Proposition 5 *Let $(X,\|\cdot\|_X)$ be a Banach space. If X^* is a UMD$^+$ space, then X is a UMD$^-$ space and $\mathfrak{s}_p(X) < \infty$ for every $p \in (1,\infty)$.*

Proof The fact that if X^* is UMD$^+$, then X is UMD$^-$ has been proved by Garling in [Gar90, Theorem 1], so we only have to prove that $\mathfrak{s}_p(X) < \infty$. Let $f_1,\ldots,f_n : \mathcal{C}_n \to X$ and $G^* : \mathcal{C}_n \times \mathcal{C}_n \to X^*$ be such that

$$(37) \quad \Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i\mathcal{E}_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p} = \frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\Big\langle G^*(\varepsilon,\delta), \sum_{i=1}^n \delta_i\mathcal{E}_i f_i(\varepsilon)\Big\rangle$$

and $\|G^*\|_{L_q(\mathcal{C}_n\times\mathcal{C}_n;X^*)} = 1$, where $\frac{1}{p}+\frac{1}{q}=1$. Let $G_i^*:\mathcal{C}_n\to X^*$ be given by

$$\forall\, \varepsilon\in\mathcal{C}_n,\quad G_i^*(\varepsilon)=\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\delta_i G^*(\varepsilon,\delta). \tag{38}$$

Then, since X^* is UMD$^+$, we deduce that X^* is also K-convex (this is proved in [Gar90] but it also follows by combining Bourgain's inequality (9) with Lemma 4) and thus

$$\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i G_i^*\Big\|_{L_q(\mathcal{C}_n;X^*)}^q\Big)^{1/q}\overset{(38)}{=}\Big(\frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\big\|\mathsf{Rad}_\delta G^*(\varepsilon,\delta)\big\|_X^q\Big)^{1/q}\leqslant \mathsf{K}_q(X^*). \tag{39}$$

Hence, we have

$$\begin{aligned}\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p} &\overset{(37)\wedge(38)}{=}\frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\Big\langle\sum_{i=1}^n\delta_iG_i^*(\varepsilon),\sum_{i=1}^n\delta_i\mathcal{E}_if_i(\varepsilon)\Big\rangle\\ &=\frac{1}{2^n}\sum_{\varepsilon\in\mathcal{C}_n}\langle G_i^*(\varepsilon),\mathcal{E}_if_i(\varepsilon)\rangle=\frac{1}{2^n}\sum_{\varepsilon\in\mathcal{C}_n}\langle\mathcal{E}_iG_i^*(\varepsilon),f_i(\varepsilon)\rangle\\ &=\frac{1}{4^n}\sum_{\varepsilon,\delta\in\mathcal{C}_n}\Big\langle\sum_{i=1}^n\delta_i\mathcal{E}_iG_i^*(\varepsilon),\sum_{i=1}^n\delta_if_i(\varepsilon)\Big\rangle\\ &\leqslant\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_iG_i^*\Big\|_{L_q(\mathcal{C}_n;X^*)}^q\Big)^{1/q}\cdot\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}.\end{aligned} \tag{40}$$

Therefore, combining (40) with (8) and (39), we deduce that

$$\begin{aligned}&\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i\mathcal{E}_i f_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\\ &\qquad\overset{(8)}{\leqslant}\mathfrak{s}_q(X^*)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_iG_i^*\Big\|_{L_q(\mathcal{C}_n;X^*)}^q\Big)^{1/q}\cdot\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p}\\ &\qquad\overset{(39)}{\leqslant}\mathfrak{s}_q(X^*)\mathsf{K}_q(X^*)\cdot\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_if_i\Big\|_{L_p(\mathcal{C}_n;X)}^p\Big)^{1/p},\end{aligned} \tag{41}$$

which shows that $\mathfrak{s}_p(X)\leqslant\mathsf{K}_q(X^*)\mathfrak{s}_q(X^*)$. ∎

We conclude by observing that spaces satisfying the assumptions of Theorem 1 are necessarily superreflexive (see [Pis16, Chapter 11] for the relevant terminology).

Lemma 6 *If a UMD$^-$ Banach space $(X,\|\cdot\|_X)$ satisfies $\mathfrak{s}_p(X)<\infty$, then X is superreflexive.*

Proof A theorem of Pisier [Pis73] asserts that a Banach space X is K-convex if and only if X has nontrivial Rademacher type. Therefore, we deduce from Lemma 4 that if $\mathfrak{s}_p(X)<\infty$ for some $p\in(1,\infty)$, then there exist $s\in(1,2]$ and $T_s(X)\in(0,\infty)$ such that

https://doi.org/10.4153/S0008439520000442 Published online by Cambridge University Press

$$\forall\, x_1,\ldots,x_n\in X,\quad \Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n \delta_i x_i\Big\|_X^s\Big)^{1/s}\leqslant T_s(X)\Big(\sum_{i=1}^n\|x_i\|_X^s\Big)^{1/s}. \tag{42}$$

Therefore, if X also satisfies the UMD$^-$ property, we deduce that for every X-valued martingale $\{\mathcal{M}_i:\Omega\to X\}_{i=0}^n$,

$$\begin{aligned}\|\mathcal{M}_n-\mathcal{M}_0\|_{L_s(\Omega,\mu;X)} &\leqslant \beta_s^-(X)\Big(\frac{1}{2^n}\sum_{\delta\in\mathcal{C}_n}\Big\|\sum_{i=1}^n\delta_i(\mathcal{M}_i-\mathcal{M}_{i-1})\Big\|_{L_s(\Omega,\mu;X)}^s\Big)^{1/s}\\ &\overset{(42)}{\leqslant} \beta_s^-(X)T_s(X)\Big(\sum_{i=1}^n\|\mathcal{M}_i-\mathcal{M}_{i-1}\|_{L_s(\Omega,\mu;X)}^s\Big)^{1/s},\end{aligned} \tag{43}$$

which means that X has martingale type s. Combining this with well-known results linking martingale type and superreflexivity (see [Pis16, Chapters 10-11]), we reach the desired conclusion. ■

Therefore, Theorem 1 establishes that $\mathfrak{P}_p^n(X)\asymp_p 1$ for X in a (strict, see [Gar90, Qiu12]) subclass of all superreflexive spaces. According to a result of the author and A. Naor (see [Esk19, Chapter 4]), the bound $\mathfrak{P}_p^n(X)=o(\log n)$ holds for every superreflexive Banach space X and $p\in(1,\infty)$.

Remark added in proofs. After the submission of this paper, Ivanisvili, van Handel and Volberg circulated a preprint [IvHV20] showing that a Banach space satisfies $\sup_n\mathfrak{P}_p^n(X)<\infty$ for every (equivalently, for some) $p\in[1,\infty)$ if and only if X has finite cotype.

Acknowledgment I would like to thank Assaf Naor for helpful discussions.

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