Euro. Jnl of Applied Mathematics (1997), *vol.* 8, *pp.* 581–594. Printed in the United Kingdom

Travelling waves for a nonlocal double-obstacle problem

PAUL C. FIFE

Mathematics Department, University of Utah, Salt Lake City, Utah 84112, USA

(*Received 7 May 1997; in revised form 9 July 1997*)

Existence, uniqueness and regularity properties are established for monotone travelling waves of a convolution double-obstacle problem

$$u_t = J * u - u - f(u),$$

the solution $u(x,t)$ being restricted to taking values in the interval $[-1,1]$. When $u = \pm 1$, the equation becomes an inequality. Here the kernel J of the convolution is nonnegative with unit integral and f satisfies $f(-1) > 0 > f(1)$. This is an extension of the theory in Bates *et al.* (1997), which deals with this same equation, without the constraint, when f is bistable. Among many other things, it is found that the travelling wave profile $u(x-ct)$ is always ± 1 for sufficiently large positive or negative values of its argument, and a necessary and sufficient condition is given for it to be piecewise constant, jumping from -1 to 1 at a single point.

1 Introduction

The bistable nonlinear diffusion equation

$$\frac{\partial}{\partial t} u = \Delta u - f(u) \tag{1.1}$$

for a function $u(x,t)$ has been the subject of an extensive literature. The normal assumption is that f is smooth and there exist exactly two stable constant solutions, say at $u = \pm 1$. In the special case that $\int_{-1}^{1} f(u)du = 0$, it was studied in the context of phase-antiphase material boundaries (Cahn & Allen, 1977; Allen & Cahn, 1979) and bears the name of Allen and Cahn, a special case of the Ginzburg–Landau equation.

The generalization of (1.1) to a parabolic variational inequality of double-obstacle type with $f(u) = -\gamma u$ for $u \in (-1,1)$ has also been studied (Chen & Elliott, 1994; Blowey & Elliott, 1993). In this, the solution u is required to take values only in the interval $[-1,1]$, and when $u = \pm 1$, the equation becomes an inequality so that effectively $f(1)$ and $f(-1)$ are set valued. The sets are of the form $[f_1, \infty)$ and $(-\infty, f_{-1}]$ respectively, where $f_1 < 0 < f_{-1}$ are the limits, as $u \uparrow 1$ and $u \downarrow -1$, respectively, of the function $f(u)$. (We shall formulate the problem slightly differently below, reserving the notation f for a function which is smooth and single-valued on the closed interval $[-1,1]$.)

Such a generalization and its analogue for the Cahn–Hilliard equation (Blowey & Elliott, 1991, 1992) has been motivated by both computational and physical considerations.

In another direction, nonlocal versions of (1.1) have been studied as well, from several points of view. Let $J(s)$ be a smooth even nonnegative function with $\int_{-\infty}^{\infty} J(s)ds = 1$. We also assume throughout that J' and $sJ(s)$ are in $L_1(-\infty, \infty)$. Let $J * u$ denote the spatial convolution of u with J. Then the equation

$$\frac{\partial}{\partial t}u = J * u - u - f(u) \tag{1.2}$$

shares important properties with (1.1), including being the L_2-gradient flow of a natural free energy functional (Bates *et al.*, 1997; Fife & Wang, in press; Orlandi & Triolo, 1997). The equation

$$\frac{\partial}{\partial t}u = \tanh\{\beta J * u - h\} - u, \tag{1.3}$$

for a similar kernel J (de Masi *et al.*, 1993; Orlandi & Triolo, 1997) also has many of the same properties; it arises as a continuum limit of Ising models.

Theories of travelling waves were given for (1.2) in the case of smooth bistable f in Bates *et al.* (1997) and for (1.3) in de Masi *et al.* (1995) and Orlandi & Triolo (1997). In Bates *et al.* (1997), smooth functions f were considered under the hypothesis that $f(u) = 0$ only at $u = \pm 1$ and $u = \alpha$, where $-1 < \alpha < 1$, and that $f'(\pm 1) > 0$. It was shown that there exists a unique (up to shifts in the independent variable z) travelling wave solution of equation (1.2) satisfying equation (1.8). Regularity and stability results were also obtained. Comments on the existence and uniqueness proof in that paper will be given at the beginning of §3.

The purpose of this paper is to extend the existence, uniqueness, and regularity parts of the travelling wave theory in Bates *et al.* (1997) to the double-obstacle analog of equation (1.2). We allow f to be a general function subject only to the restrictions given below. The extension entails a number of nontrivial considerations.

Our interest will be in monotone travelling wave double-obstacle solutions connecting the state -1 at $-\infty$ to the state $+1$ at ∞, when f satisfies the following hypothesis.

Hypothesis on f: $f \in C^1[-1, 1]$,

$$f(1) < 0 < f(-1), \tag{1.4}$$

f has only a single zero in $(-1, 1)$, and the set of values u at which $f'(u) \leqslant -1$ is empty or a single interval with positive length.

Setting $z = x - ct$, where c is an unknown velocity and abusing slightly the functional notation, we see that the travelling waves we seek are monotone functions $u(z)$ satisfying

$$cu'(z) + J * u(z) - u(z) - f(u(z)) = 0 \quad \text{where} \quad u(z) \in (-1, 1), \tag{1.5}$$

$$J * u(z) + 1 - f(-1) \leqslant 0 \quad \text{where} \quad u(z) = -1, \tag{1.6}$$

$$J * u(z) - 1 - f(1) \geqslant 0 \quad \text{where} \quad u(z) = 1, \tag{1.7}$$

$$u(-\infty) = -1, \quad u(\infty) = 1. \tag{1.8}$$

In fact, the monotonicity implies that $u = 1$ (or $u = -1$) either never or on an infinite interval, so that the derivative term is missing in (1.6) and (1.7).

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

Definition 1 (u, c) *is a double-obstacle travelling wave if* (1.5)–(1.8) *are satisfied for all but a discrete set of values of* z, *and* $u(z)$ *is continuous if* $c \neq 0$.

The requirement that u be continuous when $c \neq 0$ is natural because of the first term in (1.5). However, since that equation only holds if $|u| < 1$, the question arises about the possibility of a solution which jumps from -1 to 1, say at $z = 0$. We are excluding that possibility unless $c = 0$, suggesting that $cu'(z)$ would have a delta function singularity. It is known that the solution of the analogous problem in Bates *et al.* (1997) may be discontinuous when $c = 0$, and that is true here as well.

We prove the existence, uniqueness, and various properties of monotone double obstacle travelling waves under the above hypotheses. For example in all cases, we find that the profile u is identically 1 for large enough z, and identically -1 for large negative z. The solution assumes a particularly simple form, with $u \equiv -1$ for $z < z_0$, $u \equiv 1$ for $z > z_0$, if and only if $f(-1) \geqslant 1$ and $f(1) \leqslant -1$.

In the case $f(u) = -\gamma u$, which corresponds to the nonlinearity considered in Blowey & Elliott, 1991, 1992, 1993) and Chen & Elliott (1994), our results imply that necessarily $c = 0$, and that there exists a unique profile for each choice of $\gamma > 0$. If $0 < \gamma < 1$, the profile is continuous, but if $\gamma \geqslant 1$, it is discontinuous and piecewise constant, as described in the previous paragraph.

The regularity properties of the solutions are similar to their properties in the standard case (when f is smooth), as discovered in Bates *et al.* (1997), with the exception that $u'(z)$ will generally be discontinuous at one or two points (where $u = 1$ or $u = -1$ is first attained). If $c \neq 0$ this can happen at only a single point.

Certain *a priori* properties, such as regularity, can be deduced directly; they are given in §2. Existence and uniqueness are then proved in the succeeding two sections.

2 General properties

Let

$$g(u) = u + f(u). \tag{2.1}$$

Lemma 1 *If* $u(z)$ *is a monotone double–obstacle travelling wave profile and* u *is discontinuous at a point* $\bar{z}$, *then*

$$g(u(\bar{z} - 0)) \geqslant g(u(\bar{z} + 0)), \tag{2.2}$$

with equality holding if $u(\bar{z} \pm 0)$ *both lie in* $(-1, 1)$.

Proof By Definition 1, $c = 0$. If $u(\bar{z} \pm 0) \in (-1, 1)$, (1.5) holds on both sides of $\bar{z}$, and the result holds by observing that the convolution term is continuous. The other cases are handled in a similar way with the use of (1.5)–(1.7). □

Theorem 1 *Let* (u, c) *be a monotone solution of* (1.5)–(1.8). *Then the following properties hold:*

(a) *There are finite numbers* $z_0 \leqslant z_1$ *such that* $u(z) \equiv 1$ *for* $z > z_1$ *and* $u(z) \equiv -1$ *for* $z < z_0$.

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

(b) *If* $c > 0$, *then* $z_0 < z_1$ *and* u *is* C^1 *on the interval* $[z_0, \infty)$.

(c) *If* $c < 0$, *then* $z_0 < z_1$ *and* u *is* C^1 *on the interval* $(-\infty, z_1]$.

(d) *If* $c = 0$ *and* $g'(u) > 0$, *then* $z_0 < z_1$, $u \in C^0(-\infty, \infty)$ *and* u *is* C^1 *on* (z_0, z_1).

(e) *If* $g'(u) > 0$ *and* $\int_{-1}^{1} f(u)du = 0$, *then* $c = 0$.

Proof of (a) Suppose the contrary, specifically that $u(z) < 1$ for all large positive z. Equation (1.5) is satisfied for all large enough z. We may pass to the limit $z \to \infty$ in it, using $u' \to 0$, $J * u(z) - u(z) \to 0$, to obtain

$$f(1) = 0,$$

which contradicts (1.4). The same argument applies for large negative z.

Proof of (b) and (c) Let $c > 0$. By our requirement on the solution in Definition 1, u is continuous, hence $z_0 < z_1$. When $u \in (-1, 1)$, we have $-cu' = J * u - g(u)$; hence u is C^1 on the interval (z_0, z_1), and it is identically 1 on (z_1, ∞). It only remains to show that u' is continuous at $z = z_1$, i.e. to show that

$$u'(z_1 - 0) = 0. \tag{2.3}$$

From (1.5), (1.7) and the continuity of $J * u$, we have

$$-cu'(z_1 - 0) = J * u(z_1) - g(1) \geqslant 0. \tag{2.4}$$

Since $u' \geqslant 0$ and $c > 0$, we have

$$0 \leqslant -cu'(z_1 - 0) \leqslant 0,$$

hence (2.3). The proof of (c) is the same.

Proof of (d) Lemma 1 shows that u must be continuous; in particular, $z_0 < z_1$. Differentiate (1.5) to obtain $g'(u)u' = J' * u$ whenever $-1 < u < 1$, i.e. for $z \in (z_0, z_1)$, and observe that the convolution is continuous.

Proof of (e) By Lemma 1 again, we know that u is continuous. Multiply (1.5) by u' and integrate from z_0 to z_1. We have

$$c \int_{z_0}^{z_1} (u')^2 dz + \int_{z_0}^{z_1} u'(z)(J * u(z) - u(z))dz - \int_{-1}^{1} f(u)du = 0.$$

The integration in the second term can be extended to the entire line $(-\infty, \infty)$ since $u'(z) = 0$ outside (z_0, z_1). Let I be that integral. Integrating by parts, using the L_2 scalar product, and noting that $\langle u', u \rangle = 0$, we may express it as

$$I = -\langle u, (J * u' - u') \rangle = -\langle u, J * u' \rangle = -\langle u', J * u \rangle = -I,$$

so that $I = 0$. Therefore $c = 0$.

To proceed further, we need the concept of a null truncation of g; the analogous concept appears in Bates *et al.* (1997).

Definition 2 *The function* $\hat{g}(u)$ *is a null truncation of the function* g *if it is a continuous, nondecreasing function on* $[-1, 1]$ *satisfying* $\hat{g}(-1) \leqslant g(-1)$, $\hat{g}(1) \geqslant g(1)$, $\hat{g}(u) = g(u)$ *for all* u *such that* $\hat{g}'(u) > 0$, *and* $\int_{-1}^{1} \hat{g}(u)du = 0$.

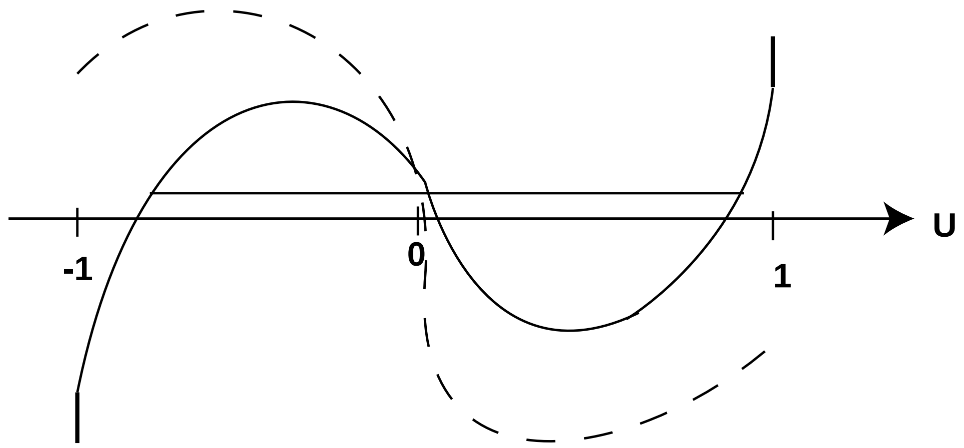

FIGURE 1. Illustration of the null truncation $\hat{g}$ of the function g. Two vertical segments have been adjoined to the graph.

This construction is illustrated in Fig. 1. The graph of the truncation $\hat{g}$ is the same as that of g except for the horizontal line segment joining the two ascending branches of g. In some cases, this segment may extend all the way to $u = -1$ and/or $u = 1$. In the former case, for instance, it would be true that $\hat{g}(-1) \leqslant g(-1)$. The level of the horizontal segment is chosen so that the integral condition holds. Note that a null truncation, if it exists, is unique, because changing the level of the horizonal segment will always cause the integral condition to be violated. Also note that it may happen that $g = \hat{g}$; this occurs when g is monotone and itself satisfies the integral condition.

Recall by the Hypothesis on f that either $g'(u) > 0$ for all $u \in (-1, 1)$ or $g'(u) \leqslant 0$ on a single interval of positive length. Theorem 1(d) established regularity in the first case; we now turn to the second one.

Theorem 2 *Let u be a monotone solution of* (1.5)–(1.8) *with* $c = 0$. *Then g has a null truncation $\hat{g}$. If $\hat{g}'(u) = 0$ on a maximal open interval (a^*, b^*) with $b^* > a^*$, then u has a single discontinuity. It is a jump discontinuity between the values a^* and b^*. Except at that discontinuity, u is continuously differentiable wherever $|u| < 1$. The jump is from -1 to 1 if and only if*

$$g(-1) \geqslant 0 \geqslant g(1). \tag{2.5}$$

Conversely if equation (2.5) *holds, then the piecewise constant function with jump from* -1 *to* 1 *is a solution with* $c = 0$.

Remark Regarding the classical case treated in Bates *et al.* (1997), it was shown that the solution profile is smooth when $c \neq 0$, and when $c = 0$ it may have a jump discontinuity under conditions similar to those outlined in Theorem 2.

Proof of Theorem 2 Let us extend the graph of the function $g(u)$ by adjoining to it the semiinfinite vertical segment which has, as lower endpoint, $(1, g(1))$. Similarly adjoin

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

the one which has upper endpoint $(-1, g(-1))$. We shall speak of these vertical segments as being parts of the right and left ascending branches of the graph of g. These segments are shown in Fig. 1.

First, we claim that when $c = 0$, u is discontinuous if and only if $g'(u) < 0$ on some interval, that it can have no more than one discontinuity, and that any discontinuity is a jump represented by a horizontal transition on the graph of the function g between two ascending branches. In fact, suppose first that u is discontinuous at some point z^*, and that its limits at that point from the left and right are both in $(-1, 1)$. Since $J * u$ is continuous and *strictly* increasing in z, it follows from (1.5) with $c = 0$ that $g(u(z))$ is also continuous at z^*. Therefore the curve $(u(z), g(u(z)))$, as z crosses z_0, undergoes a horizontal jump from one ascending branch of the graph of g to the other (there are only two, by the hypothesis on f). It follows from Lemma 1 that the same is true if u jumps from -1 and/or to $+1$. For such a jump to be possible, $g'(u) \leqslant 0$ on some interval. Since there are at most two ascending branches, there can be at most one discontinuity.

Next, suppose that $g'(u) \leqslant 0$ on an interval of positive length. Then u must be discontinuous; otherwise $g(u(z))$ would be nonincreasing somewhere. This proves the claim.

We now connect the solution with a null truncation. If it has a discontinuity, let it be at $z = 0$, and let the limiting values of u be denoted by $u(0^-)$ and $u(0^+)$. (If u is continuous, these two values are the same.) Assume for the moment that $-1 < u(0^-)$, $u(0^+) < 1$. Then of course $z_0 < z_1$. For $z \in (z_0, z_1)$, we have $J * u = g(u)$. Multiplying by u', we have that the resulting equation is valid for all z. Thus since $\int u'(z)g(u(z))dz = \int_{-1}^{u(0^-)} g(u)du + \int_{u(0^+)}^{1} g(u)du$, we have that

$$\int_{-\infty}^{0} u'J * udz + \int_{0}^{\infty} u'J * udz = \int_{-1}^{u(0^-)} g(u)du + \int_{u(0^+)}^{1} g(u)du.$$

We integrate by parts (as in Bates *et al.*, 1997, Theorem 3.1)), and let $\hat{g}$ be the truncation defined as above relative to $u(0^-)$ and $u(0^+)$ (note that $\hat{g}(u(0^-)) = \hat{g}(u(0^+))$) to obtain

$$\int_{-1}^{1} \hat{g}(u)du = 0.$$

Therefore $\hat{g}$ is a null truncation. By uniqueness of the latter, we have $u(0^-) = a^*$ and $u(0^+) = b^*$. This argument is also valid if u is continuous.

Examining this proof, we see now that in fact it is valid also if $u(0^-) = -1$ or $u(0^+) = 1$. The remaining case to consider is when $u(0^-) = -1$ and $u(0^+) = 1$. By Lemma 1, equation (2.5) holds, and therefore g has the following null truncation:

$$\hat{g}(u) \equiv 0.$$

Conversely, if equation (2.5) holds, the piecewise constant function is verified directly to be a solution.

3 Existence

The existence proof in the case of smooth f in Bates *et al.* (1997) was obtained by embedding the travelling wave problem in a family of problems

$$-cu' = \theta(J * u - u) + (1-\theta)u'' - f(u).$$

In the case $\theta = 0$, existence is a classical result obtained by a phase plane argument. A continuation argument was used to pass from $\theta = 0$ to $\theta = \theta_0$, for any $\theta_0 < 1$, incrementally in small θ-steps. At each step, the existence of a solution $(u(\theta), c(\theta))$ satisfying equation (1.8) for nearby values of θ was obtained by applying the implicit function theorem. At the same time, various estimates were used to control the properties of the solutions obtained. For example, it was shown that the velocities $c(\theta)$ obtained were uniformly bounded.

Finally, a solution for $\theta = 1$ was obtained as the weak limit of solutions for $\theta < 1$. If $c \neq 0$, the solution was shown to be smooth; otherwise it may have jump discontinuities, which can be ascertained precisely. In the context of Bates *et al.* (1997), there is a definition of null truncation which is analogous to our Definition 2. The solutions considered there have $c = 0$ if and only if g has a null truncation. The proof is based largely on arguments similar to those in the proof of Theorem 2 above.

The uniqueness was established by the rather involved construction of upper and lower solutions.

We use the existence result in Bates *et al.* (1997) to establish the existence of a solution of our double-obstacle problem (1.5)–(1.8).

Theorem 3 *There exists a monotone solution* (u, c) *of* (1.5)–(1.8).

Proof For each small enough $\epsilon > 0$ let $f_\epsilon(u)$ be a smooth function of u defined for $u \in (-2, 2)$ with the properties that $f_\epsilon(-1-\epsilon) = f_\epsilon(1+\epsilon) = 0$, $f_\epsilon(u)$ is an increasing function of u on $(-1-\epsilon, -1)$ and on $(1, 1+\epsilon)$, and is basically like $f(u)$ for $u \in [-1, 1]$. Namely,

$$f_\epsilon(u) \text{ continuous on [-1,1], uniformly in } \epsilon, \tag{3.1}$$

$$\lim_{\epsilon\to 0} f_\epsilon(u) = f(u) \tag{3.2}$$

uniformly for $u \in [-1, 1]$, and (as in the Hyp. on f) the set of values of u at which $f'_\epsilon(u) \leqslant -1$ is either empty or a single interval of positive length.

If $f'(-1+0) \geqslant 0$ and $f'(1-0) \geqslant 0$, for example, we may take $f_\epsilon(u) = f(u)$ for $u \in [-1, 1]$; and in any case to differ from f only in a vanishingly small neighbourhood of the endpoints. □

It follows from results in Bates *et al.* (1997) that for each function in this family, there exists a unique (up to translation) monotone travelling wave pair $(u_\epsilon(z), c_\epsilon)$. It satisfies

$$c_\epsilon u'_\epsilon + J * u_\epsilon - u_\epsilon - f_\epsilon(u_\epsilon) = 0, \quad u_\epsilon(\pm\infty) = \pm 1 \pm \epsilon. \tag{3.3}$$

We shall normalize these functions by translating them in a manner to be explained later.

We show that the velocities c_ϵ are bounded independently of ϵ. If they were not,

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

there would exist a sequence $\epsilon_k \to 0$ such that $c_{\epsilon_k} \to \infty$ or $-\infty$. Suppose it is the latter; the argument in the former case is the same. From equation (3.3) and the fact that the u_ϵ are uniformly bounded, it follows that $c_{\epsilon_k} u'_{\epsilon_k}$ are also; hence

$$\|u'_{\epsilon_k}\|_\infty \to 0 \qquad \text{as } k \to \infty.$$

One then sees, since $\int J(y)|y|dy < \infty$, that

$$J * u_{\epsilon_k} - u_{\epsilon_k} \to 0$$

uniformly as $k \to \infty$. Let z_k and $\delta > 0$ be such that for all k,

$$f_{\epsilon_k}(u_{\epsilon_k}(z_k)) > \delta > 0;$$

such points exist by equations (1.4) and (3.2). We then obtain from equation (3.3) again that $c_{\epsilon_k} u'_{\epsilon_k} > \delta/2$ for large enough k, hence by the monotonicity of u_ϵ, $c_{\epsilon_k} > 0$, contradicting its approach to $-\infty$.

Since the u_ϵ are monotone and the c_ϵ bounded, there exists, by Helly's theorem, a subsequence $\{\epsilon_k\}$ such that the limits

$$u(z) = \lim_{k\to\infty} u_{\epsilon_k}(z), \quad c = \lim_{k\to\infty} c_{\epsilon_k} \tag{3.4}$$

exist at every value of z. The limit function u is also monotone, with range contained in $[-1, 1]$. We shall show that (u, c) is our desired solution.

First, consider the case $c \neq 0$, so that for sufficiently small ϵ, all the c_ϵ are bounded away from zero and (by Bates *et al.*, 1997) the u_ϵ are smooth. Then (3.3) provides a uniform bound on u'_ϵ and in fact with (3.1) shows the $u'_\epsilon(z)$ to be equicontinuous on each closed interval of the z-axis on which $u(z) \in (-1, 1)$. Therefore, on a further subsequence of the original one, $u'_\epsilon(z) \to u'(z)$ uniformly on each such interval.

Let $\hat{z}$ be a fixed number such that $|u(\hat{z})| < 1$. For small enough ϵ, $|u_\epsilon(\hat{z})| < 1$ as well, and therefore $\lim_{k\to\infty} f_{\epsilon_k}(u_{\epsilon_k}(\hat{z})) = f(u(\hat{z}))$ by (3.2). We also have that $J * u_{\epsilon_k}(\hat{z}) \to J * u(\hat{z})$ by the dominated convergence theorem. We may therefore pass to the limit in equation (3.3) to obtain that u satisfies equation (1.5).

The reason this argument does not work for $u(\hat{z}) = \pm 1$ is that we cannot guarantee that $|u_\epsilon(\hat{z})| \leqslant 1$, where the convergence of f_ϵ to f holds. A separate argument must be given in that case, to show that equations (1.6) and (1.7) hold. Our argument will include the case $c = 0$. Suppose for the moment that $u(z) = 1$ for all z in an open interval I containing the point $\hat{z}$. Either (a) there exists a subsequence along which $u_\epsilon(\hat{z}) > 1$ or (b) there exists one such that $u_\epsilon(\hat{z}) \leqslant 1$.

Consider case (b) first. The argument used above in the case $-1 < u(\hat{z}) < 1$ may be repeated here; we conclude that $u'_\epsilon(\hat{z}) \to 0$, so that equation (1.7) follows.

In case (a),

$$\liminf f_\epsilon(u_\epsilon(\hat{z})) \geqslant f(1), \tag{3.5}$$

by the fact that f_ϵ is increasing for $u > 1$. We shall show that along a subsequence

$$c_\epsilon u'_\epsilon(\hat{z}) \to 0. \tag{3.6}$$

We may assume either that all $c_\epsilon \geqslant 0$ or all are $\leqslant 0$. For definiteness, take the former case. Recall that $c_\epsilon u_\epsilon(z) \to c = \text{const.}$ on I. Since $c_\epsilon u_\epsilon(z)$ is a monotone function of z,

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

the convergence is uniform on I. If it were not true that $c_\epsilon u_\epsilon'(\hat{z})\to 0$, then it may be assumed that this sequence is positive and bounded away from 0. There would then be another subsequence of the ϵ's and a corresponding sequence of points $\{z_\epsilon^1 > \hat{z}\}$ such that $c_\epsilon u_\epsilon''(z_\epsilon^1)\to -\infty$. For suppose that $c_\epsilon u_\epsilon' > \delta > 0$, and also for some $m > 0$ that $c_\epsilon u_\epsilon''(z) \geqslant -m$ for all $z \in I$. Then

$$c_\epsilon u_\epsilon(\hat{z} + \delta/m) \geqslant c_\epsilon u_\epsilon(\hat{z}) + \delta^2/2m,$$

contradicting the convergence of $c_\epsilon u_\epsilon(z)$ to a constant. Similarly, there would be a second sequence $\{z_\epsilon^2 < \hat{z}\}$ with (along the same ϵ-subsequence) $c_\epsilon u_\epsilon''(z_\epsilon^2)\to +\infty$. However, by differentiating equation (3.3) we know that

$$(c_\epsilon u_\epsilon)'' + J' * u_\epsilon = u_\epsilon' + f_\epsilon'(u_\epsilon)u_\epsilon' \geqslant 0, \tag{3.7}$$

by the monotonicity of u_ϵ and the construction of f_ϵ. Since the convolution term is bounded, we see that the second derivative term could not approach $+\infty$ along one sequence of z's and $-\infty$ along another. Therefore, equation (3.6) holds. We may therefore pass to the limit along a subsequence in equation (3.3), using equation (3.5) to obtain equation (1.7).

We have assumed that $\hat{z}$ is contained in an open interval where $u = 1$; but noting that $J * u$ is continuous, we may take a limit in equation (1.7) to show that equation (1.7) holds for all z with $u(z) = 1$. Equation (1.6) is proved the same way. Note that this argument also works in the case $c = 0$; we shall use that result below.

At this point, we have proved that equation (1.5) holds if $c \neq 0$, and equations (1.6) and (1.7) hold in any case.

We now show that equation (1.5) holds in the case $c = 0$, so that $c_\epsilon\to 0$. Either there exists a further subsequence along which $c_\epsilon \geqslant 0$, or there exists one along which $c_\epsilon \leqslant 0$. We assume the former; the proof in the other case is similar.

When $c_\epsilon \neq 0$, we know that $p_\epsilon = \max_z u_\epsilon'(z)$ exists and is finite. There are three cases to consider, namely that along some subsequence

(i) $c_\epsilon = 0$,

(ii) $c_\epsilon > 0$ and $c_\epsilon p_\epsilon \to 0$, or

(iii) $c_\epsilon > 0$ and $c_\epsilon p_\epsilon$ are bounded away from zero.

In cases (i) and (ii), we may pass to the limit in equation (3.3) to obtain equation (1.5) with $c = 0$ pointwise, as desired.

Consider now case (iii).

Lemma 2 *Assume that $c = 0$ and case* (iii) *holds. Normalize the functions $u_\epsilon(z)$ by translating the independent variable so that $u_\epsilon'(0) = p_\epsilon$. Then (after a possible redefinition at $z = 0$) u satisfies equation* (1.5)*, where $|u(z)| < 1$. If u is discontinuous, it is so only at $z = 0$.*

Proof By assumption there is a number $\delta > 0$ such that

$$p_\epsilon > \delta/c_\epsilon. \tag{3.8}$$

There exists a number $m > 0$, independent of ϵ, and points $z_\pm^*$ depending on ϵ, such

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

that

$$u'_\epsilon(z^*_\pm) < p_\epsilon/2, \quad -m/p_\epsilon < z^*_- < 0 < z^*_+ < m/p_\epsilon.$$

The reason is that otherwise the function $u_\epsilon(z)$ would attain values outside the interval $[-1, 1]$.

Since $u'_\epsilon(0) = p_\epsilon$, it follows by the mean value theorem that by possibly increasing m, we may guarantee that there exist numbers $z_\pm$ (depending on ϵ) with $-m/p_\epsilon < z_- < 0 < z_+ < m/p_\epsilon$ such that

$$u''_\epsilon(z_-) > \frac{p_\epsilon^2}{m}; \quad u''_\epsilon(z_+) < -\frac{p_\epsilon^2}{m}. \tag{3.9}$$

From equations (3.8) and (3.9), we have that at $z = z_+$,

$$g'_\epsilon(u_\epsilon(z_+))u'_\epsilon(z_+) = c_\epsilon u''_\epsilon(z_+) + J' * u_\epsilon < -\frac{c_\epsilon p_\epsilon^2}{m} + a < -\frac{\delta p_\epsilon}{m} + a,$$

where a is an upper bound on $|J' * u_\epsilon(z)|$ independent of ϵ and z. Since $p_\epsilon \to \infty$, we have

$$g'_\epsilon(u_\epsilon(z_+)) < 0 \tag{3.10}$$

for ϵ small enough. Similarly,

$$g'_\epsilon(u_\epsilon(z_-)) > 0. \tag{3.11}$$

Thus $u_\epsilon(z_-)$ is on an ascending branch of g_ϵ, and $u_\epsilon(z_+)$ is on a descending one.

Let S be the set of points z $(\neq 0)$ such that there exists a subsequence of the original sequence ϵ_k on which $u'_\epsilon(z)$ is bounded. Then S is dense; otherwise $u'_{\epsilon_k}(z) \to \infty$ for all z in an interval of positive length. Let $0 < z_1 < z_2$ be two elements in S, and $I = [z_1, z_2]$.

Still remaining on a subsequence where $u'_\epsilon(z_1)$ and $u'_\epsilon(z_2)$ are both bounded, we show that $c_\epsilon u'_\epsilon \to 0$ uniformly on I. Suppose this is not the case; then $\bar{p}_\epsilon = \max_{z \in I} u'_\epsilon(z) \to \infty$. Define $z_\epsilon \in I$ by $\bar{p}_\epsilon = u'_\epsilon(z_\epsilon)$. Then by choice of z_1, z_2 we have that the z_ϵ lie in the interior of I. We redefine the functions u_ϵ by translating the independent variable so that all the z_ϵ coincide, say $z_\epsilon = z_0$ for some $z_0 > 0$. The above argument can now be repeated to show that there are points $z'_\pm$ approaching z_0 as $\epsilon \to 0$ (on our subsequence) such that the values $u_\epsilon(z'_-)$ and $u_\epsilon(z'_+)$ lie on an ascending and a descending branch of g_ϵ, respectively. This transition between branches would happen at larger values of u than the one found before, since u is an increasing function. But such a pair of transitions cannot occur because g_ϵ has at most one descending branch, by its construction.

This contradiction shows that $c_\epsilon u'_\epsilon(z) \to 0$ for each $z \in I$, and in fact by easy extension, for all $z \neq 0$. It follows that the limit $\epsilon \to 0$ (along a subsequence) can be taken in equation (3.3) to obtain equation (1.5) with $c = 0$ for every $z \neq 0$ with $u(z) \in (-1, 1)$. (The continuity of $J * u$ can be used to show that $u(0)$ may be redefined so that equation (1.5) holds there as well.)

Essentially the same argument also holds when $u(0^-) = -1$ and/or $u(0^+) = 1$; we shall not supply the details. This completes the proof of the lemma. □

We have already shown above that equations (1.6) and (1.7) hold. This completes the existence part of the proof of the theorem, except for equation (1.8), which we now address. If u is piecewise constant, jumping from -1 to 1, then equation (1.8) is automatic, so we shall assume that u is otherwise.

We know that $\lim_{z\to\pm\infty} u(z)$ must be ± 1 or the zero of f, which is unique according to the Hypothesis on f. Call it $\alpha \in (-1, 1)$.

Our claim now is that we may, possibly through proper normalization of the u_ϵ, guarantee that $u(z)$ assumes a value either in $(-1, \alpha)$ or in $(\alpha, 1)$. First, suppose that $c = 0$ and Case (iii) holds. Then $c_\epsilon u_\epsilon'(0) \geqslant \delta > 0$. If it were true that $u(z) = \text{const}$, then we could repeat the argument following equation (3.7) to show that $c_\epsilon u_\epsilon''$ would approach ∞ and $-\infty$ on two sequences of points near 0, and obtain a contradiction as before. Therefore in this case, u assumes a value in one of the two intervals indicated.

In all other cases, we have not yet specified the normalization of the u_ϵ by translation, but shall now. Suppose that $c \geqslant 0$. Take $\bar{\alpha} \in (\alpha, 1)$. We specify the normalization so that either $u_\epsilon(0) = \bar{\alpha}$ or, if u_ϵ does not take on that value because of a discontinuity, then the discontinuuity occurs at $z = 0$. This does not affect the c_ϵ. The limiting function u therefore takes on the value $\bar{\alpha}$ or has a discontinuity straddling $\bar{\alpha}$. In the latter case if the limit from the left is $< \alpha$, (1.8) must follow and we are through. It is shown below that u cannot be identically α on an interval, so the only other case is that the limit from the left surpasses α, so that again u takes on a value between α and 1. This proves the claim; moreover if $c > 0$, we have arranged that u assumes a value between α and 1 (if $c < 0$, it is between -1 and α).

If the value assumed is $> \alpha$, we have that $\lim_{z\to\infty} u(z) = 1$, since the limit must be greater than α. If we can show that $\lim_{z\to-\infty} u(z) \neq \alpha$, that limit must be -1, and equation (1.8) will follow. Our proof of this fact follows a similar argument as in Bates *et al.* (1997, theorem 2.7).

Note that it cannot happen that $u(z) \equiv \alpha$ for large negative z, because $J * u - u > 0$ there, and equation (1.5) would be violated. Therefore, if $\lim_{z\to-\infty} u(z) = \alpha$, it must be true that $u(z) > \alpha$, so that $f(u(z)) < 0$ for all z. Therefore from (1.5), $\int_{-R}^{R} (J * u - u) dz < -k < 0$ for all large enough R. We write that inequality as

$$-k > \int_{-R}^{R} (J * u - u) dz = \int_{-\infty}^{\infty} J(y) \int_{-R}^{R} (u(z + y) - u(z)) dz dy.$$

Note the identity

$$\int_{A}^{B} (u(z + y) - u(z)) dz = y \int_{0}^{1} [u(B + ty) - u(A + ty)] dt,$$

which holds for all piecewise continuous functions u. Therefore by the dominated convergence theorem

$$-k > \int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} [u(R + ty) - u(-R + ty)] dt dy$$

$$\to (1 - \alpha) \int_{-\infty}^{\infty} yJ(y) dy = 0$$

as $R \to \infty$, which is a contradiction.

The other cases are handled in the same way. This proves equation (1.8).

We may now complete the characterization of the case when $c = 0$.

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

Theorem 4 *Let (u, c) be a double-obstacle solution. Then $c = 0$ if and only if g has a null truncation.*

Proof If $c = 0$, we know from Theorem 2 that g has a null truncation. Now assume g has a null truncation. The construction of the family f_ϵ may be arranged so that all the $g_\epsilon(u) = f_\epsilon(u) + u$ also have a null truncation. As shown in Bates *et al.* (1997), this implies the corresponding velocities $c_\epsilon = 0$ for all small ϵ. Therefore there exists a solution with $c = 0$. □

The proof of Theorem 5 (uniqueness) in the next section does not rely on the present theorem. It shows that this constructed solution is the same as that given in the theorem.

4 Uniqueness

Theorem 5 *The travelling wave double-obstacle solution constructed in Thm. 3 is unique among all such monotone solutions.*

Proof Let (u_1, c_1) and (u_2, c_2) be two solutions, with $c_1 \geqslant c_2$.

For large enough α, we have $u_1(z) \geqslant u_2(z - \alpha)$ for all z, since they are both identically ± 1 for large enough $|z|$. Let α^* be the least value of α for which this is true. This means that for all $\alpha < \alpha^*$, the function $w(z;\alpha) = u_1(z) - u_2(z - \alpha)$ will have a negative value at some number $z(\alpha)$ for which we have an a priori bound. There is a subsequence of the α's along which $\lim_{\alpha \to \alpha^*} z(\alpha) = z^*$ exists (z^* might not be unique). Then $w(z;\alpha^*) \geqslant 0$ for all z and this is not true for any smaller value of α.

For now, assume u_1 and u_2 are both smooth when $|u| \neq 1$, and neither is the piecewise constant solution. There are three cases to consider; they are not necessarily mutually exclusive, since z^* is not necessarily unique:

(i) $u_1(z^*) = 1$,

(ii) $u_1(z^*) = -1$, and

(iii) $u_1(z^*) \in (-1, 1)$.

In case (i), we claim that z^* is the least value of z at which $u_1(z) = 1$, and the same for $u_2(z - \alpha^*)$. For if it were the least for u_1 but not for u_2, it would not be true that $w(z;\alpha^*) \geqslant 0$ for all z, contrary to our construction. If it were the least for u_2 but not for u_1, we could decrease α somewhat and still have $w(z;\alpha) \geqslant 0$ for all z in some neighbourhood of z^*, contrary to the definitions of α^* and z^*. Finally, if it were the least for neither u_1 nor u_2, then it could not be the limit of $z(\alpha)$.

Therefore in case (i), when $z < z^*$ and z is close enough to z^*, we have $u_1(z)$ and $u_2(z - \alpha^*) \in (-1, 1)$, so that both functions satisfy (1.5). Moreover, we have that

$$w'(z^* - 0, \alpha^*) \leqslant 0, \tag{4.1}$$

for otherwise it would not be true that $w(z, \alpha^*) \geqslant 0$ for nearby values of z.

Analogous results hold for case (ii) (replace $z^* - 0$ by $z^* + 0$; $w' \geqslant 0$) and for case (iii) (use z^* itself; $w' = 0$).

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

Thus in all cases, equation (1.5) holds both for $u = u_1$ and $u(z) = u_2(z - \alpha^*)$ at $z = z^* \pm 0$ or z^*, according to the case.

We subtract that equation for $u = u_2$ from the equation for $u = u_1$ to obtain

$$c_2 w' + J * w + (c_1 - c_2)u_1' = 0 \tag{4.2}$$

and

$$c_1 w' + J * w + (c_1 - c_2)u_2' = 0, \tag{4.3}$$

these equations holding at the value of z appropriate to the case, as indicated above.

In all cases, the last two terms of equations (4.2) and (4.3) are nonnegative, since $w \geqslant 0$ and $u_i' \geqslant 0$. If we can show that the first term of one or the other of these two equations is also nonnegative, it will follow that all are zero. In particular, it will follow that $J * w = 0$, which since $w \geqslant 0$ is only possible if $w \equiv 0$. This means that $u_1(z) = u_2(z - \alpha^*)$ for all z, i.e. the two functions are the same up to translation. Then of course $c_1 = c_2$, and uniqueness will be established.

It therefore suffices to show that $c_2 w' \geqslant 0$ or $c_1 w' \geqslant 0$. We consider the three cases in turn:

Case (i)
Since $w' \leqslant 0$ by equation (4.1), it suffices to show $w' = 0$ or $c_2 \leqslant 0$. But it was proved in Theorem 1(b), equation (2.3), that $u_1' = u_2' = w' = 0$ if $c_1 \geqslant c_2 > 0$.

Case (ii)
We need $w' = 0$ or $c_1 \geqslant 0$. But by Theorem 1(c), we know $w' = 0$ if $c_1 < 0$.

Case (iii)
$w' = 0$.

This establishes uniqueness under the assumption that u_1 and u_2 are smooth when not equal to ± 1. We now assume the contrary, that one or both of the u_i are discontinuous. We outline the changes in the previous proof that are needed. First, a slight change in the definitions of cases (i) and (ii) is needed. If u_1 is discontinuous at z^*, case (i) is defined to be when $u_1(z^* + 0) = 1$. Similarly, the limit from the left is taken in case (ii).

Case (i): the previous proof rested on the fact that equation (1.5) holds for both functions $u_1(z)$ and $u_2(z - \alpha^*)$ at $z = z^* - 0$, and that $g(u(z))$ is continuous at $z = z^*$. This is still true if these functions are continuous at $z = z^*$. More generally, it remains true under the weaker assumption that each function assumes values in $(-1, 1)$ for $z \in (z^* - \delta, z^*)$ for some $\delta > 0$. To see this, recall that each function is either continuous or has zero velocity with $g(u(z))$ continuous.

Still assuming case (i), we see that the only other subcase to check is that when one or both of the functions jump between -1 and $+1$. If they both do, then clearly they are the same functions. If only one does, it has to be the lesser one, namely $u_2(z - \alpha^*)$. In that case, $g(-1) \geqslant 0 \geqslant g(1)$ from equation (2.5). From equation (1.5), we have

$$c_1 u_1'(z^* - 0) = g(1) - J * u_1(z^*),$$

which is strictly negative since $J * u_1(z^*) > 0$. This implies that $c_1 < 0$, which contradicts the fact that $c_1 \geqslant c_2 = 0$. This completes the revision of the proof in Case (i).

https://doi.org/10.1017/S0956792597003318 Published online by Cambridge University Press

Similar comments hold in Case (ii).

Suppose, finally, that Case (iii) holds, and that Cases (i) and (ii) do not. Then each of the functions $u_1(z)$ and $u_2(z-\alpha^*)$ takes on values in $(-1,1)$ either for $z \in (z^*-\delta, z^*)$ or for $z \in (z^*, z^*+\delta)$, for $\delta > 0$ small enough. Therefore equations (4.2) and (4.3) still hold if we interpret $w = u_1(z^* \pm 0) - u_2(z^* - \alpha^* \pm 0)$ and $w' = u_1'(z^* \pm 0) - u_2'(z^* - \alpha^* \pm 0)$ with the proper combination of signs (not necessarily both the same). Since either $c_1 = 0$ or $c_2 = 0$, we may choose one of these equations to yield $J * w = 0$, hence uniqueness. □

5 Conclusion

The problem considered in this paper is a generalization, both to nonlocal behaviour and to a variational inequality, of the classical bistable nonlinear diffusion equation, which occurs, among other places, in modelling state transitions in a solid material. There exists a complete theory of travelling waves in the classical case, and we here provide one for the existence, uniqueness and regularity of such waves for the generalization. An investigation of stability is left for future work.

References

[1] Allen, S. M. & Cahn, J. W. (1979) A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.* **27**, 1085–1095.

[2] Bates, P. W., Fife, P. C., Ren, X. & Wang, X. (1997) Travelling waves in a convolution model for phase transitions. *Arch. Rat. Mech. Anal.* **138**, 105–136.

[3] Blowey, J. F. & Elliott, C. M. (1991) The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy. Part I: Mathematical analysis. *Euro. J. Appl. Math.* **2**, 233–279.

[4] Blowey, J. F. & Elliott, C. M. (1992) The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy. Part II: Numerical analysis. *Euro. J. Appl. Math.* **3**, 147–149.

[5] Blowey, J. F. & Elliott, C. M. (1993) Curvature dependent phase boundary motion and parabolic double–obstacle problems. In: Wei-Ming Ni, Peletier, L. A. & Vazquez, J. L. (eds.), *Degenerate Diffusions*, pp. 19–60. IMA vol. 47. Springer-Verlag.

[6] Cahn, J. W. & Allen, S. M. (1977) A microscopic theory for domain wall motion and its experimental verification in Fe-Al alloy domain growth kinetics. *J. de Physique 38 Colloque* **C7**, 51–54.

[7] Chen, X. & Elliott, C. M. (1994) Asymptotics for a parabolic double-obstacle problem. *Proc. Roy. Soc. London Ser. A* **444**, 429–445.

[8] Fife, P. C. & Wang, X. (in press) A convolution model for interfacial motion: the generation and propagation of internal layers in higher space dimensions. *Advances in Diff. Equations.*

[9] Fife, P. C. (1996) Clines and material interfaces with nonlocal interaction. In: Angell, T. S., Pamela Cook, L., Kleinman, R. E & Olmstead, W. E. (eds.), *Nonlinear Problems in Applied Mathematics*, pp. 134–149. SIAM.

[10] de Masi, A., Orlandi, E., Presutti, E. & Triolo, L. (1993) Motion by curvature by scaling nonlocal evolution equations. *J. Stat. Physics* **73**, 543–570.

[11] de Masi, A., Gobron, T. & Presutti, E. (1995) Travelling fronts in nonlocal evolution equations. *Arch. Rat. Mech. Anal.* **132**, 143–205.

[12] Orlandi, E. & Triolo, L. (1997) Travelling fronts in nonlocal models for phase separation in external field. *Proc. Roy. Soc. Edinburgh.* **127A**, 823–835.