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Dirichlet's Theorem, Vojta's Inequality, and Vojta's Conjecture

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Abstract. This paper addresses questions involving the sharpness of Vojta's conjecture and Vojta's inequality for algebraic points on curves over number fields. It is shown that one may choose the approximation term $m_S(D, -)$ in such a way that Vojta's inequality is sharp in Theorem 2.3. Partial results are obtained for the more difficult problem of showing that Vojta's conjecture is sharp when the approximation term is not included (that is, when $D = 0$). In Theorem 3.7, it is demonstrated that Vojta's conjecture is the best possible with $D = 0$ for quadratic points on hyperelliptic curves. It is also shown, in Theorem 4.8, that Vojta's conjecture is sharp with $D = 0$ on a curve C over a number field when an analogous statement holds for the curve obtained by extending the base field of C to a certain function field.

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In 1955, Roth ([Ro]), building upon the work of Thue, Siegel, and Dyson, proved that for any algebraic number α and any $c > 0$ and $\epsilon > 0$, there exist only finitely many rational numbers x/y $(x, y \in \mathbb{Z})$ with

$$\left|\frac{x}{y} - \alpha\right| \leqslant \frac{c}{|y|^{2+\epsilon}}.$$

The theorem of Roth is sharp, in the sense that for any irrational algebraic number α, there are infinitely many rational numbers x/y $(x, y \in \mathbb{Z})$ such that

$$\left|\frac{x}{y} - \alpha\right| \leqslant \frac{1}{|y|^{2}}.$$

This had been shown by Dirichlet much earlier in 1842 ([Dir]).

Recently, Vojta ([V 4]) combined the technique of Roth–Thue–Siegel–Dyson with ideas from Arakelov theory to prove a vast generalization of Roth's theorem, one which encompasses Faltings' theorem for curves. To state it, we will need a bit more terminology. Let C be a curve defined over a number field k and let X be a regular model for C over the ring of integers of k. Let K be the canonical divisor of C. Suppose that S is a finite set of places of k, D is a divisor without multiple points on C, A is an ample $\mathbb{Q}$-divisor on C, ν is a positive integer, and ϵ a positive real number. Vojta ([V 4, Thm. 0.1]) shows that for all $P \in C(\bar{k})$ with $[k(P) : k] \leqslant \nu$ the following inequality holds

$$m_S(D, P) + h_K(P) \leqslant d_a(P) + \epsilon h_A(P) + \mathrm{O}(1). \tag{0.0.1}$$

Here, $d_a(P)$ is the arithmetic discriminant of P (see [V 4], p. 764]), h_A and h_K are Weil heights for A and K, respectively, and $m_S(D, P)$ is the sum

$$\sum_{v \in S} \sum_{\alpha=1}^{[k(P):k]} \frac{\lambda_{D,v}(P^{[\alpha,v]})}{[k(P) : k]}, \tag{0.0.2}$$

where λ_D is a Weil function for D and $P^{[\alpha,v]}$, $\alpha = 1, \ldots, [k(P) : k]$, are the conjugates of P in $C(\mathbb{C}_v)$. We will refer to (0.0.1) as *Vojta's inequality* throughout this paper. The A is frequently omitted; it is assumed then that A is $K_C/(2g-2)$ (unless $g = 1$ in which case it is just some fixed k-rational $\mathbb{Q}$-divisor of degree 1). Note that the choice of ample A is inconsequential: since degrees determine divisors on curves up to algebraic equivalence, changing A will only change the ϵ in (0.0.1) by the addition of an arbitrarily small factor and by multiplication by some fixed nonzero factor (see [V 4, p. 769]).

The arithmetic discriminant $d_a(P)$ is related to the normalized logarithmic field discriminant $d(P)$, which is defined as

$$d(P) := \frac{\|D_{k(P)/k}\|}{[k(P) : k]}. \tag{0.0.3}$$

Now, $d_a(P)$ is always at least as large as $d(P)$; it can be calculated by adding to $d(P)$ terms corresponding to arithmetic singularities on the arithmetic curve corresponding to P and additional data at infinite places of k. Vojta ([V 1, Conjecture 5.2.6]) conjectures that inequality (0.0.1) remains true when $d_a(P)$ is replaced by $d(P)$, that is to say, that

$$m_S(D, P) + h_K(P) \leqslant d(P) + \epsilon h_A(P) + \mathrm{O}(1) \tag{0.0.4}$$

holds, where the notation is the same as in (0.0.1). We will refer to (0.0.4) as *Vojta's conjecture*. This conjecture is quite strong. In particular, the *abc* conjecture of Masser and Oesterlé (see [V 1, 5.ABC] and [L 2, appendix]) is a corollary.

Lang conjectures that Vojta's conjecture is best possible for any curve of nonzero genus over a number field ([L 3, 2.4, p.63], see also [So]). There are a variety of ways of interpreting what it means for Vojta's conjecture to be best possible; for example, one might ask if it is true for all choices of D or only for certain choices of D. One might also ask whether or not Vojta's inequality is sharp. It is convenient then to define the following 'sharpness statements'.

In the following definitions, C is a curve defined over a number field k, D is an effective divisor without multiple components on C, and ν is an element of $\mathbb{Z}^+ \cup \{\infty\}$. The heights h_K, h_A, proximity function $m_S(D, -)$, field discriminant d, and arithmetic discriminant d_a are defined as in (0.0.1) and (0.0.4).

DEFINITION 0.1. We will say that statement $B(C, d, \nu, D)$ holds if there exists a finite extension k' of k and a finite set S of places of k' such that for any $\epsilon > 0$, there exists an infinite sequence of $P \in C(\bar{k})$ with $[k'(P) : k'] \leqslant \nu$ and $h(P) \to \infty$ satisfying

$$m_S(D, P) + h_K(P) \geqslant d(P) - \epsilon h_A(P) + \mathrm{O}(1). \tag{0.1.1}$$

Observe that statement $B(C, d, \nu, 0)$ holds if there is some k' for which there are infinitely many points P with $[k'(P) : k']$ satisfying.

$$h_K(P) \geqslant d(P) - \epsilon h_A(P) + \mathrm{O}(1). \tag{0.1.2}$$

holds. Lang conjectures that for any curve C of nonzero genus, statement B $(C, d, \infty, 0)$ holds. Note also that the condition $h(P) \to \infty$ is automatically met when ν is finite by Northcott's theorem, which asserts that there are only finitely many points of bounded degree and bounded height with respect to some fixed ample divisor.

DEFINITION 0.2. We will say that statement $B(C, d_a, \nu, D)$ holds if there exists a finite extension k' of k and a finite set S of places of k' such that for any $\epsilon > 0$, there exists an infinite sequence of $P \in C(\bar{k})$ with $[k'(P) : k'] \leqslant \nu$ and $h(P) \to \infty$ satisfying

$$m_S(D, P) + h_K(P) > d_a(P) - \epsilon h_A(P) + \mathrm{O}(1). \tag{0.2.1}$$

In this note, we show that for any curve C of nonzero genus there exists some choice of D and ν for which $B(C, d_a, \nu, D)$ holds. The proof is a simple application of Dirichlet's theorem and the basic properties of Weil functions. We also examine the problem of whether or not $B(C, d, \infty, 0)$ holds for all curves C over number fields. We describe how a certain geometric construction produces families of points with small discriminants. This construction is used to show that $B(C, d, 2, 0)$ holds for hyperelliptic curves and to relate the problem of points with small discriminants over number fields to a similar problem over function fields. The fact that Vojta has proved his conjecture, in a slightly weakened form, for curves over function fields of characteristic 0 in [V 3] makes the connection with the problem over function fields especially interesting.

Determining if $B(C, d, \nu, 0)$ holds whenever for any curve C of nonzero genus possessing infinitely many $P \in C(\bar{k})$ with $[k(P) : k] \leqslant \nu$ is beyond our grasp at this point. It is very hard to say much about which curves will contain infinitely many points of degree ν or less, for a given ν. In the case of points of degree 2 and 3 Abramovich and Harris ([A-H]) have used Faltings' theorem on subvarieties of Abelian varieties ([Fa]) to obtain results that are sufficient for our purposes. For points of higher degree, however, no such results hold: Debarre and Fahlaoui ([D-F]) have found counterexamples to the conjectures of Abramovich and Harris concerning algebraic points of degree 4 and greater.

1. Preliminaries and Statements of Results

We need to introduce a bit of Arakelov theory in order to define the arithmetic discriminant. Let C be a curve over a number field k. After taking a finite base extension of k, the curve C has a regular model over the ring of integers R of k ([Ar]). At the infinite places σ of k (which correspond to embeddings $\sigma\colon k \hookrightarrow \mathbb{C}$), we may endow $C \times_\sigma \mathbb{C}$ with an admissible Arakelov volume form (see [L 2, 4.3]). This allows us to find local and global arithmetic intersections of arithmetic divisors on X as in [V4].

An algebraic point $P \in C(\bar{k})$ gives rise to a horizontal divisor H_P on X (by taking the closure of the support of P in C). We define the *canonical* height h_K of a point $P \in C(\bar{k})$ to be

$$h_K(P) := \frac{(H_P.\omega_{X/B})}{[k(P):\mathbb{Q}]},$$

where $\omega_{X/B}$ is the canonical sheaf for X over $B = \operatorname{Spec} R$, metrized with the canonical Arakelov metric (see [L 2, 4.3 and 4.5]). Note that h_K is a Weil height for the canonical sheaf K of C. We define the arithmetic discriminant (in analogy with the arithmetic genus of a curve on a geometric surface) $d_a(P)$ to be

$$d_a(P) := \frac{(H_P.\omega_{X/B} + H_P)}{[k(P):\mathbb{Q}]}.$$

Let us also fix a horizontal $\mathbb{Q}$-divisor on F which has degree 1 on C, so that we will have a fixed height function on C. When $g \neq 1$, let $F := \omega_{X/B}/(2g-2)$. When $g = 1$, let F be the horizontal divisor coming from some k-rational divisor of degree 1 on C. We define

$$h(P) := h_F(P) := \frac{(H_P.F)}{[k(P):\mathbb{Q}]}.$$

Henceforth, we will always drop the A in the $h_A(P)$ in (0.0.1) and (0.0.4) and use $h(P)$. As noted earlier, this affects neither our sharpness statements nor the statements of Vojta's inequality and Vojta's conjecture.

Let us now say a few words about local arithmetic intersections and Weil divisors. As with algebraic points, any k-rational divisor D on C gives rise to a

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horizontal divisor H_D on X, and the local intersections with H_D, that is the functions $(-.H_D)_v$, give rise to a Weil function for D. Remember that a Weil function is actually a mapping

$$\lambda_D : \coprod_{v \in M_k} C(\mathbb{C}_v) \setminus \operatorname{Supp} D \to \mathbb{R}. \tag{1.0.1}$$

The intersections $(H_P.H_D)_v$ for points P in $C(\bar{k})$ extend to functions $g_v(Q, D)$ for points $Q \in C(\mathbb{C}_v)$; these are given by base extension of intersections in the case of finite places and are already given to us at infinity by the Green functions corresponding to our Arakelov volume form (see [V 4, p. 767]). The functions $g_v(-, D)$ form the desired Weil function for D. Vojta's inequality is stated in [V 4] using the Weil functions that arise in this way. That is to say, he defines $m_S(D, P)$ as

$$m_S(D, P) := \sum_{v \in S} \frac{(H_P.H_D)_v}{[k(P) : k]} = \sum_{v \in S} \sum_{\alpha=1}^{[k(P):k]} \frac{g_v(P^{[\alpha,v]}, D)}{[k(P) : k]}, \tag{1.0.2}$$

where $P^{[\alpha,v]}$, $\alpha = 1, \ldots, [k(P) : k]$, are the conjugates of P in $C(\mathbb{C}_v)$. We use the same definition. We note, however, that our choice of Weil functions in $m_S(D, -)$ is of no consequence. Any two Weil functions for the same divisor will differ by an M_k-constant ([L 1, Chapter 10, p. 248]); this means that they will agree at all but finitely many places and differ by O(1) at the remaining places. Thus, choice of Weil function will not affect any of our results, since all of them include an O(1) term.

We make one final note on Weil functions. For a point $P \in C(\bar{k})$, $\lambda_{E,v}(P)$ will depend on which conjugate of P in $C(\mathbb{C}_v)$ we choose. Hence, we will only speak of $\lambda_{E,v}(P^{[\alpha,v]})$, $1 \leqslant \alpha \leqslant [k(P) : k]$, where $P^{[\alpha,v]}$ is a fixed conjugate of P in $C(\mathbb{C}_v)$.

We prove the following theorem in Section 2.

THEOREM 2.3. *Let C be a curve defined over a number field k. Then there exist choices of D and ν such that statement $B(C, d_a, \nu, D)$ holds.*

This is actually stated somewhat more precisely, so that we can obtain some results that apply to any curve with infinitely many points of degrees 3 or less.

We also have the following theorem, proved in Section 3.

THEOREM 3.7. *Let C be a hyperelliptic curve defined over a number field k. Statement $B(C, d, 2, 0)$ holds.*

When discussing Vojta's conjecture with $D = 0$, we will work with curves over function fields of transcendence degree 1 as well as curves over number fields. The field discriminant is defined for function fields in the same way that it is defined

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for number fields. Moreover, we may define canonical heights and arithmetic discriminants for points on curves over function fields of transcendence degree 1. Recall that a function field k of transcendence degree 1 corresponds to a unique nonsingular, projective curve B over the field of constants of k ([Ha, 1.6]). A curve C over k then corresponds to a surface S over the field of constants of k with a natural map $\pi : S \to B$. A point in $C(\bar{k})$ corresponds to a curve in S on which π is nonconstant. We may use the relative canonical sheaf $\omega_{S/B}$ for S over B and the usual intersection pairing on surfaces ([Ha, 5.1]) to give ourselves canonical heights and arithmetic discriminants of points. Given a point $P \in C(\bar{k})$ we take the corresponding curve H_P on S and define

$$h_K(P) := \frac{H_P..\omega_{S/B}}{[k(P):k]}.$$

We also define the arithmetic discriminant of P as

$$d_a(P) := \frac{H_P.(\omega_{S/B} + H_P)}{[k(P):k]}.$$

It is also possible to define $h(P)$ as in the case of curves over number fields. We take F to be $\omega_{S/B}/(2g-2)$ when $g \neq 1$ and to be the divisor on S corresponding to some fixed k-rational $\mathbb{Q}$-divisor on C when $g = 1$. Then we define

$$h(P) := \frac{H_P.F}{[k(P):k]}.$$

With the definitions above, it is clear that it makes sense to define statements $B(C, d, \nu, 0)$ and $B(C, d_a, \nu, 0)$ for curves over function fields of transcendence degree 1 (actually, one can define Weil functions so that $B(C, d, \nu, D)$ and $B(C, d_a, \nu, D)$ make sense for $D \neq 0$ as well, but we will not consider these statements in this paper).

Now, let C be a curve of nonzero genus defined over a number field k. Let E be an elliptic curve defined over k. Let us denote as $\hat{C}$ the curve $C \times_k k(E)$ defined over the function field of E. In section 4, we obtain the result below.

THEOREM 4.8. *If statement $B(\hat{C}, d, \nu, 0)$ holds, then $B(C, d, \nu, 0)$ holds as well.*

2. The Case $D \neq 0$

Using the notation of our statement of Vojta's inequality (0.0.1), we will show that for suitable D, S, k', and ν we have for any $\epsilon > 0$ infinitely many P with $[k'(P) : k'] \leqslant \nu$ and

$$m_S(D, P) \geqslant (2\nu - \epsilon)h(P) + \mathrm{O}(1). \tag{2.0.1}$$

There is an obvious upper bound on d_a coming from a decomposition of H_P in Pic X (see [L 2, Thm. 4.2.3] and [V 4, p. 791]). This decomposition gives us

$$\nu D_P^2 = d_a(P) - h_K(P) - 2\nu h(P) + \mathrm{O}(\nu) \tag{2.0.2}$$

for D_P is an element of the Arakelov–Picard group of X corresponding to a divisor of degree 0 on C. Now, $D_P^2 \leqslant 0$ because of the positivity of the Neron–Tate height on the Jacobian of C ([L 1, Chapter 5]) and the fact that D_P^2 is negative two times the Neron–Tate of the corresponding point in the Jacobian of C ([Ch, Thm. 5.1(ii)]). Thus, we have

$$d_a(P) \leqslant h_K(P) + 2\nu h(P) + \mathrm{O}(\nu). \tag{2.0.3}$$

Since Vojta's inequality can be rewritten as

$$m_S(D, P) \leqslant d_a(P) - h_K(P) + \epsilon h(P) + \mathrm{O}(1), \tag{2.0.4}$$

the D and P satisfying (2.0.1) will satisfy statement $B(C, d_a, \nu, D)$.

We begin with a 'geometric' formulation of Dirichlet's theorem for $\mathbb{P}^1$.

THEOREM 2.1. [Dirichlet] *Let D be an effective divisor of degree at least two in* $\mathrm{Div}_{\bar{k}}(\mathbb{P}^1_k)$. *Then there exists a finite extension k' of k such that for any Weil function λ_D for D with respect to $\mathbb{P}^1_{k'}$ there exist infinitely many $P \in \mathbb{P}^1(k')$ outside of* Supp D *such that*

$$\sum_{v \in S_{k',\infty}} \lambda_{D,v}(P) \geqslant 2h(P) + \mathrm{O}(1). \tag{2.1.1}$$

Proof. Case I. The divisor D has more than one point in its support. Extend the base k to a field k' so that all the points in Supp D are rationally defined and so that k' contains a real quadratic field. This allows us to find an $\alpha \in k'$ such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$ and that for any $\sigma : k' \hookrightarrow \mathbb{C}$, $\sigma(\alpha) \in \mathbb{R}$. Let α' denote the conjugate of α over $\mathbb{Q}$ in k'. Now, writing $D = Q_1 + Q_2 + D'$ with D' an effective divisor and Q_1 and Q_2 distinct rational points, we have by basic properties of Weil functions $\lambda_{D,v} \geqslant \lambda_{Q_1,v} + \lambda_{Q_2,v} + \mathrm{O}(1)$. It will suffice then to show that (2.1.1) holds for some infinite sequence of $P \in \mathbb{P}^1(k')$ when $D = Q_1 + Q_2$. There exists a choice of coordinates on $\mathbb{P}^1_{k'}$ so that Q_1 is written as $(\alpha : 1)$ and Q_2 is written as $(\alpha' : 1)$ (since there exists an automorphism of $\mathbb{P}^1_{k'}$ sending any two distinct points to any other two distinct points). Hence, we obtain a Weil function for Q_1 away from the point at infinity by taking for points P written as $(x : y)$ $(y \neq 0)$ with respect to our choice of coordinates

$$\lambda_{Q_1,v}(P) := -\log\min(1, \|\alpha - x/y\|_v). \tag{2.1.2}$$

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Similarly for Q_2, we obtain a Weil function by setting

$$\lambda_{Q_2,v}(P) := -\log\min(1, \|\alpha' - x/y\|_v). \tag{2.1.3}$$

Note that $\lambda_{Q_1,v}$ and $\lambda_{Q_2,v}$ are always nonnegative.

Now, by the classical Dirichlet box principle ([Sch, Thm. 1A], [Dir]), it follows that for any $\sigma\colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{R}$ there exist infinitely many pairs of relatively prime integers a, b such that

$$\left|\sigma(\alpha) - \frac{a}{b}\right| \leqslant \frac{1}{|b|^2}, \tag{2.1.4}$$

where the absolute value sign denotes the usual distance in $\mathbb{R}$. Now, for any other embedding $\tau\colon\ \mathbb{Q}(\alpha) \hookrightarrow \mathbb{R}$ we have either $\tau(\alpha) = \sigma(\alpha)$ or $\tau(\alpha') = \sigma\alpha$. It follows then that at each Archimedean place v corresponding to $\tau :\ k' \hookrightarrow \mathbb{C}$ and point $(a : b)$ as above

$$\begin{aligned}\lambda_{[Q_1+Q_2],v}((a:b)) &= \lambda_{Q_1,v}((a:b)) + \lambda_{Q_1,v}((a:b))\\ &\geqslant \frac{[k'_v:\mathbb{R}]}{[k':\mathbb{Q}]} 2\log b.\end{aligned} \tag{2.1.5}$$

The fact that Weil functions are defined for points in extensions of $\mathbb{Q}$ with respect to normalized valuations (see [V 1, 1.1–1.3] and [L 1, 1.4 and 10.1–10.2]) accounts for the normalization factor preceding $2\log b$. Summing over all $v \in S_{k',\infty}$, then gives for all $(a : b)$ as above

$$\sum_{v\in S_{k',\infty}} \lambda_{[Q_1+Q_2],v}((a:b)) \geqslant 2\log b. \tag{2.1.6}$$

Now since we have infinitely many $(a : b)$ we may assume that $|\ b\ | \geqslant 2$ and that $|a/b| \leqslant \max(|\sigma\alpha|, |\sigma\alpha'|) + 1$, we see that for all the $((a : b))$, we have $h((a:b)) \leqslant \log b + \mathrm{O}(1)$. This then gives us the desired

$$\sum_{v\in S_{k',\infty}} \lambda_{[Q_1+Q_2],v}((a:b)) \geqslant 2h((a:b)) + \mathrm{O}(1).$$

Case II. The divisor D has only one point in its support. Write $D = nQ$. Extend the base to a k' for which Q is rationally defined. Choose our homogeneous coordinates so that Q is written as $(0 : 1)$. Then, $\lambda_{Q,v}(P) = \log^+ \|\frac{y}{x}\|_v$ where $(x : y)$ are the homogeneous coordinates of $P \in \mathbb{P}^1_{k'}(\bar{k}_v)$ gives us a Weil function for Q. Let P be the set of all points with coordinates $(n : 1)$. Evidently, $\lambda_{Q,v}(P) = 0$ for any nonarchimedean $v \in M_{k'}$, so $\sum_{v\in S_{k',\infty}} \lambda_{Q,v}(P) \geqslant h(P) + \mathrm{O}(1)$. It follows that

$$\sum_{v\in S_{k',\infty}} \lambda_{nQ,v}(P) \geqslant nh(P) + \mathrm{O}(1) \geqslant 2h(P) + \mathrm{O}(1). \qquad \square$$

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PROPOSITION 2.2. *Let D' be a divisor on a curve C such that there exists a nonconstant morphism $\phi\colon C \to \mathbb{P}^1_k$ of degree ν for which $D' = \phi^* D$ for D an effective divisor of degree at least two in $Div_{\bar{k}}(\mathbb{P}^1_k)$. Then there exists a finite extension k' of k such that for any Weil function $\lambda_{D'}$ for D' and any $\epsilon > 0$, there exist infinitely many $P' \in C(\bar{k}')$ with $[k'(P) : k'] \leqslant \nu$ such that*

$$\sum_{v\in S_{k',\infty}} \sum_{\alpha=1}^{[k(P'):k]} \lambda_{D',v}(P'^{[\alpha,v]}) \geqslant [k'(P') : k'](2\nu - \epsilon)h(P') + \mathrm{O}(1), \tag{2.2.1}$$

where $P'^{[\alpha,v]}$, $\alpha = 1, \ldots, [k'(P) : k']$ are the conjugates of P' in $\mathbb{C}_v$.

Proof. By Theorem 2.1, it follows that after extension to a suitable base field k' there exist infinitely many $P \in \mathbb{P}^1_{k'}(k')$ such that

$$\sum_{v\in S_{k',\infty}} \lambda_{D,v}(P) \geqslant 2h(P) + \mathrm{O}(1). \tag{2.2.2}$$

Now, λ_D pulls back to a Weil function on C. Specifically, by base extension, for any $v \in M_k$, ϕ gives us a map $\phi_{\mathbb{C}_v}\colon C(\mathbb{C}_v) \to \mathbb{P}^1(\mathbb{C}_v)$ and this allows us to define a function on $C(\mathbb{C}_v)$ by sending $Q \in (C(\mathbb{C}_v) \setminus \phi^{-1}$ (Supp D)) to $(D, \phi_{\mathbb{C}_v}(Q))_v$; the resulting map on $\coprod_{v\in M_k}(C(\mathbb{C}_v) \setminus (\phi^{-1}(\text{Supp } D))$ is a Weil function for $\phi^* D$. We will call this Weil function $\lambda_{\phi^* D}$. Let P' be a point in $C(\bar{k})$ such that $\phi(P') = P$. Note that in fact $\lambda_{\phi^* D,v}(P'^{[\alpha,v]})$ doesn't depend on the choice of conjugate $P'^{[\alpha,v]}$ since in any case it is equal to $\lambda_{D,v}(P)$. It follows that

$$\begin{aligned}\sum_{v\in S_{k',\infty}} \sum_{\alpha=1}^{[k(P'):k]} \lambda_{\phi^* D,v}(P'^{[\alpha,v]}) &= \sum_{v\in S_{k',\infty}} \sum_{\alpha=1}^{[k(P'):k]} \lambda_{D,v}(P) \\ &= [k(P') : k] \sum_{v\in S_{k',\infty}} \lambda_{D,v}(P) \\ &\geqslant [k(P') : k]2h(P) + \mathrm{O}(1). \end{aligned} \tag{2.2.3}$$

Similarly, we convert $h(P)$ to something depending on P' by functoriality of height functions. Since ϕ is a map of degree ν we have $\phi^* F = \nu A'$ for A' a divisor of degree 1. Hence $h(P) = h_{\phi^* F}(P') = \nu h_{A'}(P')$. Since A' and F (on C here) are algebraically equivalent, (2.2.3) gives

$$\sum_{v\in S_{k',\infty}} \sum_{\alpha=1}^{[k(P'):k]} \lambda_{D',v}(P'^{[\alpha,v]}) \geqslant [k'(P') : k'](2\nu - \epsilon)h(P') + \mathrm{O}(1). \tag{2.2.4}$$

□

THEOREM 2.3. *Let C be a curve defined over a number field k and let $\phi\colon C \to \mathbb{P}^1$ be a nonconstant morphism of degree ν. Then there exists a choice of D such that $B(C, d_a, \nu, D)$ holds.*

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Proof. We want to show that there is a finite extension k' of k for which there exist a k'-rational divisor D without multiple components and a finite set of places S of k' such that for any $\epsilon > 0$ there are infinitely many P with $[k'(P) : k'] \leqslant \nu$ satisfying (2.0.1). Choose E consisting of two or more distinct points in $\mathbb{P}^1(k)$ away from the ramification locus of ϕ; of course, this can be done because $\mathbb{P}^1(k)$ is infinite and the ramification locus of ϕ consists of finitely many points. Let $D = \phi^* E$. We see that D is a k-rational divisor without multiple points. Now, by Proposition 2, for any $\epsilon > 0$ and any choice of Weil function λ_D, there are infinitely many P with $[k'(P) : k'] \leqslant \nu$ such that

$$\sum_{S_{k',\infty}} \sum_{\alpha=1}^{[k'(P):k']} \lambda_{D,v}(P^{[\alpha,v]}) \geqslant [k'(P) : k'](2\nu - \epsilon)h(P) + \mathrm{O}(1). \tag{2.3.1}$$

In particular, this holds when our choice of Weil function λ_D is the one coming from the local Arakelov intersections described in Section 1, so, letting $S = S_{k',\infty}$ and recalling (1.0.2), we have $m_S(D, P) \geqslant (2\nu - \epsilon)h(P) + \mathrm{O}(1)$, which is precisely (2.0.1), and our proof is done. □

COROLLARY 2.4. *Let ν be 2 or 3. Let C be a curve of nonzero genus defined over a number field. If C has infinitely many points of degree ν or less, then there exists a choice of D such that $B(C, d_a, \nu, D)$ holds.*

Proof. Abramovich and Harris ([A-H, Thm. 1]) show that when ν is 2 or 3 a curve has infinitely many points of degree less than or equal to ν if and only if it admits a map of degree ν or less to $\mathbb{P}^1$ or an elliptic curve. Theorem 2.3 takes care of a map to $\mathbb{P}^1$ of degree ν or less. When C admits nonconstant map f of degree ν or less to an elliptic curve E, the canonical divisor K_C of C is in the linear equivalence class of the ramification divisor R_f of f, by the Riemann–Hurwitz theorem. We may extend the base field of E to a k' for which $E(k')$ is infinite. Now, $d_a(P) \leqslant \mathrm{O}(1)$ for $P \in E(k')$ (see [L 2, Cor. 4.5.6]), so by the Chevalley–Weil theorem for arithmetic discriminants ([V 2, Prop. 3.6]), for any Q with $f(Q) = P \in E(k')$, we have

$$d_a(Q) \leqslant d_a(P) + h_{R_f}(Q) + \mathrm{O}(1) \leqslant h_{K_C}(Q) + \mathrm{O}(1).$$

Since, we have infinitely many such Q and $[k'(Q) : k'] \leqslant \nu$, statement $B(C, d_a, \nu, 0)$ must hold in this case. □

3. The Case $D = 0$

We now approach the problem of producing algebraic points on curves C which demonstrate that $B(C, d, \nu, 0)$ holds for certain values of ν. This follows [So] fairly closely. The definitions we are about to make will apply to curves over function fields of transcendence degree 1 as well as curves over number fields. Let us introduce some definitions. In the definitions below, let C be a curve defined over a field K, which is either a number field or a function field of transcendence degree 1, let $\epsilon \geqslant 0$, and let $\nu \in \mathbb{Z}^+ \cup \{\infty\}$.

DEFINITION 3.1. We say that $V(C, \epsilon, d, \nu)$ holds if there exists a finite extension k' of k for which there are infinitely many $P \in C(\bar{k})$ with $[k'(P) : k'] \leqslant \nu$ such that

$$d(P) \leqslant h_{K_C}(P) + \epsilon h(P) + \mathrm{O}_\epsilon(1). \tag{3.1.1}$$

The stronger statement $V'(C, \epsilon, d, \nu)$ is said to hold if the heights of the P in (3.1.1) go to infinity.

When C is a curve over a number field and ν is finite, $V(C, \epsilon, d, \nu)$ implies $V'(C, \epsilon, d, \nu)$ by Northcott's theorem. We see that $B(C, d, \infty, 0)$ holds if $V'(C, \epsilon, d, \nu_\epsilon)$ holds for every $\epsilon > 0$ for some ν_ϵ and that $B(C, d, \nu, 0)$ holds for some finite ν if, in addition, $\nu_\epsilon \leqslant \nu$ for every ϵ.

Here are two lemmas concerning the extent to which the property $V(C, \epsilon, d, \nu)$ can be pushed forward and pulled back.

LEMMA 3.2. *Let $f : C' \to C$ be a nonconstant morphism of curves defined over a number field k. Suppose $V(C, \epsilon, d, \nu)$ holds. Then, for any $\epsilon' > 0$ $V(C', (\deg f)\epsilon + \epsilon', d, (\deg f)\nu)$ holds.*

Proof. The Chevalley–Weil theorem for fields discriminants ([V 1, Thm. 5.1.6]) states that whenever $f(P') = P$ for $P' \in C'(\bar{k})$ and $P \in C(\bar{k})$,

$$d(P') \leqslant d(P) + h_R(P') + O(1), \tag{3.2.1}$$

where R_f is the ramification divisor of f and the $\mathrm{O}(1)$ depends only on f. Let K_C and $K_{C'}$ denote the canonical divisors on C and C', respectively, and observe that $f^*K_C + R_f = K_{C'}$. Note also that for any $P' \in C(\bar{k})$, we have $h(f(P')) \leqslant (\deg f + \epsilon'')h(P') + \mathrm{O}(1)$, since a divisor of degree 1 on C pulls back to a divisor of degree $\deg f$ on C' and a divisor on a curve are defined up to algebraic equivalence by its degree. Choosing ϵ'' such that $(\deg f)\epsilon''\epsilon < \epsilon'$, applying Chevalley–Weil to the infinitely many P for which $d(P) \leqslant h_{K_C}(P) + \epsilon h(P) + \mathrm{O}(1)$, and letting P' be points for which $f(P') = P$, we obtain

$$\begin{aligned}
d(P') &\leqslant d(P) + h_{R_f}(P') + \mathrm{O}(1)\\
&\leqslant h_{K_C}(P) + \epsilon h(P) + h_{R_f}(P') + \mathrm{O}(1)\\
&\leqslant h_{f^*K_C}(P') + h_{R_f}(P') + ((\deg f)\epsilon + (\deg f)\epsilon''\epsilon)h(P') + \mathrm{O}(1)\\
&\leqslant h_{K_{C'}}(P') + ((\deg f)\epsilon + \epsilon')h(P') + \mathrm{O}(1),
\end{aligned} \tag{3.2.2}$$

using $f^*K_C + R_f = K_{C'}$ and $h(P') \leqslant (\deg f + \epsilon')h(P)$. Now, since for each P', $[k(P') : k] \leqslant (\deg f)[k(P) : k] \leqslant (\deg f)\nu$, we see that $V(C', (\deg f)\epsilon + \epsilon', d, (\deg f)\nu)$ is satisfied by the points P' on C'. $\square$

DEFINITION 3.3. Let $f: C' \to C$ be a nonconstant morphism of curves, and let R_f be the ramification divisor of f. We define

$$\delta_0(f) := \frac{\deg R_f}{\deg f}.$$

We also say that f is *δ-ramified*, for $\delta \geqslant 0$, if $\delta_0(f) \leqslant \delta$.

LEMMA 3.4. *Let $f: C' \to C$ be a nonconstant morphism of curves defined over a number field k and suppose that $V(C', \epsilon, d, \nu)$ holds. Then, for any $\epsilon' > 0$, statement $V(C, (1/\deg f)\epsilon + \delta_0(f) + \epsilon', d, \nu)$ holds as well.*

Proof. We begin with an infinite sequence of $P' \in C'(\bar{k})$ with $[k(P') : k] \leqslant \nu$ and $d(P') \leqslant h_{K_{C'}}(P') + \epsilon h(P') + \mathrm{O}(1)$. Since $k(f(P)) \subseteq k(P)$, we have $d(P) \leqslant d(P')$. Hence, we see that

$$d(P) \leqslant h_{K_{C'}}(P') + \epsilon h(P') + \mathrm{O}(1) \leqslant h_{K_C}(P) + h_{R_f}(P') + \epsilon h(P'), \qquad (3.4.1)$$

once again using the equality $f^* K_C + R_f = K_{C'}$. We may choose $\epsilon'' > 0$ small enough that $\epsilon''(1+\epsilon) < \epsilon'$, and by algebraic equivalence and the functorial property of heights, we have $h_{R_f}(P') \leqslant \delta_0(f) h(P) + \epsilon'' h(P) + \mathrm{O}(1)$ and $h(P') \leqslant 1/(\deg f) h(P) + \epsilon'' h(P) + \mathrm{O}(1)$. Substituting these inequalities into (3.4.1) then gives

$$\begin{aligned} d(P) &\leqslant h_{K_C}(P) + \delta_0(f) h(P) + \epsilon'' h(P) + \\ &\quad + \epsilon/(\deg f) h(P) + \epsilon\epsilon'' h(P) + \mathrm{O}(1) \\ &\leqslant h_{K_C}(P) + (\epsilon/(\deg f) + \delta_0(f) + \epsilon') h(P) + \mathrm{O}(1). \end{aligned} \qquad (3.4.2)$$

Naturally, $[k(P) : k] \leqslant [k(P') : k] \leqslant \nu$, and this completes the proof. □

These two lemmas combine to give us the following proposition.

PROPOSITION 3.5. *Let C be a curve with a cover $f: C' \to C$, where C' admits a nonconstant map to an elliptic curve $g: C' \to E$. Then, for any $\epsilon > 0$, $V(C, \delta_0(f) + \epsilon, d, \deg g)$ holds.*

Proof. First, we show that $V(E, 0, d, 1)$ holds. This is easy since extending k to the field of definition of some nontorsion point of $E(\bar{k})$ gives us infinitely many points in $E(k)$. Evidently, for all of these points P, $d(P) = 0 \leqslant h_{K_E}(P) + \mathrm{O}(1)$ (recall that K_E is trivial). Applying Lemma 3.2 tells us then that for any $\epsilon'' > 0$, $V(C', \epsilon'', d, \deg g)$ holds. Choosing $\epsilon', \epsilon'' > 0$ such that $\epsilon''/(\deg f) + \epsilon' < \epsilon$ and invoking Lemma 3.4 then yields $V(C, \delta_0(f) + \epsilon, d, \deg g)$. □

COROLLARY 3.6. *Let C be a curve over a number field with an étale cover $f: C' \to C$ which admits a nonconstant map to an elliptic curve, $g: C' \to E$. Then $B(C, d, \deg g, 0)$ holds.*

Proof. Since f is étale, $R_f = 0$, so $\delta_0(f) = 0$. Now apply Proposition 3.5 and Lemma 3.4. □

We can use this corollary to show that $B(C, d, 2, 0)$ holds for hyperelliptic curves C.

THEOREM 3.7. *Let C be a hyperelliptic curve defined over a number field k. Statement $B(C, d, 2, 0)$ holds.*

Proof. It will suffice to show that there exists C' which admits nonconstant maps $f\colon C' \to C$ and $g\colon C' \to E$ with E an elliptic curve and g a map of degree 2. We follow a construction of Mumford ([Mu]). Let π be a double cover of $\mathbb{P}^1$ by C, that is $\pi\colon C \xrightarrow{2:1} \mathbb{P}^1$. By the Riemann–Hurwitz theorem, π ramifies over at least four points in $\mathbb{P}^1$, all of which can be taken to be rational since we may extend the base. Let us choose exactly four of these points $Q_1, \ldots, Q_4$. By basic properties of the moduli of elliptic curves, there exists an elliptic curve E which admits a double cover of $\mathbb{P}^1$, $\phi\colon E \xrightarrow{2:1} \mathbb{P}^1$, that ramifies only at $Q_1, \ldots, Q_4$. Forming the fibre product over $\mathbb{P}^1$ of C and E with respect to π and ϕ and desingularizing yields a curve C' which admits maps $f\colon C' \xrightarrow{2:1} C$ and $g\colon C' \xrightarrow{2:1} E$ with f étale. That the degrees of f and g are two is obvious. To see that f is étale is only slightly more difficult. The maps π and ϕ give us embeddings of the function fields $k(C)$ and $k(E)$ into the algebraic closure of $k(\mathbb{P}^1)$. Their compositum is unramified over C since the only places at which it could possibly ramify are the Q_i in C since E is smooth over $\mathbb{P}^1$ away from these points. Now, C' is unramified over the points in the fibres of C over $Q_1, \ldots, Q_4$ because, whenever there points $P_i \in C(\bar{k})$ and $S_i \in E(\bar{k})$ such that $\pi(P_i) = \phi(S_i) = Q_i$, then $e(P_i/Q_i) = e(S_i/Q_i) = 2$ where $e(P_i/Q_i)$ and $e(S_i/Q_i)$ are the ramification indices of P_i and S_i over Q_i. Now, letting $k(C)_{P_i}$ and $k(E)_{S_i}$ denote the completions of $k(C)$ and $k(C')$ at P_i and S_i, respectively, we see that their compositum in $k(\mathbb{P}^1)_{Q_i}$ is unramified over $k(C)_{P_i}$ since $e(S_i/Q_i)|e(P_i/Q_i)$ by a simple result from field theory sometimes known as Abhyankar's Lemma (see, for example, [Sti, Prop. III.8.9]). Since for any $P_i' \in C'(\bar{k})$, the local field $k(C')_{P_i'}$ is the compositum of $k(C)_{P_i}$ and $k(E)_{S_i}$, it follows that P_i' is unramified over P_i. □

Remark 3.8. The same construction can be used to show that statement $B(C, d, 2, 0)$ holds for any curve C which admits a map $f\colon C \to \mathbb{P}^1$ such that there exist four points $Q_1, \ldots, Q_4$ such that for every point $P \in C(\bar{k})$ with $f(P) = Q_i$, we have $2|e(P/Q_i)$. A slight modification shows that $B(C, d, 3, 0)$ holds whenever there exists $f\colon C \to \mathbb{P}^1$ of degree 3 and three $Q_1, \ldots, Q_3$ with $3|e(P/Q_i)$ for every P such that $f(P) = Q_i$.

COROLLARY 3.9. *Statement $B(C, d, 2, 0)$ holds on any curve C of nonzero genus defined over a number field with infinitely many points of degree 2 or less.*

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Proof. The proof is similar to that of Corollary 2.4. A curve has infinitely many points of degree 2 or less if and only if it admits a map of degree 2 or less to $\mathbb{P}^1$ or an elliptic curve. Theorem 3.7 and Proposition 3.5 assert that in either case, $B(C, d, 2, 0)$ holds. $\square$

It is possibe that every curve C has an étale cover C' which admits a nonconstant map to an elliptic curve, but it appears difficult to prove this. A more approachable problem is try to show that given a curve C, there exists, for any $\delta > 0$ a curve C' which admits a nonconstant map to an elliptic curve and a δ-ramified cover (recall Definition 3.3) $f: C' \to C$. The existence of such covers for a curve C implies that statement $B(C, d, \infty, 0)$ holds (although it doesn't imply that $B(C, d, \nu, 0)$ holds for any finite ν). Here is the precise statement.

CLAIM 3.10. *Let C be a curve such that for every $\delta > 0$ there exists a curve C'_δ which admits a δ-ramified cover $f_\delta\colon\ C'_\delta \to C$ and a nonconstant map to an elliptic curve $C'_\delta \to E_\delta$. Then statement $B(C, d, \infty, 0)$ holds.*

Proof. Given ϵ, take $\delta < \epsilon$, set $\nu_\epsilon := \deg f_\delta$ and apply Proposition 3.5. Since each ν_ϵ is finite, Northcott's theorem applies and the additional restriction of $V'(C, \epsilon, d, \nu)$ is met. $\square$

One interesting feature of δ-ramified covers is that they relate in a natural way to the classical Severi problem on surfaces, specifically split elliptic surfaces.

4. δ-Ramified Covers

Let C be a curve of genus at least two defined over a number field k and let E be an elliptic curve defined over k. All curves which dominate both C and E will factor through the product surface $S := C \times_k E$. Let p_1 and p_2 be projections to C and E respectively. We will assume in this section that C itself admits no nonconstant map to E so that the Neron–Severi group of $C \times_k E$ has rank 2 by the following proposition.

PROPOSITION 4.1. *If C admits no nonconstant morphism to E, then*

(i) $\operatorname{Pic} S = p_1^*(\operatorname{Pic} C) \oplus p_2^*(\operatorname{Pic} E)$;
(ii) $\operatorname{rank}(\mathrm{NS}(S)) = \operatorname{rank}(\mathrm{Num}(S)) = 2$.

Proof. Let $(E, 0_E)$ (here 0_E is the identity element in the group structure on E) and $(J(E), 0)$ be pointed k-schemes and let $\mathcal{M}$ be the universal divisorial correspondence between these two pointed k-schemes. It is easy to see that $p_1^*(\operatorname{Pic} C) \oplus p_2^*(\operatorname{Pic} E) \subseteq \operatorname{Pic} S$. Let D_0 be an invertible sheaf in Pic S; we are going to show that $D_0 \in p_1^*(\operatorname{Pic} C) \oplus p_2^*(\operatorname{Pic} E)$. Let P be a rational point on C. We have embeddings $i: C \to S$ and $j: E \to S$ by letting i take C to $p_2^{-1}(0_E)$ and letting j take E to $p_1^{-1}(P)$. It is obvious that $D_0 - p_1^*(D_0|_{i(C)}) - p_2^*(D_0|_{j(E)})$ is a divisorial cor-

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respondence between (C, P) and $(E, 0_E)$. By the universal property of Jacobians, we have a morphism $\phi_0\colon C \to J(E)$ such that

$$(1 \times \phi_0)^*(\mathcal{M}) = D_0 - p_1^*(D_0|_{i(C)}) - p_2^*(D_0|_{j(E)}) \quad \text{and} \quad \phi_0(P) = 0.$$

By assumption $\phi_0 = 0$, and it follows that $D_0 - p_1^*(D_0|_{i(C)}) - p_2^*(D_0|_{j(E)}) = \mathcal{O}_S$ which implies $D_0 = p_1^*(D_0|_{i(C)}) + p_2^*(D_0|_{j(E)})$. The statements about NS and Num are now obvious since $\operatorname{rank}(\operatorname{Num}(S)) \geqslant 2$.

DEFINITION 4.2. Let D be a curve on an algebraic surface S. Then $\delta(D) = 2(p_a(D) - g(D))$. The quantity $\delta(D)$ measures the singularities of D (see Hartshorne [Ha, Chap. 4, Ex. 1.8]).

Let S, C, E, p_1, and p_2 be as defined at the beginning of this section and let D be an irreducible divisor on S for which $p_1|_D$ is nonconstant. By Proposition 4.1, D is numerically equivalent to $a \cdot C + b \cdot E$ for some nonnegative integers a and b. Using this notation we have the following Proposition.

PROPOSITION 4.3. *Let $q\colon \tilde{D} \to D$ be a desingularization of D and let ϕ be the composition $p_1|_D \cdot q$. Then*

$$\frac{\deg R_\phi}{\deg \phi} = 2b - \frac{\delta(D)}{a}. \tag{4.3.1}$$

Proof. We use intersection theory on elliptic surfaces. The canonical class (of differentials over k) of S is $K(S) = K(C) \times E + C \times K(E)$. By the adjunction formula for algebraic surfaces ([Ha, Chap. 4, Ex. 1.3]), we have

$$\begin{aligned} 2p_a(D) - 2 &= D^2 + D.K(S) \\ &= D^2 + 2a(g(C) - 1) + 2b(g(E) - 1) \\ &= 2ab + 2a(g(C) - 1) + 2b(g(E) - 1) \\ &= 2ab + 2a(g(C) - 1). \end{aligned} \tag{4.3.2}$$

Since $q\colon \tilde{D} \to D$ is generically injective, $\deg \phi = \deg p_1|_D = a$. Note that a is not zero, because $p_1|_D$ is nonconstant. The Riemann–Hurwitz formula tells us that

$$2g(\tilde{D}) - 2 = 2a(g(C) - 1) + \deg R_\phi. \tag{4.3.3}$$

Combining (4.3.2) and (4.3.3) then gives us

$$\frac{\deg R_\phi}{\deg \phi} = \frac{2g(\tilde{D}) - 2 - 2a(g(C) - 1)}{a}$$

$$\begin{aligned} &= \frac{2ab - (2p_a(D) - 2) + 2g(\tilde{D})}{a} - 2 \\ &= 2b - \frac{\delta(D)}{a}. \end{aligned} \tag{4.3.4}$$

□

Suppose now that we have a δ-ramified map $f: C' \to C$ where C' admits a nonconstant map g to an elliptic curve E. This gives us a map $\psi: C' \to C \times_k E$ by taking $\psi = f \times g$. Let us call the image of this map D. Naturally, D is irreducible and projection to each factor of $C \times_k E$ is nonconstant (since f and g are nonconstant). The projection $p_1|_D$ gives a map $\phi: \tilde{D} \to C$, which is equal to $p_1|_D \cdot q$, where $q: \tilde{D} \to D$ is the desingularization of D as in the proof of Proposition 4.3. Since C' is nonsingular the property of the normalization $\tilde{D}$ ensures the existence of a map $\pi: C' \to \tilde{D}$ such that $f = \phi \cdot \pi$. This is summarized in the diagram below.

$$\begin{tikzcd} & \tilde{D} \arrow[d, "q"] & \\ C' \arrow[ur, "\pi"] \arrow[r, "\psi"] \arrow[dr, "f"'] & D \arrow[r, "i"] \arrow[d, "p_1|_D"] & C \times_k E \\ & C & \end{tikzcd} \tag{4.3.9}$$

Note that i is a closed immersion and ϕ is the composite $p_1|_D \cdot q$.

LEMMA 4.4. *With the notation of* (4), $\delta_0(\phi) \leqslant \delta_0(f)$.

Proof. As in (4), $f = \phi \cdot \pi$. Now, let R_ϕ, R_π, and R_f denote the ramification divisors of ϕ, π, and f, respectively. Applying Riemann–Hurwitz to ϕ, π, and f and using the fact that $\deg f = (\deg \pi)(\deg \phi)$ yields

$$\begin{aligned} \deg R_f &= 2g(C') - 2 - \deg f(2g(C) - 2), \\ &= \deg \pi(2g(\tilde{D}) - 2) + \deg R_\pi - \frac{\deg f(2g(\tilde{D}) - 2 - \deg R_\phi)}{\deg \phi}, \\ &= \deg \pi(2g(\tilde{D}) - 2) + \deg R_\pi - \deg \phi(2g(\tilde{D}) - 2 - \deg R_\phi), \\ &= (\deg \pi)(\deg R_\phi) + \deg R_\pi. \end{aligned} \tag{4.4.1}$$

Therefore, we have

$$\frac{\deg R_\phi}{\deg \phi} \leqslant \frac{1}{\deg \phi} \frac{\deg R_f}{\deg \pi} = \frac{\deg R_f}{\deg f}. \tag{4.4.2}$$

□

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Remark 4.5. In light of Lemma 4.4 and 4.3, a cover $f: C' \to C$ can only have small $\delta_0(f)$ when its image D in $C \times_k E$ is singular. Again, we use the notation of (4). In general, $\delta(D) = 2ab - \deg R_\phi$, and if $g(C) \geqslant 1$,

$$\frac{\deg R_\phi}{2g(D) - 2} \leqslant \frac{\deg R_\phi}{\deg \phi} = 2b - \frac{\delta(D)}{a}.$$

So what we are looking for are curves on S with a lot of singularities. Further computations suggest a close relationship between our question and the classical Severi problem (see [Ha, Chap. 5, Rmk. 3.11.1] and [Sev]).

A before, let E be an elliptic curve over a number field k and let C be a curve of genus at least 2 over k that does not admit a nonconstant map to E. Denote by $\hat{C}$ the extension of C from k to $k(E)$ (the function field of E), that is $\hat{C} := C \times_k k(E)$. Of course, we may regard $\hat{C}$ as the generic fibre of the surface $S = C \times_k E$ (with respect to the projection p_2 to E). Recall from Section 1 that we may define a canonical height on $\hat{C}$ via $\omega_{S/E}$, the sheaf of relative differentials for S over E; furthermore $\omega_{S/E} = K_S - p_2^*(K_E) = K_S$ (since K_E is trivial), so we have

$$h_{K_{\hat{C}}}(P) := \frac{H_P.\omega_{S/E}}{[k(P):k(E)]} = \frac{H_P.K_S}{[k(P):k(E)]}. \tag{4.5.1}$$

We also recall the definition

$$h(P) := \frac{1}{2g(C) - 2} h_{K_{\hat{C}}}(P) \tag{4.5.2}$$

from Section 1. We see now that there is an interesting relationship between δ-ramified coverings of C which admit nonconstant maps to E and points with small discriminant on $\hat{C}$.

PROPOSITION 4.6. *With notation as above, any cover $f: C' \to C$ which admits a nonconstant map to E gives rise to a point $P \in \hat{C}(k(\bar{E}))$ with*

$$d(P) \leqslant h_{K_{\hat{C}}}(P) + \delta_0(f) h(P). \tag{4.6.1}$$

Conversely, any point P satisfying $d(P) \leqslant h_{K_{\hat{C}}}(P) + \epsilon h(P)$ and $h(P) > 0$ comes from a cover $f: C' \to C$ with $\delta_0(f) \leqslant \epsilon$ where C' admits a nonconstant map to E.

Proof. We begin with a few generalities on points P in $\hat{C}(k(\bar{E}))$ with $h(P) > 0$. First of all, we note that have

$$d(P) := \frac{2g(\tilde{H}_P) - 2}{[k(P):k(E)]}, \tag{4.6.2}$$

where $\tilde{H}_P$ is the desingularization of the curve corresponding to P. Let us denote by ϕ the mapping from $\tilde{H}_P$ induced by projection onto the first factor of S. It

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is clear that ϕ must be nonconstant, for if ϕ were constant than H_P would be in a fibre of the projection to C and $H_P.\omega_{S/E}$ would have to be zero, which would mean that $h(P)$ was zero, contrary to our assumption. We observe that using intersection theory as in Proposition 4.3 yields

$$\begin{aligned}[k(P):k(E)](d(P)-h_{K_{\hat{C}}}(P)) &= 2g(\tilde{H}_P)-2-(\deg\phi)(2g(C)-2)\\ &= \deg R_\phi. \end{aligned}\tag{4.6.3}$$

Once again $\mathrm{Num}(S)$ is generated by $C\times pt$ and $pt\times E$, which we refer to as C and E, respectively. Thus, H_P is numerically equivalent to $a\cdot C+b\cdot E$ for some nonnegative integers a and b. In fact, a must be positive since the $p_1|_D$ is nonconstant as we have already seen, and b must be positive since the projection to E is nonconstant on H_P because H_P is the closure of a point on the generic fibre $\hat{C}$. We see that the height of P has a simple formulation in terms of a and b

$$\begin{aligned} h(P) &= \frac{1}{(2g(C)-2)}\cdot\frac{K_S.H_P}{[k(P):k(E)]}\\ &= \frac{1}{(2g(C)-2)}\cdot\frac{a(g(C)-2)}{b}\\ &= \frac{a}{b}. \end{aligned}\tag{4.6.4}$$

Plugging (4.6.3) and (4.6.4) into the formula for $\delta_0(\phi)$, we obtain

$$\begin{aligned} \delta_0(\phi)h(P) &= \frac{R_\phi}{\deg\phi}h(P)\\ &= \frac{1}{\deg\phi}[k(P):k(E)](d(P)-h_{K_{\hat{C}}}(P))h(P)\\ &= \frac{b}{a}(d(P)-h_{K_{\hat{C}}}(P))\frac{a}{b}\\ &= d(P)-h_{K_{C'}}(P). \end{aligned}\tag{4.6.5}$$

Now, suppose $f\colon C'\to C$ is a nonconstant map where C' is a curve which admits a nonconstant map to E. As we saw earlier, the map from C' to E induces a point P on $\hat{C}$ and a map $\pi\colon C'\to\tilde{H}_P$ such that $f=\phi\cdot\pi$. By Lemma 4.4, $\delta_0(f)\geqslant\delta_0(\phi)$. Since $h(P)>0$, (4.6.5) then yields $d(P)\leqslant h_{K_{\hat{C}}}(P)+\delta_0(f)h(P)$.

Conversely, given a point P with $d(P)\leqslant h_{K_{\hat{C}}}(P)+\epsilon h(P)$, for some $\epsilon\geqslant 0$, taking $C'=\tilde{H}_P$ and $f=\phi$ yields $f\colon C'\to C$ with $\delta_0(f)\leqslant\epsilon$ by (4.6.5). □

COROLLARY 4.7. *Suppose $V(\hat{C},\epsilon,d,\nu)$ holds for some ν. Suppose, furthermore, that the inequality $d(P)\leqslant h_{K_{\hat{C}}}(P)+\epsilon h(P)+\mathrm{O}(1)$ is true for infinitely* *many P either with* $\mathrm{O}(1) \leqslant 0$ *and* $h(P) > 0$ *or with* $h(P) \to \infty$*. Then, for any* $\epsilon' > \epsilon$ $V(C, \epsilon', d, \nu)$*, holds as well.*

Proof. When $\mathrm{O}(1) \leqslant 0$ and $h(P) > 0$, it follows immediately from Proposition 4.6 that there exists a cover $\phi\colon C' \to C$ of degree less than or equal to ν with $\delta_0(f) \leqslant \epsilon$ and C' admits a nonconstant map to an elliptic curve. When $h(P) \to \infty$, it follows that for $\epsilon' > \epsilon$ there are infinitely many P such that $\epsilon h(P) + \mathrm{O}(1) \leqslant \epsilon' h(P)$, so again from Proposition 4.6 it follows that there exists cover $\phi\colon C' \to C$ of degree less than or equal to ν with $\delta_0(f) \leqslant \epsilon'$ and C' admits a nonconstant map to an elliptic curve. Applying Proposition 3.5 finishes the proof. □

The main result of this section is now immediate.

THEOREM 4.8. *If statement* $B(\hat{C}, d, \nu, 0)$ *holds, then* $B(C, d, \nu, 0)$ *holds as well.*

Note that the points with small field discriminants $d(P)$ constructed in Theorem 4.8 do not necessarily have small arithmetic discriminants $d_a(P)$. In fact, all of the P may be linearly equivalent as divisors on C. When this is the case, all of the corresponding D_P in (2.0.2) are the same and it follows that $d_a(P) = h_k(P) + 2[k(P) : k]h(P) + \mathrm{O}(1)$. Thus, in many cases, $d(P)$ is small because the difference $d(P)$ and $d_a(P)$ is large. This is the result of singularities on the arithmetic curve H_P corresponding to P on the arithmetic surface X corresponding to C (recall the notation of Section 1). We hope to address questions involving highly singular arithmetic curves and points with small arithmetic discriminants in future papers.

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