ORDERING PROPERTIES OF EXTREME CLAIM AMOUNTS FROM HETEROGENEOUS PORTFOLIOS

BY

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ABSTRACT

In the context of insurance, the smallest and largest claim amounts turn out to be crucial to insurance analysis since they provide useful information for determining annual premium. In this paper, we establish sufficient conditions for comparing extreme claim amounts arising from two sets of heterogeneous insurance portfolios according to various stochastic orders. It is firstly shown that the weak supermajorization order between the transformed vectors of occurrence probabilities implies the usual stochastic ordering between the largest claim amounts when the claim severities are weakly stochastic arrangement increasing. Secondly, sufficient conditions are established for the right-spread ordering and the convex transform ordering of the smallest claim amounts arising from heterogeneous dependent insurance portfolios with possibly different number of claims. In the setting of independent multiple-outlier claims, we study the effects of heterogeneity among sample sizes on the stochastic properties of the largest and smallest claim amounts in the sense of the hazard rate ordering and the likelihood ratio ordering. Numerical examples are provided to highlight these theoretical results as well. Not only can our results be applied in the area of actuarial science, but also they can be used in other research fields including reliability engineering and auction theory.

KEYWORDS

Largest claim amounts, smallest claim amounts, stochastic orders, occurrence probabilities, majorization.

1. INTRODUCTION

In the context of insurance, the annual premium, the amount paid by the policyholder on an annual basis to cover the cost of the insurance policy being purchased, is the primary cost to the policyholder of transferring the

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risk to the insurer which depends on the type of insurance. Therefore, the stochastic behaviors of claim amounts become crucial in insurance analysis since they provide useful information for determining the annual premium. Stochastic orders, which have been widely used in various areas including financial economics, actuarial science, operations research and reliability engineering, turn out to be powerful tools for exploring the stochastic behaviors of claim amounts from different viewpoints (c.f. Müller and Stoyan, 2002; Shaked and Shanthikumar, 2007).

Let $X_1, \ldots, X_n$ be a set of non-negative random variables with X_i denoting the ith claim size in a given insurance portfolio, for $i = 1, \ldots, n$, and let $I_1, \ldots, I_n$ be a group of Bernoulli random variables, independent of the X_i's, with I_i indicating whether the claim X_i occurs or not and such that $\mathbb{E}[I_i] = p_i$, for $i = 1, \ldots, n$. In the literature, there has been tremendous study on the aggregate claim number $\sum_{i=1}^{n} I_i$ and the aggregate claim amount $\sum_{i=1}^{n} I_i X_i$ by using various stochastic orders; see, for example, Ma (2000), Denuit and Frostig (2006), Khaledi and Ahmadi (2008), Barmalzan *et al.* (2015), Zhang and Zhao (2015) and Zhang *et al.* (2018b).

Let $Y_{i:n}$ and $Y^*_{i:n}$ be the ordered claim amounts from the portfolio of risks $Y_1, \ldots, Y_n$ and $Y^*_1, \ldots, Y^*_n$, where $Y_i = I_i X_i$ and $Y^*_i = I^*_i X^*_i$, for $i = 1, \ldots, n$. Since Y_i is a discrete-continuous type random variable, traditional results on stochastic comparisons for order statistics cannot be applied to the ordered claim amounts. Barmalzan *et al.* (2016) established the likelihood ratio ordering between $Y_{1:n}$ and $Y^*_{1:n}$ when X_i's and X^*_i's are independent and heterogeneous Weibull distributed claims. Barmalzan *et al.* (2017) discussed the ordering properties of $Y_{1:n}$ and $Y_{n:n}$ in the sense of the usual stochastic and hazard rate orders by employing the multivariate chain majorization order. Balakrishnan *et al.* (2018) provided sufficient conditions to compare $Y_{n:n}$ and $Y^*_{n:n}$ according to some magnitude orderings (e.g., the usual stochastic order and the hazard rate order) when the claims severities X_i's and X^*_i's have general distributions. Recently, Zhang *et al.* (2018a) stochastically compared $Y_{2:n}$ and $Y^*_{2:n}$ and applied their results in comparing the lifetimes of two fail-safe systems subjected to random shocks.

To the best of the authors' knowledge, there is little study on stochastic properties of extreme claim amounts when the claim sizes or the occurrence probabilities are dependent. For instance, the claim severities are usually positively dependent for insureds in an area suffering from serious natural disasters such as drought, floods and earthquakes. In this article, we shall show that the weak supermajorization order between the transformed vectors of occurrence probabilities implies the usual stochastic ordering between the largest claim amounts when the claim sizes are positively dependent via the weakly stochastic arrangement increasing (WSAI).

It is also worth noting that the abovementioned study only focuses on discussing magnitude orderings of the extreme claim amounts. Rare results are available for the dispersion orders (e.g., dispersive order, convex order and right-spread order) and the shape orders (e.g., convex transform order and star order). Barmalzan and Payandeh Najafabadi (2015) might be the first to discuss stochastic comparisons between $Y_{1:n}$ and $Y^*_{1:n}$ in the sense of the right-spread and the convex transform orders under certain conditions when the claims have Weibull distributions. However, their results are restricted in the setting that all of the occurrence levels are independent and the claim severities have Weibull distributions. Besides, it is assumed that the number of claims is the same for any two concerned insurance portfolios. Let $Y^*_{i:n^*}$ be the ordered claim amounts from the portfolio of risks $Y^*_1, \ldots, Y^*_{n^*}$, where $Y^*_i = I^*_i X^*_i$, for $i = 1, \ldots, n^*$. In this paper, we shall focus on the right-spread ordering and the convex transform ordering of the smallest claim amounts $Y_{1:n}$ and $Y^*_{1:n^*}$ when the occurrence levels are dependent and the claim numbers n and n^* are possibly different. To a large extent, these results generalize those corresponding ones in Barmalzan and Payandeh Najafabadi (2015).

It is also of great significance to establish sufficient conditions for some strong stochastic orders such as the hazard rate order and the likelihood ratio order, both of which play an important role in actuarial science to compare different risks and are preserved under the Tail Value-at-Risk (TVaR) measure and Esscher premium principle (c.f. Bühlmann, 1980; Van Heerwaarden *et al.*, 1989), respectively. Though Barmalzan *et al.* (2016) provided sufficient conditions for the likelihood ratio ordering between the smallest claim amounts from Weibull claims, however, their results are not correct since they misused the density functions of the smallest claim amounts, and thus neglected the stochastic behavior of the density ratio function at point 0. In this work, we shall amend this (see Remark 4.7) and provide sufficient conditions to analyze the effects of heterogeneity among sample sizes on the largest and smallest claim amounts arising from two sets of independent multiple-outlier claims in the sense of the hazard rate order and the likelihood ratio order.

It should be mentioned that our results derived here not only can be applied in the area of actuarial science but also can be used in other research fields such as reliability engineering and auction theory (c.f. Balakrishnan *et al.*, 2018; Zhang *et al.*, 2018a). In the context of first-price reverse auction, the bidders (sellers) are required to submit sealed bids to the auctioneer (buyer) who seeks for the purchase of items when the auction begins. The lowest bidder is assumed to win the bid and will be paid the amount of the lowest price from the auctioneer. However, such things may happen that some of the bidders would like to drop out of the auction before the start of the auction due to some unforeseen circumstances. For this reason, the final cost on the auction turns out to be the smallest order statistics arising from $I_1 X_1, \ldots, I_n X_n$, where I_i denotes whether the bidder i attends the auction or not, and X_i is interpreted as the bidding price for the ith bidder if he/she participates in the auction, $i = 1, \ldots, n$. The results developed in the present paper can provide quantitative analysis on the effects of number of bidders, their attending probabilities and biding price distributions on the final auction price.

The remainder of the paper is rolled out as follows. Section 2 recalls some pertinent definitions and notions used in the sequel. In Section 3, the usual stochastic ordering is firstly established between the largest claim amounts arising from two sets of heterogeneous portfolios with dependent claim severities. Secondly, stochastic comparisons are conducted on the smallest claim amounts arising from two sets of heterogeneous dependent insurance portfolios (with possibly different number of claims) in the sense of the right-spread ordering and the convex transform ordering. In Section 4, sufficient conditions for the hazard rate ordering and likelihood ratio ordering are built to compare the largest and smallest claim amounts arising from two sets of independent multiple-outlier claim models having different sample sizes. Section 5 concludes the paper. All proofs are delegated to the appendix for ease of presentation.

2. Definitions and some preliminaries

Throughout, the term increasing is used for monotone non-decreasing and decreasing is used for monotone non-increasing. All random variables are defined on a common probability space and all expectations exist whenever they appear. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\mathcal{D}_n^+ = \{\boldsymbol{x} : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$. We use '$\overset{\text{sign}}{=}$' to denote both sides of the equality have the same sign.

2.1. Stochastic orders

Assume that non-negative random variables X and Y have distribution functions F and G, survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, probability density functions f and g, and hazard rate functions h_F and h_G, respectively.

Definition 2.1. *X is said to be larger than Y in the*

(i) *usual stochastic order (denoted by $X \geq_{\text{st}} Y$) if $\overline{F}(x) \geq \overline{G}(x)$ for all $x \in \mathbb{R}_+$;*

(ii) *hazard rate order (denoted by $X \geq_{\text{hr}} Y$) if $\overline{F}(x)/\overline{G}(x)$ is increasing in $x \in \mathbb{R}_+$, or equivalently, $h_F(x) \leq h_G(x)$ for all $x \in \mathbb{R}_+$;*

(iii) *likelihood ratio order (denoted by $X \geq_{\text{lr}} Y$) if $f(x)/g(x)$ is increasing in $x \in \mathbb{R}_+$;*

(iv) *increasing convex order (denoted by $X \geq_{\text{icx}} Y$) if $\int_x^\infty \overline{F}(t)\text{d}t \geq \int_x^\infty \overline{G}(t)\text{d}t$, for all $x \in \mathbb{R}_+$;*

(v) *convex transform order (denoted by $X \geq_{\text{c}} Y$) if $F^{-1}[G(x)]$ is convex in $x \in \mathbb{R}_+$, or equivalently, $X \geq_{\text{c}} Y$ if and only if $G^{-1}[F(x)]$ is concave in $x \in \mathbb{R}_+$;*

(vi) *star order (denoted by $X \geq_\star Y$) if $F^{-1}[G(x)]$ is star-shaped in the sense that $F^{-1}[G(x)]/x$ is increasing in $x \in \mathbb{R}_+$;*

(vii) *dispersive order (denoted by $X \geq_{\text{disp}} Y$) if $F^{-1}(v) - F^{-1}(u) \geq G^{-1}(v) - G^{-1}(u)$, for $0 < u \leq v < 1$; and*

(viii) *right-spread order (denoted by $X \geq_{\text{RS}} Y$) if $\int_{F^{-1}(u)}^\infty \overline{F}(t)\text{d}t \geq \int_{G^{-1}(u)}^\infty \overline{G}(t)\text{d}t$, for all $u \in (0, 1)$.*

It is well known that the following implication always holds:

$$X \geq_{\rm lr} Y \Longrightarrow X \geq_{\rm hr} Y \Longrightarrow X \geq_{\rm st} Y \Longrightarrow X \geq_{\rm icx} Y \Longrightarrow \mathbb{E}[X] \geq \mathbb{E}[Y].$$

The convex transform order was proposed by Van Zwet (1970) to compare the skewness of different probability distributions. The star order is called the more increasing failure rate in average (IFRA) in reliability theory and is one of the partial orders which are scale invariant. It is common knowledge that

$$X \geq_{\rm c} Y \Longrightarrow X \geq_{\star} Y \Longrightarrow {\rm CV}(X) \geq {\rm CV}(Y),$$

where ${\rm CV}(X) = \sqrt{{\rm Var}[X]}/\mathbb{E}[X]$ and ${\rm CV}(Y) = \sqrt{{\rm Var}[Y]}/\mathbb{E}[Y]$ denote the coefficients of variation of X and Y, respectively. The dispersive order, which is stronger than the right-spread order, is a kind of partial orders to compare the variabilities in two probability distributions. The increasing convex order is commonly termed as stop-loss order in the context of actuarial science to measure the severity of different risks according to the stop-loss premiums. It is known that the hazard rate order, the usual stochastic order and the right-spread order (also called excess wealth order in economics) imply the increasing convex order. For comprehensive discussions on these useful orders, the reader may refer to the excellent monographs by Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

It should be mentioned that the VaR and all distortion risk measures agree with the usual stochastic order, the TVaR and distortion risk measures with concave distortion functions agree with the increasing convex order, and thus the TVaR is coherent with the hazard rate order. For more discussions on these risk measures and their applications in actuarial problems, we refer interested readers to Yaari (1987) and Wang (1996).

2.2. Majorizations

The notion of majorization is quite useful in establishing various inequalities arising from actuarial science, applied probability and reliability theory. Let $x_{1:n} \leq \cdots \leq x_{n:n}$ be the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$.

Definition 2.2. *A vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *is said to*

(i) majorize another vector $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \overset{\rm m}{\succeq} \boldsymbol{y}$*) if* $\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n}$ *for* $j = 1, \ldots, n-1$*, and* $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$*;*

(ii) weakly supermajorize another vector $\boldsymbol{y} \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \overset{\rm w}{\succeq} \boldsymbol{y}$*) if* $\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n}$ *for* $j = 1, \ldots, n$*; and*

(iii) weakly submajorize another vector $\boldsymbol{y} \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \succeq_{\rm w} \boldsymbol{y}$*) if* $\sum_{j=i}^{n} x_{j:n} \geq \sum_{j=i}^{n} y_{j:n}$ *for* $i = 1, \ldots, n$*.*

For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, it is evident that $\boldsymbol{x} \overset{\mathrm{m}}{\succeq} \boldsymbol{y}$ implies both $\boldsymbol{x} \overset{\mathrm{w}}{\succeq} \boldsymbol{y}$ and $\boldsymbol{x} \succeq_{\mathrm{w}} \boldsymbol{y}$, while the reverse is not true in general. Interested readers are referred to Marshall *et al.* (2011) for comprehensive discussions on majorization-type orders.

2.3. Stochastic versions of arrangement increasing

For any (i, j) with $1 \leq i < j \leq n$, let $\tau_{ij}(\boldsymbol{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$ and denote

$$\begin{aligned}
\mathcal{G}_{sai}^{i,j}(n) &= \{g(\boldsymbol{x}) : g(\boldsymbol{x}) \geq g(\tau_{ij}(\boldsymbol{x})) \text{ for any } x_i \leq x_j\}, \\
\mathcal{G}_{wsai}^{i,j}(n) &= \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is increasing in } x_j\}, \\
\mathcal{G}_{rwsai}^{i,j}(n) &= \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is increasing in } x_j \geq x_i\}.
\end{aligned}$$

Definition 2.3. *A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is said to be*

(i) stochastic arrangement increasing (SAI) if $\mathbb{E}[g(\boldsymbol{X})] \geq \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$ for any $g \in \mathcal{G}_{sai}^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$;

(ii) weakly stochastic arrangement increasing (WSAI) if $\mathbb{E}[g(\boldsymbol{X})] \geq \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$ for any $g \in \mathcal{G}_{wsai}^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$; and

(iii) right tail weakly stochastic arrangement increasing (RWSAI) if $\mathbb{E}[g(\boldsymbol{X})] \geq \mathbb{E}[g(\tau_{ij}(\boldsymbol{X}))]$ for any $g \in \mathcal{G}_{rwsai}^{i,j}(n)$ and any pair (i, j) such that $1 \leq i < j \leq n$.

It is clear that SAI implies RWSAI, which in turn implies WSAI. Many well-known distributions are SAI including the multivariate versions of Dirichlet distribution, inverted Dirichlet distribution, F distribution and Pareto distribution of type I. The notion of SAI has been applied in actuarial science to model the dependence among ordered random risks; see, for instance, Hua and Cheung (2008) and Zhang and Zhao (2015). RWSAI and WSAI are introduced by Cai and Wei (2014, 2015), and have been also applied in the field of financial engineering and actuarial science; see, for example, Cai and Wei (2015) and Zhang *et al.* (2018b).

3. Heterogeneous dependent portfolios

In this section, we deal with the ordering properties of the largest and smallest claim amounts coming from heterogeneous insurance portfolios consisting of dependent claim severities and/or dependent occurrence levels.

3.1. Largest claim amounts

In this subsection, we use the usual stochastic order to compare the largest claim amounts arising from two sets of heterogeneous portfolios with dependent claim sizes.

Assume that $X_1, \ldots, X_n$ are non-negative random variables denoting the claim sizes for n risks faced by the insurer. Let $I_{p_1}, \ldots, I_{p_n}[I_{p_1^*}, \ldots, I_{p_n^*}]$ be a set of independent Bernoulli random variables, independent of X_i's, such that $\mathbb{E}[I_{p_i}] = p_i[\mathbb{E}[I_{p_i^*}] = p_i^*]$, for $i = 1, 2, \ldots, n$. In the following, let $Y_{n:n}[Y_{1:n}]$ and $Y_{n:n}^*[Y_{1:n}^*]$ denote the largest[smallest] claim amount from claims $I_{p_1}X_1, \ldots, I_{p_n}X_n$ and $I_{p_1^*}X_1, \ldots, I_{p_n^*}X_n$, respectively. The following result illustrates that more heterogeneity among some specified transformed occurrence probabilities in the sense of the weak supermajorization order leads to greater tail function of the largest claim amount.

Theorem 3.1. *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be WSAI. If $h(\boldsymbol{p})$, $h(\boldsymbol{p}^*) \in \mathcal{D}_n^+$, where $h(p) = -\log p$ or $h(p) = (1-p)/p$, then it holds that*

$$(h(p_1), h(p_2), \ldots, h(p_n)) \overset{\text{w}}{\succeq} (h(p_1^*), h(p_2^*), \ldots, h(p_n^*)) \Longrightarrow Y_{n:n} \geq_{\text{st}} Y_{n:n}^*.$$

Remark 3.2. *For the case of independent claim amount $\boldsymbol{X}$, Balakrishnan et al. (2018) proved the result of Theorem 3.1 when the transformation function h is differentiable and strictly decreasing convex. It remains as an open problem to establish the result of Theorem 3.1 for other forms of the function h.*

Next, we present some numerical examples to illustrate the effectiveness of Theorem 3.1 and show that some conditions required there cannot be removed.

Example 3.3. *Let X_1, X_2 be two exponential random variables with hazard rates λ_1 and λ_2, respectively. Assume that $\boldsymbol{X} = (X_1, X_2)$ has Clayton copula (see Nelsen, 2006) with generator $\psi(x) = (x+1)^{-\frac{1}{\theta}}$ for $\theta > 0$, and its inverse $\phi(x) = \psi^{-1}(x) = x^{-\theta} - 1$, for $x \in [0, 1]$. Then, the joint distribution function of $\boldsymbol{X}$ is given by*

$$\begin{aligned}\mathbb{P}(X_1 \leq x, X_2 \leq x) &= \psi\ (\phi(\mathbb{P}(X_1 \leq x) + \phi(\mathbb{P}(X_2 \leq x))) \\ &= \left[(1-\exp\{-\lambda_1 x\})^{-\theta} + (1-\exp\{-\lambda_2 x\})^{-\theta} - 1\right]^{-\frac{1}{\theta}}, \quad x \in \mathbb{R}_+.\end{aligned}$$

Then, we can calculate that

$$\begin{aligned}&\overline{F}_{Y_{2:2}}(x) - \overline{F}_{Y_{2:2}^*}(x) \\ &= \mathbb{P}(I_{p_1^*}X_1 \leq x, I_{p_2^*}X_2 \leq x) - \mathbb{P}(I_{p_1}X_1 \leq x, I_{p_2}X_2 \leq x) \\ &= (p_1^* p_2^* - p_1 p_2)\left[(1-\exp\{-\lambda_1 x\})^{-\theta} + (1-\exp\{-\lambda_2 x\})^{-\theta} - 1\right]^{-\frac{1}{\theta}} \\ &\quad + [p_1^*(1-p_2^*) - p_1(1-p_2)](1-\exp\{-\lambda_1 x\}) \\ &\quad + [p_2^*(1-p_1^*) - p_2(1-p_1)](1-\exp\{-\lambda_2 x\}) \\ &\quad + (1-p_1^*)(1-p_2^*) - (1-p_1)(1-p_2), \quad x \in \mathbb{R}_+.\end{aligned}$$

Taking $\lambda_1 = 1.5$, $\lambda_2 = 1$ and $\theta = 0.5$, it is easy to check that $X_1 \leq_{\text{hr}} X_2$ and $x\phi'(x)$ is increasing. Then, we know that $\boldsymbol{X}$ is RWSAI according to Theorem 5.7 of Cai and Wei (2014), and thus $\boldsymbol{X}$ is WSAI.

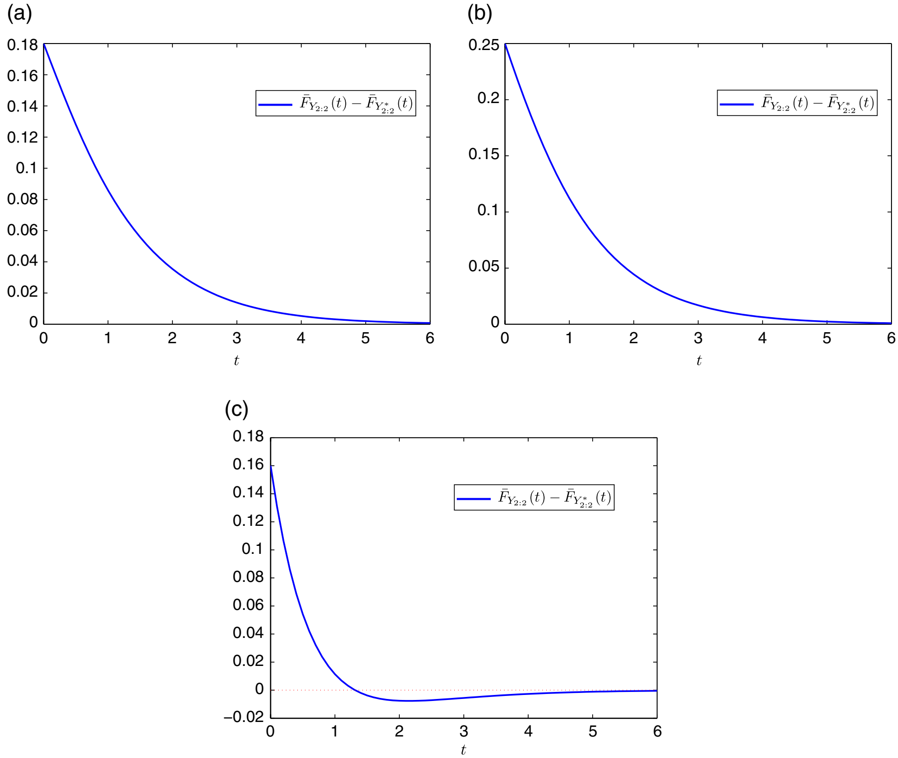

FIGURE 1: Plot of $\overline{F}_{Y_{2:2}}(x) - \overline{F}_{Y^*_{2:2}}(x)$ for $x \in \mathbb{R}_+$ in Example 3.3.

(a) *Set* $p_1 = 0.2$, $p_2 = 0.7$, $p_1^* = 0.3$, $p_2^* = 0.4$ *and* $h(x) = -\log x$ *for* $x \in (0, 1)$. *Observe that* $(h(p_1), h(p_2)), (h(p_1^*), h(p_2^*)) \in \mathcal{D}_2^+$ *and* $(h(p_1), h(p_2)) \overset{\mathrm{w}}{\succeq} (h(p_1^*), h(p_2^*))$, *which satisfy the conditions of Theorem 3.1. Figure 1(a) shows that* $\overline{F}_{Y_{2:2}}(t) - \overline{F}_{Y^*_{2:2}}(t)$ *is always non-negative for* $t \in \mathbb{R}_+$, *which is in accordance with theoretical result of Theorem 3.1.*

(b) *Set* $p_1 = 0.2$, $p_2 = 0.75$, $p_1^* = 0.25$, $p_2^* = 0.4$ *and* $h(x) = (1 - x)/x$ *for* $x \in (0, 1)$. *Clearly,* $(h(p_1), h(p_2)), (h(p_1^*), h(p_2^*)) \in \mathcal{D}_2^+$ *and* $(h(p_1), h(p_2)) \overset{\mathrm{w}}{\succeq} (h(p_1^*), h(p_2^*))$. *As seen in Figure 1(b), it appears to be* $\overline{F}_{Y_{2:2}}(t) \geq \overline{F}_{Y^*_{2:2}}(t)$ *for all* $t \in \mathbb{R}_+$. *Therefore, we have* $Y_{2:2} \geq_{\mathrm{st}} Y^*_{2:2}$, *and the effectiveness of Theorem 3.1 is verified.*

(c) *Set* $p_1 = 0.8$, $p_2 = 0.3$, $p_1^* = 0.4$ *and* $p_2^* = 0.5$. *It is easy to verify that* $(h(p_1^*), h(p_2^*)) \in \mathcal{D}_2^+$, $(h(p_1), h(p_2)) \notin \mathcal{D}_2^+$ *and* $(h(p_1), h(p_2)) \overset{\mathrm{w}}{\succeq} (h(p_1^*), h(p_2^*))$ *for* $h(x) = -\log x$. *Figure 1(c) exhibits that* $\overline{F}_{Y_{2:2}}(t) - \overline{F}_{Y^*_{2:2}}(t)$ *is not always non-negative for all* $t \in \mathbb{R}_+$, *which implies* $Y_{2:2} \ngeq_{\mathrm{st}} Y^*_{2:2}$ *and* $Y_{2:2} \nleq_{\mathrm{st}} Y^*_{2:2}$.

3.2. Smallest claim amounts

In this subsection, we turn to examining the smallest claim amounts arising from two sets of heterogeneous portfolios comprising dependent occurrence probabilities in the sense of the right-spread order and the convex transform order.

3.2.1. *Right-spread order.* Independent random variables $X_1, \ldots, X_n$ are said to follow PHR model if the survival function of X_i can be written as $\overline{F}_i(x) = [\overline{F}(x)]^{\lambda_i}$, for $i = 1, \ldots, n$, where $\overline{F}(x)$ is the survival function of some underlying random variable X. The survival function of X_i can be written as $\overline{F}_i(x) = e^{-\lambda_i R(x)}$, for $i = 1, \ldots, n$, where $R(x) = \int_0^x r(t)\mathrm{d}t$ is the cumulative hazard rate function of X and $r(\cdot)$ is the hazard rate function of X. Many well-known distributions are special cases of the PHR model such as exponential, Weibull, Pareto, Lomax, and so on.

The following useful lemma, which was given by Kochar and Xu (2010) to establish equivalent characterizations of the right-spread order in one parameter family, is needed to prove the main results.

Lemma 3.4. *Suppose $\{F_b \mid b \in \mathbb{R}\}$ is a class of distribution functions such that F_b is supported on some interval $(x_0, x_1) \subseteq (0, \infty)$ and has density f_b which does not vanish on any subinterval of (x_0, x_1). Then, $F_{b^*} \leq_{\mathrm{RS}} F_b$, for $b, b^* \in \mathbb{R}$, $b \leq b^*$, if and only if $\frac{W_b'(x)}{\overline{F}_b(x)}$ is decreasing in x, where W_b' is the derivative of $W_b(x) = \int_x^\infty \overline{F}_b(t)\mathrm{d}t$ with respect to b.*

Let $X_i[X_i^*]$ be a non-negative random variable denoting the size of the ith claim, $i = 1, 2, \ldots, n[n^*]$, and let $I_i[I_i^*]$ be a Bernoulli random variable such that $\mathbb{E}[I_i] = p_i[\mathbb{E}[I_i^*] = p_i^*]$, $i = 1, 2, \ldots, n[n^*]$. In this part, we use $Y_{1:n}$ and $Y^*_{1:n^*}$ to denote the smallest claim amounts from $I_1X_1, \ldots, I_nX_n$ and $I_1^*X_1^*, \ldots, I_{n^*}^*X_{n^*}^*$, respectively.

Next, we present sufficient conditions to compare the smallest claim amounts from two sets of heterogeneous dependent PHR claims in the sense of the right-spread order. Some results given in Barmalzan and Payandeh Najafabadi (2015) are substantially generalized.

Lemma 3.5. *Let $X_i[X_i^*]$ be independent random variables with survival function $\overline{G}^{\lambda_i}[\overline{G}^{\lambda_i^*}]$, $i = 1, 2, \ldots, n[n^*]$. If G has DHR (decreasing hazard rate) and $\sum_{i=1}^n \lambda_i \leq \sum_{i=1}^{n^*} \lambda_i^*$, then we have $X_{1:n} \geq_{\mathrm{RS}} X^*_{1:n^*}$.*

Next, sufficient conditions are given to stochastically compare the smallest claim amounts arising from two sets of heterogeneous portfolios with dependent occurrence probabilities and PHR claims in the sense of the right-spread

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order, which generalizes the result of Theorem 1 in Barmalzan and Payandeh Najafabadi (2015).

Theorem 3.6. *Let $X_i[X_i^*]$ be independent random variables with respective survival function $\overline{G}^{\lambda_i}[\overline{G}^{\lambda_i}]$, $i = 1, 2, \ldots, n[n^*]$, where $\overline{G}$ denotes the survival function of some baseline random variable. Let $\boldsymbol{I}_n[\boldsymbol{I}^*_{n^*}]$ be one multivariate Bernoulli random vector independent of $X_i[X_i^*]$'s. Assume that the following conditions hold:*

(i) *G has DHR;*
(ii) $\mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1}) = \mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1})$*; and*
(iii) $\sum_{i=1}^{n} \lambda_i \le \sum_{i=1}^{n^*} \lambda_i^*$.

Then, we have $Y_{1:n} \ge_{\mathrm{RS}} Y^*_{1:n^*}$.

It should be mentioned that the condition $\sum_{i=1}^{n} \lambda_i \le \sum_{i=1}^{n^*} \lambda_i^*$ is necessary for the result of Theorem 3.6 as addressed in Lemma 2 of Barmalzan and Payandeh Najafabadi (2015).

The following result, which strengthens Theorem 2 of Barmalzan and Payandeh Najafabadi (2015), presents equivalent characterizations for the right-spread ordering and the increasing convex ordering between the smallest claim amounts from two sets of heterogeneous portfolios with dependent occurrence probabilities and PHR claims.

Theorem 3.7. *Under the setup of Theorem 3.6, it is assumed that G has DHR and* $\mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1}) = \mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1})$. *Then, the following several statements are equivalent:*

(i) $Y_{1:n} \ge_{\mathrm{RS}} Y^*_{1:n^*}$;
(ii) $Y_{1:n} \ge_{\mathrm{icx}} Y^*_{1:n^*}$; *and*
(iii) $\sum_{i=1}^{n} \lambda_i \le \sum_{i=1}^{n^*} \lambda_i^*$.

3.2.2. *Convex transform order.* In this part, we study the convex transform ordering of the smallest claim amounts from heterogeneous and dependent portfolios with dependent occurrence levels and Weibull claims. We not only provide a complete proof for the desired result but also generalize Theorem 3 of Barmalzan and Payandeh Najafabadi (2015) to the case of heterogeneous dependent portfolios with different number of claims.

The Weibull distribution is known for its exceptional ability to fit a wide variety of data and has been widely employed in reliability engineering, lifetime testing, statistics and actuarial science. Recall that a random variable X has the Weibull distribution with shape parameter $\gamma > 0$ and scale parameter $\lambda > 0$ (denoted by $X \sim W(\gamma, \lambda)$) if its probability density function is given by

$$f(x; \gamma, \lambda) = \gamma \lambda^{\gamma} x^{\gamma - 1} e^{-(\lambda x)^{\gamma}}, \quad x > 0.$$

Of course, the distribution reduces to the exponential distribution when $\gamma = 1$. Otherwise, it is useful in modeling size of loss distributions for $0 < \gamma < 1$ and in modeling component lifetime distribution for $\gamma > 1$. Interested readers may refer to Murthy *et al.* (2004) for detailed discussions on various properties and applications of the Weibull distribution.

Theorem 3.8. *Suppose* $X_i \sim W(\gamma_1, \lambda_i)[X_i^* \sim W(\gamma_2, \lambda_i^*)]$*, for* $i = 1, \ldots, n[n^*]$*. Let* $\boldsymbol{I}_n[\boldsymbol{I}_{n^*}^*]$ *be a Bernoulli random vector, independent of* $X_i[X_i^*]$*'s, indicating whether the claims occur or not. Then, for* $\gamma_2 \geq 1$*,* $\gamma_1 \geq \frac{\gamma_2}{2\gamma_2 - 1}$ *and* $\mathbb{P}(\boldsymbol{I}_{n^*}^* = \mathbf{1}) \leq \mathbb{P}(\boldsymbol{I}_n = \mathbf{1})$*, we have* $Y_{1:n} \geq_{\rm c} Y_{1:n^*}^*$*.*

It is of importance to note that the smallest claim amount in a concerned insurance portfolio with larger occurrence probability that all claims happen is more skewed than that from a portfolio with smaller occurrence probability, where the sample size plays no role and the shape parameters in general have no relations.

Remark 3.9. *Suppose that* $I_1, \ldots, I_n[I_1^*, \ldots, I_n^*]$ *are independent Bernoulli random variables such that* $\mathbb{E}[I_i] = p_i[\mathbb{E}[I_i^*] = p_i^*]$*, for* $i = 1, \ldots, n[n^*]$*. Under the assumptions that* $n = n^*$*,* $\gamma_1 = \gamma_2 \geq 1$ *and* $\prod_{i=1}^n p_i^* \leq \prod_{i=1}^n p_i$*, the result of Theorem 3 in Barmalzan and Payandeh Najafabadi* (2015) *can be obtained from Theorem 3.8. Therefore, Theorem 3.8 substantially generalizes Theorem 3 of Barmalzan and Payandeh Najafabadi* (2015) *to the case of dependent occurrence levels, different claim numbers and different shape parameters when the claim sizes have Weibull distribution. It is worth mentioning that in the proof of Theorem 3 in Barmalzan and Payandeh Najafabadi* (2015)*, it was not discussed whether the expression* $F_{Y_{1:n}}^{-1}(F_{Y_{1:n}^*}(x))$ *is well defined or not; but fortunately, this requirement holds naturally under the assumption* $\prod_{i=1}^n p_i^* \leq \prod_{i=1}^n p_i$*.*

The next result provides sufficient conditions for the star ordering between the smallest claim amounts.

Theorem 3.10. *Suppose* $X_i \sim W(\gamma_1, \lambda_i)[X_i^* \sim W(\gamma_2, \lambda_i^*)]$*, for* $i = 1, \ldots, n[n^*]$*. Let* $\boldsymbol{I}_n[\boldsymbol{I}_{n^*}^*]$ *be a Bernoulli random vector, independent of* $X_i[X_i^*]$*'s, indicating whether the claims occur or not. Then, we have* $Y_{1:n} \geq_{\star} Y_{1:n^*}^*$ *if either of the following two conditions holds:*

(i) $\gamma_2 \geq \gamma_1$ *and* $\mathbb{P}(\boldsymbol{I}_{n^*}^* = \mathbf{1}) = \mathbb{P}(\boldsymbol{I}_n = \mathbf{1})$ *and*
(ii) $\gamma_2 \geq 1$*,* $\gamma_1 \geq \frac{\gamma_2}{2\gamma_2 - 1}$ *and* $\mathbb{P}(\boldsymbol{I}_{n^*}^* = \mathbf{1}) \leq \mathbb{P}(\boldsymbol{I}_n = \mathbf{1})$*.*

The next theorem provides sufficient conditions for comparing the smallest claim amounts from two sets of heterogeneous dependent insurance portfolios in the sense of the dispersive ordering.

Theorem 3.11. *Suppose $X_i \sim W(\gamma_1, \lambda_i)[X_i^* \sim W(\gamma_2, \lambda_i^*)]$, for $i = 1, \ldots, n[n^*]$. Let $\boldsymbol{I}_n[\boldsymbol{I}^*_{n^*}]$ be a Bernoulli random vector, independent of $X_i[X_i^*]$'s, indicating whether the claims occur or not. Then, for $\sum_{i=1}^{n} \lambda_i^{\gamma_1} \le \sum_{i=1}^{n^*} (\lambda_i^*)^{\gamma_2}$, we have $Y_{1:n} \ge_{\rm disp} Y^*_{1:n^*}$ if either of the following two conditions holds:*

(i) $\gamma_2 \ge \gamma_1$ and $\mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1}) = \mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1})$ and
(ii) $\gamma_2 \ge 1$, $\gamma_1 \ge \frac{\gamma_2}{2\gamma_2 - 1}$ and $\mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1}) \le \mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1})$.

4. Independent multiple-outlier portfolios

In practical scenarios, it may happen that some insureds have larger occurrence probabilities or claim sizes than the others in an insurance portfolio. In this regard, a natural way for describing this phenomena is to fall into the multiple-outlier model. In this section, we explore the largest and smallest claim amounts arising from two batches of independent multiple-outlier claims having different sample sizes in the sense of some strong stochastic orders including the hazard rate order and the likelihood ratio order.

4.1. Largest claim amounts

The next result establishes sufficient conditions for the likelihood ratio ordering between the largest claim amounts arising from two sets of independent multiple-outlier claims with different sample sizes.

Theorem 4.1. *Let $X_1, \ldots, X_n$ and $X_1, \ldots, X_{n^*}$ be two sets of independent random variables with common distributions G. Let $I_i[I_i^*]$ be independent Bernoulli random variables, independent of X_i's, such that $\mathbb{E}[I_i] = p_1[\mathbb{E}[I_i^*] = p_1]$ for $i = 1, 2, \ldots, n_1[n_1^*]$, and let $I_j[I_j^*]$ be another set of independent Bernoulli variables having sample size $n_2[n_2^*]$, independent of X_j's, such that $\mathbb{E}[I_j] = p_2[\mathbb{E}[I_j^*] = p_2]$ for $j = n_1[n_1^*] + 1, \ldots, n[n^*]$, where $n_1 + n_2 = n$ and $n_1^* + n_2^* = n^*$. Set $Y_i = I_i X_i [Y_i^* = I_i^* X_i]$, $i = 1, \ldots, n[n^*]$. If $p_1 \ge p_2$ and $n_2^* \ge n_2$, then*

$$n^* \le n \Longrightarrow Y_{n:n} \ge_{\rm lr} Y^*_{n^*:n^*}.$$

It is remarkable that the result established in Theorem 4.1 adheres to the intuition. Under the assumption that the insureds in a set of risk portfolio have the same claim distribution, the largest claim amount tends to be increased by reducing the number of insureds having lower occurrence probabilities while increasing the number of all insureds.

We next provide an example to illustrate the theoretical result of Theorem 4.1.

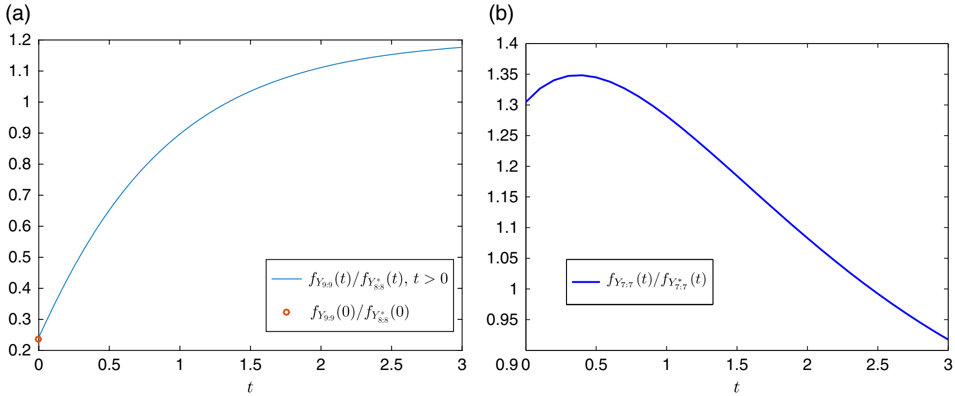

FIGURE 2: Plot of the likelihood ratio functions in Example 4.2.

Example 4.2. *Under the setup of Theorem 4.1, let* $X_1, \ldots, X_{n_1}[X_1, \ldots, X_{n_1^*}]$ *be independent non-negative random variables with common distribution* G_1*, and* $X_{n_1+1}, \ldots, X_n[X_{n_1^*+1}, \ldots, X_{n^*}]$ *be another set of independent non-negative variables having sample size* $n_2[n_2^*]$ *with common distribution* G_2*. The following two cases are considered:*

(a) $G_1(t) = G_2(t) = 1 - e^{-1.2t}$*,* $p_1 = 0.7$*,* $p_2 = 0.5$*,* $n_1^* = 2$*,* $n_2^* = 6$*,* $n_1 = 4$*,* $n_2 = 5$*. In this case,* $n_2^* > n_2$ *and* $n_1^* + n_2^* < n_1 + n_2$*. Figure 2(a) shows that the ratio* $f_{Y_{9:9}}(t)/f_{Y^*_{8:8}}(t)$ *is increasing in* $t > 0$*, which is in accordance with Theorem 4.1.*

(b) $G_1(t) = 1 - e^{-0.3t}$*,* $G_2(t) = 1 - e^{-1.2t}$*,* $p_1 = 0.4$*,* $p_2 = 0.3$*,* $n_1^* = 2$*,* $n_2^* = 5$*,* $n_1 = 3$*,* $n_2 = 4$*. It is easy to verify that* $n_2^* > n_2$ *and* $n_1^* + n_2^* = n_1 + n_2$*. However, Figure 2(b) exhibits that the ratio* $f_{Y_{7:7}}(t)/f_{Y^*_{7:7}}(t)$ *is not monotone in* $t > 0$*, which implies that the restriction* $G_1 = G_2$ *cannot be taken out.*

Let $X_1, \ldots, X_n$ be a set of non-negative independent random variables with common distributions G. Let $I_i[I_i^*]$ be independent Bernoulli random variables, independent of X_i's, such that $\mathbb{E}[I_i] = p_1[\mathbb{E}[I_i^*] = p_1^*]$ for $i = 1, 2, \ldots, n_1$, and let $I_j[I_j^*]$ be another set of independent Bernoulli variables having sample size n_2, independent of X_j's, such that $\mathbb{E}[I_j] = p_2[\mathbb{E}[I_j^*] = p_2^*]$ for $j = n_1 + 1, \ldots, n$, where $n_1 + n_2 = n$. One natural question is that whether $Y_{n:n} \geq_{\mathrm{lr}} Y^*_{n:n}$ could be derived from the majorization order between the vectors of occurrence probabilities? Unfortunately, the answer is negative by using the following counterexample.

Counterexample 4.3. *Set* $G(t) = 1 - e^{-0.9t}$*,* $p_1 = 0.9$*,* $p_2 = 0.1$*,* $p_1^* = 0.7$*,* $p_2^* = 0.3$ *and* $n_1 = n_2 = 1$*, we then have* $(p_1, p_2) \stackrel{\mathrm{m}}{\succeq} (p_1^*, p_2^*)$*. Figure 3, however, exhibits that the ratio* $f_{Y_{2:2}}(t)/f_{Y^*_{2:2}}(t)$ *is not monotone in* $t \in \mathbb{R}_+$*, which implies that* $Y_{2:2} \not\geq_{\mathrm{lr}} Y^*_{2:2}$ *and* $Y_{2:2} \not\leq_{\mathrm{lr}} Y^*_{2:2}$*.*

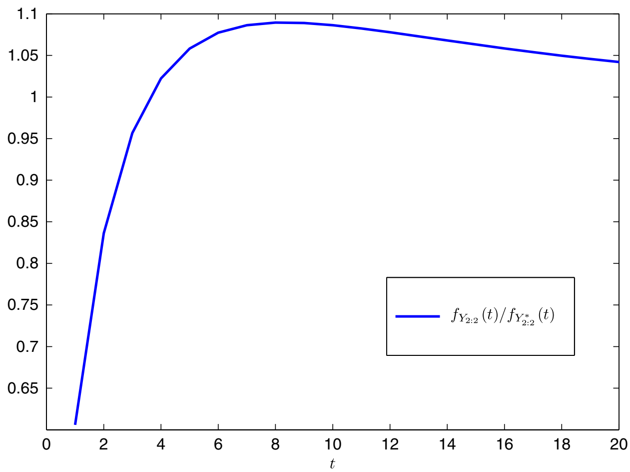

FIGURE 3: Plot of the likelihood ratio function $f_{Y_{2:2}}(t)/f_{Y^*_{2:2}}(t)$ for $t \in \mathbb{R}_+$.

4.2. Smallest claim amounts

First, we discuss the hazard rate order for the smallest claim amounts arising from two sets of independent multiple-outlier claims.

Theorem 4.4. *Let $X_i[X_i^*]$ be independent random variables with common distribution G_1, for $i = 1, \ldots, n_1[n_1^*]$, and $X_j[X_j^*]$ be another set of independent random variables having sample size $n_2[n_2^*]$ with common distribution G_2, for $j = n_1[n_1^*] + 1, \ldots, n[n^*]$, where $n_1 + n_2 = n$ and $n_1^* + n_2^* = n^*$. Let $I_i[I_i^*]$ be independent Bernoulli random variables, independent of X_i's, such that $\mathbb{E}[I_i] = p_1[\mathbb{E}[I_i^*] = p_1]$ for $i = 1, 2, \ldots, n_1[n_1^*]$, and let $I_j[I_j^*]$ be another set of independent Bernoulli variables independent of X_j's, such that $\mathbb{E}[I_j] = p_2[\mathbb{E}[I_j^*] = p_2]$ for $j = n_1[n_1^*] + 1, \ldots, n[n^*]$. If $G_1 \le_{\rm hr} G_2$, $n_2 \ge n_2^*$ and $p_1 \le p_2$, we then have*

$$n \le n^* \Longrightarrow Y_{1:n} \ge_{\rm hr} Y^*_{1:n^*}.$$

For the case of independent multiple-outlier claim models, Theorem 4.4 states that the smallest claim amount will be increased in the sense of the hazard rate order by increasing the number of insureds with larger claims and larger occurrence probabilities while decreasing the total sample size. It remains open to investigate whether if Theorem 4.4 still holds when the larger claims are accompanied with smaller occurrence probabilities.

We now move on to compare the smallest claim amounts arising from multiple-outlier models in the sense of the likelihood ratio order. In what follows, it is assumed that $\delta := \lim_{t\to 0} \frac{g_2(t)}{g_1(t)}$ always exits and is also unique to avoid technicality discussion.

Theorem 4.5. *Under the setting of Theorem 4.4, suppose that* $G_1 \leq_{\rm lr} G_2$, $n_2 \geq \max\{n_1^*, n_2^*\}$, $p_1 \leq p_2$ *and*

$$\delta\left[n_2\left(\frac{1}{p_1^{n_1^*}p_2^{n_2^*}}-1\right)-n_2^*\left(\frac{1}{p_1^{n_1}p_2^{n_2}}-1\right)\right] \geq n_1^*\left(\frac{1}{p_1^{n_1}p_2^{n_2}}-1\right)-n_1\left(\frac{1}{p_1^{n_1^*}p_2^{n_2^*}}-1\right), \tag{4.1}$$

where g_1 *and* g_2 *are probability density functions of* G_1 *and* G_2*, respectively. Then, we have*

$$(n_1, n_2) \overset{\rm w}{\succeq} (n_1^*, n_2^*) \Longrightarrow Y_{1:n} \geq_{\rm lr} Y_{1:n^*}^*.$$

Note that the condition in (4.1) looks clumsy and ugly, but it is necessary for the likelihood ratio order between the smallest claim amounts, which can be seen clearly from the proof of the density ratio function of the smallest claim amounts at the point 0. Moreover, it can be seen that the condition $(n_1, n_2) \overset{\rm w}{\succeq} (n_1^*, n_2^*)$ implies $n_1 + n_2 \leq n_1^* + n_2^*$, and (4.1) is an additional condition required for the likelihood ratio ordering compared with the hazard rate ordering result in Theorem 4.4.

Remark 4.6. *It should be mentioned that, in general,* δ *is not equal to* $g_2(0)/g_1(0)$*, which may be not even well defined (e.g.,* $g_1(0) = 0$ *for some distributions such as the gamma distribution with shape parameter strictly less than 1). For many well-known distributions ,* δ *not only exits but also is unique. For example,*

- *Under the setting of PHR model, consider* $G_1(t) = e^{-\lambda_1 R(t)}$ *and* $G_2(t) = e^{-\lambda_2 R(t)}$ *with* $\lambda_1 > \lambda_2$*. It can be checked that* $G_1 \leq_{\rm lr} G_2$ *and* $\delta = \lim_{t\to 0}(\lambda_2/\lambda_1)e^{(\lambda_1-\lambda_2)R(t)} = \lambda_2/\lambda_1$*, which is finite and unique.*
- *Suppose that* G_1 *and* G_2 *have the form of gamma distribution with respective paired shape and scale parameters* (r_1, λ) *and* (r_2, λ) *such that* $r_1 < r_2$*. Thus, we have* $G_1 \leq_{\rm lr} G_2$*. It is easy to check that* $\delta = 0$*. Then, the condition (4.1) boils down to* $n_1[1/(p_1^{n_1^*}p_2^{n_2^*}) - 1] \geq n_1^*[1/(p_1^{n_1}p_2^{n_2}) - 1]$*.*
- *Assume that* G_1 *and* G_2 *have the form of gamma distribution with respective paired shape and scale parameters* (r, λ_1) *and* (r, λ_2) *such that* $\lambda_1 > \lambda_2$*, from which we have* $G_1 \leq_{\rm lr} G_2$*. For this case, it can be seen that* $\delta = (\lambda_2/\lambda_1)^r$*.*

It can be roughly speaking from Theorem 4.5 that the smallest claim amount would be increased in the sense of the likelihood ratio order through decreasing the total sample size while increasing the number of insureds with higher claims and higher occurrence probabilities.

Remark 4.7. *Denote $Z \sim W(\gamma, \lambda)$ if Z is a Weibull distribution random variable having shape parameter $\gamma > 0$ and scale parameter $\lambda > 0$. Let $X_i \sim W(\gamma, \lambda_i)[X_i^* \sim W(\gamma, \lambda_i^*)]$, $i = 1, \ldots, n$. Suppose $I_1, \ldots, I_n[I_1^*, \ldots, I_n^*]$ is a set of independent Bernoulli random variables, independent of $X_i[X_i^*]$, with $\mathbb{E}(I_i) = p_i[\mathbb{E}(I_i^*) = p_i^*]$, $i = 1, \ldots, n$. Let $Y_i = I_i X_i$ and $Y_i^* = I_i^* X_i^*$, for $i = 1, \ldots, n$. Barmalzan et al. (2016) showed that, without any restriction on the occurrence probabilities, the condition $\sum_{i=1}^n \lambda_i^\gamma \le \sum_{i=1}^n (\lambda_i^*)^\gamma$ implies that $Y_{1:n} \ge_{\mathrm{lr}} Y_{1:n}^*$. However, this result is not correct due to the reason that they neglected the value of the density ratio function $f_{Y_{1:n}}(t)/f_{Y_{1:n}^*}(t)$ at point $t = 0$. In order to ensure the likelihood ratio order to hold, it is required that*

$$\frac{1 - \prod_{i=1}^n p_i}{1 - \prod_{i=1}^n p_i^*} \le \frac{\prod_{i=1}^n p_i \sum_{i=1}^n \lambda_i^\gamma}{\prod_{i=1}^n p_i^* \sum_{i=1}^n (\lambda_i^*)^\gamma}.$$

In this regard, we have to consider the following two cases:

(i) If $\prod_{i=1}^n p_i < \prod_{i=1}^n p_i^*$, then $Y_{1:n} \not\ge_{\mathrm{lr}} Y_{1:n}^*$.
(ii) If $\prod_{i=1}^n p_i \ge \prod_{i=1}^n p_i^*$ and

$$\frac{1/\prod_{i=1}^n p_i - 1}{1/\prod_{i=1}^n p_i^* - 1} \sum_{i=1}^n (\lambda_i^*)^\gamma \le \sum_{i=1}^n \lambda_i^\gamma \le \sum_{i=1}^n (\lambda_i^*)^\gamma, \tag{4.2}$$

then $Y_{1:n} \ge_{\mathrm{lr}} Y_{1:n}^*$.

Therefore, the sufficient conditions for the likelihood ratio order between $Y_{1:n}$ and $Y_{1:n}^*$ should be such that $\prod_{i=1}^n p_i \ge \prod_{i=1}^n p_i^*$ and (4.2).

The following numerical example is presented as an illustration of Theorem 4.5.

Example 4.8. *Under the setting of Theorem 4.5, we take $G_1(t) = 1 - \exp(-0.9t)$ and $G_2(t) = 1 - \exp(-0.1t)$, $t \ge 0$. Set $p_1 = 0.3$, $p_2 = 0.5$, $n_1 = 1$, $n_2 = 5$, $n_1^* = 2$, $n_2^* = 5$, and we then have $(n_1, n_2) \overset{\mathrm{w}}{\succeq} (n_1^*, n_2^*)$ and $G_1 \le_{\mathrm{lr}} G_2$. It can be observed from Figure 4 that $Y_{1:n} \ge_{\mathrm{lr}} Y_{1:n^*}^*$, which supports the theoretical result of Theorem 4.5.*

The following corollary can be readily derived from Theorem 4.5.

Corollary 4.9. *Under the setting of Theorem 4.5, if $G_1 = G_2$, $n_2 \ge \max\{n_1^*, n_2^*\}$ and $p_1 \le p_2$, then, we have*

$$(n_1, n_2) \overset{\mathrm{m}}{\succeq} (n_1^*, n_2^*) \Longrightarrow Y_{1:n} \ge_{\mathrm{lr}} Y_{1:n^*}^*.$$

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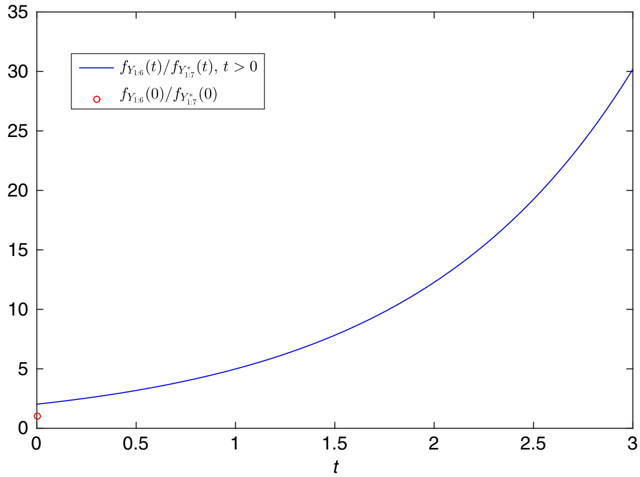

FIGURE 4: Plot of the ratio $f_{Y_{1:6}}(t)/f_{Y^*_{1:7}}(t)$ in Example 4.8.

5. Concluding Remarks

It is known that the extreme claim amounts turn out to be quite important in analyzing insurance portfolios since they provide important information for determining annual premium. In this paper, we establish sufficient conditions to stochastically compare the largest/smallest claim amounts arising from two sets of heterogeneous insurance portfolios in the sense of the usual stochastic, hazard rate, likelihood ratio, right-spread and convex transform orders. The effects of heterogeneity among sample sizes and occurrence probabilities as well as the dependence among claim sizes and/or occurrence probabilities are investigated. These results can provide insights for insurance companies to choose underwriting strategies.

It is of great importance to investigate the dispersion and shape orders for the largest claim amounts arising from two sets of heterogeneous insurance portfolios, which is still absent in the literature. We are currently working on it and some interesting findings have been found. We shall report more meaningful results in another future paper.

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APPENDIX A. PROOFS FOR ALL MAIN RESULTS

A.1. Proof of Theorem 3.1

Proof. Since the weak supermajorization order can be constructed by using the usual majorization order, the proof can be finished by conducting the following two steps:

Step 1. For real vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$, let us use $\boldsymbol{a} \circ \boldsymbol{x} = (a_1 x_1, \ldots, a_n x_n)$ to denote the Hadamard product. For any pair i, j such that $1 \le i < j \le n$, we use $x_{\{i,j\}} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}$ to represent the sub-vector with ith and jth entries removed. Let $\boldsymbol{I} = \{I_{p_1}, \ldots, I_{p_n}\}$ and $\boldsymbol{A} = \{I_{p_1} X_1, \ldots, I_{p_n} X_n\}$. By the nature of majorization, it suffices to prove that, for any increasing function u,

$$\mathbb{E}\left[u\left(\max\{I_{p_i} X_i, I_{p_j} X_j, \boldsymbol{A}_{\{i,j\}}\}\right)\right] \ge \mathbb{E}\big[u\big(\max\{I_{p_i^*} X_i, I_{p_j^*} X_j, \boldsymbol{A}_{\{i,j\}}\}\big)\big], \tag{A.1}$$

under the conditions $(h(p_i), h(p_j)) \stackrel{\text{m}}{\succeq} (h(p_i^*), h(p_j^*))$, $\big(h(p_i), h(p_j)\big) \in \mathcal{D}_2^+$ and $\big(h(p_i^*), h(p_j^*)\big) \in \mathcal{D}_2^+$ and $1 \le i < j \le n$. Note that, for any increasing function u, binary vector $\boldsymbol{a}_{\{i,j\}}$ with $a_k = 0$ or 1 and $x_k > 0$ for $k \notin \{i, j\}$,

$$
\begin{aligned}
&\mathbb{E}\left[u\left(\max\{I_{p_i}X_i, I_{p_j}X_j, \boldsymbol{A}_{\{i,j\}}\}\right)|\boldsymbol{I}_{\{i,j\}}=\boldsymbol{a}_{\{i,j\}}, \boldsymbol{X}_{\{i,j\}}=\boldsymbol{x}_{\{i,j\}}\right]\\
&\quad -\mathbb{E}\big[u\big(\max\{I_{p_i^*}X_i, I_{p_j^*}X_j, \boldsymbol{A}_{\{i,j\}}\}\big)|\boldsymbol{I}_{\{i,j\}}=\boldsymbol{a}_{\{i,j\}}, \boldsymbol{X}_{\{i,j\}}=\boldsymbol{x}_{\{i,j\}}\big]\\
&=\mathbb{E}\left[u\left(\max\{I_{p_i}X_i, I_{p_j}X_j, \{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\}\right)|\boldsymbol{X}_{\{i,j\}}=\boldsymbol{x}_{\{i,j\}}\right]\\
&\quad -\mathbb{E}\big[u\big(\max\{I_{p_i^*}X_i, I_{p_j^*}X_j, \{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\}\big)|\boldsymbol{X}_{\{i,j\}}=\boldsymbol{x}_{\{i,j\}}\big]\\
&=[p_i(1-p_j)-p_i^*(1-p_j^*)]\\
&\quad\times\left\{\mathbb{E}\left[u\left(\max\{X_i, \{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\}\right)|\boldsymbol{X}_{\{i,j\}}=\boldsymbol{x}_{\{i,j\}}\right]-u\left(\max\{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\right)\right\}\\
&\quad+[p_j(1-p_i)-p_j^*(1-p_i^*)]\\
&\quad\times\left\{\mathbb{E}\left[u\left(\max\{X_j, \{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\}\right)|\boldsymbol{X}_{\{i,j\}}=\boldsymbol{x}_{\{i,j\}}\right]-u\left(\max\{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\right)\right\}\\
&\quad+(p_ip_j-p_i^*p_j^*)\\
&\quad\times\left\{\mathbb{E}\left[u\left(\max\{X_i, X_j, \{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\}\right)|\boldsymbol{X}_{\{i,j\}}=\boldsymbol{x}_{\{i,j\}}\right]-u\left(\max\{\boldsymbol{a}\circ\boldsymbol{x}\}_{\{i,j\}}\right)\right\}.
\end{aligned}
$$

Since $h(x)$ is decreasing and convex, the condition $(h(p_i), h(p_j)) \overset{\text{m}}{\succeq} (h(p_i^*), h(p_j^*))$ implies that $p_i \le p_i^* \le p_j^* \le p_j$. Denoting by h^{-1} the inverse function of h, it is known that h^{-1} is also decreasing and convex, which implies that $p_i + p_j$ is Schur-convex in $\big(h(p_i), h(p_j)\big)$ upon applying Proposition 3.C.1 of Marshall *et al.* (2011). Thus, we have

$$p_i + p_j \ge p_i^* + p_j^*. \tag{A.2}$$

Consider the following two cases:

Case 1: $h(p) = -\log p$. Obviously, $(-\log p_i, -\log p_j) \overset{\text{m}}{\succeq} (-\log p_i^*, -\log p_j^*)$ implies $p_ip_j = p_i^*p_j^*$, it follows from inequality (A.2) that $p_i(1-p_j) + p_j(1-p_i) \ge p_i^*(1-p_j^*) + p_j^*(1-p_i^*)$.

Case 2: $h(p) = (1-p)/p$. Clearly, $((1-p_i)/p_i, (1-p_j)/p_j) \overset{\text{m}}{\succeq} ((1-p_i^*)/p_i^*, (1-p_j^*)/p_j^*)$ implies

$$\frac{1-p_i}{p_i} + \frac{1-p_j}{p_j} = \frac{1-p_i^*}{p_i^*} + \frac{1-p_j^*}{p_j^*},$$

which further implies

$$\frac{p_i+p_j}{p_ip_j} = \frac{p_i^*+p_j^*}{p_i^*p_j^*}$$

and

$$\frac{p_i(1-p_j)+p_j(1-p_i)}{p_ip_j} = \frac{p_i^*(1-p_j^*)+p_j^*(1-p_i^*)}{p_i^*p_j^*}.$$

From inequality (A.2) again, we have $p_ip_j \ge p_i^*p_j^*$ and $p_i(1-p_j) + p_j(1-p_i) \ge p_i^*(1-p_j^*) + p_j^*(1-p_i^*)$.

Hence, we can conclude that, for either $h(p) = -\log p$ or $h(p) = (1-p)/p$, we have

$$p_i p_j \geq p_i^* p_j^*, \tag{A.3}$$

$$p_j(1-p_i) - p_j^*(1-p_i^*) \geq p_i^*(1-p_j^*) - p_i(1-p_j) \geq 0. \tag{A.4}$$

Since $\boldsymbol{X}$ is non-negative and u is increasing, it must be true that

$$\mathbb{E}\left[u\left(\max\{X_j, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\}\right) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right] \geq u\left(\max\{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\right) \tag{A.5}$$

and

$$\mathbb{E}\left[u\left(\max\{X_i, X_j, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\}\right) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right] \geq u\left(\max\{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\right). \tag{A.6}$$

On the other hand, for $1 \leq i < j \leq n$, the WSAI property of $\boldsymbol{X}$ implies that of $(X_i, X_j)|\boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}$ by Proposition 3.2 of Cai and Wei (2015), and hence $[X_j|\boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}] \geq_{\text{st}} [X_i|\boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}]$. Then, for any increasing u,

$$\begin{aligned}&\mathbb{E}\left[u\left(\max\{X_j, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\}\right) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right]\\ &\quad \geq \mathbb{E}\left[u\left(\max\{X_i, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\}\right) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right].\end{aligned} \tag{A.7}$$

Thus, for any $a_k = 0$ or 1, $x_k > 0$, $k \notin \{i, j\}$ and $1 \leq i < j \leq n$, we have

$$\begin{aligned}&\mathbb{E}\left[u\left(\max\{I_{p_i}X_i, I_{p_j}X_j, \boldsymbol{A}_{\{i,j\}}\}\right) | \boldsymbol{I}_{\{i,j\}} = \boldsymbol{a}_{\{i,j\}}, \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right]\\ &\quad - \mathbb{E}\left[u\left(\max\{I_{p_i^*}X_i, I_{p_j^*}X_j, \boldsymbol{A}_{\{i,j\}}\}\right) | \boldsymbol{I}_{\{i,j\}} = \boldsymbol{a}_{\{i,j\}}, \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right]\\ &\geq [p_i^*(1-p_j^*) - p_i(1-p_j)]\Big\{\mathbb{E}\left[u\left(\max\{X_j, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\}\right) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right]\\ &\quad - \mathbb{E}\left[u\left(\max\{X_i, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\}\right) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right]\Big\}\\ &\quad + (p_i p_j - p_i^* p_j^*)\Big\{\mathbb{E}\left[u\left(\max\{X_i, X_j, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\}\right) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\right]\\ &\quad - u\left(\max\{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{i,j\}}\right)\Big\} \geq 0,\end{aligned}$$

where the first inequality follows from inequalities (A.4) and (A.5), and the second one stems from inequalities (A.3), (A.4), (A.6) and (A.7). Taking the iterated expectation, we have, for any increasing u,

$$\begin{aligned}&\mathbb{E}\left[u\left(\max\{I_{p_i}X_i, I_{p_j}X_j, \boldsymbol{A}_{\{i,j\}}\}\right)\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[u\left(\max\{I_{p_i}X_i, I_{p_j}X_j, \boldsymbol{A}_{\{i,j\}}\}\right) | \boldsymbol{I}_{\{i,j\}}, \boldsymbol{X}_{\{i,j\}}\right]\right]\\ &\geq \mathbb{E}\left[\mathbb{E}\left[u\left(\max\{I_{p_i^*}X_i, I_{p_j^*}X_j, \boldsymbol{A}_{\{i,j\}}\}\right) | \boldsymbol{I}_{\{i,j\}}, \boldsymbol{X}_{\{i,j\}}\right]\right]\\ &= \mathbb{E}\left[u\left(\max\{I_{p_i^*}X_i, I_{p_j^*}X_j, \boldsymbol{A}_{\{i,j\}}\}\right)\right],\end{aligned}$$

which completes the proof.

Step 2. Since $h(\boldsymbol{p})$, $h(\boldsymbol{p}^*) \in \mathcal{D}_n^+$, we have $h(p_1) \geq h(p_2) \geq \cdots \geq h(p_n) \geq 0$ and $h(p_1^*) \geq h_{p_2^*} \geq \cdots \geq h(p_n^*) \geq 0$. Then, it follows that $0 \leq p_1 \leq \cdots \leq p_n$ and $0 \leq p_1^* \leq \cdots \leq p_n^*$. From the definition of weak supermajorization order, it is known that $\sum_{i=j}^{n} h(p_i) \leq \sum_{i=j}^{n} h(p_i^*)$, for $1 \leq j \leq n$. Then, there must exist some p' such that

$$h(p') \geq \max\{h(p_1), h(p_1^*)\} \text{ and } (h(p'), h(p_2), \ldots, h(p_n)) \overset{\mathrm{m}}{\succeq} (h(p_1^*), h(p_2^*), \ldots, h(p_n^*)),$$

which implies that $p' \leq \min\{p_1, p_1^*\}$. Let $Z_{n:n}$ be the largest claim amount arising from portfolio of risks $I_{p'}X_1, I_{p_2}X_2, \ldots, I_{p_n}X_n$. Then, we have $Z_{n:n} \geq_{\mathrm{st}} Y_{n:n}^*$ from Step 1. In what follows, we only need to prove that $Y_{n:n} \geq_{\mathrm{st}} Z_{n:n}$, which is equivalent to showing that, for any increasing function u,

$$\mathbb{E}\left[u\left(\max\{I_{p_1}X_1, \boldsymbol{A}_{\{1\}}\}\right)\right] \geq \mathbb{E}\left[u\left(\max\{I_{p'}X_1, \boldsymbol{A}_{\{1\}}\}\right)\right], \tag{A.8}$$

where $\boldsymbol{A}_{\{1\}} = \{I_{p_2}X_2, I_{p_3}X_3, \ldots, I_{p_n}X_n\}$. Note that, for any increasing function u, binary $\boldsymbol{a}_{\{1\}}$ with $a_k = 0$ or 1 and $x_k > 0$ for $k \neq 1$,

$$\begin{aligned}
&\mathbb{E}\left[u\left(\max\{I_{p_1}X_1, \boldsymbol{A}_{\{1\}}\}\right) | \boldsymbol{I}_{\{1\}} = \boldsymbol{a}_{\{1\}}, \boldsymbol{X}_{\{1\}} = \boldsymbol{x}_{\{1\}}\right] \\
&\quad - \mathbb{E}\left[u\left(\max\{I_{p'}X_1, \boldsymbol{A}_{\{1\}}\}\right) | \boldsymbol{I}_{\{1\}} = \boldsymbol{a}_{\{1\}}, \boldsymbol{X}_{\{1\}} = \boldsymbol{x}_{\{1\}}\right] \\
&= \mathbb{E}\left[u\left(\max\{I_{p_1}X_1, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{1\}}\}\right) | \boldsymbol{X}_{\{1\}} = \boldsymbol{x}_{\{1\}}\right] \\
&\quad - \mathbb{E}\left[u\left(\max\{I_{p'}X_1, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{1\}}\}\right) | \boldsymbol{X}_{\{1\}} = \boldsymbol{x}_{\{1\}}\right] \\
&= (p_1 - p')\left\{\mathbb{E}\left[u\left(\max\{X_1, \{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{1\}}\}\right) | \boldsymbol{X}_{\{1\}} = \boldsymbol{x}_{\{1\}}\right] - u\left(\max\{\boldsymbol{a} \circ \boldsymbol{x}\}_{\{1\}}\right)\right\} \\
&\geq 0,
\end{aligned}$$

where the last inequality holds because of $p' \leq p_1$, the increasing property of u and the non-negativity of $\boldsymbol{X}$. Then, taking the double expectation leads to the inequality (A.8), and hence the proof is finished. ■

A.2. Proof of Lemma 3.5

Proof. *Step 1.* First, we consider the exponential case, that is, $\overline{G}(x) = e^{-x}$. Let $\lambda := \sum_{i=1}^{n} \lambda_i$ and $\lambda^* := \sum_{i=1}^{n^*} \lambda_i^*$. The survival function of $X_{1:n}$ can be written as

$$\overline{F}_{X_{1:n}}(x) = e^{-\sum_{i=1}^{n} \lambda_i x} = e^{-\lambda x}.$$

Denote

$$W_{X_{1:n}}(x) = \int_x^{\infty} \overline{F}_{X_{1:n}}(t)\mathrm{d}t = \int_x^{\infty} e^{-\lambda t}\mathrm{d}t.$$

Taking the derivative of $W_{X_{1:n}}(x)$ with respect to λ gives rise to

$$W'_{X_{1:n}}(x) = -\int_x^{\infty} te^{-\lambda t}\mathrm{d}t.$$

In light of Theorem 1 of Barmalzan and Payandeh Najafabadi (2015), it holds that the function $W'_{X_{1:n}}(x)/\overline{F}_{X_{1:n}}(x)$ is decreasing in $x \in \mathbb{R}_+$. Hence, the desired result follows by Lemma 3.4.

Step 2. The remaining proof is similar to that of Theorem 4.1 in Kochar and Xu (2009). Note that the cumulative hazard rate function of G is given by $H(x) = -\log \overline{G}(x)$. Then, it follows that for $x \in \mathbb{R}_+$ and $i = 1, 2, \ldots, n$,

$$\mathbb{P}(H(X_i) > x) = \mathbb{P}(X_i > H^{-1}(x)) = \overline{G}^{\lambda_i}(\overline{G}^{-1}(e^{-x})) = e^{-\lambda_i x},$$

where H^{-1} is the right inverse of H. Letting $X_i' := H(X_i)$, one can note that X_i' is exponential random variable with hazard rate λ_i, for $i = 1, \ldots, n$. Similarly, let $X_i^{*'} := H(X_i^*)$ be exponential random variable with hazard rate λ_i^*, for $i = 1, \ldots, n^*$. From Step 1, it follows that, for $\sum_{i=1}^n \lambda_i \leq \sum_{i=1}^{n^*} \lambda_i^*$, $X'_{1:n} \geq_{\text{RS}} X^{*'}_{1:n^*}$, which is equivalent to $H(X_{1:n}) \geq_{\text{RS}} H(X^*_{1:n^*})$. Since G has DHR, we know that H is increasing and concave, and thus H^{-1} is increasing and convex. Now, the required result follows from Theorem 4.1 in Kochar *et al.* (2002). ■

A.3. Proof of Theorem 3.6

Proof. Let $\lambda := \sum_{i=1}^n \lambda_i$ and $\lambda^* := \sum_{i=1}^{n^*} \lambda_i^*$. The survival function of $Y_{1:n}$ can be written as, for any $x \in \mathbb{R}_+$,

$$\begin{aligned}\overline{F}_\lambda(x) &= \prod_{i=1}^n \mathbb{P}(X_i > x)\mathbb{P}(\boldsymbol{I}_n = \mathbf{1}) \\ &= \mathbb{P}(\boldsymbol{I}_n = \mathbf{1})e^{-(\sum_{i=1}^n \lambda_i)R(x)} = \mathbb{P}(\boldsymbol{I}_n = \mathbf{1})e^{-\lambda R(x)},\end{aligned}$$

where $R(x) = \int_0^x r(t)\mathrm{d}t$ is the baseline cumulative hazard rate function. Similarly, the survival function of $Y^*_{1:n^*}$, for $x \in \mathbb{R}_+$, is given by $\overline{F}_{\lambda^*}(x) = \mathbb{P}(\boldsymbol{I}^*_{n^*} = \mathbf{1})e^{-\lambda^* R(x)}$. Let

$$W_\lambda(x) = \int_x^\infty \overline{F}_\lambda(t)\mathrm{d}t = \mathbb{P}(\boldsymbol{I}_n = \mathbf{1}) \int_x^\infty e^{-\lambda R(t)}\mathrm{d}t.$$

Taking the derivative of $W_\lambda(x)$ with respect to λ, we have

$$W'_\lambda(x) = -\mathbb{P}(\boldsymbol{I}_n = \mathbf{1}) \int_x^\infty R(t)e^{-\lambda R(t)}\mathrm{d}t.$$

The desired result boils down to showing that

$$\frac{W'_\lambda(x)}{\overline{F}_\lambda(x)} = \frac{-\int_x^\infty R(t)e^{-\lambda R(t)}\mathrm{d}t}{e^{-\lambda R(x)}}$$

is decreasing in $x \in \mathbb{R}_+$, which can be obtained from Lemmas 3.4 and 3.5 ■

A.4. Proof of Theorem 3.7

Proof. Making use of Corollary 4.A.32 of Shaked and Shanthikumar (2007), we know that the right-spread order implies the increasing convex order for non-negative random

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variables, which means that (i) implies (ii). Next, we prove that (ii) implies (iii). Suppose that $Y_{1:n} \geq_{\text{icx}} Y^*_{1:n^*}$, it follows that $\mathbb{E}[Y_{1:n}] \geq \mathbb{E}[Y^*_{1:n^*}]$, that is,

$$\mathbb{E}[Y_{1:n}] = \mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1}) \int_0^\infty e^{-(\sum_{i=1}^n \lambda_i) R(x)} \mathrm{d}x \geq \mathbb{E}[Y^*_{1:n^*}] = \mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1}) \int_0^\infty e^{-\left(\sum_{i=1}^{n^*} \lambda^*_i\right) R(x)} \mathrm{d}x. \tag{A.9}$$

If $\sum_{i=1}^n \lambda_i > \sum_{i=1}^{n^*} \lambda^*_i$, then it can be seen from (A.9) that $\mathbb{E}[Y_{1:n}] < \mathbb{E}[Y^*_{1:n^*}]$, which contradicts with the assumption (ii). Thus, (iii) is implied by (ii). On the other hand, it has been proved in Theorem 3.6 that (iii) implies (i). Therefore, the theorem follows. ■

A.5. Proof of Theorem 3.8

Proof. The distribution functions of $Y_{1:n}$ and $Y^*_{1:n}$ can be written as

$$F_{Y_{1:n}}(x) = 1 - \mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1}) e^{-\left(\sum_{i=1}^n \lambda_i^{\gamma_1}\right) x^{\gamma_1}}$$

and

$$F_{Y^*_{1:n^*}}(x) = 1 - \mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1}) e^{-\left(\sum_{i=1}^{n^*} (\lambda^*_i)^{\gamma_2}\right) x^{\gamma_2}}.$$

Note that

$$F^{-1}_{Y_{1:n}}(x) = \left(-\frac{1}{\sum_{i=1}^n \lambda_i^{\gamma_1}} \ln\left(\frac{1-x}{\mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1})}\right)\right)^{1/\gamma_1}, \quad x \geq 1 - \mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1}).$$

First, in order to make sure that $F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n}}(x))$ is well defined, we need that $F_{Y^*_{1:n^*}}(x) \geq F_{Y_{1:n}}(0)$, which holds naturally from the condition $\mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1}) \leq \mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1})$. Note that

$$\begin{aligned} F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x)) &= \left(-\frac{1}{\sum_{i=1}^n \lambda_i^{\gamma_1}} \ln\left(\frac{1-F_{Y^*_{1:n^*}}(x)}{\mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1})}\right)\right)^{1/\gamma_1} \\ &= \left(-\frac{1}{\sum_{i=1}^n \lambda_i^{\gamma_1}} \ln\left(\frac{\mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1})}{\mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1})}\right) + \frac{\sum_{i=1}^{n^*} (\lambda^*_i)^{\gamma_2}}{\sum_{i=1}^n \lambda_i^{\gamma_1}} x^{\gamma_2}\right)^{1/\gamma_1}. \end{aligned}$$

The desired result can be reached if we can show that $F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x))$ is convex in $x > 0$. By taking the derivative of $F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x))$ with respect to x, we have

$$\frac{\mathrm{d}F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x))}{\mathrm{d}x} \overset{\text{sign}}{=} x^{\gamma_2 - 1} \left(-\frac{1}{\sum_{i=1}^n \lambda_i^{\gamma_1}} \ln\left(\frac{\mathbb{P}(\boldsymbol{I}^*_{n^*} = \boldsymbol{1})}{\mathbb{P}(\boldsymbol{I}_n = \boldsymbol{1})}\right) + \frac{\sum_{i=1}^{n^*} (\lambda^*_i)^{\gamma_2}}{\sum_{i=1}^n \lambda_i^{\gamma_1}} x^{\gamma_2}\right)^{1/\gamma_1 - 1},$$

based on which we can obtain

$$\frac{\mathrm{d}^2 F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x))}{\mathrm{d}x^2}$$

$$\stackrel{\text{sign}}{=} (\gamma_2-1)x^{\gamma_2-2}\left(-\frac{1}{\sum_{i=1}^n \lambda_i^{\gamma_1}}\ln\left(\frac{\mathbb{P}(\boldsymbol{I}^*_{n^*}=\mathbf{1})}{\mathbb{P}(\boldsymbol{I}_n=\mathbf{1})}\right)+\frac{\sum_{i=1}^{n^*}(\lambda_i^*)^{\gamma_2}}{\sum_{i=1}^n \lambda_i^{\gamma_1}}x^{\gamma_2}\right)^{1/\gamma_1-1}$$

$$+\gamma_2\left(\frac{1}{\gamma_1}-1\right)\frac{\sum_{i=1}^{n^*}(\lambda_i^*)^{\gamma_2}}{\sum_{i=1}^n \lambda_i^{\gamma_1}}x^{2\gamma_2-2}$$

$$\times\left(-\frac{1}{\sum_{i=1}^n \lambda_i^{\gamma_1}}\ln\left(\frac{\mathbb{P}(\boldsymbol{I}^*_{n^*}=\mathbf{1})}{\mathbb{P}(\boldsymbol{I}_n=\mathbf{1})}\right)+\frac{\sum_{i=1}^{n^*}(\lambda_i^*)^{\gamma_2}}{\sum_{i=1}^n \lambda_i^{\gamma_1}}x^{\gamma_2}\right)^{1/\gamma_1-2}$$

$$\stackrel{\text{sign}}{=} -\frac{\gamma_2-1}{\sum_{i=1}^n \lambda_i^{\gamma_1}}\ln\left(\frac{\mathbb{P}(\boldsymbol{I}^*_{n^*}=\mathbf{1})}{\mathbb{P}(\boldsymbol{I}_n=\mathbf{1})}\right)+\frac{\sum_{i=1}^{n^*}(\lambda_i^*)^{\gamma_2}}{\sum_{i=1}^n \lambda_i^{\gamma_1}}\left(2\gamma_2-1-\frac{\gamma_2}{\gamma_1}\right)x^{\gamma_2}. \tag{A.10}$$

From the assumptions that $\gamma_2 \geq 1$, $\gamma_1 \geq \frac{\gamma_2}{2\gamma_2-1}$ and $\mathbb{P}(\boldsymbol{I}^*_{n^*}=\mathbf{1}) \leq \mathbb{P}(\boldsymbol{I}_n=\mathbf{1})$, we can conclude that (A.10) is non-negative. Thus, the proof is finished. ■

A.6. Proof of Theorem 3.10

Proof. We only give the proof for (i) since the result for the case of (ii) can be obtained directly from Theorem 3.8. According to $\mathbb{P}(\boldsymbol{I}^*_{n^*}=\mathbf{1})=\mathbb{P}(\boldsymbol{I}_n=\mathbf{1})$, we know that $F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x))$ is well defined. The desired result is equivalent to showing that $F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x))/x$ is increasing in $x>0$. According to the proof of Theorem 3.8, we have

$$\frac{F^{-1}_{Y_{1:n}}(F_{Y^*_{1:n^*}}(x))}{x}=\left(\frac{\sum_{i=1}^{n^*}(\lambda_i^*)^{\gamma_2}}{\sum_{i=1}^n \lambda_i^{\gamma_1}}x^{\gamma_2-\gamma_1}\right)^{1/\gamma_1},$$

which is obviously increasing in $x>0$ due to $\gamma_2 \geq \gamma_1$. ■

A.7. Proof of Theorem 3.11

Proof. We only give the proof for the case of (i) since the proof for (ii) can be completed in a similar manner. It is easy to check that $Y_{1:n} \geq_{\text{st}} Y^*_{1:n^*}$ by using $\sum_{i=1}^n \lambda_i^{\gamma_1} \leq \sum_{i=1}^{n^*}(\lambda_i^*)^{\gamma_2}$ and $\mathbb{P}(\boldsymbol{I}^*_{n^*}=\mathbf{1})=\mathbb{P}(\boldsymbol{I}_n=\mathbf{1})$. On the other hand, we have $Y_{1:n} \geq_{\star} Y^*_{1:n^*}$ by applying Theorem 3.10. According to the statement on Page 473 in Bartoszewicz (1985) (see also Theorem 2.3 in Ahmed *et al.*, 1986), we must have $Y_{1:n} \geq_{\text{disp}} Y^*_{1:n^*}$. ■

A.8. Proof of Theorem 4.1

Proof. The distribution function of $Y_{n:n}$ is given by

$$F_{Y_{n:n}}(t)=\left(1-p_1\overline{G}(t)\right)^{n_1}\left(1-p_2\overline{G}(t)\right)^{n_2}, \quad t \geq 0,$$

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and this leads to the density function of $Y_{n:n}$ given as

$$f_{Y_{n:n}}(t)=\begin{cases}(1-p_1)^{n_1}(1-p_2)^{n_2}+n_1p_1g(0)(1-p_1)^{n_1-1}(1-p_2)^{n_2} & \\ \qquad +n_2p_2g(0)(1-p_1)^{n_1}(1-p_2)^{n_2-1} & \text{for } t=0,\\ n_1p_1g(t)\left(1-p_1\overline{G}(t)\right)^{n_1-1}\left(1-p_2\overline{G}(t)\right)^{n_2} & \\ \qquad +n_2p_2g(t)\left(1-p_2\overline{G}(t)\right)^{n_2-1}\left(1-p_1\overline{G}(t)\right)^{n_1} & \text{for } t>0,\end{cases}$$

where g is the density function of G. Similarly, we can write the density function of $Y^*_{n^*:n^*}$ as

$$f_{Y^*_{n^*:n^*}}(t)=\begin{cases}(1-p_1)^{n_1^*}(1-p_2)^{n_2^*}+n_1^*p_1g(0)(1-p_1)^{n_1^*-1}(1-p_2)^{n_2^*} & \\ \qquad +n_2^*p_2g(0)(1-p_1)^{n_1^*}(1-p_2)^{n_2^*-1} & \text{for } t=0,\\ n_1^*p_1g(t)\left(1-p_1\overline{G}(t)\right)^{n_1^*-1}\left(1-p_2\overline{G}(t)\right)^{n_2^*} & \\ \qquad +n_2^*p_2g(t)\left(1-p_2\overline{G}(t)\right)^{n_2^*-1}\left(1-p_1\overline{G}(t)\right)^{n_1^*} & \text{for } t>0.\end{cases}$$

For the sake of convenience, denote

$$S_{n_1,n_2}(t)=\left(1-p_1\overline{G}(t)\right)^{n_1}\left(1-p_2\overline{G}(t)\right)^{n_2}\quad\text{and}\quad S_{n_1^*,n_2^*}(t)=(1-p_1\overline{G}(t))^{n_1^*}(1-p_2\overline{G}(t))^{n_2^*}.$$

First, it is necessary to require that $S_{n_1,n_2}(0)\le S_{n_1^*,n_2^*}(0)$, that is, $(1-p_1)^{n_1}(1-p_2)^{n_2}\le(1-p_1)^{n_1^*}(1-p_2)^{n_2^*}$. From the assumptions $n_2^*\ge n_2$, $n_1^*+n_2^*\le n_1+n_2$ and $p_1\ge p_2$, we immediately have $(1-p_1)^{n_1-n_1^*}\le(1-p_1)^{n_2^*-n_2}\le(1-p_2)^{n_2^*-n_2}$, which proves the desired inequality.

Next, we show that $f_{Y_{n:n}}(t)/f_{Y^*_{n^*:n^*}}(t)$ is increasing in $t\in\mathbb{R}_+$.

Case 1: $t>0$. For this case, the problem falls into proving that

$$\varphi(t):=\frac{n_1p_1S_{n_1-1,n_2}(t)+n_2p_2S_{n_1,n_2-1}(t)}{n_1^*p_1S_{n_1^*-1,n_2^*}(t)+n_2^*p_2S_{n_1^*,n_2^*-1}(t)}$$

is increasing in $t>0$. Taking the derivative of $\varphi(t)$ with respect to t, we have

$$\begin{aligned}\varphi'(t)&\overset{\text{sign}}{=}\left[n_1p_1S'_{n_1-1,n_2}(t)+n_2p_2S'_{n_1,n_2-1}(t)\right]\times\left[n_1^*p_1S_{n_1^*-1,n_2^*}(t)+n_2^*p_2S_{n_1^*,n_2^*-1}(t)\right]\\&\quad-\left[n_1p_1S_{n_1-1,n_2}(t)+n_2p_2S_{n_1,n_2-1}(t)\right]\times\left[n_1^*p_1S'_{n_1^*-1,n_2^*}(t)+n_2^*p_2S'_{n_1^*,n_2^*-1}(t)\right]\\&=n_1n_1^*p_1^2S'_{n_1-1,n_2}(t)S_{n_1^*-1,n_2^*}(t)-n_1n_1^*p_1^2S'_{n_1^*-1,n_2^*}(t)S_{n_1-1,n_2}(t)\\&\quad+n_2n_1^*p_1p_2S'_{n_1,n_2-1}(t)S_{n_1^*-1,n_2^*}(t)-n_1^*n_2p_1p_2S'_{n_1^*-1,n_2^*}(t)S_{n_1,n_2-1}(t)\\&\quad+n_1n_2^*p_1p_2S'_{n_1-1,n_2}(t)S_{n_1^*,n_2^*-1}(t)-n_1n_2^*p_1p_2S'_{n_1^*,n_2^*-1}(t)S_{n_1-1,n_2}(t)\\&\quad+n_2n_2^*p_2^2S'_{n_1,n_2-1}(t)S_{n_1^*,n_2^*-1}(t)-n_2^*n_2p_2^2S'_{n_1^*,n_2^*-1}(t)S_{n_1,n_2-1}(t)\\&=:A_1+A_2+A_3+A_4,\quad\text{say,}\end{aligned}$$

where

$$\begin{aligned}A_1&=n_1n_1^*p_1^2S'_{n_1-1,n_2}(t)S_{n_1^*-1,n_2^*}(t)-n_1n_1^*p_1^2S'_{n_1^*-1,n_2^*}(t)S_{n_1-1,n_2}(t)\\&\overset{\text{sign}}{=}\left(\frac{S_{n_1-1,n_2}(t)}{S_{n_1^*-1,n_2^*}(t)}\right)'=\left[\frac{\left(1-p_1\overline{G}(t)\right)^{n_1-n_1^*}}{\left(1-p_2\overline{G}(t)\right)^{n_2^*-n_2}}\right]',\end{aligned}$$

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$$A_2 = n_2 n_1^* p_1 p_2 S'_{n_1,n_2-1}(t) S_{n_1^*-1,n_2^*}(t) - n_1^* n_2 p_1 p_2 S'_{n_1^*-1,n_2^*}(t) S_{n_1,n_2-1}(t)$$

$$\overset{\text{sign}}{=} \left(\frac{S_{n_1,n_2-1}(t)}{S_{n_1^*-1,n_2^*}(t)}\right)' = \left[\frac{\left(1-p_1\overline{G}(t)\right)^{n_1-n_1^*+1}}{\left(1-p_2\overline{G}(t)\right)^{n_2^*-n_2+1}}\right]',$$

$$A_3 = n_1 n_2^* p_1 p_2 S'_{n_1-1,n_2}(t) S_{n_1^*,n_2^*-1}(t) - n_1 n_2^* p_1 p_2 S'_{n_1^*,n_2^*-1}(t) S_{n_1-1,n_2}(t)$$

$$\overset{\text{sign}}{=} \left(\frac{S_{n_1-1,n_2}(t)}{S_{n_1^*,n_2^*-1}(t)}\right)' = \left[\frac{\left(1-p_1\overline{G}(t)\right)^{n_1-n_1^*-1}}{\left(1-p_2\overline{G}(t)\right)^{n_2^*-n_2-1}}\right]'$$

and

$$A_4 = n_2 n_2^* p_2^2 S'_{n_1,n_2-1}(t) S_{n_1^*,n_2^*-1}(t) - n_2^* n_2 p_2^2 S'_{n_1^*,n_2^*-1}(t) S_{n_1,n_2-1}(t)$$

$$\overset{\text{sign}}{=} \left(\frac{S_{n_1,n_2-1}(t)}{S_{n_1^*,n_2^*-1}(t)}\right)' = \left[\frac{\left(1-p_1\overline{G}(t)\right)^{n_1-n_1^*}}{\left(1-p_2\overline{G}(t)\right)^{n_2^*-n_2}}\right]'.$$

The assumptions $n_2^* \geq n_2$ and $n_1^* + n_2^* \leq n_1 + n_2$ imply that $n_1 - n_1^* \geq n_2^* - n_2 \geq 0$. Note that the desired result is clearly true for the case when $n_1 - n_1^* = n_2^* - n_2 = 0$, and hence we need to show the result when $n_1 - n_1^* \geq 1$. Since $p_1 \geq p_2$, we know $\dfrac{1-p_1\overline{G}(t)}{1-p_2\overline{G}(t)}$ is increasing in $t > 0$, that is, $\left(\dfrac{1-p_1\overline{G}(t)}{1-p_2\overline{G}(t)}\right)' \geq 0$. Then, it follows that $\left(\dfrac{1-p_1\overline{G}(t)}{1-p_2\overline{G}(t)}\right)^{n_2^*-n_2}$ is increasing in $t > 0$. On the other hand, since $1 - p_i\overline{G}(t)$ is increasing in $t > 0$ for $i = 1, 2$, we can conclude $A_i \geq 0$ for $i = 1, 2, 3, 4$. Therefore, $\varphi'(t) \geq 0$, for $t > 0$, and thus the proof is finished.

Case 2: $t = 0$. Based on Case 1, it suffices to show that $\frac{f_{Y_{n:n}}(0)}{f_{Y^*_{n^*:n^*}}(0)} \leq \varphi(0) := \lim_{t\to 0}\varphi(t)$, that is,

$$\frac{(1-p_1)^{n_1}(1-p_2)^{n_2} + n_1 p_1 g(0)(1-p_1)^{n_1-1}(1-p_2)^{n_2} + n_2 p_2 g(0)(1-p_1)^{n_1}(1-p_2)^{n_2-1}}{(1-p_1)^{n_1^*}(1-p_2)^{n_2^*} + n_1^* p_1 g(0)(1-p_1)^{n_1^*-1}(1-p_2)^{n_2^*} + n_2^* p_2 g(0)(1-p_1)^{n_1^*}(1-p_2)^{n_2^*-1}}$$
$$\leq \frac{n_1 p_1 g(0)(1-p_1)^{n_1-1}(1-p_2)^{n_2} + n_2 p_2 g(0)(1-p_1)^{n_1}(1-p_2)^{n_2-1}}{n_1^* p_1 g(0)(1-p_1)^{n_1^*-1}(1-p_2)^{n_2^*} + n_2^* p_2 g(0)(1-p_1)^{n_1^*}(1-p_2)^{n_2^*-1}},$$

which holds by proving that

$$\frac{(1-p_1)^{n_1}(1-p_2)^{n_2}}{(1-p_1)^{n_1^*}(1-p_2)^{n_2^*}} \leq \frac{n_1 p_1 g(0)(1-p_1)^{n_1-1}(1-p_2)^{n_2} + n_2 p_2 g(0)(1-p_1)^{n_1}(1-p_2)^{n_2-1}}{n_1^* p_1 g(0)(1-p_1)^{n_1^*-1}(1-p_2)^{n_2^*} + n_2^* p_2 g(0)(1-p_1)^{n_1^*}(1-p_2)^{n_2^*-1}}. \tag{A.11}$$

Arranging both sides of (A.11), we have

$$1 \leq \frac{n_1 p_1(1-p_2) + n_2 p_2(1-p_1)}{n_1^* p_1(1-p_2) + n_2^* p_2(1-p_1)},$$

which is equivalent to showing that

$$p_1(1-p_2)(n_1 - n_1^*) \geq p_2(1-p_1)(n_2^* - n_2). \tag{A.12}$$

Since $p_1 \geq p_2$, $n_2^* \geq n_2$ and $n_1 + n_2 \geq n_1^* + n_2^*$, inequality (A.12) can be obtained immediately. Hence, the proof is finished. ■

A.9. Proof of Theorem 4.4

Proof. Note that the survival functions of $Y_{1:n}$ and $Y^*_{1:n^*}$ are given by

$$\overline{F}_{Y_{1:n}}(t) = \left(p_1\overline{G}_1(t)\right)^{n_1} \left(p_2\overline{G}_2(t)\right)^{n_2}, \quad t \geq 0,$$

and

$$\overline{F}_{Y^*_{1:n^*}}(t) = \left(p_1\overline{G}_1(t)\right)^{n_1^*} \left(p_2\overline{G}_2(t)\right)^{n_2^*}, \quad t \geq 0.$$

The desired result is equivalent to showing that $\overline{F}_{Y_{1:n}}(t)/\overline{F}_{Y^*_{1:n^*}}(t)$ in increasing in $t \in \mathbb{R}_+$ and $\overline{F}_{Y_{1:n}}(t)/\overline{F}_{Y^*_{1:n^*}}(t) \geq 1$ for all $t \in \mathbb{R}_+$. First, observe that

$$\begin{aligned}\frac{\overline{F}_{Y_{1:n}}(t)}{\overline{F}_{Y^*_{1:n^*}}(t)} &= \frac{\left(p_1\overline{G}_1(t)\right)^{n_1} \left(p_2\overline{G}_2(t)\right)^{n_2}}{\left(p_1\overline{G}_1(t)\right)^{n_1^*} \left(p_2\overline{G}_2(t)\right)^{n_2^*}} \\ &= \frac{p_1^{n_1}p_2^{n_2}}{p_1^{n_1^*}p_2^{n_2^*}} \left(\frac{\overline{G}_2(t)}{\overline{G}_1(t)}\right)^{n_2-n_2^*} [\overline{G}_1(t)]^{n-n^*}\end{aligned}$$

is increasing in $t \in \mathbb{R}_+$ by using $G_1 \leq_{\text{hr}} G_2$, $n_2 \geq n_2^*$ and $n_1 + n_2 \leq n_1^* + n_2^*$. On the other hand, it is required that $\overline{F}_{Y_{1:n}}(0)/\overline{F}_{Y^*_{1:n^*}}(0) \geq 1$ since $\overline{F}_{Y_{1:n}}(t)/\overline{F}_{Y^*_{1:n^*}}(t)$ in increasing in $t \in \mathbb{R}_+$. In other words, we need that $p_1^{n_1}p_2^{n_2} \geq p_1^{n_1^*}p_2^{n_2^*}$, that is, $p_1^{n_1-n_1^*} \geq p_2^{n_2^*-n_2}$. Since $p_1 \leq p_2$, $n_2 \geq n_2^*$ and $n \leq n^*$, we can obtain that $p_1^{n_1-n_1^*} \geq p_1^{n_2^*-n_2} \geq p_2^{n_2^*-n_2}$. Thus, the proof is finished. ■

A.10. Proof of Theorem 4.5

Proof. The density function of $Y_{1:n}$ can be written as

$$f_{Y_{1:n}}(t) = \begin{cases} 1 - p_1^{n_1}p_2^{n_2} + p_1^{n_1}p_2^{n_2}(n_1g_1(0) + n_2g_2(0)) & \text{for } t = 0, \\ p_1^{n_1}p_2^{n_2}\Big[n_1g_1(t)\left(\overline{G}_1(t)\right)^{n_1-1}\left(\overline{G}_2(t)\right)^{n_2} & \\ \qquad + n_2g_2(t)\left(\overline{G}_2(t)\right)^{n_2-1}\left(\overline{G}_1(t)\right)^{n_1}\Big] & \text{for } t > 0. \end{cases}$$

Similarly, the density function of $Y^*_{1:n^*}$ is given by

$$f_{Y^*_{1:n^*}}(t) = \begin{cases} 1 - p_1^{n_1^*}p_2^{n_2^*} + p_1^{n_1^*}p_2^{n_2^*}(n_1^*g_1(0) + n_2^*g_2(0)) & \text{for } t = 0, \\ p_1^{n_1^*}p_2^{n_2^*}\Big[n_1^*g_1(t)\left(\overline{G}_1(t)\right)^{n_1^*-1}\left(\overline{G}_2(t)\right)^{n_2^*} & \\ \qquad + n_2^*g_2(t)\left(\overline{G}_2(t)\right)^{n_2^*-1}\left(\overline{G}_1(t)\right)^{n_1^*}\Big] & \text{for } t > 0. \end{cases}$$

According to the proof of Theorem 4.4, we know that it should be required that $\overline{F}_{Y_{1:n}}(0)/\overline{F}_{Y^*_{1:n^*}}(0) \geq 1$ since the likelihood ratio order implies the hazard rate order and the usual stochastic order. This holds obviously from the assumptions $p_1 \le p_2$, $n_2 \ge n_2^*$ and $(n_1, n_2) \overset{w}{\succeq} (n_1^*, n_2^*)$.

Next, we need to show that $f_{Y_{1:n}}(t)/f_{Y^*_{1:n^*}}(t)$ is increasing in $t \in \mathbb{R}_+$.

Case 1: $t > 0$. Write

$$T_{n_1,n_2}(t) = \left(\overline{G}_1(t)\right)^{n_1} \left(\overline{G}_2(t)\right)^{n_2} \quad \text{and} \quad T_{n_1^*,n_2^*}(t) = (\overline{G}_1(t))^{n_1^*}(\overline{G}_2(t))^{n_2^*}.$$

It suffices to show that

$$\frac{f_{Y_{1:n}}(t)}{f_{Y^*_{1:n^*}}(t)} = \frac{p_1^{n_1} p_2^{n_2}\left[n_1 g_1(t) T_{n_1-1,n_2}(t) + n_2 g_2(t) T_{n_1,n_2-1}(t)\right]}{p_1^{n_1^*} p_2^{n_2^*}\left[n_1^* g_1(t) T_{n_1^*-1,n_2^*}(t) + n_2^* g_2(t) T_{n_1^*,n_2^*-1}(t)\right]}$$

is increasing in $t > 0$, which reduces to prove

$$\psi(t) = \frac{n_1 g_1(t) T_{n_1-1,n_2}(t) + n_2 g_2(t) T_{n_1,n_2-1}(t)}{n_1^* g_1(t) T_{n_1^*-1,n_2^*}(t) + n_2^* g_2(t) T_{n_1^*,n_2^*-1}(t)}$$

is increasing in $t > 0$. Taking the derivative of $\psi(t)$ with respect to t, we have

$$\begin{aligned}
\psi'(t) \overset{\text{sign}}{=} & \Big[n_1 g_1'(t) T_{n_1-1,n_2}(t) + n_1 g_1(t) T'_{n_1-1,n_2}(t) + n_2 g_2'(t) T_{n_1,n_2-1}(t) \\
& + n_2 g_2(t) T'_{n_1,n_2-1}(t)\Big] \times \Big[n_1^* g_1(t) T_{n_1^*-1,n_2^*}(t) + n_2^* g_2(t) T_{n_1^*,n_2^*-1}(t)\Big] \\
& - \left[n_1 g_1(t) T_{n_1-1,n_2}(t) + n_2 g_2(t) T_{n_1,n_2-1}(t)\right] \times \Big[n_1^* g_1'(t) T_{n_1^*-1,n_2^*}(t) \\
& + n_1^* g_1(t) T'_{n_1^*-1,n_2^*}(t) + n_2^* g_2'(t) T_{n_1^*,n_2^*-1}(t) + n_2^* g_2(t) T'_{n_1^*,n_2^*-1}(t)\Big] \\
= \; & n_1 n_1^* g_1^2(t) \Big[T'_{n_1-1,n_2}(t) T_{n_1^*-1,n_2^*}(t) - T'_{n_1^*-1,n_2^*}(t) T_{n_1-1,n_2}(t)\Big] \\
& + n_2 n_1^* g_1(t) g_2(t) \Big[T'_{n_1,n_2-1}(t) T_{n_1^*-1,n_2^*}(t) - T'_{n_1^*-1,n_2^*}(t) T_{n_1,n_2-1}(t)\Big] \\
& + n_1 n_2^* g_1(t) g_2(t) \Big[T'_{n_1-1,n_2}(t) T_{n_1^*,n_2^*-1}(t) - T'_{n_1^*,n_2^*-1}(t) T_{n_1-1,n_2}(t)\Big] \\
& + n_2 n_2^* g_2^2(t) \Big[T'_{n_1,n_2-1}(t) T_{n_1^*,n_2^*-1}(t) - T'_{n_1^*,n_2^*-1}(t) T_{n_1,n_2-1}(t)\Big] \\
& + \left[g_2'(t) g_1(t) - g_2(t) g_1'(t)\right] \Big[n_1^* n_2 T_{n_1,n_2-1}(t) T_{n_1^*-1,n_2^*}(t) - n_1 n_2^* T_{n_1-1,n_2}(t) T_{n_1^*,n_2^*-1}(t)\Big] \\
=: \; & B_1 + B_2 + B_3 + B_4 + B_5, \quad \text{say,}
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= n_1 n_1^* g_1^2(t) \Big[T'_{n_1-1,n_2}(t) T_{n_1^*-1,n_2^*}(t) - T'_{n_1^*-1,n_2^*}(t) T_{n_1-1,n_2}(t)\Big] \\
&\overset{\text{sign}}{=} \left(\frac{T_{n_1-1,n_2}(t)}{T_{n_1^*-1,n_2^*}(t)}\right)' = \left[\frac{\left(\overline{G}_2(t)\right)^{n_2-n_2^*}}{\left(\overline{G}_1(t)\right)^{n_1^*-n_1}}\right]',
\end{aligned}$$

$$\begin{aligned}
B_2 &= n_2 n_1^* g_1(t) g_2(t) \Big[T'_{n_1,n_2-1}(t) T_{n_1^*-1,n_2^*}(t) - T'_{n_1^*-1,n_2^*}(t) T_{n_1,n_2-1}(t)\Big] \\
&\overset{\text{sign}}{=} \left(\frac{T_{n_1,n_2-1}(t)}{T_{n_1^*-1,n_2^*}(t)}\right)' = \left[\frac{\left(\overline{G}_2(t)\right)^{n_2-n_2^*-1}}{\left(\overline{G}_1(t)\right)^{n_1^*-n_1-1}}\right]',
\end{aligned}$$

$$\begin{aligned} B_3 &= n_1 n_2^* g_1(t) g_2(t) \left[T'_{n_1-1,n_2}(t) T_{n_1^*,n_2^*-1}(t) - T'_{n_1^*,n_2^*-1}(t) T_{n_1-1,n_2}(t) \right] \\ &\overset{\text{sign}}{=} \left(\frac{T_{n_1-1,n_2}(t)}{T_{n_1^*,n_2^*-1}(t)} \right)' = \left[\frac{\left(\overline{G}_2(t)\right)^{n_2-n_2^*+1}}{\left(\overline{G}_1(t)\right)^{n_1^*-n_1+1}} \right]', \\ B_4 &= n_2 n_2^* g_2^2(t) \left[T'_{n_1,n_2-1}(t) T_{n_1^*,n_2^*-1}(t) - T'_{n_1^*,n_2^*-1}(t) T_{n_1,n_2-1}(t) \right] \\ &\overset{\text{sign}}{=} \left(\frac{T_{n_1,n_2-1}(t)}{T_{n_1^*,n_2^*-1}(t)} \right)' = \left[\frac{\left(\overline{G}_2(t)\right)^{n_2-n_2^*}}{\left(\overline{G}_1(t)\right)^{n_1^*-n_1}} \right]' \end{aligned}$$

and

$$\begin{aligned} B_5 &= \left[n_1^* n_2 T_{n_1,n_2-1}(t) T_{n_1^*-1,n_2^*}(t) - n_1 n_2^* T_{n_1-1,n_2}(t) T_{n_1^*,n_2^*-1}(t) \right] \\ &\quad \times \left[g_2'(t) g_1(t) - g_2(t) g_1'(t) \right] \\ &= \left(\overline{G}_1(t)\right)^{n_1+n_1^*-1} \left(\overline{G}_2(t)\right)^{n_2+n_2^*-1} (n_1^* n_2 - n_1 n_2^*) \left[g_2'(t) g_1(t) - g_2(t) g_1'(t) \right]. \end{aligned}$$

Both $n_2 \geq \max\{n_1^*, n_2^*\}$ and $(n_1, n_2) \overset{\text{w}}{\succeq} (n_1^*, n_2^*)$ imply that $n_1 \leq \min\{n_1^*, n_2^*\} \leq \max\{n_1^*, n_2^*\} \leq n_2$ and $n_1^* - n_1 \geq n_2 - n_2^* \geq 0$. Also, we have $n_1^* n_2 \geq n_2^* n_1$. We only need to deal with the case when $n_1^* - n_1 \geq 1$ since it is trivially true if $n_1^* - n_1 = n_2 - n_2^* = 0$. $G_1 \leq_{\text{lr}} G_2$ means that $g_2(t)/g_1(t)$ is increasing in $t > 0$, which further implies $g_2'(t) g_1(t) \geq g_2(t) g_1'(t)$ for $t > 0$. Thus, we have $B_5 \geq 0$. On the other hand, it follows from $G_1 \leq_{\text{lr}} G_2$ that $\overline{G}_2(t)/\overline{G}_1(t)$ is increasing in $t > 0$, that is, $\left(\overline{G}_2(t)/\overline{G}_1(t)\right)' \geq 0$ for $t > 0$. Upon using an argument similar to that of Theorem 4.1, we can readily get that $B_i \geq 0$ for $i = 1, 2, 3, 4$; now we can conclude that $\psi'(t) \geq 0$ for $t > 0$. Hence, the proof is finished.

Case 2: $t = 0$. It suffices to show that

$$\frac{1 - p_1^{n_1} p_2^{n_2} + p_1^{n_1} p_2^{n_2} (n_1 g_1(0) + n_2 g_2(0))}{1 - p_1^{n_1^*} p_2^{n_2^*} + p_1^{n_1^*} p_2^{n_2^*} (n_1^* g_1(0) + n_2^* g_2(0))} \leq \frac{p_1^{n_1} p_2^{n_2} (n_1 g_1(0) + n_2 g_2(0))}{p_1^{n_1^*} p_2^{n_2^*} (n_1^* g_1(0) + n_2^* g_2(0))},$$

which can be guaranteed by showing that

$$\frac{1 - p_1^{n_1} p_2^{n_2}}{1 - p_1^{n_1^*} p_2^{n_2^*}} \leq \frac{p_1^{n_1} p_2^{n_2}}{p_1^{n_1^*} p_2^{n_2^*}} \lim_{t \to 0} \frac{n_1 g_1(t) + n_2 g_2(t)}{n_1^* g_1(t) + n_2^* g_2(t)} = \frac{p_1^{n_1} p_2^{n_2}}{p_1^{n_1^*} p_2^{n_2^*}} \frac{n_1 + n_2 \delta}{n_1^* + n_2^* \delta}.$$

This is obvious by arranging (4.1). Hence, the proof is finished. ■

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