Combinatorics, Probability and Computing (2016) 25, 766–767.
doi:10.1017/S0963548316000055

On the Normalized Shannon Capacity of a Union

PETER KEEVASH[1†] and EOIN LONG[2]

[1] Mathematical Institute, University of Oxford, Andrew Wiles Building, Woodstock Rd, Oxford OX2 6GG, UK
(e-mail: keevash@maths.ox.ac.uk)
[2] School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
(e-mail: eoinlong@post.tau.ac.il)

Received 13 January 2016; first published online 3 March 2016

Let $G_1 \times G_2$ denote the strong product of graphs G_1 and G_2, that is, the graph on $V(G_1) \times V(G_2)$ in which (u_1, u_2) and (v_1, v_2) are adjacent if for each $i = 1, 2$ we have $u_i = v_i$ or $u_i v_i \in E(G_i)$. The Shannon capacity of G is $c(G) = \lim_{n\to\infty} \alpha(G^n)^{1/n}$, where G^n denotes the n-fold strong power of G, and $\alpha(H)$ denotes the independence number of a graph H. The normalized Shannon capacity of G is

$$C(G) = \frac{\log c(G)}{\log |V(G)|}.$$

Alon [1] asked whether for every $\epsilon > 0$ there are graphs G and G' satisfying $C(G), C(G') < \epsilon$ but with $C(G + G') > 1 - \epsilon$. We show that the answer is no.

2010 *Mathematics subject classification*: Primary 05C35
Secondary 94A05

Despite much impressive work (*e.g.*, [1, 3, 4, 5, 7]) since the introduction of the Shannon capacity in [8], many natural questions regarding this parameter remain wide open (see [2, 6] for surveys). Let $G_1 + G_2$ denote the disjoint union of the graphs G_1 and G_2. It is easy to see that $c(G_1 + G_2) \geqslant c(G_1) + c(G_2)$. Shannon [8] conjectured that $c(G_1 + G_2) = c(G_1) + c(G_2)$, but this was disproved in a strong form by Alon [1], who showed that there are n-vertex graphs G_1, G_2 with $c(G_i) < e^{c\sqrt{\log n \log\log n}}$ but $c(G_1 + G_2) \geqslant \sqrt{n}$. In terms of the normalized Shannon capacity, this implies that for any $\epsilon > 0$, there exist graphs G_1, G_2 with $C(G_i) < \epsilon$ but $C(G_1 + G_2) > 1/2 - \epsilon$. Alon [1] asked whether '1/2' can be changed to '1' here. In this short note we will give a negative answer to this question. In fact, the following result implies that '1/2' is tight.

† Research supported in part by ERC Consolidator Grant 647678.

Theorem. If $C(G_1) \leqslant \epsilon$ and $C(G_2) \leqslant \epsilon$, then

$$C(G_1 + G_2) \leqslant \frac{1+\epsilon}{2} + \frac{1-\epsilon}{2\log_2(|V(G_1)| + |V(G_2)|)}.$$

Proof. Let $N_i = |V(G_i)|$ for $i = 1, 2$. Fix a maximum size independent set I in $(G_1 + G_2)^n$ for some $n \in \mathbb{N}$. We write $|I| = \sum_{S \subset [n]} |I_S|$, where

$$I_S = \{x = (x_1, \ldots, x_n) \in I \, : x_i \in V(G_1) \Leftrightarrow i \in S\}.$$

To bound $|I_S|$, we may suppose that $S = [m]$ for some $0 \leqslant m \leqslant n$. Then I_S is an independent set in $G_1^m \times G_2^{n-m}$. As $C(G_1) \leqslant \epsilon$, by supermultiplicativity $\alpha(G_1^m) \leqslant N_1^{\epsilon m}$; similarly, $\alpha(G_2^{n-m}) \leqslant N_2^{\epsilon(n-m)}$. For any $x \in V(G_1)^m$, the set of $y \in V(G_2)^{n-m}$ such that $(x, y) \in I_S$ is independent in G_2^{n-m}, so $|I_S| \leqslant N_1^m N_2^{\epsilon(n-m)}$. Similarly, $|I_S| \leqslant N_1^{\epsilon m} N_2^{n-m}$.

We multiply these bounds: $|I_S|^2 \leqslant (N_1^m N_2^{n-m})^{1+\epsilon}$. Writing

$$\gamma = \frac{N_1}{N_1 + N_2},$$

we have

$$\begin{aligned}
\alpha((G_1 + G_2)^n) = |I| = \sum_{S \subset [n]} |I_S| &\leqslant \sum_{m=0}^{n} \binom{n}{m} (N_1^{(1+\epsilon)/2})^m (N_2^{(1+\epsilon)/2})^{n-m} \\
&= (N_1^{(1+\epsilon)/2} + N_2^{(1+\epsilon)/2})^n \\
&= (\gamma^{(1+\epsilon)/2} + (1-\gamma)^{(1+\epsilon)/2})^n (N_1 + N_2)^{(1+\epsilon)n/2} \\
&\leqslant 2^{(1-\epsilon)n/2} (N_1 + N_2)^{(1+\epsilon)n/2},
\end{aligned}$$

as $\gamma^b + (1-\gamma)^b$ is maximized at $\gamma = 1/2$ for $0 < b < 1$ and $0 \leqslant \gamma \leqslant 1$. Therefore

$$C(G_1 + G_2) = \lim_{n \to \infty} \frac{\log \alpha((G_1 + G_2)^n)}{n \log (N_1 + N_2)} \leqslant \frac{1+\epsilon}{2} + \frac{1-\epsilon}{2\log_2(N_1 + N_2)}.$$ □

References

[1] Alon, N. (1998) The Shannon capacity of a union. *Combinatorica* **18** 301–310.

[2] Alon, N. (2002) Graph powers, *Contemporary Combinatorics*, Vol. 10 of *Bolyai Society Mathematical Studies*, János Bolyai Mathematical Society, pp. 11–28.

[3] Alon, N. and Lubetzky, E. (2006) The Shannon capacity of a graph and the independence numbers of its powers. *IEEE Trans. Inform. Theory* **52** 2172–2176.

[4] Alon, N. and Orlitsky, A. (1995) Repeated communication and Ramsey graphs. *IEEE Trans. Inform. Theory* **41** 1276–1289.

[5] Haemers, W. (1979) On some problems of Lovász concerning the Shannon capacity of a graph. *IEEE Trans. Inform. Theory* **25** 231–232.

[6] Körner, J. and Orlitsky, A. (1998) Zero-error information theory. *IEEE Trans. Inform. Theory* **44** 2207–2229.

[7] Lovász, L. (1979) On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory* **25** 1–7.

[8] Shannon, C. E. (1956) The zero-error capacity of a noisy channel *IRE Trans. Inform. Theory* **2** 8–19.

https://doi.org/10.1017/S0963548316000055 Published online by Cambridge University Press