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Local well-posedness for Frémond's model of complete damage in elastic solids

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We consider a model for the evolution of damage in elastic materials originally proposed by Michel Frémond. For the corresponding PDE system, we prove existence and uniqueness of a local in time strong solution. The main novelty of our result stands in the fact that, differently from previous contributions, we assume no occurrence of any type of regularising terms.

Key words: Degenerate elliptic–parabolic system, damage phenomena, local existence, *a priori* estimates

2020 Mathematics Subject Classification: Primary: 35K55, 35Q74; Secondary: 35K86, 35M33, 74R05

1 Introduction

We consider a basic model for the evolution of damage in an elastic material subject to an external load under the approach originally proposed by Frémond and coauthors in a number of papers [13, 12, 14, 15] (see also the monographs [10, 11, 21] for a general presentation of related models as well as a detailed mechanical background).

We will give here an overview of the model in its generality; we notice however from the very beginning that, in order to reduce technical complications, a simplified formulation will be addressed for the purpose of a mathematical analysis. Let us consider a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ occupied by the elastic medium over some given reference time interval $(0, T)$. The material is subject to an external load $\boldsymbol{g}$ leading to elastic deformations represented by means of the displacement variable $\boldsymbol{u}$. As a response to deformations, the material undertakes stresses and strains, the latter representing a source of *damage*. At a microscopic level, this phenomenon can be thought as a progressive failure of elastic bonds; as a consequence, the material loses stiffness and microcracks tend to develop.

A description of the progression of damage at the microscopic level is however very difficult, especially because the micro-breaks are very small compared to the scale of macroscopic displacements. For this reason, in this type of *continuum models*, the damage is rather described by means of a *macroscopic* variable z, i.e. an order parameter that represents the locally averaged evolution of damage at any point $x \in \Omega$ and $t \in (0, T)$. For simplicity, z is normalised in such a way that, for $z=1$, the material is completely integer, i.e. no damage has yet occurred, whereas for $z=0$, all the elastic bonds have been broken. We speak then of *complete damage* at that point, meaning that the material has completely lost its elastic properties and a (macroscopic) fracture has occurred. According to such an interpretation, the values of z below $z=0$, as well as those above $z=1$ have no physical significance and should be somehow penalised in the mathematical formulation of the model.

We will assume a quasi-static regime; namely, the damage process occurs at a much slower scale compared to the elastic response, which can thus be represented by an *elliptic* equation of the form

$$-(\mathbb{C}_{ijkl}(z)\varepsilon(\boldsymbol{u})_{kl})_{,j} = g_i, \qquad \text{in } (0,T)\times\Omega. \tag{1.1}$$

Here, $\varepsilon(\boldsymbol{u}) = (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t)/2$ is the strain tensor, $\boldsymbol{g} = (g_i)$ represents the action of the (given) external forces, and the elastic tensor $\mathbb{C}$ may be assumed to satisfy proper symmetry and ellipticity conditions and to degenerate as $z=0$ (the precise hypotheses will be presented below). Here and below, we are assuming Einstein's convention for summation over repeated indices. It is worth noting that *dynamical* models for damage evolution are also significant and have been studied mathematically in a number of contributions. We may quote, with no claim of completeness, [6, 13, 16, 18, 19] (see also the references therein) for models including inertial and/or viscosity effects.

Relation (1.1) is complemented with the following parabolic equation describing the evolution of the damage variable z:

$$\alpha(z_t) + \delta_1 z_t - \delta_2\Delta z + f'(z) \ni w - \frac{1}{2}\mathbb{C}'_{ijkl}(z)\varepsilon(\boldsymbol{u})_{kl}\varepsilon(\boldsymbol{u})_{ij}, \qquad \text{in } (0,T)\times\Omega. \tag{1.2}$$

Here and below, z_t represents the time derivative of the damage variable z. Moreover, $\alpha = \partial I_{(-\infty,0]}$, i.e. the subdifferential of the *indicator function* of the interval $(-\infty, 0]$. We refer the reader to the monographs [2, 8] for the underlying background material from convex analysis. Here, we just recall that α is a multivalued mapping; indeed, we have $\alpha(0) = [0,+\infty)$, $\alpha(r) = \{0\}$ for $r<0$ and $\alpha(r) = \emptyset$ for $r>0$. This motivates the occurrence of the inclusion sign in (1.2). The presence of α is aimed at enforcing the *irreversibility* (or *unidirectionality*) constraint on the evolution of z. Namely, any solution must satisfy $z_t \le 0$, which means that once some amount of damage has been created, it cannot be repaired. Note that this fact implies in turn that, once $z_0 \le 1$, then z can never exceed 1 at any point in the evolution, implying that the unphysical states $z>1$ are automatically excluded. Irreversibility is a reasonable physical ansatz in many real-world applications; on the other hand, it is worth observing that also *reversible* models (i.e. such that the broken bonds may be at least partially restored) are significant and have been extensively studied in the literature (see, e.g. [4] and the references quoted there). It is also worth noticing that (1.2) subsumes a *rate-dependent* evolution of z; *rate-independent* damage models are equally interesting and have been addressed in several works (see, e.g. [9, 20, 22, 23, 24, 25] and the references therein).

The coefficients $\delta_1, \delta_2 > 0$ in (1.2) are related to the timescale of the damaging process (the smaller δ_1 the faster it occurs) and to the 'thickness' of the (diffuse) interface between damaged and sound areas (which depends on the scale length of the micro-breaks and goes like $\delta_2^{1/2}$). The positive constant $w>0$ on the right-hand side has the significance of a threshold: let us explain

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this fact by assuming $f \equiv 0$, which, physically speaking, can be seen as the 'model case'. In this situation, if the forcing term $\mathbb{C}'_{ijkl}(z)\varepsilon(\boldsymbol{u})_{kl}\varepsilon(\boldsymbol{u})_{ij}$ does not exceed $2w$, the right-hand side of (1.2) is positive, which basically indicates that no damage is being created. In the converse situation, i.e. in presence of large deformation gradients, a source of damage occurs. In the case $f \not\equiv 0$, this damaging effect can be thought to vary a little depending on the actual value of z; nevertheless one expects that, in practice, $f'(z)$ is small compared to w. Hence, if we set $\psi'(r) = f'(r) - w$ (as we will do in the sequel), we expect in particular ψ' be strictly negative or, in other words, the *configuration potential* ψ to be decreasing, meaning that, in some measure, the body tends to oppose resistance to the damaging effects which, as said, will occur only if the strains are large.

In order to present our mathematical results, let us assume for simplicity $\boldsymbol{g}$ independent of time and take homogeneous Dirichlet boundary conditions for $\boldsymbol{u}$ and no-flux (i.e. homogeneous Neumann) boundary conditions for z. Moreover, let us assume (at least) the symmetry property $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$. Then, testing (1.1) by $\boldsymbol{u}_t$ and (1.2) by z_t and integrating over Ω permit us to (formally) deduce the *energy equality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \delta_1 \|z_t\|^2_{L^2(\Omega)} = 0 \tag{1.3}$$

with the energy functional

$$\mathcal{E}(t) = \int_\Omega \left(\frac{1}{2}\mathbb{C}_{ijkl}(z)\varepsilon(\boldsymbol{u})_{kl}\varepsilon(\boldsymbol{u})_{ij} - \boldsymbol{g}\cdot\boldsymbol{u} + \frac{\delta_2}{2}|\nabla z|^2 + f(z) - wz \right), \tag{1.4}$$

where it is worth noting that the product between z_t and $\alpha(z_t)$ is a.e. equal to 0, in view of the fact that $\alpha(z_t)$ (or, to be precise, any element of such a set) may be different from 0 only when $z_t = 0$. The energy relation (1.3) is the basic source of the *a prior* estimates needed for attempting a mathematical analysis of system (1.1)–(1.2).

On the other hand, there are several reasons why the information provided by the above relation is not sufficient in order to obtain a satisfactory mathematical result. An important point stands of course in the fact that, even if the body is completely integer at the beginning (i.e. $z_0 \equiv 1$ in Ω), it is expected that after some time, due to progression of damage, z becomes 0 at some point $x \in \Omega$. In such a situation, the elastic tensor $\mathbb{C}(z)$ degenerates and the energy $\mathcal{E}$ is no longer coercive. Consequently, it becomes impossible to control the quadratic term in $\varepsilon(\boldsymbol{u})$ on the right-hand side of (1.2) and the model somehow loses significance. This is an intrinsic feature of this system (and of related ones) and, actually, for such models of *complete* damage, it seems natural to look for *local in time solutions*, namely those defined on a 'small' time interval $(0, T_0)$ with possibly $T_0 < T$, where degeneration does not occur. This type of *local existence* result is what is proved in several related papers (see, e.g. [5, 13, 12]) and will also be the object of the present note. Indeed, it seems that the description of *complete* damaging of the material, i.e. of what happens after the onset of some macroscopic fracture, requires a different modelling approach, see, e.g. [7, 17, 22].

There is, however, a second relevant difficulty; indeed, in order to prevent degeneration of z at least in a short time interval $(0, T_0)$, one needs a quantitative estimate of the form

$$\|z\|_{L^\infty(0,T_0;X(\Omega))} \le c, \tag{1.5}$$

where $T_0 > 0$ may depend on the prescribed data and $X = X(\Omega)$ is a Banach space such that $X \subset C^0(\overline{\Omega})$ with continuous embedding. This corresponds to a (local) control of z in the *uniform* norm, in such a way that degeneration cannot occur at any point in the short time span. On the other hand, if the energy has the expression (1.4), an estimate like (1.5) follows directly from (1.3) only in space dimension one (this is, indeed, the spirit of the pioneering results proved in [13, 12]), whereas, in the present three-dimensional setting, (1.5) may be obtained only by performing higher regularity estimates. Here, however, two additional difficulties arise: (i) the combined occurrence in (1.2) of the *nonsmooth* function α and of the *quadratic gradient term* on the right-hand side, and (ii) the poor regularity of $\boldsymbol{u}$ provided by the *elliptic* equation (1.1) characterised by a z-dependent (hence nonsmooth) diffusion coefficient. For these reasons, at least up to our knowledge, local existence has been obtained so far only in presence of additional smoothing terms. Actually, common regularisations considered in the literature are: *viscoelastic* (rather than purely elastic) behaviour for $\boldsymbol{u}$ [3, 6, 16, 19], presence of inertial effects in (1.1) [6, 16, 18, 19], and replacement of the Laplacian in (1.2) by a more regularising operator like the fractional Laplacian $(-\Delta)^s$ with suitable $s > 1$ [20] or the p-Laplacian $-\Delta_p$ with suitable $p > 2$ [17, 18, 19].

In this work, we will consider the 'original' system (1.1)–(1.2) with *no occurrence of any regularising term*. We will actually prove that an estimate of the form (1.5) can be obtained also in such a setting, so filling the gap of a long-standing regularity problem. Our argument is based on a more careful control of the L^∞-, H^1- and H^2-norms of the difference between $z(t)$ at $t > 0$ and the initial datum z_0 in terms of the parameters of the system. As an outcome of our procedure, we will be able to prove existence and uniqueness of strong solutions to the initial boundary value problem for system (1.1)–(1.2) on a time span $(0, T_0)$, with T_0 explicitly computable in terms of the data, where z does not degenerate to 0 at any point.

In order to avoid unessential technicalities, proceeding in the spirit of [5, 6] we will actually consider a simplified version of the model, where the displacement $\boldsymbol{u}$ is replaced by a scalar variable u and some quantities and parameters are normalised. We point out that these simplifications are not restrictive and are taken only for the sake of clarity. Indeed, our results could be easily extended to the 'original' system (1.1)–(1.2) by applying some more or less standard tools (like, e.g. Korn's inequality) and doing a little more technical work.

The paper is organised as follows. In the next section, we provide a detailed presentation of our assumptions and state our main result. The *a priori* estimates that are at the core of the proof are given in the subsequent Section 3. A possible regularisation of the system compatible with the *a priori* estimates is sketched in Section 4, where a number of additional comments are also given. Some final consideration is then provided in the conclusive Section 5.

2 Main result

First of all, we introduce a simplified version of system (1.1)–(1.2). As said, we replace the vector-valued displacement $\boldsymbol{u}$ by a scalar one u, and correspondingly assume that the elasticity tensor $\mathbb{C}(z)$ is replaced by a scalar function $c(z)$. Moreover, in order to take the simplest example of a strictly positive function that degenerates at 0, we just choose $c(z) = z$. Actually, we expect that other types of degeneration for c, like, e.g. the case $c(z) \sim z^m$, $m > 0$, considered in [24, 25], may be addressed by using a procedure similar to ours. The technical details, however, may be rather different, especially for what concerns the regularisation and truncation parts of the argument given below.

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We also normalise the parameters δ_1, δ_2 to 1 and incorporate the positive constant w into the function f' so introducing a new configuration potential $\psi(r) = f(r) - wr$. With these choices, system (1.1)–(1.2) reduces to

$$-\operatorname{div}(z\nabla u) = g, \qquad \text{in } (0, T) \times \Omega, \tag{2.1}$$

$$\alpha(z_t) + z_t - \Delta z + \psi'(z) \ni -\frac{1}{2}|\nabla u|^2, \qquad \text{in } (0, T) \times \Omega. \tag{2.2}$$

The above equations are complemented with the boundary conditions (which are a rather standard choice for this class of models)

$$u = \partial_{\boldsymbol{n}} z = 0, \qquad \text{in } (0, T) \times \Gamma, \tag{2.3}$$

where $\Gamma = \partial\Omega$, $\partial_{\boldsymbol{n}} = \boldsymbol{n} \cdot \nabla$ and $\boldsymbol{n}$ denotes the outer unit normal vector to Γ. System (2.1)–(2.2) is stated over an assigned reference interval $(0, T)$; however, as said, we will prove existence on a possibly smaller interval $(0, T_0)$. Finally, we assume the initial condition

$$z|_{t=0} = z_0, \qquad \text{in } \Omega. \tag{2.4}$$

In order to fix a concept of strong solution and formulate our related existence result, we need to introduce some preparatory material. Letting Ω be a smooth bounded domain of $\mathbb{R}^3$, we set $H := L^2(\Omega)$, $V := H^1(\Omega)$ and $V_0 := H^1_0(\Omega)$. We will often write H in place of $H \times H \times H$ (with similar notation for other spaces), in case vector-valued functions are considered. We denote by $(\cdot, \cdot)$ the standard scalar product of H and by $\|\cdot\|$ the associated Hilbert norm. Moreover, we equip V and V_0 with norms $\|\cdot\|_V = \|\cdot\| + \|\nabla \cdot\|$ and $\|\cdot\|_{V_0} = \|\nabla \cdot\|$, respectively. Identifying H with its dual space H' by means of the above scalar product, we obtain the chains of continuous and dense embeddings $V \subset H \subset V'$ and $V_0 \subset H \subset V_0'$. We may indicate by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V, or, more generally, between X' and X where X is a Banach space continuously and densely embedded into H. Recalling that $\boldsymbol{n}$ stands for the outer unit normal vector to Γ, we also set

$$W := \left\{ v \in H^2(\Omega) : \ \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma \right\} \subset C^0(\overline{\Omega}). \tag{2.5}$$

Then, W is a closed subspace of $H^2(\Omega)$. We equip W with the norm

$$\|v\|_W^2 := \|v\|^2 + \|\Delta v\|^2, \tag{2.6}$$

which (on W) is equivalent to the usual H^2-norm in view of well-known elliptic regularity results.

Next, we can fix our basic hypotheses on coefficients and data:

Assumption 2.1 *(A1)* $\psi \in C^2(\mathbb{R}; \mathbb{R})$.

(A2) $g \in L^p(\Omega)$ *for some* $p \geq 3$.

(A3) $z_0 \in W$ *with* $z_0 \leq 1$ *at every point of* Ω. *Moreover, denoting by* c_Ω *an embedding constant of* W *into* $C^0(\overline{\Omega})$, *i.e. a constant such that* $\|v\|_{C^0(\overline{\Omega})} \leq c_\Omega \|v\|_W$ *for all* $v \in W$, *we assume that* $\varepsilon = \varepsilon(z_0) := c_\Omega \|1 - z_0\|_W \leq 1/2$.

It is worth commenting a bit about the above assumptions. First of all, since we will prove that the z-component of the local solution takes values in $(0, 1]$, the behaviour of $\psi(r)$ for large r is in fact irrelevant. On the other hand, it may be useful to assume that

$$\psi(r) = r^2 \quad \text{for every } |r| \ge 2, \tag{2.7}$$

whence it also follows that

$$\psi(r) \ge \frac{r^2}{2} - c \quad \text{for every } r \in \mathbb{R}. \tag{2.8}$$

Actually, such a free 'extra-coercivity' property will help us in the approximation and for writing the *a priori* estimates in a simpler way.

We may also observe that (A3) implies

$$\|1 - z_0\|_{C^0(\overline{\Omega})} \le c_\Omega \|1 - z_0\|_W = \varepsilon. \tag{2.9}$$

Since $\varepsilon \le 1/2$, we have $z_0 \ge 1 - \varepsilon \ge 1/2$ a.e. in Ω, i.e. the initial amount of damage is less than one half at (almost) any point. Of course, the ideal, and simplest, situation occurs when $z_0 \equiv 1$, i.e. the body is completely integer at the initial time. Note that the condition $z_0 \le 1$ is used only to respect the physical significance of the model. Of course, under such an assumption, any hypothetical solution satisfies $z \le 1$ also for $t > 0$ due to the irreversibility constraint embedded into equation (2.2).

We can now state the main result of this paper:

Theorem 2.2 *Let Assumption 2.1 hold. Let $\delta \in (0, 1/12]$. Then there exist a time $T_0 \in (0, T]$ depending only on ψ, g, ε and δ and at least a triple (u, z, ξ) of functions defined over $(0, T_0) \times \Omega$ and satisfying the regularity properties*

$$u \in C^0([0, T_0]; W^{2,\rho}(\Omega) \cap V_0) \ \textit{for any } \rho \in [1, p] \cap [1, 6), \tag{2.10}$$

$$z \in H^1(0, T_0; V) \cap C_w([0, T_0]; W), \tag{2.11}$$

$$\xi \in L^2(0, T_0, H), \tag{2.12}$$

$$c_\Omega \|1 - z(t)\|_W \le 1 - 3\delta, \quad \textit{for all } t \in [0, T_0], \tag{2.13}$$

where $C_w([0, T_0]; X)$ stands for the space of weakly continuous functions defined on $[0, T_0]$ with values in a Banach space X. Moreover, the triple (u, z, ξ) satisfies the equations

$$-\operatorname{div}(z\nabla u) = g, \tag{2.14}$$

$$\xi + z_t - \Delta z + \psi'(z) = -\frac{1}{2}|\nabla u|^2, \tag{2.15}$$

$$\xi \in \alpha(z_t) \tag{2.16}$$

almost everywhere in $(0, T_0) \times \Omega$, with the boundary conditions (2.3) and the initial condition (2.4) in the sense of traces. In addition, if $p > 3$ in (A2), then (u, z) is uniquely determined by initial data z_0 and continuously depends on z_0. More precisely, for $i = 1, 2$, let (u_i, z_i) be solutions on $[0, T_0]$ satisfying (2.10)–(2.16). Then

$$\|(z_1 - z_2)(t)\|_V + \|(u_1 - u_2)(t)\|_{V_0} \le C\|(z_1 - z_2)(0)\|_V$$

for every $t \in [0, T_0]$.

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Note that relation (2.13) entails in particular

$$\|1 - z(t)\|_{C^0(\overline{\Omega})} \leq c_\Omega \|1 - z(t)\|_W \leq 1 - 3\delta. \tag{2.17}$$

Hence, for any $t \in [0, T_0]$, we have $z(t, x) \geq 3\delta > 0$ for every $x \in \overline{\Omega}$. In this sense, we are able to compute a time before which complete damage cannot occur at any point. In such a time span, the system remains nondegenerate and existence of strong solutions can be proved. Of course, condition $\delta \leq 1/12$ combined with assumption (A3) implies

$$c_\Omega \|1 - z_0\|_{C^0(\overline{\Omega})} = \varepsilon \leq 1/2 < 3/4 \leq 1 - 3\delta, \tag{2.18}$$

namely there is a gap of at least $1/4$ between $1 - \varepsilon$ and 3δ. Of course, the magnitude of such a gap is somehow an arbitrary choice of ours; on the other hand, keeping it as a given value permits us to write the estimates in a computationally simpler way.

3 Proofs

We start with introducing a truncated version of system (2.1)–(2.2) in the same spirit as in [5]. To this aim, for $\delta \in (0, 1/12]$ we consider a mapping $T_\delta \in C^{1,1}(\mathbb{R}; \mathbb{R})$ such that

$$T_\delta(r) = \begin{cases} r & \text{if } r \geq 3\delta, \\ 2\delta & \text{if } r \leq \delta \end{cases} \tag{3.1}$$

and T_δ is monotone and convex in the interval $(\delta, 3\delta)$ and fulfills

$$|T_\delta'(r)| \leq 1, \quad |T_\delta''(r)| \leq c\delta^{-1} \quad \text{for almost all } r \in \mathbb{R} \tag{3.2}$$

and for some $c > 0$. A possible explicit choice could be

$$T_\delta(r) = 2\delta + (4\delta)^{-1}(r - \delta)^2 \quad \text{for } r \in (\delta, 3\delta), \tag{3.3}$$

but other options may be equally allowed. Then, the truncated system may be stated as follows:

$$-\operatorname{div}(T_\delta(z)\nabla u) = g, \qquad \text{in } (0, T) \times \Omega; \tag{3.4}$$

$$\alpha(z_t) + z_t - \Delta z + \psi'(z) \ni -\frac{T_\delta'(z)}{2}|\nabla u|^2, \qquad \text{in } (0, T) \times \Omega, \tag{3.5}$$

where, as before, the differential inclusion (3.5) may be interpreted as the equality

$$\xi + z_t - \Delta z + \psi'(z) = -\frac{T_\delta'(z)}{2}|\nabla u|^2, \tag{3.6}$$

for a suitable ξ satisfying (2.16) at almost every point of the parabolic cylinder.

We postpone to the next section a proof of the fact that a global in time solution (u, z) to (3.4)–(3.5) plus the initial and boundary conditions exists in a suitable regularity class. In this part, we just show that such a solution complies with a number of *a priori* estimates. The compatibility of the estimates with the approximation will also be discussed later on. In this procedure, we will denote by c a generic positive constant depending only on the assigned data of the problem, including ε and the final time T. On the other hand, c will *not* be allowed to depend on δ (so when δ appears in the computations, it will be kept explicit).

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Our purpose is to construct in a computable way a time interval $(0, T_0)$, with $T_0 > 0$ possibly smaller than T and depending on the given constants δ and ε, such that $z(t,x) \geq 3\delta$ for a.e. $(t,x) \in (0, T_0) \times \Omega$. In this way, due to (3.1), (u,z) will turn out to solve the original system (2.1)–(2.2) in that time span.

To start, we perform the analogue of the energy estimate described in the introduction. Testing (3.4) by u_t, (3.5) by z_t, and performing standard manipulations (note in particular that the product ξz_t is a.e. equal to 0 since $\alpha(z_t)$ may contain non-zero values only at $z_t = 0$), we easily arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\delta(t) + \|z_t\|^2 = 0, \tag{3.7}$$

with the truncated energy functional

$$\mathcal{E}_\delta(t) = \int_\Omega \left(\frac{T_\delta(z)}{2}|\nabla u|^2 - gu + \frac{1}{2}|\nabla z|^2 + \psi(z) \right). \tag{3.8}$$

Note that the use of test functions and the integrations by parts performed to deduce this estimate and the subsequent ones will be justified as far as one works with the regularised solutions (see the next section for details). Now, as we integrate (3.7) over some time interval $(0,t)$, we see that $\mathcal{E}_\delta(0)$ also depends on the 'initial value' $u_0 = u|_{t=0}$. However, in view of the quasi-static nature of the system, u_0 is not a datum, but has to be computed by evaluating (3.4) at the time $t = 0$. Namely, u_0 corresponds to the (unique) solution to the elliptic problem

$$-\operatorname{div}(T_\delta(z_0)\nabla u_0) = g, \qquad \text{in } \Omega, \tag{3.9}$$

complemented with the homogeneous Dirichlet boundary condition. In view of Assumption (A3) and of the fact $3\delta \leq 1 - \varepsilon$, we actually have $T_\delta(z_0) = z_0 \geq 1/2$. Hence, testing (3.9) by u_0, we obtain

$$\frac{1}{2}\|\nabla u_0\|^2 \leq \int_\Omega T_\delta(z_0)|\nabla u_0|^2 = (g, u_0) \leq \|g\|\|u_0\| \leq \frac{1}{4}\|\nabla u_0\|^2 + c, \tag{3.10}$$

where Poincaré's inequality has also been used. This fact implies in particular that

$$\begin{aligned} \left|\mathcal{E}_\delta(0)\right| &= \left| \int_\Omega \left(\frac{T_\delta(z_0)}{2}|\nabla u_0|^2 - gu_0 + \frac{1}{2}|\nabla z_0|^2 + \psi(z_0) \right) \right| \\ &= \left| \int_\Omega \left(-\frac{T_\delta(z_0)}{2}|\nabla u_0|^2 + \frac{1}{2}|\nabla z_0|^2 + \psi(z_0) \right) \right| \leq c(1 + \|z_0\|_V^2), \end{aligned} \tag{3.11}$$

with c independent of δ. Hence, recalling that $z_0 \in V$, $z_0 \leq 1$ almost everywhere (cf. Assumption (A3)), we see in particular that our assumptions on the initial data imply the finiteness of the energy at $t = 0$.

Integrating (3.7) over the generic time interval $(0,t)$ (where the choice of the admissible 'small' time $t > 0$ will be made clear later on), we then infer that

$$\mathcal{E}_\delta(t) + \int_0^t \|z_t\|^2 = \mathcal{E}_\delta(0) \leq c\left(1 + \|z_0\|_V^2\right). \tag{3.12}$$

Now, using Poincaré's inequality, we arrive at

$$\left| \int_\Omega gu \right| \leq \|g\|\|u\| \leq c\|g\|\|\nabla u\| \leq \frac{\delta}{2}\|\nabla u\|^2 + \frac{c}{\delta}. \tag{3.13}$$

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As a consequence of the above relations (2.8) and (3.1), we have

$$\mathcal{E}_\delta(t) \geq \frac{\delta}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|z(t)\|_V^2 - \frac{c}{\delta}. \tag{3.14}$$

Combining (3.12) with (3.14), we then obtain the *a priori* estimates,

$$\|z\|_{L^\infty(0,t;V)} \leq c\big(\delta^{-1/2} + \|z_0\|_V\big), \tag{3.15}$$

$$\|u\|_{L^\infty(0,t;V_0)} \leq c\delta^{-1/2}\big(\delta^{-1/2} + \|z_0\|_V\big), \tag{3.16}$$

$$\|z_t\|_{L^2(0,t;H)} \leq c\big(\delta^{-1/2} + \|z_0\|_V\big). \tag{3.17}$$

Next, evaluating (3.4) at the generic time t and testing it by u, applying once more Poincaré's inequality, we obtain

$$\begin{aligned}\|\nabla u\|^2 &= \int_\Omega |\nabla u|^2 = \int_\Omega \frac{T_\delta(z)}{T_\delta(z)}|\nabla u|^2 \leq \left\|\frac{1}{T_\delta(z)}\right\|_{L^\infty(\Omega)} \int_\Omega T_\delta(z)|\nabla u|^2 \\ &= \left\|\frac{1}{T_\delta(z)}\right\|_{L^\infty(\Omega)} (g,u) \leq c\left\|\frac{1}{T_\delta(z)}\right\|_{L^\infty(\Omega)} \|g\|\|\nabla u\|,\end{aligned} \tag{3.18}$$

whence

$$\|\nabla u\| \leq c\left\|\frac{1}{T_\delta(z)}\right\|_{L^\infty(\Omega)}, \tag{3.19}$$

with computable $c > 0$ also depending on g.

Now let us define, for $r \in \mathbb{R}$,

$$\phi_\delta(r) := \frac{1}{T_\delta(1-r)}, \qquad \text{so that } \frac{1}{T_\delta(r)} = \frac{1}{T_\delta(1-(1-r))} = \phi_\delta(1-r). \tag{3.20}$$

In other words, for $r \in \mathbb{R}$, the function $\phi_\delta(r)$ is a regularisation of the function $r \mapsto 1/(1-r)_+$; in particular, $\phi_\delta(r) = (1-r)^{-1}$ for $r \leq 1 - 3\delta$. Notice also that ϕ_δ is non-decreasing on $\mathbb{R}$.

By the use of (3.20), (3.19) can be rewritten as

$$\|\nabla u\| \leq c\|\phi_\delta(1-z)\|_{L^\infty(\Omega)} = c\phi_\delta\big(\|1-z\|_{L^\infty(\Omega)}\big). \tag{3.21}$$

Next, let us observe that (3.4) may be equivalently rewritten as

$$-T_\delta(z)\Delta u = g + T'_\delta(z)\nabla z\cdot\nabla u. \tag{3.22}$$

We now compute the L^2- and L^3-norms of both sides of the above relation. Observing that $T_\delta(r) \geq 2\delta$ with $|T'_\delta(r)| \leq 1$ for every $\delta \in (0, 1/12]$ and $r \in \mathbb{R}$, and using elementary interpolation and embedding inequalities along with (2.6), we first find that

$$\begin{aligned}2\delta\|\Delta u\| &\leq \|g\| + \|\nabla z\|_{L^6(\Omega)}\|\nabla u\|_{L^3(\Omega)} \\ &= \|g\| + \|\nabla(z-1)\|_{L^6(\Omega)}\|\nabla u\|_{L^3(\Omega)} \\ &\leq \|g\| + c\|z-1\|_W\|\nabla u\|^{1/2}\|\Delta u\|^{1/2} \\ &\leq c + c\delta^{-1/2}\big(\|z-1\| + \|\Delta z\|\big)\|\nabla u\|^{1/2}\delta^{1/2}\|\Delta u\|^{1/2} \\ &\leq c + c\delta^{-1}\big(\|z-1\|^2 + \|\Delta z\|^2\big)\|\nabla u\| + \delta\|\Delta u\|.\end{aligned} \tag{3.23}$$

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Analogously, combining the Gagliardo–Nirenberg inequality [26] with standard elliptic regularity results of L^p-type, we infer that

$$\|\nabla v\|_{L^6(\Omega)} \le c\|\Delta v\|_{L^3(\Omega)}^{2/3}\|\nabla v\|^{1/3}, \tag{3.24}$$

which holds for every $v \in V_0 \cap W^{2,3}(\Omega)$. Using such a relation, we deduce that

$$\begin{aligned} 2\delta\|\Delta u\|_{L^3(\Omega)} &\le \|g\|_{L^3(\Omega)} + \|\nabla z\|_{L^6(\Omega)}\|\nabla u\|_{L^6(\Omega)} \\ &= \|g\|_{L^3(\Omega)} + \|\nabla(z-1)\|_{L^6(\Omega)}\|\nabla u\|^{1/3}\|\Delta u\|_{L^3(\Omega)}^{2/3} \\ &\le c + c\delta^{-2/3}\big(\|z-1\| + \|\Delta z\|\big)\|\nabla u\|^{1/3}\delta^{2/3}\|\Delta u\|_{L^3(\Omega)}^{2/3} \\ &\le c + c\delta^{-2}\big(\|z-1\|^3 + \|\Delta z\|^3\big)\|\nabla u\| + \delta\|\Delta u\|_{L^3(\Omega)}. \end{aligned} \tag{3.25}$$

Hence, recalling also (3.21), (3.23) and (3.25) imply, respectively

$$\|\Delta u\| \le c\delta^{-1} + c\delta^{-2}\big(\|z-1\|^2 + \|\Delta z\|^2\big)\phi_\delta\big(\|1-z\|_{L^\infty(\Omega)}\big), \tag{3.26}$$

$$\|\Delta u\|_{L^3(\Omega)} \le c\delta^{-1} + c\delta^{-3}\big(\|z-1\|^3 + \|\Delta z\|^3\big)\phi_\delta\big(\|1-z\|_{L^\infty(\Omega)}\big). \tag{3.27}$$

As a next step, we test (3.5) by $-\Delta z_t$. Then, using the monotonicity of α and the no-flux boundary conditions, we would expect that

$$(\alpha(z_t), -\Delta z_t) = \int_\Omega \alpha'(z_t)|\nabla z_t|^2 \ge 0. \tag{3.28}$$

On the other hand, the above computation is formal. Indeed, α is a nonsmooth maximal monotone graph (and $\alpha(z_t)$ has to be interpreted as a selection ξ (cf. (2.16)). Nevertheless, the inequality $(\xi, -\Delta z_t) \ge 0$ is valid anyway, and it could be rigorously proved by proceeding, e.g. along the lines of [28, Lemma 2.4] (see also Remark 4.1 below for a further justification of this procedure). Hence, we deduce that

$$\|\nabla z_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta z\|^2 \le \int_\Omega |\psi''(z)\nabla z\cdot\nabla z_t| + \frac{1}{2}\int_\Omega \big|\nabla\big(T_\delta'(z)|\nabla u|^2\big)\cdot\nabla z_t\big| =: I_1 + I_2 \tag{3.29}$$

and we need to control the terms on the right-hand side. First of all, by (A1) and (2.7) we have that $\psi'' \in L^\infty(\mathbb{R})$, whence

$$I_1 \le c\|\nabla z\|\|\nabla z_t\| \le \frac{1}{6}\|\nabla z_t\|^2 + c\|1-z\|_V^2. \tag{3.30}$$

Next, recalling (3.2), we easily obtain

$$I_2 \le c\delta^{-1}\int_\Omega |\nabla u|^2|\nabla z\cdot\nabla z_t| + c\int_\Omega |D^2u||\nabla u||\nabla z_t| =: I_{2,1} + I_{2,2}. \tag{3.31}$$

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Furthermore, using also (3.26), we infer that

$$\begin{aligned}
I_{2,1} &\le c\delta^{-1}\|\nabla u\|^2_{L^6(\Omega)}\|\nabla z\|_{L^6(\Omega)}\|\nabla z_t\| \\
&\le c\delta^{-1}\|\Delta u\|^2\big(\|z-1\|+\|\Delta z\|\big)\|\nabla z_t\| \\
&\le \frac{1}{6}\|\nabla z_t\|^2 + c\delta^{-2}\|\Delta u\|^4\big(\|z-1\|^2+\|\Delta z\|^2\big) \\
&\le \frac{1}{6}\|\nabla z_t\|^2 + c\delta^{-6}\big(\|z-1\|^2+\|\Delta z\|^2\big) \\
&\qquad + c\delta^{-10}\big(\|z-1\|^{10}+\|\Delta z\|^{10}\big)\phi_\delta^4\big(\|1-z\|_{L^\infty(\Omega)}\big).
\end{aligned} \tag{3.32}$$

Similarly, using elliptic regularity along with (3.26) and (3.27) as well as Young's inequality, we obtain

$$\begin{aligned}
I_{2,2} &\le c\|D^2u\|_{L^3(\Omega)}\|\nabla u\|_{L^6(\Omega)}\|\nabla z_t\| \\
&\le \frac{1}{6}\|\nabla z_t\|^2 + c\|\Delta u\|^2_{L^3(\Omega)}\|\Delta u\|^2 \\
&\le \frac{1}{6}\|\nabla z_t\|^2 + c\delta^{-10}\big(\|z-1\|^{10}+\|\Delta z\|^{10}\big)\phi_\delta^4\big(\|1-z\|_{L^\infty(\Omega)}\big) \\
&\quad + c\delta^{-6}\big(\|z-1\|^4+\|\Delta z\|^4\big)\phi_\delta^2\big(\|1-z\|_{L^\infty(\Omega)}\big) \\
&\quad + c\delta^{-8}\big(\|z-1\|^6+\|\Delta z\|^6\big)\phi_\delta^2\big(\|1-z\|_{L^\infty(\Omega)}\big) + c\delta^{-4} \\
&\le \frac{1}{6}\|\nabla z_t\|^2 + c\delta^{-10}\big(1+\|z-1\|_W^{10}\big)\left[1+\phi_\delta^4\big(\|1-z\|_{L^\infty(\Omega)}\big)\right]
\end{aligned} \tag{3.33}$$

for $\delta \in (0, 1/12]$. Notice that this is actually the only point in the existence proof where we need the control on the L^3-norm of Δu (and, in turn, the assumption $g \in L^3(\Omega)$).

Collecting (3.29)–(3.33) gives

$$\begin{aligned}
\|\nabla z_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\Delta z\|^2 &\le c\|1-z\|_V^2 + c\delta^{-6}\|z-1\|_W^2 \\
&\quad + c\delta^{-10}\left(1+\|z-1\|_W^{10}\right)\left[1+\phi_\delta^4\big(\|1-z\|_{L^\infty(\Omega)}\big)\right].
\end{aligned} \tag{3.34}$$

In order to deduce some useful information from the above relation, we observe the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|1-z\|^2 \le 2|(1-z, z_t)| \le c\|1-z\|^4 + c\|z_t\|^{4/3}. \tag{3.35}$$

Adding it to (3.34) and rearranging terms, with the aid of Young's inequality, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|1-z\|_W^2 + \|\nabla z_t\|^2 \le c\delta^{-10}\left(1+\|z-1\|_W^{10}\right)\left[1+\phi_\delta^4\big(\|1-z\|_{L^\infty(\Omega)}\big)\right] + c\|z_t\|^{4/3}. \tag{3.36}$$

Let us now multiply the above by c_Ω^2, the embedding constant of $H^2(\Omega)$ into $C^0(\overline{\Omega})$ as introduced before. Then, setting

$$y(t) := c_\Omega^2\|1-z(t)\|_W^2 \overset{(2.17)}{\ge} \|1-z(t)\|^2_{L^\infty(\Omega)}, \tag{3.37}$$

and temporarily neglecting the non-negative term $\|\nabla z_t\|^2$ on the left-hand side, we deduce the differential inequality

$$y'(t) \le c_1\delta^{-10}\left[1+y^5(t)\right]\left[1+\phi_\delta^4(y^{1/2}(t))\right] + c_2\|z_t\|^{4/3},$$

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where it is worth noting that $y_0 := y(0) = \varepsilon^2 \le 1/4$ by assumption (A3) and c_1, c_2 are computable positive constants independent of δ. Dividing both sides by $[1+y^5(t)][1+\phi_\delta^4(y^{1/2}(t))]$, which is clearly larger than 1, we then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}B_\delta(y) := \frac{1}{[1+y^5(t)]\left[1+\phi_\delta^4(y^{1/2}(t))\right]}y' \le c_1\delta^{-10} + c_2\|z_t\|^{4/3}, \tag{3.38}$$

where the function B_δ is defined by the left-hand side, namely we have set

$$B_\delta(s) := \int_0^s \frac{\mathrm{d}r}{(1+r^5)\left[1+\phi_\delta^4(r^{1/2})\right]}. \tag{3.39}$$

Here, we note that the function B_δ, as far as δ is a fixed number in the given range $(0, 1/12]$, is well defined and strictly increasing on $\mathbb{R}$. Now, it is clear that, for $s \in [0, 1]$,

$$\frac{1}{2}\int_0^s \frac{\mathrm{d}r}{1+\phi_\delta^4(r^{1/2})} \le B_\delta(s) \le \int_0^s \frac{\mathrm{d}r}{1+\phi_\delta^4(r^{1/2})}. \tag{3.40}$$

Moreover, from (3.20), we observe that, for $r^{1/2} \in [0, 1-3\delta]$, or equivalently $r \in [0, (1-3\delta)^2]$,

$$\frac{1}{1+\phi_\delta^4(r^{1/2})} = \frac{(1-r^{1/2})^4}{(1-r^{1/2})^4+1}, \tag{3.41}$$

whence, we can notice that, as far as s lies in the range $[0, (1-3\delta)^2]$, the expression of $B_\delta(s)$ is independent of δ so that, for such s, we can simply write $B(s)$ in place of $B_\delta(s)$. Notice also that, at largest, $\delta = 1/12$; hence $(1-3\delta)^2$ is always at least $9/16$.

Integrating (3.38) in time and using (3.17) with Hölder's inequality, we obtain

$$\begin{aligned} B_\delta(y(t)) &\le B_\delta(y_0) + \int_0^t \left(c_1\delta^{-10} + c_2\|z_t\|^{4/3}\right) \le B_\delta(\varepsilon^2) + c_1\delta^{-10}t + c_2 t^{1/3}\left(\delta^{-1} + \|z_0\|_V^2\right)^{2/3} \\ &\le B_\delta(\varepsilon^2) + c_3\delta^{-10}t^{1/3}, \end{aligned} \tag{3.42}$$

where the new constant c_3 may also depend on z_0 and T.

On the other hand, due to (2.18) along with the strict increase of B_δ, (3.42) can be rewritten as

$$y(t) \le B_\delta^{-1}\left(B_\delta(\varepsilon^2) + c_3\delta^{-10}t^{1/3}\right) = B_\delta^{-1}\left(B(\varepsilon^2) + c_3\delta^{-10}t^{1/3}\right), \tag{3.43}$$

where we used that $\varepsilon^2 \le 1/4 < 9/16 \le (1-3\delta)^2$.

Now, since δ is assigned and c_3 is a computable constant depending only on the given parameters of the system, using that B_δ is strictly monotone (hence such is its inverse B_δ^{-1}), we deduce that there exists $T_0 \in (0, T]$ so small that, for every $t \in [0, T_0]$, there holds

$$B(\varepsilon^2) + c_3\delta^{-10}t^{1/3} \le B((1-3\delta)^2) = B_\delta((1-3\delta)^2). \tag{3.44}$$

In other words, T_0 can be defined as the largest time $t \in (0, T]$ such that $B(\varepsilon^2) + c_3\delta^{-10}t^{1/3} \le B((1-3\delta)^2)$, that is,

$$T_0 = \left(\frac{B((1-3\delta)^2) - B(\varepsilon^2)}{c_3\delta^{-10}}\right)^3 \wedge T \in (0, T].$$

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As a consequence, in the range $[0, T_0]$ the expression of B_δ is independent of δ and (3.43) reduces to

$$y(t) \le B^{-1}\big(B(\varepsilon^2) + c_3\delta^{-10}t^{1/3}\big) \le B^{-1}\big(B((1-3\delta)^2)\big), \quad \text{for all } t \in [0, T_0], \tag{3.45}$$

which in turn implies

$$\|1 - z(t)\|_{C^0(\overline{\Omega})} \le c_\Omega \|1 - z(t)\|_W = y^{1/2}(t) \le 1 - 3\delta \tag{3.46}$$

and consequently

$$z(t, x) \ge 3\delta \quad \text{for all } t \in [0, T_0],\ x \in \overline{\Omega}. \tag{3.47}$$

This entails in particular that, for every $t \in [0, T_0]$, there holds $T_\delta(z(t)) = z(t)$ a.e. in Ω, whence (u, z) turns out to solve the original system (2.1)–(2.2).

We finally prove the regularity properties (2.10)–(2.12). First of all, we shall check (2.11); the fact $z \in C_w([0, T_0]; W)$ comes from (3.46), while $z \in H^1(0, T_0; V)$ follows from integration of (3.36) over $(0, T_0)$.

Next, we prove (2.10) which is a bit more tricky. First of all, let (z_i, u_i) be two solutions for (2.1), (2.2) on $[0, T_0]$ such that $z_1 \ge 3\delta$. Then by subtraction, we have

$$-\operatorname{div}\,[z_1(\nabla u_1 - \nabla u_2) + (z_1 - z_2)\nabla u_2] = 0, \quad \text{in } \Omega.$$

Test it by $u_1 - u_2$. We see that

$$\begin{aligned}\int_\Omega z_1|\nabla(u_1 - u_2)|^2 &= -\int_\Omega (z_1 - z_2)\nabla u_2 \cdot \nabla(u_1 - u_2)\\ &\le \|z_1 - z_2\|_{L^4(\Omega)}\|\nabla u_2\|_{L^4(\Omega)}\|\nabla(u_1 - u_2)\|,\end{aligned}$$

which entails

$$3\delta\|\nabla(u_1 - u_2)\| \le \|z_1 - z_2\|_{L^4(\Omega)}\|\nabla u_2\|_{L^4(\Omega)}.$$

Hence, we may conclude in particular that

$$3\delta\|\nabla u(t) - \nabla u(s)\| \le \|z(t) - z(s)\|_{L^4(\Omega)} \sup_{\tau\in[0,T_0]} \|u(\tau)\|_{H^2(\Omega)}, \quad \text{for } t, s \in [0, T_0], \tag{3.48}$$

and, therefore, $t \mapsto u(t)$ turns out to be continuous on $[0, T_0]$ with values in V_0. Furthermore, (2.1) implies

$$-\Delta u = \frac{g}{z} + \frac{\nabla z}{z} \cdot \nabla u \ \text{ in } (0, T_0) \times \Omega. \tag{3.49}$$

Note that $t \mapsto 1/z(t)$ is continuous with values in $L^\infty(\Omega)$ on $[0, T_0]$ (indeed, $H^1(0, T_0; V) \cap L^\infty(0, T_0; W)$ is embedded in $C^0([0, T_0]; L^\infty(\Omega))$ and z is uniformly away from zero in $(0, T_0) \times \Omega$). Since $u(t)$ is also bounded in $W^{2,3}(\Omega)$ for any $t \in [0, T_0]$ and $u \in C^0([0, T_0]; V_0)$, the map $t \mapsto \nabla u(t)$ is continuous on $[0, T_0]$ strongly in $L^q(\Omega)$ for any $q \in [1, +\infty)$. On the other hand, thanks to an Aubin–Lions type embedding (see, e.g. [27]), we may observe that

$$L^\infty(0, T_0; H^1(\Omega)) \cap H^1(0, T_0; H) \hookrightarrow C^0([0, T_0]; L^q(\Omega)), \quad \text{for any } q \in [1, 6).$$

Applying this to ∇z, we can verify that $t \mapsto \nabla z(t)$ is of class $C^0([0, T_0]; L^q(\Omega))$ for $q \in [1, 6)$. Combining the above facts, we deduce that $t \mapsto z^{-1}(t)\nabla z(t) \cdot \nabla u(t)$ is continuous strongly in

$L^q(\Omega)$ for any $q \in [1, 6)$. Thus the (strong) continuity of $t \mapsto \Delta u(t)$ in $L^\rho(\Omega)$ for any $\rho \in [1, p] \cap [1, 6)$ on $[0, T_0]$ follows from (A2), (3.49) and the facts observed so far.

Concerning the continuous dependence of solutions on the initial data, let (u_i, z_i) for $i = 1, 2$ be two solutions on $[0, T_0]$ such that either z_1 or z_2 is not less than 3δ and assume (A2) holds for $p > 3$. Then, setting $Z = z_1 - z_2$ and $U = u_1 - u_2$, by subtraction, we have

$$\alpha(\partial_t z_1) - \alpha(\partial_t z_2) + Z_t - \Delta Z + \psi'(z_1) - \psi'(z_2) \ni -\frac{1}{2}\left(|\nabla u_1|^2 - |\nabla u_2|^2\right).$$

Test both sides by Z_t and employ the monotonicity of α. Moreover, note that $u_i \in L^\infty(0, T_0; W^{2,\rho}(\Omega))$, $i = 1, 2$, where now $\rho > 3$, and the embedding $W^{1,\rho}(\Omega) \hookrightarrow L^\infty(\Omega)$. We then obtain

$$\frac{1}{2}\|Z_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Z\|^2 \le \|\psi'(z_1) - \psi'(z_2)\|^2 + \frac{1}{4}\|\nabla(u_1 + u_2)\|^2_{L^\infty(\Omega)}\|\nabla U\|^2 \tag{3.50}$$

by using

$$\left|\int_\Omega \left(|\nabla u_1|^2 - |\nabla u_2|^2\right) Z_t\right| \le \frac{1}{2}\|Z_t\|^2 + \frac{1}{2}\|(\nabla u_1 + \nabla u_2)\cdot \nabla U\|^2$$
$$\le \frac{1}{2}\|Z_t\|^2 + \frac{1}{2}\|\nabla u_1 + \nabla u_2\|^2_{L^\infty(\Omega)}\|\nabla U\|^2.$$

Next, notice that

$$\|\psi'(z_1) - \psi'(z_2)\| \le c\|Z\|$$

for some constant $c > 0$. Hence, (3.50) implies

$$\frac{1}{2}\|Z_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Z\|^2 \le c\left(\|Z\|^2 + \|\nabla U\|^2\right),$$

which, along with (3.48), implies

$$\frac{1}{2}\|Z_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Z\|^2 \le c\left(\|Z\|^2 + \|Z\|^2_{L^4(\Omega)}\right) \le c\|Z\|^2_V.$$

Summing the elementary inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|^2 \le \frac{1}{4}\|Z_t\|^2 + \|Z\|^2 \tag{3.51}$$

in order to recover the full V-norm on the left-hand side and subsequently using Gronwall's lemma, we conclude that

$$\|Z(t)\|^2_V \le c\|Z(0)\|^2_V \quad \text{for } t \in [0, T_0].$$

Moreover, (3.48) yields

$$\|U(t)\|^2_{V_0} \le C\|Z(t)\|^2_V \quad \text{for } t \in [0, T_0].$$

The uniqueness follows immediately under the assumption $Z(0) = 0$, i.e. when the initial data are the same.

Finally, (2.12) follows from (2.10)–(2.11) and a comparison of terms in (2.15). This concludes the proof of Theorem 2.2 provided that we can exhibit a regularisation of the system for which:

- we can prove existence of sufficiently smooth solutions on the time interval $(0, T)$;
- we can show compatibility of the regularisation with the *a priori* estimates performed above.

This will be the purpose of the next section.

4 Approximation

We introduce here a regularisation of system (2.14)–(2.16) for which existence can be proved by means of a fixed point argument. Namely, letting $\epsilon \in (0, 1)$ be a regularisation parameter intended to go to 0 in the limit, we introduce the system

$$\epsilon \Delta^2 u - \operatorname{div}(T_\delta(z)\nabla u) = g, \qquad \text{in } (0, T) \times \Omega, \tag{4.1}$$

$$\alpha(z_t) + z_t - \Delta z + \psi'(z) \ni -\frac{T'_\delta(z)}{2}|\nabla u|^2, \qquad \text{in } (0, T) \times \Omega, \tag{4.2}$$

(for brevity, here we avoid to introduce the notation ξ for the representative of $\alpha(z_t)$, cf. (2.16)). It is worth observing that, in this approximation, we *do not need* to smooth out the operator α. Hence, the irreversibility constraint and the related property will hold also for solutions to (4.1)–(4.2).

The above relations are complemented with the same initial and boundary condition considered before and with the additional boundary condition

$$\Delta u = 0, \quad \text{on } (0, T) \times \Gamma. \tag{4.3}$$

It is worth noting from the very beginning that the system above is fully compatible with the local *a priori* estimates performed in the previous section. Indeed, as we test (4.1) by u_t we obtain an additional (positive) term in the energy functional, namely we have

$$\mathcal{E}_{\epsilon,\delta}(t) = \int_\Omega \Big(\frac{\epsilon}{2}|\Delta u|^2 + \frac{T_\delta(z)}{2}|\nabla u|^2 - gu + \frac{1}{2}|\nabla z|^2 + \psi(z)\Big), \tag{4.4}$$

and the new term is a source of additional *a priori* regularity. On the other hand, the elliptic regularisation is also compatible with the procedure used to get the differential inequality (3.38). Actually, the key estimates (3.23) and (3.25) can still be obtained similarly as before. Namely, to get the analogue of (3.23) we now need to test (4.1) by $-\Delta u$, whereas for (3.25), we test (4.1) by $-|\Delta u|\Delta u$ and notice that

$$\int_\Omega -\epsilon \Delta^2 u(|\Delta u|\Delta u) = 2\epsilon \int_\Omega |\Delta u||\nabla \Delta u|^2 \geq 0, \tag{4.5}$$

also in view of the additional boundary condition (4.3).

On the other hand, the new term provides additional compactness and it may help to solve (4.1)–(4.2) by means of a fixed point argument. We now sketch a possible procedure (which, in some sense, is inspired by the argument given in [5]), leaving the details to the reader.

(1) We take a prescribed function $\overline{u}$ instead of u in (4.2). More precisely, we choose

$$\overline{u} \in L^4(0, T; W^{2,3}(\Omega) \cap V_0). \tag{4.6}$$

This in particular implies that

$$|\nabla \overline{u}|^2 \in L^2(0, T; V) \tag{4.7}$$

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as a direct check shows. The corresponding equation

$$\alpha(z_t) + z_t - \Delta z + \psi'(z) \ni -\frac{T'_\delta(z)}{2}|\nabla \overline{u}|^2 \tag{4.8}$$

is a parabolic equation with the Lipschitz non-linearity $T'_\delta(z)$ and the nonsmooth term $\alpha(z_t)$. For this type of equation the regularity theory is well-established. For instance, one can test it by $-\Delta z_t$ (see also Remark 4.1 below). Then, using the monotonicity of α, condition (4.6), the Lipschitz continuity of T'_δ, and Gronwall's lemma, one may deduce the existence of at least one solution z in the same regularity class of Theorem 2.2, namely

$$z \in H^1(0, T; V) \cap L^\infty(0, T; W). \tag{4.9}$$

Moreover, such a solution is readily seen to be unique. To check this fact it suffices to take a couple of solutions (with the same proposed $\overline{u}$), compute correspondingly the difference of (4.8), and test it by the difference of the z_t's. Then, exploiting the monotonicity of α one can easily obtain a contraction estimate.

(2) We plug the function z obtained at the previous step into (4.1). This gives rise to a fourth-order elliptic equation, whose leading term is linear, with the boundary conditions $u = \Delta u = 0$ on $(0, T) \times \Gamma$. Hence, it has a unique weak solution $u \in L^\infty(0, T; H^2(\Omega) \cap V_0)$. Moreover, we can also prove that

$$u \in L^\infty(0, T; H^4(\Omega)). \tag{4.10}$$

Indeed, rewrite (4.1) as

$$\epsilon \Delta^2 u - T_\delta(z)\Delta u = T'_\delta(z)\nabla z \cdot \nabla u + g \ \text{ in } (0, T) \times \Omega, \tag{4.11}$$

which is complemented with the homogeneous Dirichlet boundary conditions and where the right-hand side lies at least on $L^\infty(0, T; L^2(\Omega))$. Hence, the L^2-regularity theory for higher order elliptic operators entails $u(\cdot, t) \in H^4(\Omega)$ for a.e. $t \in (0, T)$. More precisely, we can set $v = -\Delta u$ and apply the L^2 elliptic regularity of second order type. Then we have

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \left|\partial^2_{ij} v\right|^2 \le C,$$

where $\partial_{ij} = \partial^2/\partial x_i \partial x_j$ for $i, j = 1, 2, 3$. Here, we used $u \in L^\infty(0, T; V_0)$ and (4.9) along with $W \subset L^\infty(\Omega)$. Using relation $v = -\Delta u$ and integrating by parts, the above can be rewritten as

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \left|\partial^4_{ijkl} u\right|^2 \le C \quad \text{ for } \ i, j, k, l = 1, 2, 3,$$

which yields $u \in L^\infty(0, T; H^4(\Omega))$.

(3) We finally consider the mapping $\overline{u} \mapsto u$ and we aim to apply the Schauder fixed point theorem to this map in order to get existence of at least one local in time solution to the initial boundary value problem for (4.1)–(4.2). The most delicate point is proving compactness, because the system is quasi-stationary and we have no information on u_t. On the other hand, by (4.9) and the Aubin–Lions theorem, one can easily obtain that the mapping $\overline{u} \mapsto z$ is completely continuous from the space (4.6) to the space

$$C^0([0, T]; H^\alpha(\Omega)), \quad \text{ for every } \ \alpha \in (3/2, 2), \tag{4.12}$$

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which is continuously embedded into $C^0([0, T] \times \overline{\Omega})$. Hence, one can repeat the argument in **(2)** by taking the space (4.12) for z. No modification is required and one can see that the mapping $z \mapsto u$ is continuous from the space (4.12) to the space in (4.10). Note that the space in (4.10) is continuously (though not compactly) embedded into the space in (4.6). Hence, $\overline{u} \mapsto u$ is completely continuous because it is the composition of a compact map and a continuous one. Thus, to apply Schauder's theorem it just remains to choose a proper ball B of the space in (4.6) and prove that there exists a small time $T_1 \leq T$ such that the image of B is contained in B. This fact can be verified by a number of simple checkings. In particular, we may use the fact that

$$\|v\|_{L^4(0,T_1;W^{2,3}(\Omega))} \leq c\|v\|_{L^4(0,T_1;H^4(\Omega))} \leq cT_1^{1/4}\|v\|_{L^\infty(0,T_1;H^4(\Omega))} \tag{4.13}$$

for any $v \in L^\infty(0, T_1; H^4(\Omega))$ (cf. (4.10)). As a consequence, Schauder's theorem provides existence of a solution to (4.1)–(4.2) with the initial and boundary conditions (including (4.3)) over the time interval $(0, T_1)$. Note that, actually, T_1 may be strictly smaller than T_0. On the other hand, performing the *a priori* estimates by keeping $\delta > 0$ fixed at a first stage, we can easily see that the resulting bounds are uniform over the interval $(0, T)$. Hence, by standard extension arguments, the solution to the regularised problem can be thought to be defined over the whole of $(0, T)$.

Remark 4.1 *One can see in particular that the additional regularity on u obtained in the framework of the regularised problem is sufficient to justify the a priori estimates of the previous part. Concerning z there is just a point that needs to be clarified a bit. Indeed, in the above part we have used the test function $-\Delta z_t$ in a parabolic equation having the following structure:*

$$\alpha(z_t) + z_t - \Delta z \ni \eta, \tag{4.14}$$

where one can easily check that

$$\eta = -\psi'(z) - \frac{T'_\delta(z)}{2}|\nabla \overline{u}|^2 \in L^2(0, T; V). \tag{4.15}$$

On the other hand, if α is not regularised, up to our knowledge no L^2-regularity theory is available for equation (4.14)*, i.e. the single summands on the left-hand side of* (4.14) *are not expected to lie separately in L^2, nor it does the test function $-\Delta z_t$, which is then not directly admissible. To overcome this issue, one should, at the step* **(1)***, first consider a further regularisation of* (4.14)*, namely*

$$\alpha_\lambda(z_t) + z_t - \Delta z = \eta, \tag{4.16}$$

where α_λ is the Yosida approximation of α of order $\lambda > 0$ (cf. [2, 8]), and notice that (4.16) *is well-posed in L^2. Then, one can first test* (4.16) *by $-\Delta z_t$ (which is allowed thanks to better regularity holding for $\lambda > 0$) and then take $\lambda \searrow 0$ before proceeding with the fixed point argument. Indeed, the obtained a priori bound is preserved in the limit $\lambda \searrow 0$ by semicontinuity. The details, based on standard convex analysis tools, are left to the reader (see also [1, Lemma 3.10 and Proof of Theorem 3.1] for a similar procedure).*

Remark 4.2 *It is worth observing that our choice of performing an elliptic regularisation of* (3.4) *is also motivated by the fact that a parabolic regularisation (obtained for instance by plugging a term ϵu_t or $-\epsilon \Delta u_t$ in place of our $\epsilon \Delta^2 u$) would not be fully compatible with the estimates of the previous section. In particular, we need to estimate (cf.* (3.18)*) the L^2-norm* *of ∇u at any fixed time t, and that argument does not seem to work due to the presence of an additional term depending on u_t.*

5 Conclusion

We have considered a model for the evolution of damage in an elastic medium subject to an external load. We have analysed the case of 'complete' damage, namely we have assumed that the elastic tensor may degenerate after complete damage has occurred at some point, i.e. a macroscopic fracture has appeared. Correspondingly, we have provided a quantitative estimate of a 'small' time $T_0 > 0$, depending on the problem data, such that elastic degeneration certainly *does not* occur in the time interval $[0, T_0]$, provided the material is sound enough at the initial time. We have also assumed *irreversibility* of the damage phenomenon; namely, microfractures, once they are created, can never be repaired.

Models for 'complete' damage have been extensively studied in the recent scientific literature, often in connection with other phenomena (e.g. thermal diffusion) or in more specific contexts (e.g. delamination or contact with adhesion). Nevertheless, if one considers a quasi-static regime, the basic structure of most damage or delamination models appears to be strongly related to system (2.1)–(2.2); on the other hand, in the literature, such a system is often regularised (e.g. by viscosity terms) either in the elastic equation, or in the damage evolution equation, or in both these relations.

In the present work, we have been able to prove a local well-posedness result for system (2.1)–(2.2), coupled with the initial and boundary condition, with no need for adding any smoothing term. Our technique may open the way to applications to more complex models, whose mathematical analysis may also become possible in a non-regularised setting.

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Conflict of interest

The authors declare that they have no conflict of interest.

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