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VANISHING THEOREMS FOR HYPERSURFACES IN THE UNIT SPHERE

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Abstract. Let M^n, $n \geq 3$, be a complete hypersurface in $\mathbb{S}^{n+1}$. When M^n is compact, we show that M^n is a homology sphere if the squared norm of its traceless second fundamental form is less than $\frac{2(n-1)}{n}$. When M^n is non-compact, we show that there are no non-trivial L^2 harmonic p-forms, $1 \leq p \leq n-1$, on M^n under pointwise condition. We also show the non-existence of L^2 harmonic 1-forms on M^n provided that M^n is minimal and $\frac{n-1}{n}$-stable. This implies that M^n has only one end. Finally, we prove that there exists an explicit positive constant C such that if the total curvature of M^n is less than C, then there are no non-trivial L^2 harmonic p-forms on M^n for all $1 \leq p \leq n-1$.

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1. Introduction. Let M^n be a complete hypersurface in a Riemannian manifold N^{n+1}. Fix a point $x \in M$ and a local orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ of N^{n+1} such that $\{e_1, \ldots, e_n\}$ are tangent fields at x. In the following, we shall use the following convention on the ranges of indices: $1 \leq i, j, k, \ldots \leq n$. The second fundamental form A is defined by $\langle AX, Y\rangle = \langle \overline{\nabla}_X Y, e_{n+1}\rangle$ for any tangent fields X, Y. Here, $\overline{\nabla}$ is the Riemannian connection of N^{n+1}. Denote by $h_{ij} = \langle Ae_i, e_j\rangle$, then $|A|^2 = \sum_{i,j}(h_{ij})^2$, and the mean curvature vector H is defined by $H = \frac{1}{n}\sum_i h_{ii}e_{n+1}$. The traceless second fundamental form ϕ is defined by

$$\phi(X, Y) = \langle AX, Y\rangle - \langle X, Y\rangle H.$$

It is easy to see that

$$|\phi|^2 = |A|^2 - n|H|^2,$$

which measures how much the immersion deviates from being totally umbilical. For $0 < \delta \leq 1$, a minimal hypersurface M^n in the sphere $\mathbb{S}^{n+1}$ is called δ-stable if

$$\delta \int_M (n + |A|^2)f^2)dv \leq \int_M |\nabla f|^2 dv, \quad \forall f \in C_0^\infty(M).$$

When $\delta = 1$, M is also said to be stable.

We recall that the classification of stable constant mean curvature surfaces in $\mathbb{S}^3$ is completely known. It is well-known that there is no stable complete minimal surface in $\mathbb{S}^3$ (this can be proved by Theorem 4 in [**13**]) and Theorem 5.1.1 in [**16**]). In

[6], Frensel proved that there is no weakly stable complete non-compact surface with constant mean curvature in $\mathbb{S}^3$. For the higher dimensional case, very little is known about complete non-compact stable hypersurfaces with constant mean curvature in the sphere $\mathbb{S}^{n+1}$, $n > 2$.

In [2], Cao–Shen–Zhu showed that a complete immersed stable minimal hypersurface M^n in $\mathbb{R}^{n+1}$ with $n \geq 3$ must have only one end. This result was generalized by Li–Wang [15], they proved that if a complete minimal hypersurface M^n in $\mathbb{R}^{n+1}$ has finite index, then the dimension of the space of L^2 harmonic 1-forms on M^n is finite, and M^n must have finitely many ends. In [19], Yun proved that for a complete-oriented minimal hypersurface M^n in $\mathbb{R}^{n+1}$ with $n \geq 3$, if the L^n-norm of its second fundamental form is less than an explicit constant, then there are no non-trivial L^2 harmonic 1-forms on M^n, which implies that M^n has only one end. Fu–Xu [8] proved that if an oriented complete submanifold M^n ($n \geq 3$) in $\mathbb{R}^{n+m}$ has finite total curvature and finite total mean curvature, then the space of L^2 harmonic 1-form on M^n has finite dimension and M^n has finitely many ends. Recently, Cavalcante–Mirandola–Vitório [4] proved vanishing and finiteness theorems for L^2 harmonic 1-forms on a complete non-compact submanifold in a Hadamard manifold with finite total curvature, without any additional hypothesis on the mean curvature. Later, Zhu–Fang [20] obtained a generalized version of Cavalcante–Mirandola–Vitorio's results for submanifolds in $\mathbb{S}^{n+m}$. On the other hand, for the case of L^2 harmonic p-forms of higher order, Tanno [17] proved that if M^n is a complete-oriented stable minimal hypersurface in $\mathbb{R}^{n+1}$, $n \leq 4$, then there exist no non-trivial L^2 harmonic p-forms on M^n for all $0 \leq p \leq n$. In [11, 12], the author proved vanishing and finiteness theorems for L^2 harmonic p-forms, $0 \leq p \leq n$, on submanifolds of Euclidean space, under pointwise or integral conditions.

In this paper, we investigate vanishing theorems for harmonic p-forms on complete submanifold of $\mathbb{S}^{n+1}$. We denote the space of all L^2 harmonic p-forms on a Riemannian manifold M^n by $H^p(L^2(M))$. These spaces have a (reduced) L^2-cohomology interpretation. For more results concerning L^2 harmonic p-forms on complete non-compact manifolds, one can consult [3].

Our main results in this paper are stated as follows.

THEOREM 1.1. *Let M^n, $n \geq 3$, be a compact hypersurface in $\mathbb{S}^{n+1}$. Assume that*

$$|\phi|^2 < \frac{2(n-1)}{n}.$$

Then, the Betti number $\beta_p(M) = 0$ for all $1 \leq p \leq n-1$, and M is a homology sphere.

THEOREM 1.2. *Let M^n, $n \geq 3$, be a complete non-compact hypersurface in $\mathbb{S}^{n+1}$. Assume that*

$$|\phi|^2 \leq \frac{2p(n-p)}{n} + \frac{1}{n}\min\{p, n-p\}|A|^2$$

for $1 \leq p \leq n-1$. Then, every harmonic p-form ω on M with $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^{2\beta} dv = 0$, $\beta > 1 - K_p$, vanishes identically. In particular, $H^p(L^2(M)) = \{0\}$.

THEOREM 1.3. *Let M^n, $n \geq 3$, be a complete non-compact $\frac{n-1}{n}$-stable minimal hypersurface in $\mathbb{S}^{n+1}$. Then, $H^1(L^2(M)) = \{0\}$, and M has only one end.*

THEOREM 1.4. *Let M^n, $n \geq 3$, be a complete non-compact hypersurface in $\mathbb{S}^{n+1}$. Then, there exists a positive constant C such that if*

$$\int_M |\phi|^n dv < C,$$

then every harmonic p-form ω, $1 \leq p \leq n-1$, on M with $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ vanishes identically. In particular, $H^p(L^2(M)) = \{0\}$ for all $1 \leq p \leq n-1$.

REMARK 1.1. Zhu–Fang [**20**] and Zhu [**21**] proved vanishing theorems for L^2 harmonic 1-forms or 2-forms on submanifolds of $\mathbb{S}^{n+m}$. Theorem 1.2 can be seen as generalizations of their results.

2. Estimates for the Weitzenböck curvature operator. Let M^n be an n-dimensional complete hypersurface in $\mathbb{S}^{n+1}$, and let $\triangle$ be the Hodge Laplace–Beltrami operator of M^n acting on the space of differential p-forms. Given two p-forms ω and θ, we define a pointwise inner product

$$\langle \omega, \theta \rangle = \sum_{i_1,\ldots,i_p=1}^{n} \omega(e_{i_1}, \ldots, e_{i_p})\theta(e_{i_1}, \ldots, e_{i_p}).$$

Note that we omit the normalizing factor $1/p!$. Denote by R_{ij} and R_{ijkl} the components of the Ricci tensor and the curvature tensor of M^n, respectively, then the Weitzenböck formula [**18**] gives

$$\begin{aligned} \frac{1}{2}\triangle|\omega|^2 &= |\nabla\omega|^2 + \langle \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega\rangle \\ &= |\nabla\omega|^2 + pW(\omega), \end{aligned} \tag{2.1}$$

where

$$W(\omega) = R_{ij}\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} - \frac{p-1}{2}R_{ijkl}\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p}. \tag{2.2}$$

Here, repeated indices are contracted and summed£‹ and the indices $1 \leq i_1, i_2, \ldots, i_n \leq n$ are distinct with each other in the following discussion.

To estimate $W(\omega)$, noting that M^n has flat normal bundle, we can choose an orthonormal frame $\{e_i\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. Then, the Gauss equation implies that

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \lambda_i\lambda_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Substituting into (2.2) yields

$$\begin{aligned} W(\omega) &= (n-1)\delta_{ij}\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} - \frac{p-1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p} \\ &\quad + \lambda_i\lambda_k(\delta_{kk}\delta_{ij} - \delta_{ik}\delta_{jk})\omega^{ii_2\ldots i_p}\omega^j_{i_2\ldots i_p} - \frac{p-1}{2}\lambda_i\lambda_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\omega^{iji_3\ldots i_p}\omega^{kl}_{i_3\ldots i_p} \\ &= (n-p)|\omega|^2 + nH\lambda_i\omega^{ii_2\ldots i_p}\omega^i_{i_2\ldots i_p} - \lambda_i^2\omega^{ii_2\ldots i_p}\omega^i_{i_2\ldots i_p} - (p-1)\lambda_i\lambda_j\omega^{iji_3\ldots i_p}\omega^{ij}_{i_3\ldots i_p} \end{aligned}$$

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$$= (n-p)|\omega|^2 + \frac{nH}{p}(\lambda_i + \lambda_{i_2} + \lambda_{i_3} + \cdots + \lambda_{i_p})\omega^{ii_2\ldots i_p}\omega^i_{i_2\ldots i_p}$$
$$- \frac{1}{p}(\lambda_i + \lambda_j + \lambda_{i_3} + \cdots + \lambda_{i_p})^2\omega^{iji_3\ldots i_p}\omega^{ij}_{i_3\ldots i_p}$$
$$= (n-p)|\omega|^2 + \frac{1}{p}\left[nH(\lambda_{i_1} + \cdots + \lambda_{i_p}) - (\lambda_{i_1} + \cdots + \lambda_{i_p})^2\right]\omega^{i_1 i_2\ldots i_p}\omega^{i_1 i_2\ldots i_p}$$
$$\geq (n-p)|\omega|^2 + \frac{1}{p}\inf_{i_1,\ldots,i_n}\left[(\lambda_{i_1} + \cdots + \lambda_{i_p})(\lambda_{i_{p+1}} + \cdots + \lambda_{i_n})\right]|\omega|^2. \tag{2.3}$$

By a direct computation, we have

$$(\lambda_{i_1} + \cdots + \lambda_{i_p})(\lambda_{i_{p+1}} + \cdots + \lambda_{i_n})$$
$$= \frac{1}{2}\left[(\lambda_{i_1} + \cdots + \lambda_{i_n})^2 - (\lambda_{i_1} + \cdots + \lambda_{i_p})^2 - (\lambda_{i_{p+1}} + \cdots + \lambda_{i_n})^2\right]$$
$$\geq \frac{1}{2}\left(n^2|H|^2 - \max\{p, n-p\}|A|^2\right). \tag{2.4}$$

Substituting (2.4) into (2.3), and combining with (2.1) yields

$$\frac{1}{2}\triangle|\omega|^2 \geq |\nabla\omega|^2 + p(n-p)|\omega|^2 + \frac{1}{2}\left(n^2|H|^2 - \max\{p, n-p\}|A|^2\right)|\omega|^2$$
$$= |\nabla\omega|^2 + p(n-p)|\omega|^2 - \frac{n}{2}|\phi|^2|\omega|^2 + \frac{1}{2}\min\{p, n-p\}|A|^2|\omega|^2. \tag{2.5}$$

Using Kato's inequality [**1**], it follows from (2.5) that

$$|\omega|\triangle|\omega| \geq K_p|\nabla|\omega||^2 + p(n-p)|\omega|^2 - \frac{n}{2}|\phi|^2|\omega|^2 + \frac{1}{2}\min\{p, n-p\}|A|^2|\omega|^2, \tag{2.6}$$

where $K_p = \frac{1}{n-p}$ if $1 \leq p \leq n/2$, and $K_p = \frac{1}{p}$ if $n/2 \leq p \leq n-1$.

3. Proof of Theorems 1.1–1.3. By using the relation (2.5) for harmonic p-forms, we have the following general vanishing theorem.

THEOREM 3.1. *Let M^n, $n \geq 3$, be a compact hypersurface of $\mathbb{S}^{n+1}$. Assume that*

$$|\phi|^2 \leq \frac{2p(n-p)}{n} + \frac{1}{n}\min\{p, n-p\}|A|^2 \tag{3.1}$$

for $1 \leq p \leq n-1$. Then, every harmonic p-form ω on M is parallel. Assume further that the inequality (3.1) *is strict at a point, then the Betti number $\beta_p(M) = 0$.*

Proof. Given a harmonic p-form ω on M. By (2.5) and the hypothesis (3.1), we conclude that

$$\frac{1}{2}\triangle|\omega|^2 \geq |\nabla\omega|^2 + \left[p(n-p) + \frac{1}{2}\min\{p, n-p\}|A|^2| - \frac{n}{2}|\phi|^2\right]|\omega|^2 \geq 0. \tag{3.2}$$

By the compactness of M and the maximum principle, $|\omega| =$ constant. Hence, (3.2) implies that $|\nabla\omega| = 0$, which means that ω is parallel. If (3.1) is strict at some point $x_0 \in M$, it follows from (3.2) that $\omega(x_0) = 0$. Since ω is parallel, $\omega = 0$ on M. □

Since $\min_{1\le p\le n-1} \frac{2p(n-p)}{n} = \frac{2(n-1)}{n}$, the conclusion of Theorem 1.1 follows immediately from Theorem 3.1.

Proof of Theorem 1.2. Let ω be a harmonic p-form satisfying $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^{2\beta} dv = 0$, $\beta > 1 - K_p$. It follows from (2.6) and the hypothesis that

$$|\omega|\triangle|\omega| \ge K_p|\nabla|\omega||^2 + \Big[p(n-p) + \frac{1}{2}\min\{p, n-p\}|A|^2 - \frac{n}{2}|\phi|^2\Big]|\omega|^2 \\ \ge K_p|\nabla|\omega||^2. \tag{3.3}$$

Following a calculation in [7], for any $\alpha > 0$, we have

$$\begin{aligned} |\omega|^\alpha\triangle|\omega|^\alpha &= |\omega|^\alpha\big[\alpha(\alpha-1)|\omega|^{\alpha-2}|\nabla|\omega||^2 + \alpha|\omega|^{\alpha-1}\triangle|\omega|\big] \\ &= \frac{\alpha-1}{\alpha}|\nabla|\omega|^\alpha|^2 + \alpha|\omega|^{2\alpha-2}|\omega|\triangle|\omega| \\ &\ge \Big(1 - \frac{1-K_p}{\alpha}\Big)|\nabla|\omega|^\alpha|^2. \end{aligned} \tag{3.4}$$

Let $\eta \in C_0^\infty(M)$. Multiplying both sides of (3.4) by $\eta^2|\omega|^{2q\alpha}$, $q > 0$, and integrating over M, we find

$$\begin{aligned} &\Big[2(q+1) - \frac{1-K_p}{\alpha}\Big]\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv \\ &\le -2\int_M \eta|\omega|^{(2q+1)\alpha}\langle\nabla\eta, \nabla|\omega|^\alpha\rangle dv \\ &\le \epsilon\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv + \frac{1}{\epsilon}\int_M |\omega|^{2(1+q)\alpha}|\nabla\eta|^2 dv \end{aligned}$$

for any $\epsilon > 0$, which gives

$$\Big[2(q+1) - \frac{1-K_p}{\alpha} - \epsilon\Big]\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv \le \frac{1}{\epsilon}\int_M |\omega|^{2(1+q)\alpha}|\nabla\eta|^2 dv. \tag{3.5}$$

Let $\beta = 2(q+1)\alpha$. Since $\beta > 1 - K_p$, we can choose $\epsilon > 0$ small enough such that $2(q+1) - \frac{1-K_p}{\alpha} - \epsilon > 0$. Hence, it follows from (3.5) that

$$\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv \le C\int_M |\omega|^\beta|\nabla\eta|^2 dv \tag{3.6}$$

for some constant $C > 0$.

Fix a point $x_0 \in M$ and let $\rho(x)$ be the geodesic distance on M from x_0 to x. Let us choose $\eta_r \in C_0^\infty(M)$ satisfying

$$\eta_r(x) = \begin{cases} 1 & \text{if } \rho(x) \le r, \\ 0 & \text{if } 2r < \rho(x) \end{cases}$$

and

$$|\nabla\eta_r|(x) \le \frac{2}{r} \ \text{ if } r < \rho(x) \le 2r$$

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for $r > 0$. Substituting $\eta = \eta_r$ into (3.6) yields

$$\int_{B_{x_0}(R)} |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 dv \leq \frac{4C}{R^2} \int_{B_{x_0}(2R)} |\omega|^\beta dv.$$

Letting $R \to \infty$, we conclude that

$$\int_M |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 dv \leq 0,$$

which gives $|\omega| =$ constant. Substituting this fact into (3.3), we find that $\omega = 0$.

To prove Theorem 1.3, we first consider the following vanishing theorem for harmonic p-forms of general degrees.

THEOREM 3.2. *Let M^n, $n \geq 3$, be a complete non-compact minimal hypersurface immersed in $\mathbb{S}^{n+1}$. Assume that $\lambda_1(\Delta + \frac{p(n-p)}{n}|A|^2) \geq 0$ for $1 \leq p \leq n-1$. Then, every harmonic p-form ω on M with $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B_{x_0}(r)} |\omega|^{2\beta} dv = 0$, $1 - \sqrt{K_p} < \beta < 1 + \sqrt{K_p}$, vanishes identically. In particular, $H^p(L^2(M)) = \{0\}$.*

Proof. Let $\omega \in H^p(L^2(M))$ with $1 \leq p \leq n-1$. It follows from the assumption $H = 0$ that

$$\lambda_{i_1} + \cdots + \lambda_{i_p} = -(\lambda_{i_{p+1}} + \cdots + \lambda_{i_n}).$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|A|^2 &= (\lambda_{i_1})^2 + \cdots + (\lambda_{i_p})^2 + \left[(\lambda_{i_{p+1}})^2 + \cdots + (\lambda_{i_n})^2\right] \\ &\geq \frac{1}{p}(\lambda_{i_1} + \cdots + \lambda_{i_p})^2 + \frac{1}{n-p}(\lambda_{i_{p+1}} + \cdots + \lambda_{i_n})^2 \\ &= \frac{n}{p(n-p)}(\lambda_{i_1} + \cdots + \lambda_{i_p})^2.\end{aligned}$$

Thus,

$$(\lambda_{i_1} + \cdots + \lambda_{i_p})(\lambda_{i_{p+1}} + \cdots + \lambda_{i_n}) = -(\lambda_{i_1} + \cdots + \lambda_{i_p})^2 \geq -\frac{p(n-p)}{n}|A|^2.$$

Substituting into (2.3) and combining (2.5), we conclude that

$$|\omega|\Delta|\omega| + \frac{p(n-p)}{n}|A|^2|\omega|^2 \geq K_p|\nabla|\omega||^2 + p(n-p)|\omega|^2$$

for all $1 \leq p \leq n-2$. For any $\alpha > 0$, we compute

$$\begin{aligned}|\omega|^\alpha \Delta|\omega|^\alpha &= |\omega|^\alpha\left[\alpha(\alpha-1)|\omega|^{\alpha-2}|\nabla|\omega||^2 + \alpha|\omega|^{\alpha-1}\Delta|\omega|\right] \\ &\geq \left(1 - \frac{1-K_p}{\alpha}\right)|\nabla|\omega|^\alpha|^2 - \frac{\alpha p(n-p)}{n}|A|^2|\omega|^{2\alpha} + \alpha p(n-p)|\omega|^{2\alpha}. \qquad (3.7)\end{aligned}$$

Let $\eta \in C_0^\infty(M)$. Multiplying both sides of (3.7) by $\eta^2|\omega|^{2q\alpha}$, $q > 0$, and integrating over M, we get

$$\begin{aligned}&\left[1 - \frac{1-K_P}{\alpha}\right]\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv + \alpha p(n-p)\int_M \eta^2|\omega|^{2(1+q)\alpha} dv \\ &\leq \int_M \eta^2|\omega|^{(2q+1)\alpha}\Delta|\omega|^\alpha dv + \frac{\alpha p(n-p)}{n}\int_M \eta^2|A|^2|\omega|^{2(1+q)\alpha} dv\end{aligned}$$

$$= -(2q+1)\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv - 2\int_M \eta|\omega|^{(2q+1)\alpha}\langle\nabla\eta, \nabla|\omega|^\alpha\rangle dv$$
$$+ \frac{\alpha p(n-p)}{n}\int_M \eta^2|A|^2|\omega|^{2(1+q)\alpha} dv,$$

which gives

$$\left[2(q+1) - \frac{1-K_p}{\alpha}\right]\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv + \alpha p(n-p)\int_M \eta^2|\omega|^{2(1+q)\alpha} dv$$
$$\leq -2\int_M \eta|\omega|^{(2q+1)\alpha}\langle\nabla\eta, \nabla|\omega|^\alpha\rangle dv + \frac{\alpha p(n-p)}{n}\int_M \eta^2|A|^2|\omega|^{2(1+q)\alpha} dv. \quad (3.8)$$

On the other hand, the variational principle for $\lambda_1(\triangle + \frac{p(n-p)}{n}|A|^2) \geq 0$ asserts the validity of the following inequality

$$\frac{p(n-p)}{n}\int_M |A|^2 f^2 dv \leq \int_M |\nabla f|^2 dv, \quad \forall f \in C_0^\infty(M). \quad (3.9)$$

By choosing $f = \eta|\omega|^{(1+q)\alpha}$ in (3.9), we have

$$\frac{p(n-p)}{n}\int_M \eta^2|A|^2|\omega|^{2(1+q)\alpha} dv \leq (1+q)^2\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv + \int_M |\omega|^{2(1+q)\alpha}|\nabla\eta|^2 dv$$
$$+ 2(1+q)\int_M \eta|\omega|^{(1+2q)\alpha}\langle\nabla\eta, \nabla|\omega|^\alpha\rangle dv. \quad (3.10)$$

Substituting (3.10) into (3.8) yields

$$\frac{1}{\alpha}\left[2(q+1)\alpha - (1-K_p) - (1+q)^2\alpha^2\right]\int_M \eta^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 dv$$
$$\leq 2[(1+q)\alpha - 1]\int_M \eta|\omega|^{(2q+1)\alpha}\langle\nabla\eta, \nabla|\omega|^\alpha\rangle dv + \alpha\int_M |\omega|^{2(1+q)\alpha}|\nabla\eta|^2 dv$$
$$- \alpha p(n-p)\int_M \eta^2|\omega|^{2(1+q)\alpha} dv. \quad (3.11)$$

Take $\beta = (1+q)\alpha$. Using the Cauchy–Schwarz inequality, it follows from (3.11) that

$$\frac{1}{\alpha}\left[2\beta - (1-K_p) - \beta^2 - |\beta-1|\epsilon\right]\int_M \eta^2|\omega|^{2\beta-2\alpha}|\nabla|\omega|^\alpha|^2 dv$$
$$\leq \left(\alpha + \frac{|\beta-1|}{\epsilon}\right)\int_M |\omega|^{2\beta}|\nabla\eta|^2 dv - \alpha p(n-p)\int_M \eta^2|\omega|^{2\beta} dv \quad (3.12)$$

for all $\epsilon > 0$. Since $1 - \sqrt{K_p} < \beta < 1 + \sqrt{K_p}$, we choose sufficiently small $\epsilon > 0$ such that $2\beta - (1-K_p) - \beta^2 - |\beta-1|\epsilon > 0$. Hence, it follows from (3.12) that

$$C_1\int_M \eta^2|\omega|^{2\beta-2\alpha}|\nabla|\omega|^\alpha|^2 dv \leq C_2\int_M |\omega|^{2\beta}|\nabla\eta|^2 dv - \alpha p(n-p)\int_M \eta^2|\omega|^{2\beta} dv \quad (3.13)$$

for some constants $C_1 > 0$ and $C_2 > 0$.

Let $\eta_r \in C_0^\infty(M)$ be the cut-off function defined as before. Substituting $\eta = \eta_r$ into (3.13) yields

$$C_1 \int_{B_{x_0}(r)} |\omega|^{2\beta-2\alpha} |\nabla|\omega|^\alpha|^2 dv \le C_1 \int_M \eta^2 |\omega|^{2\beta-2\alpha} |\nabla|\omega|^\alpha|^2 dv$$
$$\le \frac{4C_2}{r^2} \int_{B_{x_0}(2r)} |\omega|^{2\beta} dv - \alpha p(n-p) \int_{B_{x_0}(r)} |\omega|^{2\beta} dv.$$

Since $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B_{x_0}(r)} |\omega|^{2\beta} dv = 0$, letting $r \to \infty$ in the above inequality, we have $\omega = 0$. □

Let us recall that an end E of a complete manifold M is non-parabolic if E admits a positive Green's function with Neumann boundary condition. To discuss the number of ends of complete submanifolds, we recall the following basic lemma.

Lemma 3.1 [**14**]. *Let M be a complete Riemannian manifold. Let $\mathcal{H}_D^0(M)$ be the space of bounded harmonic functions with finite Dirichlet integral and denote by $H^1(L^2(M))$ the space of L^2 harmonic 1-forms on M. Then, the number of non-parabolic ends of M is bounded from above by* $\dim \mathcal{H}_D^0(M) \le \dim H^1(L^2(M)) + 1$.

By using Theorem 3.2 together with Lemma 3.1, we now give the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $\frac{n-1}{n}$-stability implies $\lambda_1(\triangle + \frac{n-1}{n}|A|^2) \ge 0$, by Theorem 3.2, we have $H^1(L^2(M)) = \{0\}$. It also follows from $\frac{n-1}{n}$-stability that $\lambda_1(M) \ge n-1$, which implies that each end E of M satisfies a Sobolev type inequality of the form as

$$\int_E f^2 dv \le \frac{1}{n-1} \int_E |df|^2 dv, \quad \forall f \in C_0^\infty(M).$$

Since M is minimal, by Proposition 2.1 of [**5**], each end of M has infinite volume. Hence, according to Corollary 4 in [**15**], each end of M is non-parabolic. Therefore, by Lemma 3.1, M must have only one end.

4. Proofs of Theorem 1.4. Obviously, through the composition of isometric immersions

$$M^n \to \mathbb{S}^{n+1} \to \mathbb{R}^{n+2},$$

M^n can be considered as a submanifold in $\mathbb{R}^{n+2}$. Denote by $\bar{H}$ the mean curvature vector of M^n in $\mathbb{R}^{n+2}$, then we have

$$|\bar{H}|^2 = |H|^2 + 1. \tag{4.1}$$

It is known that the following Sobolev inequality due to Hoffman and Spruck [**9**]

$$\left(\int_M |f|^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \le c(n) \int_M (|\nabla f|^2 + |\bar{H}|^2 f^2) dv, \quad \forall f \in C_0^\infty(M) \tag{4.2}$$

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holds on M^n for some $c(n) > 0$. Substituting (4.1) into (4.2) yields

$$\left(\int_M |f|^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \leq c(n)\int_M |\nabla f|^2 dv + c(n)\int_M (1+|H|^2)f^2 dv \tag{4.3}$$

for any $f \in C_0^\infty(M)$.

Now, we are in the position to prove Theorem 1.4. It is obvious that Theorem 1.4 can be deduced immediately from the following result.

THEOREM 4.1. *Let M^n, $n \geq 3$, be a complete non-compact hypersurface of $\mathbb{S}^{n+1}$. Assume that*

$$\left(\int_M |\phi|^n dv\right)^{\frac{2}{n}} < \frac{2(1+K_p)}{nc(n)} \tag{4.4}$$

for $1 \leq p \leq n-1$. Then, every harmonic p-form ω on M with $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$ vanishes identically. In particular, $H^p(L^2(M)) = \{0\}$.

Proof. Given a harmonic p-form ω satisfying $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$. Let $\eta \in C_0^\infty(M)$ be a smooth function on M^n with compact support. Multiplying (2.6) by η^2 and integrating over M^n, we obtain

$$\begin{aligned}\int_M \eta^2|\omega|\triangle|\omega| dv \geq & K_p\int_M \eta^2|\nabla|\omega||^2 dv + p(n-p)\int_M \eta^2|\omega|^2 dv - \frac{n}{2}\int_M |\phi|^2\eta^2|\omega|^2 dv \\ & + \frac{1}{2}\min\{p, n-p\}\int_M |A|^2\eta^2|\omega|^2 dv.\end{aligned} \tag{4.5}$$

Integrating by parts and using the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}\int_M \eta^2|\omega|\triangle|\omega| dv = & -2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv - \int_M \eta^2|\nabla|\omega||^2 dv \\ & \leq (b-1)\int_M \eta^2|\nabla|\omega||^2 dv + \frac{1}{b}\int_M |\omega|^2|\nabla\eta|^2 dv\end{aligned} \tag{4.6}$$

for all $b > 0$. Using (4.3) together with the Hölder and Cauchy–Schwarz inequalities, we have

$$\begin{aligned}\int_M |\phi|^2\eta^2|\omega|^2 dv \leq & \left(\int_{\mathrm{supp}(\eta)} |\phi|^n dv\right)^{\frac{2}{n}}\left(\int_M |\eta|\omega||^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \\ & \leq c(n)\left(\int_{\mathrm{supp}(\eta)} |\phi|^n dv\right)^{\frac{2}{n}}\int_M \left[|\nabla(\eta|\omega|)|^2 + (1+|H|^2)\eta^2|\omega|^2\right] dv \\ & = E(\eta)\int_M \left[\eta^2|\nabla|\omega||^2 + |\omega|^2|\nabla\eta|^2 + (1+|H|^2)\eta^2|\omega|^2\right] dv \\ & \quad + 2E(\eta)\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle dv \\ & \leq E(\eta)(1+\gamma)\int_M \eta^2|\nabla|\omega||^2 dv + E(\eta)\big(1+\frac{1}{\gamma}\big)\int_M |\omega|^2|\nabla\eta|^2 dv \\ & \quad + E(\eta)\int_M (1+|H|^2)\eta^2|\omega|^2 dv.\end{aligned} \tag{4.7}$$

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for all $\gamma > 0$, where $E(\eta) = c(n)(\int_{\mathrm{supp}(\eta)} |\phi|^n dv)^{\frac{2}{n}}$. Substituting (4.6) and (4.7) into (4.5), we conclude that

$$C\int_M \eta^2|\nabla|\omega||^2 dv \leq D\int_M |\omega|^2|\nabla\eta|^2 dv + \frac{n}{2}E(\eta)\int_M (1+|H|^2)\eta^2|\omega|^2 dv - \frac{1}{2}\min\{p, n-p\}\int_M |A|^2\eta^2|\omega|^2 dv - p(n-p)\int_M \eta^2|\omega|^2 dv, \tag{4.8}$$

where

$$C = 1 + K_p - b - \frac{n}{2}E(\eta)(1+\gamma) \quad \text{and} \quad D = \frac{1}{b} + \frac{n(1+\gamma)}{2\gamma}E(\eta). \tag{4.9}$$

It follows from (4.4) that

$$E(\eta) = c(n)\Big(\int_{\mathrm{supp}(\eta)} |\phi|^n dv\Big)^{\frac{2}{n}} < \frac{2(1+K_p)}{n}, \tag{4.10}$$

which implies that $1 + K_p - \frac{nE(\eta)}{2} > 0$. Hence, we can choose γ and b small enough such that

$$C = 1 + K_p - b - \frac{n}{2}E(\eta)(1+\gamma) > 0.$$

Fix a point $x_0 \in M$ and let $\rho(x)$ be the geodesic distance on M from x_0 to x. Let us choose $\eta \in C_0^\infty(M)$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{if } \rho(x) \leq r, \\ 0 & \text{if } 2r < \rho(x) \end{cases}$$

and

$$|\nabla\eta|(x) \leq \frac{2}{r} \text{ if } r < \rho(x) \leq 2r$$

for $r > 0$. By (4.8), we have

$$\begin{aligned} 0 &\leq C\int_{B_{x_0}(r)} \eta^2|\nabla|\omega||^2 dv \\ &\leq D\int_M |\omega|^2|\nabla\eta|^2 dv + \frac{n}{2}E(\eta)\int_M (1+|H|^2)\eta^2|\omega|^2 dv \\ &\quad - \frac{1}{2}\min\{p, n-p\}\int_M |A|^2\eta^2|\omega|^2 dv - p(n-p)\int_M \eta^2|\omega|^2 dv \\ &\leq \frac{4D}{r^2}\int_M |\omega|^2 dv + \int_M \Big[\frac{n}{2}E(\eta)|H|^2 - \frac{1}{2}\min\{p, n-p\}|A|^2\Big]\eta^2|\omega|^2 dv \\ &\quad + \Big[\frac{n}{2}E(\eta) - p(n-p)\Big]\int_M \eta^2|\omega|^2 dv. \end{aligned} \tag{4.11}$$

It follows from (4.10) that

$$\frac{n}{2}E(\eta)|H|^2 - \frac{1}{2}\min\{p, n-p\}|A|^2 \le \frac{1}{2}\big(E(\eta) - \min\{p, n-p\}\big)|A|^2 \le 0,$$

and

$$\frac{n}{2}E(\eta) - p(n-p) < 0.$$

Letting $r \to \infty$ in (4.11), and noting that $\liminf_{r\to\infty} \frac{1}{r^2}\int_{B_{x_0}(r)} |\omega|^2 dv = 0$, we have $\omega = 0$. □

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