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Formulations of the $\mathcal{F}$-functional calculus and some consequences

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In this paper we introduce the two possible formulations of the $\mathcal{F}$-functional calculus that are based on the Fueter–Sce mapping theorem in integral form and we introduce the pseudo-$\mathcal{F}$-resolvent equation. In the case of dimension 3 we prove the $\mathcal{F}$-resolvent equation and we study the analogue of the Riesz projectors associated with this calculus. The case of dimension 3 is also useful to study the quaternionic version of the $\mathcal{F}$-functional calculus.

Keywords: Fueter–Sce mapping theorem in integral form; $\mathcal{F}$-spectrum; formulations of the $\mathcal{F}$-functional calculus for n-tuples of operators; projectors; quaternionic version of the $\mathcal{F}$-functional calculus; quaternionic $\mathcal{F}$-resolvent equation

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1. Introduction

Probably the most important functional calculus for linear operators acting on a Banach space is the Riesz–Dunford functional calculus (see [26]). For the case of n-tuples of operators we quote the paper by Anderson [9], who developed the Weyl functional calculus, and the work of Taylor and Taylor [43–46], who defined a functional calculus for n-tuples of operators using the theory of holomorphic functions of several variables. Since then, the literature on this topic has been developed in different directions using different notions of hyperholomorphicity.

In the past few years the theory of slice hyperholomorphic functions (see [11, 12, 17, 19, 22]) has been the underlying function theory on which two new functional calculuses have been developed for several operators and for quaternionic operators. These calculuses are the $\mathcal{S}$-functional calculus [4, 7, 12, 13, 16] and the $\mathcal{F}$-functional calculus [4, 14, 18]. The former is based on the Cauchy formula of the slice hyperholomorphic functions (see [11, 12]), and it applies to n-tuples of not necessarily commuting operators and to quaternionic operators. When applied to quaternionic operators, the S-functional calculus is also called quaternionic functional calculus. For an overview on slice hyperholomorphic functions and for the $\mathcal{S}$-functional calculus see the monograph [21]. A continuous version of the $\mathcal{S}$-functional calculus can be found in [28].

In the recent paper [7] it was shown that, in order to have a full description of the $\mathcal{S}$-functional calculus, we need both formulations of this calculus (more precisely, both the formulation based on the right $\mathcal{S}$-resolvent operator and the one based on the left $\mathcal{S}$-resolvent operator). This is necessary because the $\mathcal{S}$-resolvent equation involves both resolvent operators. This equation is a key tool with which to study the Riesz projectors associated with the $\mathcal{S}$-functional calculus.

The aim of this work is to introduce the formulations of the $\mathcal{F}$-functional calculus and to extend some of the results proved in [7] to the $\mathcal{F}$-functional calculus. For example, the Riesz-type projectors are based on the two formulations of the $\mathcal{F}$-functional calculus and on the $\mathcal{F}$-resolvent equation. To define the two formulations of the $\mathcal{F}$-functional calculus we need the two integral versions of the Fueter–Sce mapping theorem in integral form.

The Fueter–Sce mapping theorem is one of the deepest results in hypercomplex analysis (see [27]). It gives a procedure to generate Cauchy–Fueter regular functions starting from holomorphic functions of a complex variable. In the case of Clifford-algebra-valued functions, see [10, 15, 25, 30]. The proof of the analogue of the Fueter mapping theorem is due to Sce [41] for n odd and to Qian [38] for the general case. Fueter's theorem has been generalized to the case in which a function f is multiplied by a monogenic homogeneous polynomial of degree k (see [34, 36, 37, 42]) and to the case in which the function f is defined on an open set U not necessarily chosen in the upper complex plane (see [38–40]).

The problem of the inversion of the Fueter–Sce mapping theorem as been investigated in a series of papers (see [20, 23, 24]).

The $\mathcal{F}$-functional calculus is based on the Fueter–Sce mapping theorem in integral form. This is an integral transform obtained by applying suitable powers of the Laplace operator to the Cauchy kernel of slice hyperholomorphic functions.

The case of left-slice monogenic functions was studied in [18], where the $\mathcal{F}$-functional calculus was introduced.

We observe that, to study the Riesz projectors, in the classical case, we need the resolvent equation

$$(\lambda I - A)^{-1}(\mu I - A)^{-1} = \frac{(\lambda I - A)^{-1} - (\mu I - A)^{-1}}{\mu - \lambda}, \quad \lambda, \mu \in \mathbb{C} \setminus \sigma(A), \tag{1.1}$$

where A is a complex operator on a complex Banach space.

In fact, to study the classical Riesz projectors, we use the fact that the product of the resolvent operators $(\lambda I - A)^{-1}(\mu I - A)^{-1}$ can be written in terms of the difference $(\lambda I - A)^{-1} - (\mu I - A)^{-1}$ multiplied by the Cauchy kernel of holomorphic functions. As a consequence of the holomorphicity, one can prove that

$$P_\Omega = \int_{\partial\Omega} (\lambda I - A)^{-1} \, \mathrm{d}\lambda,$$

where Ω contains part of the spectrum of A, is a projector, i.e. $P_\Omega^2 = P_\Omega$.

In the case of the $\mathcal{S}$-functional calculus (and, in particular, its commutative version, the $\mathcal{SC}$-functional calculus) we can follow the same strategy but in this case the $\mathcal{SC}$-resolvent equation contains both the $\mathcal{SC}$-resolvent operators, as has recently been shown in [7].

Precisely, we consider a paravector operator $T = T_0 + e_1T_1 + \cdots + e_nT_n$, where T_j, $j = 0, 1, \dots, n$, are real bounded operators commuting among themselves, acting on a real Banach space and e_j, $j = 1, \dots, n$, are the units of the real Clifford algebra $\mathbb{R}_n$. The $\mathcal{F}$-spectrum of T is defined as

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} \colon s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T} \text{ is not invertible}\},$$

where we have set $\bar{T} := T_0 - e_1T_1 - \cdots - e_nT_n$. The left $\mathcal{SC}$-resolvent operator is defined as

$$S_{C,\mathrm{L}}^{-1}(s,T) := (s\mathcal{I} - \bar{T})(s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T})^{-1}, \quad s \notin \sigma_{\mathcal{F}}(T), \tag{1.2}$$

and the right $\mathcal{SC}$-resolvent operator is

$$S_{C,\mathrm{R}}^{-1}(s,T) := (s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T})^{-1}(s\mathcal{I} - \bar{T}), \quad s \notin \sigma_{\mathcal{F}}(T). \tag{1.3}$$

In this case, for $s, p \notin \sigma_{\mathcal{F}}(T)$, the $\mathcal{SC}$-resolvent equation is

$$\begin{aligned} S_{C,\mathrm{R}}^{-1}(s,T)S_{C,\mathrm{L}}^{-1}(p,T) = ((&S_{C,\mathrm{R}}^{-1}(s,T) - S_{C,\mathrm{L}}^{-1}(p,T))p \\ &- \bar{s}(S_{C,\mathrm{R}}^{-1}(s,T) - S_{C,\mathrm{L}}^{-1}(p,T)))(p^2 - 2s_0p + |s|^2)^{-1}, \end{aligned} \tag{1.4}$$

where p and s are paravectors, i.e. $s = s_0 + s_1e_1 + \cdots + s_ne_n$ and $|s|^2 = s_0^2 + s_1^2 + \cdots + s_n^2$. This equation is the key tool for studying the Riesz projectors for the $\mathcal{S}$-functional calculus and, in particular, for its commutative version, the $\mathcal{SC}$-functional calculus. In the case of the $\mathcal{F}$-functional calculus, let n be an odd number. We define the left $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^{\mathrm{L}}(s,T) := \gamma_n(s\mathcal{I} - \bar{T})(s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T})^{-(n+1)/2}, \quad s \notin \sigma_{\mathcal{F}}(T), \tag{1.5}$$

and the right $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^{\mathrm{R}}(s,T) := \gamma_n(s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T})^{-(n+1)/2}(s\mathcal{I} - \bar{T}), \quad s \notin \sigma_{\mathcal{F}}(T), \tag{1.6}$$

where γ_n are suitable constants. In this case we cannot expect an $\mathcal{F}$-resolvent equation, as in the case of the Riesz–Dunford functional calculus or as in the case of the $\mathcal{SC}$-functional calculus, because the $\mathcal{F}$-functional calculus in based on an integral transform and not on a Cauchy formula.

Now we are able to show that, at least in the case when $n = 3$, there is an $\mathcal{F}$-resolvent equation, but in addition to the two $\mathcal{F}$-resolvent operators it contains the two $\mathcal{SC}$-resolvent operators. Precisely, for $s, p \notin \sigma_{\mathcal{F}}(T)$,

$$\begin{aligned} &\mathcal{F}_3^{\mathrm{R}}(s,T)S_{C,\mathrm{L}}^{-1}(p,T) + S_{C,\mathrm{R}}^{-1}(s,T)\mathcal{F}_3^{\mathrm{L}}(p,T) + \gamma_3^{-1}(s\mathcal{F}_3^{\mathrm{R}}(s,T)\mathcal{F}_3^{\mathrm{L}}(p,T)p \\ &\quad - s\mathcal{F}_3^{\mathrm{R}}(s,T)T\mathcal{F}_3^{\mathrm{L}}(p,T) - \mathcal{F}_3^{\mathrm{R}}(s,T)T\mathcal{F}_3^{\mathrm{L}}(p,T)p + \mathcal{F}_3^{\mathrm{R}}(s,T)T^2\mathcal{F}_3^{\mathrm{L}}(p,T)) \\ &\qquad = [(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))p - \bar{s}(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))](p^2 - 2s_0p + |s|^2)^{-1}. \end{aligned}$$

Even though it is more complicated than the $\mathcal{SC}$-resolvent equation, this is the correct tool to study the analogue of the Riesz projectors for the $\mathcal{F}$-functional calculus. Moreover, the case when $n = 3$ allows the detailed study of the quaternionic version of the $\mathcal{F}$-functional calculus. We conclude by observing that important applications of the quaternionic functional calculus and of its $\mathcal{S}$-resolvent operators can be found in Schur analysis in the slice hyperholomorphic setting (see [2, 3, 5]; see also [1] for the classical case). This is a very active field of research that started a few years ago.

Outline of the paper. The remainder of the paper contains nine sections (two on hypercomplex analysis, six on operator theory and a section for conclusions and future directions of research) and an appendix. In § 2 we state the main results on the theory of slice hyperholomorphic functions that are necessary for § 3, in which we consider the two formulations of the Fueter–Sce mapping theorem in integral form. These are the key tools for the formulation of the $\mathcal{F}$-functional calculus for bounded commuting operators studied in § 4. Section 5 contains some relations between the two $\mathcal{F}$-resolvent operators and the operators that are the candidates to be the analogues of the Riesz projectors in this setting.

Section 6 contains the main results on the commutative version of the $\mathcal{S}$-functional calculus, where we recall the $\mathcal{SC}$-resolvent operators. These operators are involved in the $\mathcal{F}$-resolvent equation that is stated in § 7 for the case when $n = 3$ and is of fundamental importance for studying the Riesz projectors for the $\mathcal{F}$-functional calculus.

In § 8 we formulate the quaternionic version of the $\mathcal{F}$-functional calculus. The results of this section are deduced from the case $n = 3$ studied previously and fully describe the $\mathcal{F}$-functional calculus in the quaternionic setting. Section 9 concludes and gives future directions of research. Finally, Appendix A gives the details of the technical proof of lemma 5.7.

2. Preliminaries on slice monogenic functions

In this section we recall some results on slice monogenic functions that will be useful in the rest of the paper. We refer the reader to [21] for more details.

The setting in which we shall work is the real Clifford algebra $\mathbb{R}_n$ over n imaginary units $e_1, \dots, e_n$ satisfying the relations $e_ie_j + e_je_i = -2\delta_{ij}$. An element of the Clifford algebra will be denoted by $\sum_{A\subset\{1,\dots,n\}} e_Ax_A$ with $x_A \in \mathbb{R}$, where $e_\emptyset = 1$ and $e_A = e_{i_1}\cdots e_{i_r}$ for $A = \{i_1, \dots, i_r\}$ with $i_1 < \dots < i_r$. When $n = 1$, we have that $\mathbb{R}_1$ is the algebra of complex numbers $\mathbb{C}$ (the only case in which the Clifford algebra is commutative), while when $n = 2$ we obtain the division algebra of real quaternions $\mathbb{H}$. As is well known, for $n > 2$, the Clifford algebras $\mathbb{R}_n$ have zero divisors.

In the Clifford algebra $\mathbb{R}_n$, we can identify some specific elements with the vectors in the Euclidean space $\mathbb{R}^n$: an element $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ can be identified with a so-called 1-vector in the Clifford algebra through the map $(x_1, x_2, \dots, x_n) \mapsto \underline{x} = x_1e_1 + \dots + x_ne_n$.

An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element

$$x = x_0 + \underline{x} = x_0 + \sum_{j=1}^{n} x_je_j$$

called, in short, a paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The real part x_0 of x will also be denoted by $\mathrm{Re}(x)$. A function

$f\colon U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is seen as a function $f(x)$ of x (and similarly for a function $f(\underline{x})$ of $\underline{x} \in U \subset \mathbb{R}^n$). We shall denote by $\mathbb{S}$ the sphere of unit 1-vectors in $\mathbb{R}^n$, i.e.

$$\mathbb{S} = \{\underline{x} = e_1x_1 + \cdots + e_nx_n \colon x_1^2 + \cdots + x_n^2 = 1\}.$$

Definition 2.1. Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f\colon U \to \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$ and denote by $u + Iv$ an element on $\mathbb{C}_I$. We say that f is a left-slice monogenic (or *s-monogenic*) function if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_I(u+Iv) + I\frac{\partial}{\partial v} f_I(u+Iv)\right) = 0$$

on $U \cap \mathbb{C}_I$. We shall denote by $\mathcal{SM}(U)$ (or by $\mathcal{SM}^{\mathrm{L}}(U)$ when confusion may arise) the set of left-slice monogenic functions on the open set U. We say that f is a right-slice monogenic (or *right s-monogenic*) function if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_I(u+Iv) + \frac{\partial}{\partial v} f_I(u+Iv)I\right) = 0$$

on $U \cap \mathbb{C}_I$. We shall denote by $\mathcal{SM}^{\mathrm{R}}(U)$ the set of right-slice monogenic functions on the open set U.

Definition 2.2. Given an element $x \in \mathbb{R}^{n+1}$, we define

$$[x] = \{y \in \mathbb{R}^{n+1} \colon y = \mathrm{Re}(x) + I|\underline{x}|,\ I \in \mathbb{S}\}.$$

The set $[x]$ is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. When $x \in \mathbb{R}$, $[x]$ contains only x. In this case, the $(n-1)$-dimensional sphere has radius equal to zero. The domains on which slice hyperholomorphic functions have a Cauchy formula are the so-called slice domains and axially symmetric domains.

Definition 2.3.

(i) Let $U \subseteq \mathbb{R}^{n+1}$ be a domain. We say that U is a *slice domain (or s-domain)* if $U \cap \mathbb{R}$ is non-empty and if $\mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.

(ii) Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is *axially symmetric* if, for all $u + Iv \in U$, the whole $(n-1)$-sphere $[u + Iv]$ is contained in U.

It is important to point out that a key tool in our theory is the Cauchy formula for slice monogenic functions. If $x = x_0 + e_1x_1 + \cdots + e_nx_n$, $s = s_0 + e_1s_1 + \cdots + e_ns_n$ are paravectors in $\mathbb{R}^{n+1}$, then we have the following facts. The following identity follows by a direct computation as stated in [12, proposition 2.8].

Proposition 2.4. *Suppose that x and $s \in \mathbb{R}^{n+1}$ are such that $x \notin [s]$. Then*

$$-(x^2 - 2x\,\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}) = (s - \bar{x})(s^2 - 2\,\mathrm{Re}(x)s + |x|^2)^{-1}. \tag{2.1}$$

This fact justifies the following definition.

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DEFINITION 2.5. Let $x, s \in \mathbb{R}^{n+1}$ such that $x \notin [s]$.

- We say that $S_{\mathrm{L}}^{-1}(s,x)$ is written in form I if

$$S_{\mathrm{L}}^{-1}(s,x) := -(x^2 - 2x\,\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}).$$

- We say that $S_{\mathrm{L}}^{-1}(s,x)$ is written in form II if

$$S_{\mathrm{L}}^{-1}(s,x) := (s - \bar{x})(s^2 - 2\,\mathrm{Re}(x)s + |x|^2)^{-1}.$$

The following identity follows by a direct computation as in proposition 2.4.

PROPOSITION 2.6. *Suppose that x and $s \in \mathbb{R}^{n+1}$ are such that $x \notin [s]$. The following identity holds:*

$$(s^2 - 2\,\mathrm{Re}(x)s + |x|^2)^{-1}(s - \bar{x}) = -(x - \bar{s})(x^2 - 2\,\mathrm{Re}(s)x + |s|^2)^{-1}. \tag{2.2}$$

This fact justifies the following definition.

DEFINITION 2.7. Let $x, s \in \mathbb{R}^{n+1}$ such that $x \notin [s]$.

- We say that $S_{\mathrm{R}}^{-1}(s,x)$ is written in form I if

$$S_{\mathrm{R}}^{-1}(s,x) := -(x - \bar{s})(x^2 - 2\,\mathrm{Re}(s)x + |s|^2)^{-1}.$$

- We say that $S_{\mathrm{R}}^{-1}(s,x)$ is written in form II if

$$S_{\mathrm{R}}^{-1}(s,x) := (s^2 - 2\,\mathrm{Re}(x)s + |x|^2)^{-1}(s - \bar{x}).$$

THEOREM 2.8 (the Cauchy formula). *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Suppose that $\partial(U \cap \mathbb{C}_I)$ is a finite union of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. Set $\mathrm{d}s_I = -\mathrm{d}sI$ for $I \in \mathbb{S}$. If f is a (left) slice monogenic function on a set that contains $\bar{U}$, then*

$$f(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_{\mathrm{L}}^{-1}(s,x)\,\mathrm{d}s_I\, f(s) \tag{2.3}$$

and the integral depends neither on U nor on the imaginary unit $I \in \mathbb{S}$.

If f is a right-slice monogenic function on a set that contains $\bar{U}$, then

$$f(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I S_{\mathrm{R}}^{-1}(s,x) \tag{2.4}$$

and the integral depends neither on U nor on the imaginary unit $I \in \mathbb{S}$.

The deepest property of slice monogenic functions on axially symmetric slice domains is the representation formula (also called the structure formula; see [11]).

THEOREM 2.9 (representation formula). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain.*

- *Let $f \in \mathcal{SM}^{\mathrm{L}}(U)$. Then, for any vector $x = u + I_x v \in U$, we have*

$$f(x) = \tfrac{1}{2}[1 - I_x I]f(u + Iv) + \tfrac{1}{2}[1 + I_x I]f(u - Iv) \tag{2.5}$$

and

$$f(x) = \tfrac{1}{2}[f(u + Iv) + f(u - Iv)] + \tfrac{1}{2}I_x I[f(u - Iv) - f(u + Iv)]]. \tag{2.6}$$

- *Let $f \in \mathcal{SM}^{\mathrm{R}}(U)$. Then, for any vector $x = u + I_x v \in U$, we have*

$$f(x) = \tfrac{1}{2}[1 - I I_x]f(u + Iv) + \tfrac{1}{2}[1 + I I_x]f(u - Iv) \tag{2.7}$$

and

$$f(x) = \tfrac{1}{2}[f(u + Iv) + f(u - Iv)] + \tfrac{1}{2}[[f(u - Iv) - f(u + Iv)]I I_x]. \tag{2.8}$$

As we shall see in the following, the representation formula shows that it is possible to apply the Fueter–Sce mapping theorem to slice monogenic functions to obtain monogenic functions, i.e. functions in the kernel of the Dirac operator.

3. The Fueter–Sce mapping theorem in integral form

For completeness we recall the notion of monogenic functions.

DEFINITION 3.1 (monogenic functions). Let U be an open set in $\mathbb{R}^{n+1}$. A real differentiable function $f\colon U \to \mathbb{R}_n$ is left monogenic if

$$\frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} f(x) = 0.$$

It is right monogenic if

$$\frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x) e_i = 0.$$

The representation formula shows that a slice monogenic function

$$f\colon U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$$

is of the form

$$f(x + Iy) = \alpha(x, y) + I\beta(x, y),$$

where α and β are suitable $\mathbb{R}_n$-valued functions satisfying the Cauchy–Riemann system and I is a 1-vector in the Clifford algebra $\mathbb{R}_n$ such that $I^2 = -1$. The Fueter–Sce theorem states that, for n odd, if $f(x + Iy) = \alpha(x, y) + I\beta(x, y)$ is slice monogenic, then the function

$$\breve{f}(x_0, |\underline{x}|) = \Delta^{(n-1)/2}\left(\alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|)\right)$$

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is monogenic, i.e. it is in the kernel of the Dirac operator. Here Δ is the Laplace operator in dimension $n+1$. This means that, using the Cauchy formula of the slice monogenic functions, we apply the operator $\Delta^{(n-1)/2}$ to the Cauchy kernels to obtain an integral version of the Fueter–Sce mapping theorem.

The crucial point, as observed in [14], is that we can get an elegant formula only when we apply the operator $\Delta^{(n-1)/2}$ to the Cauchy kernel in form II. This means that when we define the $\mathcal{F}$-functional calculus using this integral transform we are restricted to the case of commuting operators.

THEOREM 3.2. *Let $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$ and let*

$$\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$$

be the Laplace operator in the variable x.

(a) *Consider the left-slice monogenic Cauchy kernel $S_{\mathrm{L}}^{-1}(s,x)$ written in form II, i.e.*

$$S_{\mathrm{L}}^{-1}(s,x) := (s-\bar{x})(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-1}.$$

Then, for $h \geqslant 1$, we have

$$\Delta^h S_{\mathrm{L}}^{-1}(s,x) = (-1)^h \prod_{\ell=1}^{h}(2\ell) \prod_{\ell=1}^{h}(n-(2\ell-1))(s-\bar{x})(s^2-2\operatorname{Re}[x]s+|x|^2)^{-(h+1)}. \tag{3.1}$$

(b) *Consider the right-slice monogenic Cauchy kernel $S_{\mathrm{R}}^{-1}(s,x)$ written in form II, i.e.*

$$S_{\mathrm{R}}^{-1}(s,x) := (s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-1}(s-\bar{x}).$$

Then, for $h \geqslant 1$, we have

$$\Delta^h S_{\mathrm{R}}^{-1}(s,x) = (-1)^h \prod_{\ell=1}^{h}(2\ell) \prod_{\ell=1}^{h}(n-(2\ell-1))(s^2-2\operatorname{Re}[x]s+|x|^2)^{-(h+1)}(s-\bar{x}). \tag{3.2}$$

Proof. We shall only consider the right-slice monogenic case, as the left-slice monogenic case has already been proved in [18]. We shall prove the formula by induction. We have

$$\begin{aligned}\frac{\partial^2}{\partial x_0^2} S_{\mathrm{R}}^{-1}(s,x) &= 2(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-3}(-2s+2x_0)^2(s-\bar{x}) \\ &\quad - 2(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-2}(s-\bar{x}) \\ &\quad + 2(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-2}(-2s+2x_0)\end{aligned}$$

and, for $i = 1, \dots, n$,

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2} S_{\mathrm{R}}^{-1}(s,x) &= 8x_i^2(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-3}(s-\bar{x}) \\ &\quad - 2(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-2}(s-\bar{x}) \\ &\quad - 4x_i(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-2}e_i.\end{aligned}$$

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Thus, we obtain

$$\begin{aligned}\Delta S_{\mathrm{R}}^{-1}(s,x) &= 2(s^2-2\operatorname{Re}(x)s+|x|^2)^{-3}(-2s+2x_0)^2(s-\bar{x})\\ &\quad+2(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}(-2s+2x_0)\\ &\quad+\sum_{i=1}^{n}8x_i^2(s^2-2\operatorname{Re}(x)s+|x|^2)^{-3}(s-\bar{x})\\ &\quad-\sum_{i=1}^{n}4x_i(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}e_i\\ &\quad-2(n+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}(s-\bar{x}),\end{aligned}$$

and since $(s^2-2\operatorname{Re}(x)s+|x|^2)^{-1}$ and $(-2s+2x_0)$ commute we have

$$\begin{aligned}\Delta S_{\mathrm{R}}(s,x)^{-1} &= \left(2(-2s+2x_0)^2+\sum_{i=1}^{n}8x_i^2\right)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-3}(s-\bar{x})\\ &\quad+(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}\left(2(-2s+2x_0)-\sum_{i=1}^{n}4x_ie_i\right)\\ &\quad-2(n+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}(s-\bar{x}).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\Delta S_{\mathrm{R}}^{-1}(s,x) &= 8(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}(s-\bar{x})\\ &\quad-4(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}(s-\bar{x})\\ &\quad-2(n+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}(s-\bar{x})\\ &= -2(n-1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-2}(s-\bar{x}),\end{aligned}$$

which corresponds to (3.2) for $h=1$.

Let us assume that (3.2) holds for some $h\in\mathbb{N}$, and show that it holds for $h+1$. In order to avoid the constants, we consider the function

$$G_h(s,x) := (s^2-2\operatorname{Re}(x)s+|x|^2)^{-(h+1)}(s-\bar{x}). \tag{3.3}$$

We have

$$\begin{aligned}\frac{\partial^2}{\partial x_0^2}G_h &= (h+2)(h+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-(h+3)}(-2s+2x_0)^2(s-\bar{x})\\ &\quad-2(h+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-(h+2)}(s-\bar{x})\\ &\quad+2(h+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-(h+2)}(-2s+2x_0)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2}G_h(s,x) &= 4(h+2)(h+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-(h+3)}x_i^2(s-\bar{x})\\ &\quad-2(h+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-(h+2)}(s-\bar{x})\\ &\quad-4(h+1)(s^2-2\operatorname{Re}(x)s+|x|^2)^{-(h+2)}x_ie_i.\end{aligned}$$

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Thus, we obtain

$$\Delta G_h(s,x) = -2(h+1)(n-(2h+1))(s^2 - 2\,\mathrm{Re}(x)s + |x|^2)^{-(h+2)}(s-\bar{x}),$$

and so, taking into account the fact that

$$\Delta S_{\mathrm{R}}^{-1}(s,x) = (-1)^h \prod_{l=1}^{h}(2l)\prod_{l=1}^{h}(n-(2l-1))G_h(s,x),$$

we obtain that (3.2) holds for $h+1$. By induction we get the statement. □

Proposition 3.3. *Let $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. The function $\Delta^h S_{\mathrm{L}}^{-1}(s,x)$ is a right-slice monogenic function in the variable s for all $h \geqslant 0$. The function $\Delta^h S_{\mathrm{R}}^{-1}(s,x)$ is a left-slice monogenic function in the variable s for all $h \geqslant 0$.*

Proof. We shall only consider the right-slice monogenic case, as the left-slice monogenic case has already been proved in [18]. For $h = 0$ the statement is well known. If $h \geqslant 1$, we set $s = u + Iv$ for $I \in \mathbb{S}$ and we consider the function G_h introduced in (3.3) to avoid the constants. We have

$$\begin{aligned}\frac{\partial}{\partial u}G_h(u+vI,x) &= (u^2 - v^2 + 2Iuv + 2\,\mathrm{Re}(x)(u+Iv) + |x|^2)^{-(h+1)}\\ &\quad - (h+1)(u^2 - v^2 + 2Iuv + 2\,\mathrm{Re}(x)(u+Iv) + |x|^2)^{-(h+2)}\\ &\quad \times (2u + 2Iv - 2\,\mathrm{Re}(x))(u+vI-\bar{x})\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial v}G_h(u+vI,x) &= (u^2 - v^2 + 2Iuv + 2\,\mathrm{Re}(x)(u+Iv) + |x|^2)^{-(h+1)}I\\ &\quad - (h+1)(u^2 - v^2 + 2Iuv + 2\,\mathrm{Re}(x)(u+Iv) + |x|^2)^{-(h+2)}I\\ &\quad \times (-2v + 2Iu - 2I\,\mathrm{Re}(x))(u+vI-\bar{x}).\end{aligned}$$

As I and $(u^2 - v^2 + 2Iuv + 2\,\mathrm{Re}(x)(u+Iv) + |x|^2)^{-1}$ commute, it follows immediately that

$$\frac{\partial}{\partial u}G_h(u+Iv,x) + I\frac{\partial}{\partial v}G_h(u+Iv,x) = 0.$$

Therefore, $\Delta S_{\mathrm{R}}^{-1}(s,x)$ is left-slice monogenic in its first variable. □

Proposition 3.4. *Let n be an odd number and let $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function $\Delta^{(n-1)/2}S_{\mathrm{L}}^{-1}(s,x)$ is a left monogenic function in the variable x, and the function $\Delta^{(n-1)/2}S_{\mathrm{R}}^{-1}(s,x)$ is a right monogenic function in the variable x.*

Proof. We shall only consider the right-slice monogenic case, as the left-slice monogenic case has already been proved in [18]. Again we consider the function $G_{(n-1)/2}$ as defined in (3.3) to avoid the constants. We have

$$\begin{aligned}\frac{\partial}{\partial x_0}G_{(n-1)/2} &= -\frac{n+1}{2}(s^2 - 2\,\mathrm{Re}(x)s + |x|^2)^{-(n+1)/2-1}(-2s + 2x_0)(s-\bar{x})\\ &\quad - (s^2 - 2\,\mathrm{Re}(x)s + |x|^2)^{-(n+1)/2}\end{aligned}$$

and

$$\frac{\partial}{\partial x_i} G_{(n-1)/2} = -\frac{n+1}{2}(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2-1}(2x_i)(s-\bar{x}) + (s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2} e_i$$

for $i = 1, \dots, n$. If we consider

$$\partial_r G_{(n-1)/2} := \frac{\partial}{\partial x_0} G_{(n-1)/2} + \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} G_{(n-1)/2} \right) e_i,$$

we get

$$\begin{aligned}
\partial_r G_{(n-1)/2} &= -\frac{n+1}{2}(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2-1}(-2s + 2x_0)(s-\bar{x}) \\
&\quad - (s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2} \\
&\quad - \frac{n+1}{2}(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2-1}(s-\bar{x})\left(\sum_{i=1}^{n} 2x_i e_i\right) \\
&\quad - n(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2} \\
&= (n+1)(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2-1}(s(s-\bar{x}) - (s-\bar{x})x) \\
&\quad - (n+1)(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2} \\
&= 0
\end{aligned}$$

as $s(s-\bar{x}) - (s-\bar{x})x = s^2 - 2\operatorname{Re}(x)s + |x|^2$. Therefore, $G_{(n-1)/2}(s,x)$ and $S_{\mathrm{R}}^{-1}(s,x)$ are right monogenic in x. □

Definition 3.5 (the $\mathcal{F}_n$-kernel). Let n be an odd number and let $x, s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the $\mathcal{F}_n^{\mathrm{L}}$-kernel as

$$\mathcal{F}_n^{\mathrm{L}}(s,x) := \Delta^{(n-1)/2} S_{\mathrm{L}}^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2},$$

and the $\mathcal{F}_n^{\mathrm{R}}$-kernel as

$$\mathcal{F}_n^{\mathrm{R}}(s,x) := \Delta^{(n-1)/2} S_{\mathrm{R}}^{-1}(s,x) = \gamma_n (s^2 - 2\operatorname{Re}(x)s + |x|^2)^{-(n+1)/2}(s-\bar{x}),$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{n-1} [(\tfrac{1}{2}(n-1))!]^2. \tag{3.4}$$

Remark 3.6. Observe that the constants γ_n are obtained from the identity

$$(-1)^{(n-1)/2} \prod_{\ell=1}^{(n-1)/2} (2\ell) \prod_{\ell=1}^{(n-1)/2} (n - (2\ell - 1)) = (-1)^{(n-1)/2} 2^{n-1} [(\tfrac{1}{2}(n-1))!]^2.$$

Theorem 3.7 (the Fueter–Sce mapping theorem in integral form).
Let n be an odd number. Set $\mathrm{d}s_I = \mathrm{d}s/I$. Let $W \subset \mathbb{R}^{n+1}$ be an open set. Let U be a bounded axially symmetric s-domain such that $\bar{U} \subset W$. Suppose that the boundary of $U \cap \mathbb{C}_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$.

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(a) *If $x \in U$ and $f \in \mathcal{SM}^{\mathrm{L}}(W)$, then $\breve{f}(x) = \Delta^{(n-1)/2} f(x)$ is left monogenic and it admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n^{\mathrm{L}}(s,x)\,\mathrm{d}s_I f(s). \tag{3.5}$$

(b) *If $x \in U$ and $f \in \mathcal{SM}^{\mathrm{R}}(W)$, then $\breve{f}(x) = \Delta^{(n-1)/2} f(x)$ is right monogenic and it admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,x). \tag{3.6}$$

The integrals depend neither on U nor on the imaginary unit $I \in \mathbb{S}$.

Proof. This follows from the Fueter–Sce mapping theorem, from the Cauchy formulae and from proposition 3.4. □

We point out that the Fueter–Sce mapping theorem in integral form can be proved for more general open sets (more precisely, for open sets that are only axially symmetric); see [18]. For the application to the $\mathcal{F}$-functional calculus we can take axially symmetric s-domains. This is why we work directly with slice monogenic functions. In this case we consider just axially symmetric open sets. We have to change the definition of slice monogenicity and consider holomorphic functions of a paravector variable. Precisely, the definition has to be changed as follows. Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} \subseteq \mathbb{R}\times\mathbb{R}$ be such that $x = u + Iv \in U$ for all $(u,v) \in \mathcal{U}$. We consider functions on U of the form

$$f(x) = \alpha(u,v) + I\beta(u,v),$$

where α, β are $\mathbb{R}_n$-valued differentiable functions such that

$$\alpha(u,v) = \alpha(u,-v), \quad \beta(u,v) = \beta(u,-v) \quad \text{for all } (u,v) \in \mathcal{U}$$

and, moreover, α and β satisfy the Cauchy–Riemann system

$$\partial_u\alpha - \partial_v\beta = 0, \qquad \partial_v\alpha + \partial_u\beta = 0.$$

On axially symmetric s-domains this class of functions coincides with the class of slice monogenic functions.

4. The formulations of the $\mathcal{F}$-functional calculus

In this section we recall some definitions and results to be used later. By V we denote a real Banach space over $\mathbb{R}$ with norm $\|\cdot\|$. It is possible to endow V with an operation of multiplication by elements of $\mathbb{R}_n$ that gives a two-sided module over $\mathbb{R}_n$. We denote by V_n the two-sided Banach module $V \otimes \mathbb{R}_n$. An element of V_n is of the form $\sum_{A\subset\{1,\dots,n\}} e_A v_A$ with $v_A \in V$, where $e_\emptyset = 1$ and $e_A = e_{i_1}\cdots e_{i_r}$ for $A = \{i_1,\dots,i_r\}$ with $i_1 < \cdots < i_r$. The multiplications (right and left) of an element $v \in V_n$ with a scalar $a \in \mathbb{R}_n$ are defined as

$$va = \sum_A v_A \otimes (e_A a) \quad \text{and} \quad av = \sum_A v_A \otimes (a e_A).$$

For short, we shall write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. Moreover, we define

$$\|v\|_{V_n} = \sum_A \|v_A\|_V.$$

Let $\mathcal{B}(V)$ be the space of bounded $\mathbb{R}$-homomorphisms of the Banach space V into itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$.

If $T_A \in \mathcal{B}(V)$, we can define the operator $T = \sum_A T_A e_A$ and its action on

$$v = \sum_B v_B e_B$$

as

$$T(v) = \sum_{A,B} T_A(v_B) e_A e_B.$$

The set of all such bounded operators is denoted by $\mathcal{B}(V_n)$. The norm is defined by

$$\|T\|_{\mathcal{B}(V_n)} = \sum_A \|T_A\|_{\mathcal{B}(V)}.$$

In the following, we shall only consider operators of the form $T = T_0 + \sum_{j=1}^n e_j T_j$, where $T_j \in \mathcal{B}(V)$ for $j = 0, 1, \dots, n$, and we recall that the conjugate is defined by

$$\bar{T} = T_0 - \sum_{j=1}^n e_j T_j.$$

The set of such operators in $\mathcal{B}(V_n)$ will be denoted by $\mathcal{B}^{0,1}(V_n)$. In this section we shall always consider n-tuples of bounded commuting operators, in paravector form, and we shall denote the set of such operators as $\mathcal{BC}^{0,1}(V_n)$. We recall some results proved in [13, 18].

Definition 4.1 (the $\mathcal{F}$-spectrum and the $\mathcal{F}$-resolvent sets). Let $T \in \mathcal{BC}^{0,1}(V_n)$. We define the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ of T as

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} \colon s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T} \text{ is not invertible}\}.$$

The $\mathcal{F}$-resolvent set $\rho_{\mathcal{F}}(T)$ is defined by

$$\rho_{\mathcal{F}}(T) = \mathbb{R}^{n+1} \setminus \sigma_{\mathcal{F}}(T).$$

Here we state two important properties of the $\mathcal{F}$-spectrum. The first is its axial symmetry, and the second is that, for the case of bounded operators, the $\mathcal{F}$-spectrum is a compact and non-empty set.

Theorem 4.2 (structure of the $\mathcal{F}$-spectrum). *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and let*

$$p = p_0 + p_1 I \in [p_0 + p_1 I] \subset \mathbb{R}^{n+1} \setminus \mathbb{R},$$

such that $p \in \sigma_{\mathcal{F}}(T)$. Then all the elements of the $(n-1)$-sphere $[p_0 + p_1 I]$ belong to $\sigma_{\mathcal{F}}(T)$.

Thus, the $\mathcal{F}$-spectrum consists of real points and/or $(n-1)$-spheres.

Theorem 4.3 (compactness of the $\mathcal{F}$-spectrum). *Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ is a compact non-empty set. Moreover, $\sigma_{\mathcal{F}}(T)$ is contained in $\{s \in \mathbb{R}^{n+1} \colon |s| \leqslant \|T\|\}$.*

The definition of the $\mathcal{F}$-resolvent operator is suggested by the Fueter–Sce mapping theorem in integral form.

Definition 4.4 ($\mathcal{F}$-resolvent operators). Let n be an odd number and let $T \in \mathcal{BC}^{0,1}(V_n)$. For $s \in \rho_{\mathcal{F}}(T)$ we define the left $\mathcal{F}$-resolvent operator by

$$\mathcal{F}_n^{\mathrm{L}}(s,T) := \gamma_n(s\mathcal{I} - \bar{T})(s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T})^{-(n+1)/2}, \tag{4.1}$$

and the right $\mathcal{F}$-resolvent operator by

$$\mathcal{F}_n^{\mathrm{R}}(s,T) := \gamma_n(s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T})^{-(n+1)/2}(s\mathcal{I} - \bar{T}), \tag{4.2}$$

where the constants γ_n are given by (3.4).

Now we have to say which are the functions that are defined on suitable open sets that contain the $\mathcal{F}$-spectrum. For this class of slice monogenic functions it is possible to define the $\mathcal{F}$-functional calculus for bounded operators.

Definition 4.5. Let $T \in \mathcal{BC}^{0,1}(V_n)$ and let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain.

(a) We say that U is admissible for T if it contains the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$, and if $\partial(U \cap \mathbb{C}_I)$ is the union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$.

(b) Let W be an open set in $\mathbb{R}^{n+1}$. A function $f \in \mathcal{SM}^{\mathrm{L}}(W)$ (respectively, $f \in \mathcal{SM}^{\mathrm{R}}(W)$) is said to be locally left (respectively, right) slice monogenic on $\sigma_{\mathcal{F}}(T)$ if there exists an admissible domain $U \subset \mathbb{R}^{n+1}$ such that $\bar{U} \subset W$, on which f is left (respectively, right) slice monogenic.

(c) We shall denote by $\mathcal{SM}^{\mathrm{L}}_{\sigma_{\mathcal{F}}(T)}$ (respectively, $\mathcal{SM}^{\mathrm{R}}_{\sigma_{\mathcal{F}}(T)}$) the set of locally left (respectively, right) slice monogenic functions on $\sigma_{\mathcal{F}}(T)$.

Finally, the following theorem is crucial for the well posedness of the $\mathcal{F}$-functional calculus.

Theorem 4.6. *Let n be an odd number, let $T \in \mathcal{BC}^{0,1}(V_n)$ and set $\mathrm{d}s_I = \mathrm{d}s/I$. Then the integrals*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n^{\mathrm{L}}(s,T)\,\mathrm{d}s_I f(s), \quad f \in \mathcal{SM}^{\mathrm{L}}_{\sigma_{\mathcal{F}}(T)}, \tag{4.3}$$

and

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T), \quad f \in \mathcal{SM}^{\mathrm{R}}_{\sigma_{\mathcal{F}}(T)}, \tag{4.4}$$

depend neither on the imaginary unit $I \in \mathbb{S}$ nor on the set U.

Proof. The case of left-slice monogenic functions has been treated in [18]. We only consider the right-slice monogenic case. For every continuous linear functional $\phi \in V_n'$ and $v \in V_n$, we define the function

$$g_{\phi,v}(s) := \langle \phi, \mathcal{F}_n^{\mathrm{R}}(s,T)v \rangle.$$

Since $\Delta^{(n-1)/2} S_{\mathrm{R}}^{-1}(s,x)$ is left s-monogenic in s by proposition 3.3, the function $g_{\phi,v}$ is also left-slice monogenic on $\rho_{\mathcal{F}}(T)$. Furthermore, as $\lim_{s\to\infty} g_{\phi,v}(s) = 0$, it is also left-slice monogenic at ∞.

To prove that the integral (4.4) does not depend on the open set U, we consider another open set U' as in definition 4.5 with $\sigma_{\mathcal{F}}(T) \subset U' \subset U$. As every $g_{\phi,v}$ is left-slice monogenic on $U'^{\mathrm{c}} \subset \rho_{\mathcal{F}}(T)$ and $g_{\phi,v}(\infty) = 0$, we can apply the Cauchy formula and obtain

$$g_{\phi,v}(x) = \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)^-} S_{\mathrm{L}}^{-1}(s,x)\,\mathrm{d}s_I g_{\phi,v}(s) = -\frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)} S_{\mathrm{L}}^{-1}(s,x)\,\mathrm{d}s_I g_{\phi,v}(s), \tag{4.5}$$

where $\partial(U'\cap\mathbb{C})^-$ is the border of $U'\cap\mathbb{C}$ oriented in a way that includes U'^{c}. As $S_{\mathrm{R}}^{-1}(s,x) = -S_{\mathrm{L}}^{-1}(x,s)$ and as f is right-slice monogenic on $\bar{U}$, we get

$$\begin{aligned}
&\left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T)\right] v \right\rangle \\
&\qquad = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I g_{\phi,v}(s) \\
&\qquad = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \left[-\frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)^-} S_{\mathrm{L}}^{-1}(t,s)\,\mathrm{d}t_I g_{\phi,v}(t)\right] \\
&\qquad = \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)} \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I S_{\mathrm{R}}^{-1}(s,t)\right] \mathrm{d}t_I g_{\phi,v}(t) \\
&\qquad = \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)} f(t)\,\mathrm{d}t_I g_{\phi,v}(t) \\
&\qquad = \left\langle \phi, \left[\frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T)\right] v \right\rangle.
\end{aligned}$$

This equality holds for every $\phi \in V_n'$ and $v \in V_n$. Therefore, by the Hahn–Banach theorem, it follows that

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T) = \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T).$$

Now let $\tilde{U}$ be an arbitrary set as in definition 4.5 that is not necessarily a subset of U. Then we can find an admissible set U' with $\sigma_{\mathcal{F}}(T) \subset U' \subset U \cap \tilde{U}$ and therefore the integrals over all three sets must agree.

The proof of the independence of the imaginary unit I works analogously. Again we consider an admissible open set U' with $\sigma_{\mathcal{F}}(T) \subset U' \subset U$. As (4.5) is indepen-

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dent of the imaginary unit, for an arbitrary $J \in \mathbb{S}$ we have

$$\begin{aligned}\frac{1}{2\pi}&\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I g_{\phi,v}(s)\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I\left[-\frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)^-} S_{\mathrm{L}}^{-1}(t,s)\,\mathrm{d}t_J g_{\phi,v}(t)\right]\\ &= \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)}\left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I S_{\mathrm{R}}^{-1}(s,t)\right]\mathrm{d}t_J g_{\phi,v}(t)\\ &= \frac{1}{2\pi}\int_{\partial(U'\cap\mathbb{C}_J)} f(t)\,\mathrm{d}t_J g_{\phi,v}(t).\end{aligned}$$

As we already know that these integrals are independent of the set U, for every $\phi \in V_n'$ and $v \in V_n$ we therefore have

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I g_{\phi,v}(s) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} f(t)\,\mathrm{d}t_J g_{\phi,v}(t),$$

and from the Hahn–Banach theorem it follows that the integral (4.4) does not depend on the imaginary unit. □

Definition 4.7 ($\mathcal{F}$-functional calculus for bounded operators). Let n be an odd number, let $T \in \mathcal{BC}^{0,1}(V_n)$ and set $\mathrm{d}s_I = \mathrm{d}s/I$. We define the $\mathcal{F}$-functional calculus as

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n^{\mathrm{L}}(s,T)\,\mathrm{d}s_I f(s), \quad f \in \mathcal{SM}^{\mathrm{L}}_{\sigma_{\mathcal{F}}(T)}, \tag{4.6}$$

and

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T), \quad f \in \mathcal{SM}^{\mathrm{R}}_{\sigma_{\mathcal{F}}(T)}, \tag{4.7}$$

where U is admissible for T.

5. Some relations between the $\mathcal{F}$-resolvent operators

With the position

$$\mathcal{Q}_s(T) := (s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T})^{-1}, \quad s \in \rho_{\mathcal{F}}(T),$$

we can write the left $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^{\mathrm{L}}(s,T) := \gamma_n(s\mathcal{I} - \bar{T})\mathcal{Q}_s(T)^{(n+1)/2}, \tag{5.1}$$

and the right $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^{\mathrm{R}}(s,T) := \gamma_n\mathcal{Q}_s(T)^{(n+1)/2}(s\mathcal{I} - \bar{T}). \tag{5.2}$$

Theorem 5.1 (the left and right $\mathcal{F}$-resolvent equations). *Let n be an odd number and let $T \in \mathcal{BC}^{0,1}(V_n)$. Let $s \in \rho_{\mathcal{F}}(T)$. Then the $\mathcal{F}$-resolvent operators satisfy the equations*

$$\mathcal{F}_n^{\mathrm{L}}(s,T)s - T\mathcal{F}_n^{\mathrm{L}}(s,T) = \gamma_n\mathcal{Q}_s(T)^{(n-1)/2} \tag{5.3}$$

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and

$$s\mathcal{F}_n^{\mathrm{R}}(s,T) - \mathcal{F}_n^{\mathrm{R}}(s,T)T = \gamma_n \mathcal{Q}_s(T)^{(n-1)/2}. \tag{5.4}$$

Proof. Relation (5.3) was proved in [4]. Relation (5.4) follows from

$$\mathcal{F}_n(s,T)s = \gamma_n(s\mathcal{I} - \bar{T})s\mathcal{Q}_s(T)^{(n+1)/2}$$

and

$$T\mathcal{F}_n(s,T) = \gamma_n(Ts - T\bar{T})\mathcal{Q}_s(T)^{(n+1)/2}.$$

Taking the difference, we obtain

$$\mathcal{F}_n(s,T)s - T\mathcal{F}_n(s,T) = \gamma_n(s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T})\mathcal{Q}_s(T)^{(n+1)/2} = \gamma_n \mathcal{Q}_s(T)^{(n+1)/2}.$$

□

Theorem 5.2 (left and right generalized $\mathcal{F}$-resolvent equations). *Let n be an odd number, $T \in \mathcal{BC}^{0,1}(V_n)$ and set*

$$\begin{aligned}\mathcal{M}_m^{\mathrm{L}}(s,T) &:= \gamma_n \sum_{i=0}^{m-1} T^i \mathcal{Q}_s(T)^{(n-1)/2} s^{m-1-i} \\ &= \gamma_n(\mathcal{Q}_s(T)^{(n-1)/2}s^{m-1} + T\mathcal{Q}_s(T)^{(n-1)/2}s^{m-2} + \cdots \\ &\qquad + T^{m-2}Q_s(T)^{(n-1)/2}s + T^{m-1}Q_s(T)^{(n-1)/2})\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_m^{\mathrm{R}}(s,T) &:= \gamma_n \sum_{i=0}^{m-1} s^{m-1-i} \mathcal{Q}_s(T)^{(n-1)/2} T^i \\ &= \gamma_n(s^{m-1}\mathcal{Q}_s(T)^{(n-1)/2} + s^{m-2}\mathcal{Q}_s(T)^{(n-1)/2}T + \cdots \\ &\qquad + sQ_s(T)^{(n-1)/2}T^{m-2} + T^{m-1}Q_s(T)^{(n-1)/2}).\end{aligned}$$

Then, for $m \in \mathbb{N}_0$ and $s \in \rho_{\mathcal{F}}(T)$, the following equations hold:

$$T^m\mathcal{F}_n^{\mathrm{L}}(s,T) = \mathcal{F}_n^{\mathrm{L}}(s,T)s^m - \mathcal{M}_m^{\mathrm{L}}(s,T), \tag{5.5}$$

$$\mathcal{F}_n^{\mathrm{R}}(s,T)T^m = s^m\mathcal{F}_n^{\mathrm{R}}(s,T) - \mathcal{M}_m^{\mathrm{R}}(s,T). \tag{5.6}$$

Proof. The proof works by induction. We shall only show (5.5), as the proof is analogous for (5.6). For $m = 0$ the statement is trivial. For $m = 1$ it is the $\mathcal{F}$-resolvent equation (theorem 5.1). Assume that (5.5) holds for $m-1$. Then we have

$$\begin{aligned}T^m\mathcal{F}_n^{\mathrm{L}}(s,T) &= T(\mathcal{F}_n^{\mathrm{L}}(s,T)s^{m-1} - \mathcal{M}_{m-1}(s,T)) \\ &= T\mathcal{F}_n^{\mathrm{L}}(s,T)s^{m-1} - T\mathcal{M}_{m-1}(s,T).\end{aligned} \tag{5.7}$$

As

$$\begin{aligned}T\mathcal{M}_{m-1}(s,T) &= \gamma_n T \sum_{i=0}^{m-2} T^i \mathcal{Q}_s(T)^{(n-1)/2} s^{m-2-i} \\ &= \gamma_n \sum_{i=1}^{m-1} T^i \mathcal{Q}_s(T)^{(n-1)/2} s^{m-1-i},\end{aligned}$$

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by substituting the resolvent equation (5.3) in (5.7), we obtain

$$\begin{aligned}
&T^m\mathcal{F}_N^{\mathrm{L}}(s,T)\\
&\quad=\mathcal{F}_n^{\mathrm{L}}(s,T)s^m-\gamma_n Q_s(T)^{(n-1)/2}s^{m-1}-\gamma_n\sum_{i=1}^{m-1}T^i\mathcal{Q}_s(T)^{(n-1)/2}s^{m-1-i}\\
&\quad=\mathcal{F}_n^{\mathrm{L}}(s,T)s^m-\gamma_n\sum_{i=0}^{m-1}T^i\mathcal{Q}_s(T)^{(n-1)/2}s^{m-1-i}\\
&\quad=\mathcal{F}_n^{\mathrm{L}}(s,T)s^m-\mathcal{M}_m(s,T).
\end{aligned}$$

□

THEOREM 5.3 (the pseudo-$\mathcal{F}$-resolvent equation). *Let n be an odd number and let $T\in\mathcal{BC}^{0,1}(V_n)$. Then, for $p,s\in\rho_{\mathcal{F}}(T)$, the following equation holds:*

$$\begin{aligned}
&\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)\\
&\quad=[(\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T))p\\
&\qquad-\bar{s}(\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T))](p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}$$

Proof. We prove the statement by showing that

$$\begin{aligned}
\mathcal{F}_n^{\mathrm{R}}(s,T)&\mathcal{F}_n^{\mathrm{L}}(p,T)(p^2-2s_0p+|s|^2)\\
&=(\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T))p\\
&\quad-\bar{s}(\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T)).
\end{aligned}$$

If we apply the generalized $\mathcal{F}$-resolvent equations (5.5) and (5.6), we obtain

$$\begin{aligned}
&\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)p^2\\
&\quad=\mathcal{F}_n^{\mathrm{R}}(s,T)[T^2\mathcal{F}_n^{\mathrm{L}}(p,T)+T\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)+\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)p]\\
&\quad=\mathcal{F}_n^{\mathrm{R}}(s,T)T^2\mathcal{F}_n^{\mathrm{L}}(p,T)+\mathcal{F}_n^{\mathrm{R}}(s,T)T\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)+\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)p\\
&\quad=[s^2\mathcal{F}_n^{\mathrm{R}}(s,T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)T-s\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)]\mathcal{F}_n^{\mathrm{L}}(p,T)\\
&\qquad+[s\mathcal{F}_n^{\mathrm{R}}(s,T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)]\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)+\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)p\\
&\quad=s^2\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)T\mathcal{F}_n^{\mathrm{L}}(p,T)-s\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T)\\
&\qquad+s\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)-\gamma_n^2\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{Q}_p^{(n-1)/2}(T)\\
&\qquad+\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)p.
\end{aligned}$$

In a similar way, by applying the $\mathcal{F}$-resolvent equations (5.3) and (5.4), we obtain

$$\begin{aligned}
\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)2s_0p&=2s_0\mathcal{F}_n^{\mathrm{R}}(s,T)[T\mathcal{F}_n^{\mathrm{L}}(p,T)+\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)]\\
&=2s_0[s\mathcal{F}_n^{\mathrm{R}}(s,T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)]\mathcal{F}_n^{\mathrm{L}}(p,T)\\
&\qquad+2s_0\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)\\
&=2s_0s\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)-2s_0\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T)\\
&\qquad+2s_0\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T).
\end{aligned}$$

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Therefore, we have

$$\begin{aligned}
&\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)(p^2-2s_0s+|s|^2)\\
&\quad=(s^2-2s_0s+|s|^2)\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)T\mathcal{F}_n^{\mathrm{L}}(p,T)\\
&\qquad+(2s_0-s)\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T)+(s-2s_0)\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_nQ_p^{(n-1)/2}(T)\\
&\qquad-\gamma_n^2\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{Q}_p^{(n-1)/2}(T)+\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)p.
\end{aligned}$$

As $2s_0-s=\bar{s}$ and $s^2-2s_0s+|s|^2=0$, by applying the $\mathcal{F}$-resolvent equation (5.3) once again to $\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)T\mathcal{F}_n^{\mathrm{L}}(p,T)$, we get

$$\begin{aligned}
&\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)(p^2-2s_0s+|s|^2)\\
&\quad=-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)[\mathcal{F}_n^{\mathrm{L}}(p,T)p-\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)]\\
&\qquad-\bar{s}(\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_nQ_p^{(n-1)/2}(T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T))\\
&\qquad-\gamma_n^2\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{Q}_p^{(n-1)/2}(T)+\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_n\mathcal{Q}_p^{(n-1)/2}(T)p\\
&\quad=[\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_nQ_p^{(n-1)/2}(T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T)]p\\
&\qquad-\bar{s}[\mathcal{F}_n^{\mathrm{R}}(s,T)\gamma_nQ_p^{(n-1)/2}(T)-\gamma_n\mathcal{Q}_s^{(n-1)/2}(T)\mathcal{F}_n^{\mathrm{L}}(p,T)].
\end{aligned}$$

□

The pseudo-$\mathcal{F}$-resolvent equation can be written in terms of the $\mathcal{F}$-resolvent operators by only using the left and the right $\mathcal{F}$-resolvent equations.

Corollary 5.4 (the pseudo-$\mathcal{F}$-resolvent equation, form II). *Let n be an odd number and let $T\in\mathcal{BC}^{0,1}(V_n)$. Then, for $p,s\in\rho_{\mathcal{F}}(T)$, the following equation holds:*

$$\begin{aligned}
&\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)\\
&\quad=[(\mathcal{F}_n^{\mathrm{R}}(s,T)(\mathcal{F}_n^{\mathrm{L}}(p,T)p-T\mathcal{F}_n^{\mathrm{L}}(p,T))-(s\mathcal{F}_n^{\mathrm{R}}(s,T)-\mathcal{F}_n^{\mathrm{R}}(s,T)T)\mathcal{F}_n^{\mathrm{L}}(p,T))p\\
&\qquad-\bar{s}(\mathcal{F}_n^{\mathrm{R}}(s,T)(\mathcal{F}_n^{\mathrm{L}}(p,T)p-T\mathcal{F}_n^{\mathrm{L}}(p,T))-(s\mathcal{F}_n^{\mathrm{R}}(s,T)-\mathcal{F}_n^{\mathrm{R}}(s,T)T)\mathcal{F}_n^{\mathrm{L}}(p,T))]\\
&\qquad\times(p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}$$

Proof. This is a direct consequence of theorems 5.3 and 5.1. □

Remark 5.5. We conclude this section with an important property of the operators $\breve{P}_j$, defined in (5.8): in dimension $n=3$ they are the Riesz projectors associated to a given paravector operator T with commuting components, as proved in § 7. The case $n>3$ is still under investigation, and it is related to the structure of the $\mathcal{F}$-resolvent equation in dimension $n>3$.

We begin by recalling the definition of projectors and some of their basic properties.

Definition 5.6. Let V be a Banach module and let $P\colon V\to V$ be a linear operator. If $P^2=P$, we say that P is a projector.

For neatness, we state the proof of the following lemma in Appendix A.

Lemma 5.7. *Let n be an odd number and let $\mathcal{P}_{n-1,n}=\Delta^{(n-1)/2}x^{n-1}$ be the monogenic polynomial defined on $\mathbb{R}^{n+1}$. Then we have $\mathcal{P}_{n-1,n}\equiv\gamma_n$.*

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Lemma 5.7 motivates the definition of the operators $\breve{P}_j$.

THEOREM 5.8. *Let n be an odd number, $T \in \mathcal{BC}^{0,1}(V_n)$ and $f \in \mathcal{SM}^{\mathrm{L}}_{\sigma_{\mathcal{F}}(T)}$. Let $\sigma_{\mathcal{F}}(T) = \sigma_{1\mathcal{F}} \cup \sigma_{2\mathcal{F}}$, with $\mathrm{dist}(\sigma_{1\mathcal{F}}, \sigma_{2\mathcal{F}}) > 0$. Let U_1 and U_2 be two admissible sets for T such that $\sigma_{1\mathcal{F}} \subset U_1$ and $\sigma_{2\mathcal{F}} \subset U_2$, with $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. For $j = 1, 2$, set*

$$\breve{P}_j := \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} \mathcal{F}_n^{\mathrm{L}}(s,T)\, \mathrm{d}s_I s^{n-1}, \tag{5.8}$$

$$\breve{T}_j := \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} \mathcal{F}_n^{\mathrm{L}}(s,T)\, \mathrm{d}s_I s^{n} - \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} \mathcal{Q}_s(T)^{(n-1)/2}\, \mathrm{d}s_I s^{n-1}. \tag{5.9}$$

Then the following properties hold.

(1) *$T\breve{P}_j = \breve{P}_j T = T_j$ for $j = 1, 2$.*

(2) *For $\lambda \in \rho_{\mathcal{F}}(T)$ we have*

$$\breve{P}_j \mathcal{F}_n^{\mathrm{L}}(\lambda,T)\lambda - \breve{T}_j \mathcal{F}_n^{\mathrm{L}}(\lambda,T) = \breve{P}_j \gamma_n \mathcal{Q}_\lambda(T)^{(n-1)/2}, \quad j = 1, 2, \tag{5.10}$$

$$\lambda \mathcal{F}_n^{\mathrm{R}}(\lambda,T)\breve{P}_j - \mathcal{F}_n^{\mathrm{R}}(\lambda,T)\breve{T}_j = \gamma_n \mathcal{Q}_\lambda(T)^{(n-1)/2}\breve{P}_j, \quad j = 1, 2. \tag{5.11}$$

Proof. Note that the operators $\breve{P}_j$ and $\breve{T}_j$ can also be written using the right $\mathcal{F}$-resolvent operator. To prove (1), we apply the $\mathcal{F}$-resolvent equation and get

$$\begin{aligned}
T\breve{P}_j &= T\frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} \mathcal{F}_n^{\mathrm{L}}(s,T)\, \mathrm{d}s_I s^{n-1} \\
&= \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} T\mathcal{F}_n^{\mathrm{L}}(s,T)\, \mathrm{d}s_I s^{n-1} \\
&= \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} (\mathcal{F}_n^{\mathrm{L}}(s,T)s - \gamma_n \mathcal{Q}_s(T)^{(n-1)/2})\, \mathrm{d}s_I s^{n-1} \\
&= \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} \mathcal{F}_n^{\mathrm{L}}(s,T)\, \mathrm{d}s_I s^{n} - \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} \mathcal{Q}_s(T)^{(n-1)/2}\, \mathrm{d}s_I s^{n-1} \\
&= \breve{T}_j.
\end{aligned}$$

As s, $\mathrm{d}s_I$ and $\mathcal{Q}_s(T)$ commute on $\mathbb{C}_I$, we also have

$$\begin{aligned}
\breve{P}_j T &= \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} s^{n-1}\, \mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T)T \\
&= \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} s^{n-1}\, \mathrm{d}s_I (s\mathcal{F}_n^{\mathrm{R}}(s,T) - \gamma_n \mathcal{Q}_s(T)^{(n-1)/s}) \\
&= \frac{\gamma_n^{-1}}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} s^{n}\, \mathrm{d}s_I \mathcal{F}_n^{\mathrm{R}}(s,T) - \frac{1}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} s^{n-1}\, \mathrm{d}s_I \mathcal{Q}_s(T)^{(n-1)/s} \\
&= \breve{T}_j.
\end{aligned}$$

To prove (5.10), for $\lambda \in \rho_{\mathcal{F}}(T)$ we apply the $\mathcal{F}$-resolvent equation (5.3) and obtain

$$\begin{aligned}\breve{P}_j \mathcal{F}_n^{\mathrm{L}}(\lambda, T)\lambda &= \breve{P}_j (T\mathcal{F}_n^{\mathrm{L}}(\lambda, T) - \gamma_n \mathcal{Q}_\lambda(T)^{(n-1)/2}) \\ &= \breve{P}_j T\mathcal{F}_n^{\mathrm{L}}(\lambda, T) - \breve{P}_j \gamma_n \mathcal{Q}_\lambda(T)^{(n-1)/2} \\ &= \breve{T}_j \mathcal{F}_n^{\mathrm{L}}(\lambda, T) - \breve{P}_j \gamma_n \mathcal{Q}_\lambda(T)^{(n-1)/2}.\end{aligned}$$

The identity (5.11) can be proved analogously. □

6. Preliminary results on the $\mathcal{SC}$-functional calculus

As we mentioned in § 1, the $\mathcal{F}$-resolvent equation for the $\mathcal{F}$-functional calculus also involves the $\mathcal{SC}$-resolvent operators. In this section we recall some results on the $\mathcal{SC}$-functional calculus (for more details see [13]).

Definition 6.1 (the $\mathcal{SC}$-resolvent operators). Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $s \in \rho_{\mathcal{F}}(T)$. We define the left $\mathcal{SC}$-resolvent operator as

$$S_{C,\mathrm{L}}^{-1}(s, T) := (s\mathcal{I} - \bar{T})(s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T})^{-1}, \tag{6.1}$$

and the right $\mathcal{SC}$-resolvent operator as

$$S_{C,\mathrm{R}}^{-1}(s, T) := (s^2\mathcal{I} - s(T + \bar{T}) + T\bar{T})^{-1}(s\mathcal{I} - \bar{T}). \tag{6.2}$$

Theorem 6.2. *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $s \in \rho_{\mathcal{F}}(T)$. Then $S_{C,\mathrm{L}}^{-1}(s, T)$ satisfies the left $\mathcal{SC}$-resolvent equation,*

$$S_{C,\mathrm{L}}^{-1}(s, T)s - TS_{C,\mathrm{L}}^{-1}(s, T) = \mathcal{I}, \tag{6.3}$$

and $S_{C,\mathrm{R}}^{-1}(s, T)$ satisfies the right $\mathcal{SC}$-resolvent equation,

$$sS_{C,\mathrm{R}}^{-1}(s, T) - S_{C,\mathrm{R}}^{-1}(s, T)T = \mathcal{I}.$$

The following crucial results is proved in [7].

Theorem 6.3 (the $\mathcal{SC}$-resolvent equation). *Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $s, p \in \rho_{\mathcal{F}}(T)$. Then we have*

$$\begin{aligned}S_{C,\mathrm{R}}^{-1}(s, T)S_{C,\mathrm{L}}^{-1}(p, T) = ((&S_{C,\mathrm{R}}^{-1}(s, T) - S_{C,\mathrm{L}}^{-1}(p, T))p \\ &- \bar{s}(S_{C,\mathrm{R}}^{-1}(s, T) - S_{C,\mathrm{L}}^{-1}(p, T)))(p^2 - 2s_0 p + |s|^2)^{-1}.\end{aligned} \tag{6.4}$$

Moreover, the resolvent equation can also be written as

$$\begin{aligned}S_{C,\mathrm{R}}^{-1}(s, T)S_{C,\mathrm{L}}^{-1}(p, T) = (s^2 - 2p_0 s + |p|^2)^{-1}(&s(S_{C,\mathrm{R}}^{-1}(s, T) - S_{C,\mathrm{L}}^{-1}(p, T)) \\ &- (S_{C,\mathrm{R}}^{-1}(s, T) - S_{C,\mathrm{L}}^{-1}(p, T))\bar{p}).\end{aligned} \tag{6.5}$$

Definition 6.4 (the $\mathcal{SC}$-functional calculus). Let $T \in \mathcal{BC}^{0,1}(V_n)$. Let $U \subset \mathbb{R}^{n+1}$ be admissible for T and set $\mathrm{d}s_I = \mathrm{d}s/I$ for $I \in \mathbb{S}$. We define

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_{C,\mathrm{L}}^{-1}(s, T)\,\mathrm{d}s_I\, f(s) \quad \text{for } f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)} \tag{6.6}$$

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and

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I S_{C,\mathrm{R}}^{-1}(s,T) \quad \text{for } f\in\mathcal{SM}^{\mathrm{R}}_{\sigma_{\mathcal{F}}(T)}. \tag{6.7}$$

Finally, we need a technical lemma.

DEFINITION 6.5. Let $f\colon U\to\mathbb{R}_n$ be a slice monogenic function, where U is an open set in $\mathbb{R}^{n+1}$. We define

$$\mathcal{N}(U) = \{f\in\mathcal{SM}(U)\colon f(U\cap\mathbb{C}_I)\subseteq\mathbb{C}_I\ \forall I\in\mathbb{S}\}.$$

First, let us observe that functions in the subclass $\mathcal{N}(U)$ are both left- and right-slice hyperholomorphic. When we take the power series expansion of this class of functions at a point on the real line the coefficients of the expansion are real numbers.

Now observe that, for functions in $f\in\mathcal{N}(U)$, we can define $f(T)$ using the left and the right $\mathcal{SC}$-functional calculus, as follows:

$$\begin{aligned} f(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S_{C,\mathrm{L}}^{-1}(s,T)\,\mathrm{d}s_I f(s)\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I S_{C,\mathrm{R}}^{-1}(s,T).\end{aligned}$$

The following lemma is proved in [7].

LEMMA 6.6. *Let $B\in\mathcal{B}(V_n)$. Let G be an axially symmetric s-domain, and assume that $f\in\mathcal{N}(G)$. Then, for $p\in G$, we have*

$$\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I(\bar{s}B - Bp)(p^2 - 2s_0p + |s|^2)^{-1} = Bf(p).$$

7. Projectors for the dimension $n = 3$

The proof that the operators $\breve{P}_j$ defined in (5.8) are projectors is based on a suitable $\mathcal{F}$-resolvent equation that establishes a link between the product, $\mathcal{F}_n^{\mathrm{R}}(s,T)\mathcal{F}_n^{\mathrm{L}}(p,T)$, and the difference, $\mathcal{F}_n^{\mathrm{R}}(s,T)-\mathcal{F}_n^{\mathrm{L}}(p,T)$. For the case $n=3$ we are able to show that such a relation exists and we can prove that the operators $\breve{P}_j$ are projectors.

7.1. The $\mathcal{F}$-resolvent equations for $n = 3$

We start with a preliminary lemma.

LEMMA 7.1. *Let $T\in\mathcal{BC}(V_3)$. Then for $p,s\in\rho_{\mathcal{F}}(T)$ the following equation holds:*

$$\begin{aligned} &\mathcal{F}_3^{\mathrm{R}}(s,T)S_{C,\mathrm{L}}^{-1}(p,T) + S_{C,\mathrm{R}}^{-1}(s,T)\mathcal{F}_3^{\mathrm{L}}(p,T) + \gamma_3\mathcal{Q}_s(T)\mathcal{Q}_p(T)\\ &\quad = [(\mathcal{F}_3^{\mathrm{R}}(s,T)-\mathcal{F}_3^{\mathrm{L}}(p,T))p - \bar{s}(\mathcal{F}_3^{\mathrm{R}}(s,T)-\mathcal{F}_3^{\mathrm{L}}(p,T))](p^2-2s_0p+|s|^2)^{-1}.\end{aligned}$$

Proof. Let us consider the $\mathcal{SC}$-resolvent equation. Multiplying it on the left by $\gamma_3\mathcal{Q}_s(T)$, we get

$$\begin{aligned} \mathcal{F}_3^{\mathrm{R}}(s,T)S_{C,\mathrm{L}}^{-1}(p,T) &= [(\mathcal{F}_3^{\mathrm{R}}(s,T) - \gamma_3\mathcal{Q}_s(T)S_{\mathrm{L}}^{-1}(p,T))p\\ &\quad - \bar{s}(\mathcal{F}_3^{\mathrm{R}}(s,T) - \gamma_3\mathcal{Q}_s(T)S_{\mathrm{L}}^{-1}(p,T))](p^2-2s_0p+|s|^2)^{-1}.\end{aligned}$$

Now, multiplying the $\mathcal{S}$-resolvent equation on the right by $\gamma_3 \mathcal{Q}_p(T)$, we get

$$\begin{aligned} S^{-1}_{C,\mathrm{R}}(s,T)\mathcal{F}^{\mathrm{L}}_3(p,T) = [(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \mathcal{F}^{\mathrm{L}}_3(p,T))p \\ - \bar{s}(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \mathcal{F}^{\mathrm{L}}_3(p,T))](p^2 - 2s_0 p + |s|^2)^{-1}. \end{aligned}$$

Now we add the two equations above to get

$$\begin{aligned} &\mathcal{F}^{\mathrm{R}}_3(s,T)S^{-1}_{C,\mathrm{L}}(p,T) + S^{-1}_{C,\mathrm{R}}(s,T)\mathcal{F}^{\mathrm{L}}_3(p,T) \\ &\quad = [(\mathcal{F}^{\mathrm{R}}_3(s,T) - \mathcal{F}^{\mathrm{L}}_3(p,T))p - \bar{s}(\mathcal{F}^{\mathrm{R}}_3(s,T) - \mathcal{F}^{\mathrm{L}}_3(p,T))](p^2 - 2s_0 p + |s|^2)^{-1} \\ &\qquad + [(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \gamma_3\mathcal{Q}_s(T)S^{-1}_{C,\mathrm{L}}(p,T))p \\ &\qquad\quad - \bar{s}(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \gamma_3\mathcal{Q}_s(T)S^{-1}_{C,\mathrm{L}}(p,T))](p^2 - 2s_0 p + |s|^2)^{-1}. \end{aligned}$$

Finally, we verify that

$$\begin{aligned} &[(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \gamma_3\mathcal{Q}_s(T)S^{-1}_{C,\mathrm{L}}(p,T))p \\ &\qquad - \bar{s}(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \gamma_3\mathcal{Q}_s(T)S^{-1}_{C,\mathrm{L}}(p,T))](p^2 - 2s_0 p + |s|^2)^{-1} \\ &\qquad\qquad = -\gamma_3\mathcal{Q}_s(T)\mathcal{Q}_p(T). \end{aligned}$$

This follows from

$$\begin{aligned} &(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \gamma_3\mathcal{Q}_s(T)S^{-1}_{C,\mathrm{L}}(p,T))p \\ &\qquad - \bar{s}(S^{-1}_{C,\mathrm{R}}(s,T)\gamma_3\mathcal{Q}_p(T) - \gamma_3\mathcal{Q}_s(T)S^{-1}_{C,\mathrm{L}}(p,T)) \\ &\qquad\quad = \gamma_3[(\mathcal{Q}_s(T)(s\mathcal{I} - \bar{T})\mathcal{Q}_p(T) - \mathcal{Q}_s(T)(p\mathcal{I} - \bar{T})\mathcal{Q}_p(T))p \\ &\qquad\qquad - \bar{s}(\mathcal{Q}_s(T)(s\mathcal{I} - \bar{T})\mathcal{Q}_p(T) - \mathcal{Q}_s(T)(p\mathcal{I} - \bar{T})\mathcal{Q}_p(T))] \\ &\qquad\quad = \gamma_3[\mathcal{Q}_s(T)(s-p)\mathcal{Q}_p(T)p - \bar{s}\mathcal{Q}_s(T)(s-p)\mathcal{Q}_p(T)] \\ &\qquad\quad = \gamma_3[\mathcal{Q}_s(T)(sp - p^2)\mathcal{Q}_p(T) - \mathcal{Q}_s(T)(\bar{s}s - \bar{s}p)\mathcal{Q}_p(T)] \\ &\qquad\quad = \gamma_3[\mathcal{Q}_s(T)(sp - p^2)\mathcal{Q}_p(T) - \mathcal{Q}_s(T)(\bar{s}s - \bar{s}p)\mathcal{Q}_p(T)] \\ &\qquad\quad = -\gamma_3\mathcal{Q}_s(T)\mathcal{Q}_p(T)(p^2 - 2s_0 p + |s|^2). \end{aligned}$$

□

THEOREM 7.2 (the $\mathcal{F}$-resolvent equation for $n = 3$). *Let $T \in \mathcal{BC}(V_3)$. Then, for $p, s \in \rho_{\mathcal{F}}(T)$, the following equation holds:*

$$\begin{aligned} &\mathcal{F}^{\mathrm{R}}_3(s,T)S^{-1}_{C,\mathrm{L}}(p,T) + S^{-1}_{C,\mathrm{R}}(s,T)\mathcal{F}^{\mathrm{L}}_3(p,T) + \gamma_3^{-1}(s\mathcal{F}^{\mathrm{R}}_3(s,T)\mathcal{F}^{\mathrm{L}}_3(p,T)p \\ &\quad - s\mathcal{F}^{\mathrm{R}}_3(s,T)T\mathcal{F}^{\mathrm{L}}_3(p,T) - \mathcal{F}^{\mathrm{R}}_3(s,T)T\mathcal{F}^{\mathrm{L}}_3(p,T)p + \mathcal{F}^{\mathrm{R}}_3(s,T)T^2\mathcal{F}^{\mathrm{L}}_3(p,T)) \\ &\qquad = [(\mathcal{F}^{\mathrm{R}}_3(s,T) - \mathcal{F}^{\mathrm{L}}_3(p,T))p - \bar{s}(\mathcal{F}^{\mathrm{R}}_3(s,T) - \mathcal{F}^{\mathrm{L}}_3(p,T))](p^2 - 2s_0 p + |s|^2)^{-1}. \end{aligned}$$

Proof. We now replace the term $\mathcal{Q}_s(T)\mathcal{Q}_p(T)$ in the right and left $\mathcal{F}$-resolvent equations, respectively, for $p, s \in \rho_{\mathcal{F}}(T)$, by

$$\begin{aligned} s\mathcal{F}^{\mathrm{R}}_3(s,T) - \mathcal{F}^{\mathrm{R}}_3(s,T)T = \gamma_3\mathcal{Q}_s(T), \\ \mathcal{F}^{\mathrm{L}}_3(p,T)p - T\mathcal{F}^{\mathrm{L}}_3(p,T) = \gamma_3\mathcal{Q}_p(T). \end{aligned} \tag{7.1}$$

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So we get

$$\begin{aligned}\gamma_3^2 \mathcal{Q}_s(T)\mathcal{Q}_p(T) &= (s\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{R}}(s,T)T)(\mathcal{F}_3^{\mathrm{L}}(p,T)p - T\mathcal{F}_3^{\mathrm{L}}(p,T)) \\ &= s\mathcal{F}_3^{\mathrm{R}}(s,T)\mathcal{F}_3^{\mathrm{L}}(p,T)p - s\mathcal{F}_3^{\mathrm{R}}(s,T)T\mathcal{F}_3^{\mathrm{L}}(p,T) \\ &\quad - \mathcal{F}_3^{\mathrm{R}}(s,T)T\mathcal{F}_3^{\mathrm{L}}(p,T)p + \mathcal{F}_3^{\mathrm{R}}(s,T)T^2\mathcal{F}_3^{\mathrm{L}}(p,T).\end{aligned}$$

□

Later we shall need the following lemma, which is based on the monogenic functional calculus (for more details see the book [31], or the papers [32,33,35] in which the calculus was introduced).

LEMMA 7.3. *Let $T \in \mathcal{BC}(V_3)$. Suppose that G contains just some points of the $\mathcal{F}$-spectrum of T and assume that the closed smooth curve $\partial(G \cap \mathbb{C}_I)$ belongs to the $\mathcal{F}$-resolvent set of T, for every $I \in \mathbb{S}$. Then*

$$\int_{\partial(G\cap\mathbb{C}_I)} \mathrm{d}s_I\, s\mathcal{F}_3^{\mathrm{R}}(s,T) = 0 \quad \textit{and} \quad \int_{\partial(G\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p\,\mathrm{d}p_I = 0.$$

Proof. As $\Delta x \equiv 0$, we have

$$\int_{\partial(G\cap\mathbb{C}_I)} \mathrm{d}s_I\, s\mathcal{F}_3^{\mathrm{R}}(s,x) = 0 \quad \text{and} \quad \int_{\partial(G\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,x)p\,\mathrm{d}p_I = 0$$

for all x such that $x \notin [s]$ if $s \in \partial(G \cap \mathbb{C}_I)$ (respectively, for all x such that $x \notin [p]$ if $p \in \partial(G \cap \mathbb{C}_I)$). We consider the case of $F_3^{\mathrm{L}}(p,x)$; the other case can be treated in a similar way. We now recall that $F_3^{\mathrm{L}}(p,x)$ is left monogenic in x for every p, such that $x \notin [p]$. Therefore, using the monogenic functional calculus (see [31]), we write

$$\mathcal{F}_3^{\mathrm{L}}(p,T) = \int_{\partial\Omega} \mathcal{G}_\omega(T)\boldsymbol{n}(\omega)\mathcal{F}_3^{\mathrm{L}}(p,\omega)\,\mathrm{d}\mu(\omega),$$

where the open set Ω contains the monogenic spectrum of T, $\mathcal{G}_\omega(T)$ is the monogenic resolvent operator, $\boldsymbol{n}(\omega)$ is the unit normal vector to $\partial\Omega$ and $\mathrm{d}\mu(\omega)$ is the surface element. Using the vector-valued Fubini theorem (see [26, theorem 9, p. 190]) we have

$$\begin{aligned}\int_{\partial(G\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p\,\mathrm{d}p_I &= \int_{\partial(G\cap\mathbb{C}_I)} \int_{\partial\Omega} (\mathcal{G}_\omega(T)\boldsymbol{n}(\omega)\mathcal{F}_3^{\mathrm{L}}(p,\omega)\,\mathrm{d}\mu(\omega))p\,\mathrm{d}p_I \\ &= \int_{\partial\Omega} \mathcal{G}_\omega(T)\boldsymbol{n}(\omega)\bigg(\int_{\partial(G\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,\omega)p\,\mathrm{d}p_I\bigg)\,\mathrm{d}\mu(\omega) \\ &= 0,\end{aligned}$$

which concludes the proof. □

THEOREM 7.4. *Let $T \in \mathcal{BC}^{0,1}(V_3)$ and let $\sigma_{\mathcal{F}}(T) = \sigma_{\mathcal{F},1}(T) \cup \sigma_{\mathcal{F},2}(T)$ with*

$$\mathrm{dist}(\sigma_{\mathcal{F},1}(T), \sigma_{\mathcal{F},2}(T)) > 0.$$

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Let $G_1, G_2 \subset \mathbb{H}$ be two admissible sets for T such that $\sigma_{\mathcal{F},1}(T) \subset G_1$ and $\bar{G}_1 \subset G_2$ and such that $\operatorname{dist}(G_2, \sigma_{\mathcal{F},2}(T)) > 0$*. Then the operator*

$$\breve{P} := \frac{\gamma_3^{-1}}{2\pi} \int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)\,\mathrm{d}p_I p^2 = \frac{\gamma_3^{-1}}{2\pi} \int_{\partial(G_2\cap\mathbb{C}_I)} s^2\,\mathrm{d}s_I \mathcal{F}_3^{\mathrm{R}}(s,T)$$

is a projector, i.e. we have

$$\breve{P}^2 = \breve{P}.$$

Proof. If we multiply the $\mathcal{F}$-resolvent equation in Theorem 7.2 by s on the left and by p on the right, we get

$$\begin{aligned} &s\mathcal{F}_3^{\mathrm{R}}(s,T)S_{C,\mathrm{L}}^{-1}(p,T)p + sS_{C,\mathrm{R}}^{-1}(s,T)\mathcal{F}_3^{\mathrm{L}}(p,T)p \\ &\quad + \gamma_3^{-1}(s^2\mathcal{F}_3^{\mathrm{R}}(s,T)\mathcal{F}_3^{\mathrm{L}}(p,T)p^2 - s^2\mathcal{F}_3^{\mathrm{R}}(s,T)T\mathcal{F}_3^{\mathrm{L}}(p,T)p \\ &\qquad\qquad - s\mathcal{F}_3^{\mathrm{R}}(s,T)T\mathcal{F}_3^{\mathrm{L}}(p,T)p^2 + s\mathcal{F}_3^{\mathrm{R}}(s,T)T^2\mathcal{F}_3^{\mathrm{L}}(p,T)p) \\ &\qquad = s[(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))p - \bar{s}(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))](p^2 - 2s_0p + |s|^2)^{-1}p. \end{aligned}$$

If we multiply this equation by $\mathrm{d}s_I$ on the left, integrate it over $\partial(G_2 \cap \mathbb{C}_I)$ with respect to $\mathrm{d}s_I$ and then multiply it by $\mathrm{d}p_I$ on the right and integrate over $\partial(G_1 \cap \mathbb{C}_I)$ with respect to $\mathrm{d}p_I$, we obtain

$$\begin{aligned} &\int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I\, s\mathcal{F}_3^{\mathrm{R}}(s,T) \int_{\partial(G_1\cap\mathbb{C}_I)} S_{C,\mathrm{L}}^{-1}(p,T)p\,\mathrm{d}p_I \\ &\quad + \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I\, sS_{C,\mathrm{R}}^{-1}(s,T) \int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p\,\mathrm{d}p_I \\ &\quad + \gamma_3^{-1}\bigg(\int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I\, s^2\mathcal{F}_3^{\mathrm{R}}(s,T) \int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p^2\,\mathrm{d}p_I \\ &\qquad\quad - \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I\, s^2\mathcal{F}_3^{\mathrm{R}}(s,T)T \int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p\,\mathrm{d}p_I \\ &\qquad\quad - \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I s\mathcal{F}_3^{\mathrm{R}}(s,T)T \int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p^2\,\mathrm{d}p_I \\ &\qquad\quad + \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I\, s\mathcal{F}_3^{\mathrm{R}}(s,T)T^2 \int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p\,\mathrm{d}p_I \bigg) \\ &\quad = \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I \int_{\partial(G_1\cap\mathbb{C}_I)} s[(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))p - \bar{s}(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (p^2 - 2s_0p + |s|^2)^{-1}p\,\mathrm{d}p_I. \end{aligned}$$

Using lemma 7.3 we have

$$\begin{aligned} &\gamma_3^{-1} \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I\, s^2\mathcal{F}_3^{\mathrm{R}}(s,T) \int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p^2\,\mathrm{d}p_I \\ &\quad = \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I \int_{\partial(G_1\cap\mathbb{C}_I)} s[(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))p - \bar{s}(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (p^2 - 2s_0p + |s|^2)^{-1}\,\mathrm{d}p_I p. \end{aligned}$$

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This is equal to

$$\frac{(2\pi)^2}{\gamma_3^{-1}}\breve{P}^2 = \int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I \times \int_{\partial(G_1\cap\mathbb{C}_I)} s[(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))p - \bar{s}(\mathcal{F}_3^{\mathrm{R}}(s,T) - \mathcal{F}_3^{\mathrm{L}}(p,T))] \times (p^2 - 2s_0p + |s|^2)^{-1}\,\mathrm{d}p_I p. \tag{7.2}$$

Let us observe the integral on the right-hand side. As $\bar{G}_1 \subset G_2$, for any $s \in \partial(G_2\cap\mathbb{C}_I)$ the functions

$$p \mapsto p(p^2 - 2s_0p + |s|^2)^{-1}p \quad \text{and} \quad p \mapsto (p^2 - 2s_0p + |s|^2)^{-1}p$$

are slice monogenic on $\bar{G}_1$. Therefore, we have

$$\int_{\partial(G_1\cap\mathbb{C}_I)} p(p^2 - 2s_0p + |s|^2)^{-1}\,\mathrm{d}p_I p = 0$$

and

$$\int_{\partial(G_1\cap\mathbb{C}_I)} (p^2 - 2s_0p + |s|^2)^{-1}p\,\mathrm{d}p_I = 0,$$

and it follows that

$$\int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I \int_{\partial(G_1\cap\mathbb{C}_I)} s\mathcal{F}_3^{\mathrm{R}}(s,T)p(p^2 - 2s_0p + |s|^2)^{-1}\,\mathrm{d}p_I p = 0$$

and

$$\int_{\partial(G_2\cap\mathbb{C}_I)} \mathrm{d}s_I \int_{\partial(G_1\cap\mathbb{C}_I)} s\bar{s}\mathcal{F}_3^{\mathrm{R}}(s,T)(p^2 - 2s_0p + |s|^2)^{-1}\,\mathrm{d}p_I p = 0.$$

Thus, (7.2) simplifies to

$$\breve{P}^2 = \frac{\gamma_3^{-1}}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_I)} s\,\mathrm{d}s_I \times \int_{\partial(G_1\cap\mathbb{C}_I)} [(\bar{s}\mathcal{F}_3^{\mathrm{L}}(p,T) - \mathcal{F}_3^{\mathrm{L}}(p,T)p)](p^2 - 2s_0p + |s|^2)^{-1}\,\mathrm{d}p_I p,$$

and, by applying lemma 6.6, we finally obtain

$$\breve{P}^2 = \frac{\gamma_3^{-1}}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_I)} \mathcal{F}_3^{\mathrm{L}}(p,T)p\,\mathrm{d}p_I p = \breve{P}.$$

□

8. Formulations of the quaternionic $\mathcal{F}$-functional calculus

We point out that, even though the $\mathcal{F}$-resolvent equation is known only for $n = 3$, this case is of particular importance because it also allows us to study the quaternionic version of the $\mathcal{F}$-functional calculus. In this section we shall state the main

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results related to the quaternionic $\mathcal{F}$-functional calculus, without details, since they very similar to the Clifford setting for $n = 3$.

We denote by $\mathbb{H}$ the algebra of quaternions. The imaginary units in $\mathbb{H}$ are denoted by i, j and k, respectively, and an element in $\mathbb{H}$ is of the form $q = x_0 + \mathrm{i}x_1 + \mathrm{j}x_2 + \mathrm{k}x_3$, for $x_\ell \in \mathbb{R}$, $\ell = 0, 1, 2, 3$. The real part, the imaginary part and the modulus of a quaternion are defined as $\mathrm{Re}(q) = x_0$, $\underline{q} = \mathrm{Im}(q) = \mathrm{i}x_1 + \mathrm{j}x_2 + \mathrm{k}x_3$, $|q|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. The conjugate of the quaternion q is defined by $\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q) = x_0 - \mathrm{i}x_1 - \mathrm{j}x_2 - \mathrm{k}x_3$ and it satisfies $|q|^2 = q\bar{q} = \bar{q}q$. Let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e. $\mathbb{S} = \{q = \mathrm{i}x_1 + \mathrm{j}x_2 + \mathrm{k}x_3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. The Fueter mapping theorem consists in applying the Laplace operator in dimension 4 to functions of the form

$$f(q) = \alpha(x_0, |\underline{q}|) + I\beta(x_0, |\underline{q}|),$$

where α and β are suitable functions satisfying the Cauchy–Riemann system and $q = x_0 + \underline{q}$ is a quaternion. Functions of this form are slice regular (see [21] for more details).

Definition 8.1 (slice regular functions). Let U be an open set in $\mathbb{H}$ and consider a real differentiable function $f : U \to \mathbb{H}$. Denote by f_I the restriction of f to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$.

- We say that f is (left) slice regular if, for every $I \in \mathbb{S}$, on $U \cap \mathbb{C}_I$ we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

  The set of left-slice-regular functions on the open set U is denoted by $\mathcal{SR}^{\mathrm{L}}(U)$.

- We say that f is right slice regular if, for every $I \in \mathbb{S}$, on $U \cap \mathbb{C}_I$ we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy) I\right) = 0.$$

  The set of right-slice-regular functions on the open set U is denoted by $\mathcal{SR}^{\mathrm{R}}(U)$.

Definition 8.2 (Fueter regular functions). Let U be an open set in $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is left Cauchy–Fueter (for brevity 'Fueter') regular if

$$\frac{\partial}{\partial x_0} f(q) + \mathrm{i}\frac{\partial}{\partial x_1} f(q) + \mathrm{j}\frac{\partial}{\partial x_2} f(q) + \mathrm{k}\frac{\partial}{\partial x_3} f(q) = 0, \quad q \in U.$$

It is right Fueter regular if

$$\frac{\partial}{\partial x_0} f(q) + \frac{\partial}{\partial x_1} f(q)\mathrm{i} + \frac{\partial}{\partial x_2} f(q)\mathrm{j} + \frac{\partial}{\partial x_3} f(q)\mathrm{k} = 0, \quad q \in U.$$

Definition 8.3 (the $\mathcal{F}$-kernel). Let $q, s \in \mathbb{H}$. We define, for $s \notin [q]$, the $\mathcal{F}^{\mathrm{L}}$-kernel as

$$\mathcal{F}^{\mathrm{L}}(s, q) := -4(s - \bar{q})(s^2 - 2\,\mathrm{Re}(q)s + |q|^2)^{-2},$$

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and the F^{R}-kernel as

$$\mathcal{F}^{\mathrm{R}}(s,q) := -4(s^2 - 2\operatorname{Re}(q)s + |q|^2)^{-2}(s - \bar{q}).$$

With the above notation, the Fueter mapping theorem in integral form becomes the following.

Theorem 8.4 (the Fueter mapping theorem in integral form).
Set $\mathrm{d}s_I = \mathrm{d}s/I$. *Let* $W \subset \mathbb{H}$ *be an open set and let* U *be a bounded axially symmetric s-domain such that* $\bar{U} \subset W$. *Suppose that the boundary of* $U \cap \mathbb{C}_I$ *consists of a finite number of rectifiable Jordan curves for any* $I \in \mathbb{S}$.

(a) *If* $q \in U$ *and* $f \in \mathcal{SR}^{\mathrm{L}}(W)$, *then* $\breve{f}(q) = \Delta f(q)$ *is left Fueter regular and it admits the integral representation*

$$\breve{f}(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}^{\mathrm{L}}(s,q)\,\mathrm{d}s_I f(s). \tag{8.1}$$

(b) *If* $q \in U$ *and* $f \in \mathcal{SR}^{\mathrm{L}}(W)$, *then* $\breve{f}(q) = \Delta f(q)$ *is right Fueter regular and it admits the integral representation*

$$\breve{f}(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}^{\mathrm{R}}(s,q). \tag{8.2}$$

The integrals depend neither on U *nor on the imaginary unit* $I \in \mathbb{S}$.

We now consider the formulations of the $\mathcal{F}$-functional calculus in the quaternionic setting for right linear quaternionic operators. The same formulation also holds for left linear operators with a suitable interpretation of the symbols.

Definition 8.5. Let V be a right vector space on $\mathbb{H}$. A map $T\colon V \to V$ is said to be a right linear operator if

$$T(u+v) = T(u) + T(v), \qquad T(us) = T(u)s,$$

for all $s \in \mathbb{H}$ and all $u, v \in V$.

In the following, we shall consider only two-sided vector spaces V, otherwise the set of right linear operators is not a (left or right) vector space. With this assumption, the set $\operatorname{End}(V)$ of right linear operators on V is both a left and a right vector space on $\mathbb{H}$ with respect to the operations

$$(aT)(v) := aT(v), \quad (Ta)(v) := T(av).$$

Definition 8.6. Let V be a bilateral quaternionic Banach space. We shall denote by $\mathcal{B}(V)$ the bilateral Banach space of all right linear bounded operators $T\colon V \to V$.

We shall denote by $\mathcal{BC}(V)$ the subclass of $\mathcal{B}(V)$ that consists of those quaternionic operators T that can be written as $T = T_0 + \mathrm{i}T_1 + \mathrm{j}T_2 + \mathrm{k}T_3$, where the operators T_ℓ, $\ell = 0, 1, 2, 3$, commute among themselves.

It is easy to verify that $\mathcal{B}(V)$ is a Banach space endowed with its natural norm.

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Definition 8.7 (the $\mathcal{F}$-spectrum and the $\mathcal{F}$-resolvent sets). Let $T \in \mathcal{BC}(V)$. We define the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ of T as

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{H}\colon s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T} \text{ is not invertible}\}.$$

The $\mathcal{S}$-resolvent set $\rho_S(T)$ is defined as

$$\rho_{\mathcal{F}}(T) = \mathbb{H} \setminus \sigma_{\mathcal{F}}(T).$$

Theorem 8.8 (structure of the $\mathcal{F}$-spectrum). *Let $T \in \mathcal{BC}(V)$ and let $p = p_0 + p_1 I \in p_0 + p_1\mathbb{S} \subset \mathbb{H} \setminus \mathbb{R}$, such that $p \in \sigma_{\mathcal{F}}(T)$. Then all the elements of the sphere $p_0 + p_1\mathbb{S}$ belong to $\sigma_{\mathcal{F}}(T)$.*

Theorem 8.9 (compactness of the $\mathcal{F}$-spectrum). *Let $T \in \mathcal{BC}(V)$. Then the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ is a compact non-empty set.*

Definition 8.10 ($\mathcal{F}$-resolvent operators). Let $T \in \mathcal{BC}(V)$. For $s \in \rho_{\mathcal{F}}(T)$ we define the left $\mathcal{F}$-resolvent operator as

$$\mathcal{F}^{\mathrm{L}}(s,T) := -4(s\mathcal{I} - \bar{T})(s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T})^{-2},$$

and the right $\mathcal{F}$-resolvent operator as

$$\mathcal{F}^{\mathrm{R}}(s,T) := -4(s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T})^{-2}(s\mathcal{I} - \bar{T}).$$

The definition of the T-admissible set U and of the locally left- and right-slice-regular functions on the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ can be obtained by rephrasing definition 4.5.

We shall denote by $\mathcal{SR}^{\mathrm{L}}_{\sigma_{\mathcal{F}}(T)}$ (respectively, $\mathcal{SR}^{\mathrm{R}}_{\sigma_{\mathcal{F}}(T)}$) the set of locally left (respectively, right) slice regular functions on $\sigma_{\mathcal{F}}(T)$.

Definition 8.11 (quaternionic $\mathcal{F}$-functional calculus for bounded operators). Let $T \in \mathcal{BC}(V)$ and set $\mathrm{d}s_I = \mathrm{d}s/I$. We define the formulations of the quaternionic $\mathcal{F}$-functional calculus as

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}^{\mathrm{L}}(s,T)\,\mathrm{d}s_I f(s), \quad f \in \mathcal{SR}^{\mathrm{L}}_{\sigma_{\mathcal{F}}(T)}, \tag{8.3}$$

and

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(s)\,\mathrm{d}s_I \mathcal{F}^{\mathrm{R}}(s,T), \quad f \in \mathcal{SR}^{\mathrm{R}}_{\sigma_{\mathcal{F}}(T)}, \tag{8.4}$$

where U is T-admissible.

Theorem 8.12 (the quaternionic $\mathcal{F}$-resolvent equation). *Let $T \in \mathcal{BC}(V)$. Then, for $p, s \in \rho_{\mathcal{F}}(T)$, the following equation holds:*

$$\begin{aligned}\mathcal{F}^{\mathrm{R}}(s,T)S^{-1}_{C,\mathrm{L}}(p,T) + S^{-1}_{C,\mathrm{R}}(s,T)\mathcal{F}^{\mathrm{L}}(p,T) + \tfrac{1}{4}(s\mathcal{F}^{\mathrm{R}}(s,T)\mathcal{F}^{\mathrm{L}}(p,T)p \\ - s\mathcal{F}^{\mathrm{R}}(s,T)T\mathcal{F}^{\mathrm{L}}(p,T) - \mathcal{F}^{\mathrm{R}}(s,T)T\mathcal{F}^{\mathrm{L}}(p,T)p + \mathcal{F}^{\mathrm{R}}(s,T)T^2\mathcal{F}^{\mathrm{L}}(p,T)) \\ = [(\mathcal{F}^{\mathrm{R}}(s,T) - \mathcal{F}^{\mathrm{L}}(p,T))p - \bar{s}(\mathcal{F}^{\mathrm{R}}(s,T) - \mathcal{F}^{\mathrm{L}}(p,T))](p^2 - 2s_0p + |s|^2)^{-1},\end{aligned}$$

where the quaternionic $\mathcal{SC}$-resolvent operators are defined as

$$S_{C,\mathrm{L}}^{-1}(s,T) := (s\mathcal{I}-\bar{T})(s^2\mathcal{I}-s(T+\bar{T})+T\bar{T})^{-1} \tag{8.5}$$

and

$$S_{C,\mathrm{R}}^{-1}(s,T) := (s^2\mathcal{I}-s(T+\bar{T})+T\bar{T})^{-1}(s\mathcal{I}-\bar{T}). \tag{8.6}$$

As a consequence of the quaternionic $\mathcal{F}$-resolvent equations, we can study the Riesz projectors associated with the quaternionic $\mathcal{F}$-functional calculus. As a consequence of theorem 5.8 (in this quaternionic version) and of the quaternionic $\mathcal{F}$-resolvent equation we have the following.

Theorem 8.13. *Let $T \in \mathcal{BC}(V)$. Let*

$$\sigma_{\mathcal{F}}(T) = \sigma_{1\mathcal{F}}(T) \cup \sigma_{2\mathcal{F}}(T) \quad \textit{with} \ \operatorname{dist}(\sigma_{1\mathcal{F}}(T), \sigma_{2\mathcal{F}}(T)) > 0.$$

Let U_1 and U_2 be two T-admissible sets such that $\sigma_{1\mathcal{F}}(T) \subset U_1$ and $\sigma_{2\mathcal{F}}(T) \subset U_2$, with $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Set

$$\breve{P}_j := \frac{C}{2\pi} \int_{\partial(U_j \cap \mathbb{C}_I)} \mathcal{F}^{\mathrm{L}}(s,T)\, \mathrm{d}s_I\, s^2, \quad j = 1,2,$$

where $C := \Delta q^2$. Then, for $j = 1,2$, the following properties hold:

(1) $\breve{P}_j^2 = \breve{P}_j$,

(2) $T\breve{P}_j = \breve{P}_j T$.

9. Conclusions and future directions of research

The theory of slice hyperholomorphic functions is the main tool with which to study quaternionic linear operators and n-tuples of linear operators. Its Cauchy formula suggests the definition of the notion of the $\mathcal{S}$-spectrum. In the quaternionic setting, the $\mathcal{S}$-spectrum, and the $\mathcal{F}$-spectrum, which is its commutative version, turned out to be the correct objects to use to study the quaternionic version of spectral analysis. The foundations of this theory are now complete and we can summarize the three main directions.

- The quaternionic version of the $\mathcal{S}$-functional calculus, also called quaternionic functional calculus: this is the quaternionic analogue of the Riesz–Dunford functional calculus.
- The $\mathcal{F}$-functional calculus for quaternionic linear operators: this is a monogenic functional calculus, in the spirit of the one introduced by Alan McIntosh and his collaborators, but it is based on the theory of slice hyperholomorphic functions.
- The spectral theorem for quaternionic normal operators on a quaternionic Hilbert space based on the $\mathcal{S}$-spectrum (see [6, 8, 29]): this is the analogue of the classical spectral theorem for complex normal operators and it plays an important role in the quaternionic formulation of quantum mechanics.

Even though the foundation of quaternionic spectral analysis is understood there are still many problems that have to be investigated from the operator theoretic point of view. Moreover, there are also several problems that are still open regarding the hypercomplex analysis setting that is behind the $\mathcal{F}$-functional calculus and the calculus itself.

The first problem is to show that the Fueter–Sce mapping theorem can be extended to the case of even dimensions. In this case the classical Fueter–Sce mapping theorem has been studied by Tao Qian [38] using the Fourier transform. His approach to extend the integral version of the Fueter–Sce mapping theorem to even dimensions is under investigation. We point out that the even-dimensional case is very different from the odd-dimensional one, because the fractional powers of the Laplacian give rise to non-local operators.

If the Fueter–Sce mapping theorem in integral form can be proved for the even dimensions, we have to extend the $\mathcal{F}$-functional calculus to this case. Finally, we observe that much effort has to be made to understand the structure of the $\mathcal{F}$-resolvent equation for $n \neq 3$. These cases do not seem to be similar to the case $n = 3$, which is also the quaternionic case.

Appendix A. Proof of lemma 5.7

For the proof, we need the following identity.

Proposition A.1. *Let $m \geqslant 0$. Then the following identity holds:*

$$\sum_{k=0}^{j}(-1)^k\binom{m+k-1}{k}\binom{m+j}{m+k} = 1, \quad j = 0, 1, 2, \ldots. \tag{A 1}$$

Proof. Let

$$\Lambda(j,k) := (-1)^k\binom{m+k-1}{k}\binom{m+j}{m+k}.$$

It is easy to check that the following recurrence relation is satisfied:

$$-\frac{m+j+1}{j+2}\Lambda(j,k) + \frac{m+j+1}{j+2}\Lambda(j+1,k) + \frac{m+j+1}{j+2}\Lambda(j,k+1) - \frac{m+2j+3}{j+2}\Lambda(j+1,k+1) + \Lambda(j+2,k+1) = 0$$

for all $j, k \in \mathbb{Z}$. Note that $\Lambda(j,k) = 0$ if $k < 0$ or $k > j$. Thus, by taking the sum over all $k \in \mathbb{Z}$, we obtain

$$-\frac{m+j+1}{j+2}\sum_{k=0}^{j}\Lambda(j,k) + \frac{m+j+1}{j+2}\sum_{k=0}^{j+1}\Lambda(j+1,k) + \frac{m+j+1}{j+2}\sum_{k=-1}^{j-1}\Lambda(j,k+1) - \frac{m+2j+3}{j+2}\sum_{k=-1}^{j}\Lambda(j+1,k+1) + \sum_{k=-1}^{j+1}\Lambda(j+2,k+1) = 0.$$

If we define

$$S(j) := \sum_{k=0}^{j}(-1)^k\binom{m+k-1}{k}\binom{m+j}{m+k},$$

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this equation turns into

$$-\frac{m+j+1}{j+2}S(j)+\frac{m+j+1}{j+2}S(j+1)+\frac{m+j+1}{j+2}S(j)\\ -\frac{m+2j+3}{j+2}S(j+1)+S(j+2)=0,$$

which simplifies to

$$-S(j+1)+S(j+2)=0.$$

Thus, as $S(0)=1$, by induction, we obtain (A 1). □

Proof of lemma 5.7. Let $x\in\mathbb{R}^{n+1}$, and let $U\subset\mathbb{R}^{n+1}$ be a ball centred at 0 such that $x\in U$. By (3.5), for any $I\in\mathbb{S}$, we have

$$\begin{aligned}\mathcal{P}_{n-1,n}(x)&=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)}\mathcal{F}_n^{\mathrm{L}}(s,x)\,\mathrm{d}s_I s^{n-1}\\&=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)}\gamma_n(s-\bar{x})(s^2-2\operatorname{Re}(x)s+|x|^2)^{-(n+1)/2}\,\mathrm{d}s_I s^{n-1}.\end{aligned}$$

Let $I=I_x$ and $m=\frac{1}{2}(n-1)$. Then s and x commute, and by applying the residue theorem we obtain

$$\mathcal{P}_{n-1,n}(x)=\frac{-I\gamma_n}{2\pi}\int_{\partial(U\cup\mathbb{C}_I)}\frac{1}{(s-x)^{m+1}}\frac{1}{(s-\bar{x})^m}s^{2m}\,\mathrm{d}s=\gamma_n(\operatorname{Res}_x(f)+\operatorname{Res}_{\bar{x}}(f)),$$

where

$$f(s):=\frac{1}{(s-x)^{m+1}}\frac{1}{(s-\bar{x})^m}s^{2m}.$$

It is easy to see that

$$\begin{aligned}\frac{\partial^k}{\partial s^k}s^{2m}&=\frac{(2m)!}{(2m-k)!}s^{2m-k},\\ \frac{\partial^k}{\partial s^k}\frac{1}{(s-\bar{x})^m}&=(-1)^k\frac{(m+k-1)!}{(m-1)!}\frac{1}{(s-\bar{x})^{m+k}},\\ \frac{\partial^k}{\partial s^k}\frac{1}{(s-x)^{m+1}}&=(-1)^k\frac{(m+k)!}{m!}\frac{1}{(s-x)^{m+1+k}}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\operatorname{Res}_x(f)&=\frac{1}{m!}\lim_{s\to x}\frac{\partial^m}{\partial s^m}((s-x)^{m+1}f(s))=\frac{1}{m!}\lim_{s\to x}\frac{\partial^m}{\partial s^m}\left(\frac{1}{(s-\bar{x})^m}s^{2m}\right)\\&=\frac{1}{m!}\lim_{s\to x}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{\partial^k}{\partial s^k}\frac{1}{(s-\bar{x})^m}\right)\left(\frac{\partial^{m-k}}{\partial s^{m-k}}s^{2m}\right)\\&=\frac{1}{m!}\lim_{s\to x}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{(m+k-1)!}{(m-1)!}\frac{1}{(s-\bar{x})^{m+k}}\frac{(2m)!}{(m+k)!}s^{m+k}\end{aligned}$$

$$= \sum_{k=0}^{m} (-1)^k \binom{2m}{m-k} \binom{m+k-1}{k} \frac{x^{m+k}}{(x-\bar{x})^{m+k}}$$
$$= \frac{1}{(x-\bar{x})^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m}{m-k} \binom{m+k-1}{k} x^{m+k} (x-\bar{x})^{m-k}.$$

But, as

$$(x-\bar{x})^{m-k} = \sum_{j=0} m-k \binom{m-k}{j} x^j (-\bar{x})^{m-k-j},$$

we have

$$\begin{aligned}
&\mathrm{Res}_x(f) \\
&= \frac{1}{(x-\bar{x})^{2m}} \sum_{k=0}^{m} \sum_{j=0}^{m-k} (-1)^k \binom{2m}{m-k} \binom{m+k-1}{k} \binom{m-k}{j} x^{m+k+j} (-\bar{x})^{m-k-j} \\
&= \frac{1}{(x-\bar{x})^{2m}} \sum_{k=0}^{m} \sum_{j=k}^{m} (-1)^k \binom{2m}{m-k} \binom{m+k-1}{k} \binom{m-k}{j-k} x^{m+j} (-\bar{x})^{m-j} \\
&= \frac{1}{(x-\bar{x})^{2m}} \sum_{j=0}^{m} \sum_{k=0}^{j} (-1)^k \binom{2m}{m-k} \binom{m+k-1}{k} \binom{m-k}{j-k} x^{m+j} (-\bar{x})^{m-j}.
\end{aligned}$$

For the coefficients, we obtain

$$\begin{aligned}
&\sum_{k=0}^{j} (-1)^k \binom{2m}{m-k} \binom{m+k-1}{k} \binom{m-k}{j-k} \\
&\qquad = \sum_{k=0}^{j} (-1)^k \frac{(2m)!}{(m-k)!(m+k)!} \frac{(m+k-1)!}{k!(m-1)!} \frac{(m-k)!}{(j-k)!(m-j)!} \\
&\qquad = \frac{(2m)!}{(m-j)!(m+j)!} \sum_{k=0}^{j} (-1)^k \frac{(m+k-1)!}{k!(m-1)!} \frac{(m+j)!}{(j-k)!(m+k)!} \\
&\qquad = \binom{2m}{m+j} \sum_{k=0}^{j} (-1)^k \binom{m+k-1}{k} \binom{m+j}{m+k} = \binom{2m}{m+j},
\end{aligned}$$

where the last equation follows from (A 1). Therefore, we finally get

$$\begin{aligned}
\mathrm{Res}_x(f) &= \frac{1}{(x-\bar{x})^{2m}} \sum_{j=0}^{m} \binom{2m}{m+j} x^{m+j} \bar{x}^{m-j} \\
&= \frac{1}{(x-\bar{x})^{2m}} \sum_{j=m}^{2m} \binom{2m}{j} x^j \bar{x}^{2m-j}.
\end{aligned}$$

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Similarly, we have

$$\begin{aligned}\operatorname{Res}_{\bar{x}}(f) &= \frac{1}{(m-1)!}\lim_{s\to\bar{x}}\frac{\partial^{m-1}}{\partial s^{m-1}}((s-\bar{x})^m f(s))\\ &= \frac{1}{(m-1)!}\lim_{s\to\bar{x}}\frac{\partial^{m-1}}{\partial s^{m-1}}\left(\frac{1}{(s-x)^{m+1}}s^{2m}\right)\\ &= \frac{1}{(m-1)!}\lim_{s\to\bar{x}}\sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{\partial^k}{\partial s^k}\frac{1}{(s-x)^{m+1}}\right)\left(\frac{\partial^{m-1-k}}{\partial s^{m-1-k}}s^{2m}\right)\end{aligned}$$

and also

$$\begin{aligned}&\operatorname{Res}_{\bar{x}}(f)\\ &= \frac{1}{(m-1)!}\lim_{s\to\bar{x}}\sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^k\frac{(m+k)!}{m!}\frac{1}{(s-x)^{m+1+k}}\frac{(2m)!}{(m+k+1)!}s^{m+k+1}\\ &= \sum_{k=0}^{m-1}(-1)^k\binom{2m}{m-k-1}\binom{m+k}{k}\frac{\bar{x}^{m+k+1}}{(\bar{x}-x)^{m+1+k}}\\ &= \frac{1}{(x-\bar{x})^{2m}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{m-k-1}\binom{m+k}{k}(-\bar{x})^{m+k+1}(x-\bar{x})^{m-k-1}.\end{aligned}$$

As we have

$$(x-\bar{x})^{m-k-1} = \sum_{j=0}^{m-k-1}\binom{m-k-1}{j}x^j(-\bar{x})^{m-k-1-j},$$

this equals

$$\begin{aligned}&\operatorname{Res}_{\bar{x}}(f)\\ &= \frac{1}{(x-\bar{x})^{2m}}\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1}(-1)^k\binom{2m}{m-k-1}\binom{m+k}{k}\binom{m-k-1}{j}x^j(-\bar{x})^{2m-j}\\ &= \frac{1}{(x-\bar{x})^{2m}}\sum_{j=0}^{m-1}\sum_{k=0}^{m-j-1}(-1)^k\binom{2m}{m-k-1}\binom{m+k}{k}\binom{m-k-1}{j}x^j(-\bar{x})^{2m-j}.\end{aligned}$$

For the coefficients, we again obtain

$$\begin{aligned}&\sum_{k=0}^{m-j-1}(-1)^k\binom{2m}{m-k-1}\binom{m+k}{k}\binom{m-k-1}{j}\\ &\qquad= \sum_{k=0}^{m-j-1}(-1)^k\frac{(2m)!}{(m-k-1)!(m+k+1)!}\frac{(m+k)!}{k!m!}\frac{(m-k-1)!}{j!(m-k-j-1)!}\end{aligned}$$

$$= \frac{2m!}{j!(2m-j)!} \sum_{k=0}^{m-j-1} (-1)^k \frac{(2m-j)!}{(m+k+1)!(m-k-j-1)!} \frac{(m+k)!}{k!m!}$$

$$= \binom{2m}{j} \sum_{k=0}^{m-j-1} (-1)^k \binom{2m-j}{m+k+1} \binom{m+k}{k} = \binom{2m}{j},$$

where the last equation again follows from (A 1). Thus, we finally obtain

$$\operatorname{Res}_{\bar{x}}(f) = \frac{1}{(x-\bar{x})^{2m}} \sum_{j=0}^{m-1} \binom{2m}{j} x^j (-\bar{x})^{2m-j}.$$

Putting all this together, we get

$$\begin{aligned} \mathcal{P}_{n-1,n}(x) & \\ &= \gamma_n (\operatorname{Res}_x(f) + \operatorname{Res}_{\bar{x}}(f)) \\ &= \gamma_n \left(\frac{1}{(x-\bar{x})^{2m}} \sum_{j=m}^{2m} \binom{2m}{j} x^j \bar{x}^{2m-j} + \frac{1}{(x-\bar{x})^{2m}} \sum_{j=0}^{m-1} \binom{2m}{j} x^j (-\bar{x})^{2m-j} \right) \\ &= \frac{\gamma_n}{(x-\bar{x})^{2m}} \sum_{j=0}^{2m} \binom{2m}{j} x^j \bar{x}^{2m-j} = \frac{\gamma_n}{(x-\bar{x})^{2m}} (x-\bar{x})^{2m} = \gamma_n. \end{aligned}$$

□

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References

1 D. Alpay. *The Schur algorithm, reproducing kernel spaces and system theory* (English transl.) (Providence, RI: American Mathematical Society, 2001).

2 D. Alpay, F. Colombo and I. Sabadini. Schur functions and their realizations in the slice hyperholomorphic setting. *Integ. Eqns Operat. Theory* **72** (2012), 253–289.

3 D. Alpay, F. Colombo and I. Sabadini. Pontryagin–de Branges–Rovnyak spaces of slice hyperholomorphic functions. *J. Analyse Math.* **121** (2013), 87–125.

4 D. Alpay, F. Colombo and I. Sabadini. On some notions of convergence for n-tuples of operators. *Math. Meth. Appl. Sci.* **37** (2014), 2363–2371.

5 D. Alpay, F. Colombo and I. Sabadini. Krein–Langer factorization and related topics in the slice hyperholomorphic setting. *J. Geom. Analysis* **24** (2014), 843–872.

6 D. Alpay, F. Colombo and D. P. Kimsey. The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum. *J. Math. Phys.* (2016), 023503.

7 D. Alpay, F. Colombo, J. Gantner and I. Sabadini. A new resolvent equation for the $\mathcal{S}$-functional calculus. *J. Geom. Analysis* **25** (2015), 1939–1968.

8 D. Alpay, F. Colombo, D. P. Kimsey and I. Sabadini. The spectral theorem for unitary operators based on the S-spectrum. *Milan J. Math.* (2015), doi:10.1007/s00032-015-0249-7.

9 R. F. V. Anderson. The Weyl functional calculus. *J. Funct. Analysis* **4** (1969), 240–267.

10 F. Brackx, R. Delanghe and F. Sommen. *Clifford analysis*. Pitman Research Notes in Mathematics, vol. 76 (London: Longman, 1982).

11 F. Colombo and I. Sabadini. A structure formula for slice monogenic functions and some of its consequences. In *Hypercomplex analysis*, Trends in Mathematics, pp. 101–114 (Birkhäuser, 2009).

12 F. Colombo and I. Sabadini. The Cauchy formula with s-monogenic kernel and a functional calculus for noncommuting operators. *J. Math. Analysis Applic.* **373** (2011), 655–679.

13 F. Colombo and I. Sabadini. The $\mathcal{F}$-spectrum and the $\mathcal{SC}$-functional calculus. *Proc. R. Soc. Edinb.* A **142** (2012), 479–500.

14 F. Colombo and I. Sabadini. The $\mathcal{F}$-functional calculus for unbounded operators. *J. Geom. Phys.* **86** (2014), 392–407.

15 F. Colombo, I. Sabadini, F. Sommen and D. C. Struppa. *Analysis of Dirac systems and computational algebra*. Progress in Mathematical Physics, vol. 39 (Birkhäuser, 2004).

16 F. Colombo, I. Sabadini and D. C. Struppa. A new functional calculus for noncommuting operators. *J. Funct. Analysis* **254** (2008), 2255–2274.

17 F. Colombo, I. Sabadini and D. C. Struppa. Slice monogenic functions. *Israel J. Math.* **171** (2009), 385–403.

18 F. Colombo, I. Sabadini and F. Sommen. The Fueter mapping theorem in integral form and the $\mathcal{F}$-functional calculus. *Math. Meth. Appl. Sci.* **33** (2010), 2050–2066.

19 F. Colombo, I. Sabadini and D. C. Struppa. An extension theorem for slice monogenic functions and some of its consequences. *Israel J. Math.* **177** (2010), 369–389.

20 F. Colombo, I. Sabadini and F. Sommen. The inverse Fueter mapping theorem. *Commun. Pure Appl. Analysis* **10** (2011), 1165–1181.

21 F. Colombo, I. Sabadini and D. C. Struppa. *Noncommutative functional calculus: theory and applications of slice hyperholomorphic functions*. Progress in Mathematics, vol. 289 (Birkhäuser, 2011).

22 F. Colombo, J. O. Gonzales-Cervantes and I. Sabadini. A nonconstant coefficients differential operator associated to slice monogenic functions. *Trans. Am. Math. Soc.* **365** (2013), 303–318.

23 F. Colombo, I. Sabadini and F. Sommen. The inverse Fueter mapping theorem in integral form using spherical monogenics. *Israel J. Math.* **194** (2013), 485–505.

24 F. Colombo, I. Sabadini and F. Sommen. The Fueter primitive of biaxially monogenic functions. *Commun. Pure Appl. Analysis* **13** (2014), 657–672.

25 R. Delanghe, F. Sommen and V. Soucek. *Clifford algebra and spinor-valued functions: a function theory for the Dirac operator*. Mathematics and Its Applications, vol. 53 (Dordrecht: Kluwer Academic, 1992).

26 N. Dunford and J. Schwartz. *Linear operators, part I: general theory* (Wiley, 1988).

27 R. Fueter. Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen. *Comment. Math. Helv.* **7** (1934), 307–330.

28 R. Ghiloni, V. Moretti and A. Perotti. Continuous slice functional calculus in quaternionic Hilbert spaces. *Rev. Math. Phys.* **25** (2013), 1350006.

29 R. Ghiloni, V. Moretti and A. Perotti. Spectral properties of compact normal quaternionic operators. In *Hypercomplex analysis: new perspectives and applications trends in mathematics*, pp. 133–143 (Springer, 2014).

30 K. Gürlebeck, K. Habetha and W. Sprößig. *Holomorphic functions in the plane and n-dimensional space* (Birkhäuser, 2008).

31 B. Jefferies. *Spectral properties of noncommuting operators*. Lecture Notes in Mathematics, vol. 1843 (Springer, 2004).

32 B. Jefferies and A. McIntosh. The Weyl calculus and Clifford analysis. *Bull. Austral. Math. Soc.* **57** (1998), 329–341.

33 B. Jefferies, A. McIntosh and J. Picton-Warlow. The monogenic functional calculus. *Studia Math.* **136** (1999), 99–119.

34 K. I. Kou, T. Qian and F. Sommen. Generalizations of Fueter's theorem. *Meth. Applic. Analysis* **9** (2002), 273–290.

35 A. McIntosh and A. Pryde. A functional calculus for several commuting operators. *Indiana Univ. Math. J.* **36** (1987), 421–439.

36 D. Peña-Peña. Cauchy–Kowalevski extensions, Fueter's theorems and boundary values of special systems in Clifford analysis. PhD thesis, University of Gent, 2008.

37 D. Peña-Peña, T. Qian and F. Sommen. An alternative proof of Fueter's theorem. *Complex Var. Elliptic Eqns* **51** (2006), 913–922.

38 T. Qian. Generalization of Fueter's result to $\mathbb{R}^{n+1}$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **8** (1997), 111–117.

39 T. Qian. Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space. *Math. Annalen* **310** (1998), 601–630.

40 T. Qian. Fourier analysis on starlike Lipschitz surfaces. *J. Funct. Analysis* **183** (2001), 370–412.

41 M. Sce. Osservazioni sulle serie di potenze nei moduli quadratici. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **23** (1957), 220–225.

42 F. Sommen. On a generalization of Fueter's theorem. *Z. Analysis Anwend.* **19** (2000), 899–902.

43 J. L. Taylor. A joint spectrum for several commuting operators. *J. Funct. Analysis* **6** (1970), 172–191.

44 J. L. Taylor. The analytic-functional calculus for several commuting operators. *Acta Math.* **125** (1970), 1–38.

45 J. L. Taylor. A general framework for a multi-operator functional calculus. *Adv. Math.* **9** (1972), 183–252.

46 M. E. Taylor. Functions of several self-adjoint operators. *Proc. Am. Math. Soc.* **19** (1968), 91–98.