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EMBEDDING THEOREM OF THE WEIGHTED SOBOLEV–LORENTZ SPACES*

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Abstract. Weight criteria for embedding of the weighted Sobolev–Lorentz spaces to the weighted Besov–Lorentz spaces built upon certain mixed norms and iterated rearrangement are investigated. This gives an improvement of some known Sobolev embedding. We achieve the result based on different norm inequalities for the weighted Besov–Lorentz spaces defined in some mixed norms.

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1. Introduction. This paper is dedicated to the study of some inequalities for functions in the Sobolev–Lorentz spaces and Besov–Lorentz spaces.

The Sobolev space $W_p^1(\mathbb{R}^n)(1 \le p < \infty)$ is defined as the class of all functions $f \in L^p(\mathbb{R}^n)$ for which every first-order weak derivative exists and belongs to $L^p(\mathbb{R}^n)$. The classical Sobolev theorem (see [**31**, Chapter V]) illustrates as follows.

THEOREM 1.1. *Let $n \ge 2$, $1 \le p < n$, and $p^* = np/(n-p)$. Then for any $f \in W_p^1(\mathbb{R}^n)$*

$$\|f\|_{p^*} \le C\|\nabla f\|_p. \tag{1.1}$$

The Lebesgue norm at the left side of (1.1) can be substituted with the stronger Lorentz norm, i.e., for any $f \in W_p^1(\mathbb{R}^n)$, $n \ge 2$, $1 \le p < n$,

$$\|f\|_{p^*,p} \le C\|\nabla f\|_p \tag{1.2}$$

(see [**1**, **25**, **26**, **29**]).

Let a function f be defined on $\mathbb{R}^n$, $k \in \{1, \ldots, n\}$ and

$$\Delta_k(h)f(x) = f(x + he_k) - f(x), \ x \in \mathbb{R}^n, \ h \in \mathbb{R}$$

(e_k is the kth unit coordinate vector). The following holds.

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THEOREM 1.2. *([13, 15, 16]) Let $n \in \mathbb{N}$. Assume that $1 < p < \infty$ and $n \geq 1$ or $p = 1$ and $n \geq 2$. If $p < q < \infty$ and $s = 1 - n(1/p - 1/q) > 0$, then for any $f \in W_p^1(\mathbb{R}^n)$*

$$\sum_{k=1}^{n} \left(\int_0^\infty h^{-sp} \|\Delta_k(h)f\|_{q,p}^p \frac{dh}{h} \right)^{1/p} \leq C \sum_{k=1}^{n} \|D_k f\|_p. \tag{1.3}$$

Let $x = (x_1, \ldots, x_n)$. Denote by $\hat{x}_k$ the $(n-1)$-dimensional vector obtained from the n-tuple x by removal of its kth coordinate. Write $x = (x_k, \hat{x}_k)$.

If $X(\mathbb{R})$ and $Y(\mathbb{R}^{n-1})$ are Banach function spaces (see [5]), and $k \in \{1, \ldots, n\}$, we denote by $Y[X]_k$ the mixed norm space obtained by taking first the norm in X with respect to x_k, and then the norm in Y with respect to $\hat{x}_k \in \mathbb{R}^{n-1}$. As a rule, $f \downarrow$ indicates that f is a nonnegative decreasing function in $\mathbb{R}_+$.

Kolyada [19] refined Theorem 1.2 by using mixed norm replacing the Lorentz spaces.

THEOREM 1.3. *Let $n \in \mathbb{N}$. Assume that $1 < p < \infty$ and $n \geq 2$. If $p < q < \infty$ and $\alpha = 1 - (n-1)(1/p - 1/q) > 0$, then for any $f \in W_p^1(\mathbb{R}^n)$*

$$\sum_{k=1}^{n} \left(\int_0^\infty h^{-\alpha p} \|\Delta_k(h)f\|_{L^{q,p}[L^p]_k} \frac{dh}{h} \right)^{1/p} \leq C \sum_{k=1}^{n} \|D_k f\|_p.$$

The Sobolev embedding can refer to, for instance, [16, 21, 23, 24, 27, 28, 33, 34] and their references.

In this paper, we consider the improvement and generalization of Sobolev-embedding Theorem 1.3 through introducing weighted Lorentz spaces $\Lambda^{p,q}(w)$ and multidimensional Lorentz spaces $\Lambda^p_{x_j \hat{x}_j}(v)$. The paper is organized as follows. Some definitions and auxiliary results are given in Section 2. In Section 3, we prove inequalities between Besov norms built upon the spaces $\Lambda^{p,\nu}(w)(\mathbb{R}^n)$ and $\Lambda^{p,\nu}(w)(\mathbb{R}^{n-1})[L^r(\mathbb{R})]$. In Section 4, we prove the main result, Theorem 4.1.

Throughout the paper, we agree on the convention that the expressions of the form $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are equal to zero.

2. Preliminaries. Let (X, μ) be a σ-finite measure space and $\mathcal{M}(X, \mu)$ be the space of all μ-measurable real-valued functions on X. For an $f \in \mathcal{M}(X, \mu)$, its decreasing rearrangement f_μ^* is defined by [5]

$$f_\mu^*(t) = \inf\{s : \lambda_f^\mu(s) \leq t\}, \quad t \geq 0,$$

where

$$\lambda_f^\mu(s) = \mu\{x \in X : |f(x)| > s\}, \ s \geq 0$$

is the distribution function of f. The function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is called a weight function, or simply a weight, whenever w is Lebesgue measurable, not identically equal to zero and integrable on sets of finite measure. If w is a weight on $\mathbb{R}_+$, then we denote $W(t) = \int_0^t w(s)\, ds$ and we easily check that $W(t) < \infty$ for all $t > 0$. Let u be a weight on $\mathbb{R}^n$ or $\mathbb{R}_+$. If $(X, \mu) = (\mathbb{R}^n, dx)$, then we denote $\lambda_f^\mu = \lambda_f$, $f_\mu^* = f^*$ and $\mu(E) = |E|$ for every Lebesgue measurable subset E of $\mathbb{R}^n$. Especially, we write $f_u^* = f^*$ and $\lambda_f^u = \lambda_f$ if $u = 1$. In what follows the operator A is defined by for any nonnegative measurable function f on $\mathbb{R}_+$

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$$Af(t)=\frac{1}{t}\int_0^t f(s)ds$$

and write

$$f^{**}(t)=Af^*(t),\quad t>0.$$

Let $0<p,\ q<\infty$. We say that $f\in\mathcal{M}(X,\mu)$ belongs to the Lorentz space $L^{p,q}(X)$ [5, 14] if

$$\|f\|_{L^{p,q}(X)}=\left(\int_0^\infty (t^{1/p}f_\mu^*(t))^q\frac{dt}{t}\right)^{1/q}<\infty.$$

If $(X,\mu)=(\mathbb{R}^n,dx)$, we use the notation $L^{p,q}(X)=L^{p,q}$.

Let w be a weight on $\mathbb{R}_+$. Define, for $0<p,\ q<\infty$, the weighted Lorentz space $\Lambda_X^{p,q}(w)$ (see [7, 9]) as a class of $f\in\mathcal{M}(X,\mu)$ such that

$$\|f\|_{\Lambda_X^{p,q}(w)}=\|f_\mu^*\|_{L^{p,q}(w)}<\infty.$$

Also, the weighted Lorentz space $\Lambda_X^{p,\infty}(w)$ consists of all $f\in\mathcal{M}(X,\mu)$ satisfying

$$\|f\|_{\Lambda_X^{p,\infty}(w)}=\|f_\mu^*\|_{L^{p,\infty}(w)}<\infty.$$

Denote $\Lambda_X^p(w)=\Lambda_X^{p,p}(w)$. Note that if we use the notation

$$\|g\|_{L^q(\frac{dy}{y})}=\left(\int_0^\infty |g(y)|^q\frac{dy}{y}\right)^{1/q},$$

then for $0<p,\ q<\infty$,

$$\|f\|_{\Lambda_X^{p,q}(w)}=p^{1/q}\left\|y\left(\int_0^{\lambda_f^\mu(y)}w(t)dt\right)^{\frac{1}{p}}\right\|_{L^q(\frac{dy}{y})},$$

and $\Lambda_X^{p,q}(w)=\Lambda_X^q(\bar{w})$, where $\bar{w}=W^{\frac{q}{p}-1}w$.

If $(X,\mu)=(\mathbb{R}^n,dx)$, denote $\Lambda_X^{p,q}(w)=\Lambda^{p,q}(w)$. By the definition, it is obvious $\Lambda_X^{p,q}(1)=L^{p,q}(X)$. In the remaining of this paper, without loss of generality, we always assume that weights w in $\mathbb{R}_+$ vanish on the interval $[\mu(X),\infty)$ if $\mu(X)<\infty$.

Let $L^p_{dec}(w)$ be the cone of all decreasing functions in $L^p(w)$, $0<p<\infty$. Ariño and Muckenhoupt [2] gave a characterization of the boundedness of $A:\ L^p_{dec}(w)\to L^p(w)$ in terms of the inequality on w called condition B_p. Carro and Soria [8] obtained similar characterization of boundedness of $A:\ L^p_{dec}(w)\to L^{p,\infty}(w)$, showing that A is bounded whenever $w\in B_{p,\infty}$. It is worth indicating that $B_p=B_{p,\infty}$ if $p>1$ and $w\in B_{1,\infty}$ if w is decreasing. We know in [9, Theorem 2.5.8] that $\Lambda_X^p(w)$ is normable, that is there exists a norm in $\Lambda_X^p(w)$ equivalent to the expression $\|\cdot\|_{\Lambda_X^p(w)}$, if and only if $p\geq 1$ and $w\in B_{p,\infty}$. We say that a function $G:[0,\infty)\to[0,\infty)$ satisfies condition Δ_2, in symbol $G\in\Delta_2$, whenever $\sup_{t>0}G(2t)/G(t)<\infty$.

The following is the classical Hardy's inequality [5].

PROPOSITION 2.1. *Let* $-\infty<\lambda<1$ *and* $1\leq p<\infty$. *Then*

$$\left(\int_0^\infty\left(t^{\lambda-1}\int_0^t\phi(u)du\right)^p\frac{dt}{t}\right)^{1/p}\leq\frac{1}{1-\lambda}\left(\int_0^\infty(t^\lambda\phi(u))^p\frac{dt}{t}\right)^{1/p}. \tag{2.1}$$

for all $\phi\geq 0$.

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There are many generalized forms for $p \geq 1$. We can refer to, e.g., [11, 12, 22].

PROPOSITION 2.2. *Let $1 \leq p < \infty$, v be a nonnegative measurable function on $\mathbb{R}_+$. Then*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p v(x)dx\right)^{1/p} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{1/p}, \quad \textit{for all } f \geq 0 \tag{2.2}$$

if and only if

$$A := \sup_{a>0}\left(\int_0^a v(x)^{1-p'}dx\right)^{1/p'}\left(\int_a^\infty \frac{v(x)}{x^p}dx\right)^{1/p} < \infty,$$

and moreover, $C_0 \approx A$, where C_0 is the least possible constant C in (2.2).

PROPOSITION 2.3. *Let $1 \leq p < \infty$, v be a nonnegative measurable function on $\mathbb{R}_+$. Then*

$$\left(\int_0^\infty \left(\int_x^\infty \frac{f(t)}{t}dt\right)^p v(x)dx\right)^{1/p} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{1/p}, \quad \textit{for all } f \geq 0 \tag{2.3}$$

if and only if

$$B := \sup_{a>0}\left(\int_a^\infty \frac{v(x)^{1-p'}}{x^{p'}}dx\right)^{1/p'}\left(\int_0^a v(x)dx\right)^{1/p} < \infty,$$

and moreover, $C_1 \approx B$, where C_1 is the least possible constant C in (2.3).

We say that a measurable function ϕ on $(0, \infty)$ is quasi-decreasing if there exists a constant $c > 0$ such that $\phi(t_1) \leq c\phi(t_2)$, whenever $0 < t_2 < t_1 < \infty$. It is well known that in the case $0 < p < 1$, Hardy-type inequalities hold true for quasi-decreasing functions [20, Proposition 2.2]. The next Proposition generalize it by replacing power functions with nonnegative measurable functions.

PROPOSITION 2.4. *Suppose also that nonnegative functions v, w satisfy $0 < \int_0^u v(s)ds < \infty$ and $\int_u^\infty w(s)ds < \infty$ for all $u > 0$. Let ψ be a nonnegative, quasi-decreasing function on $(0, \infty)$ and $0 < p < 1$. Then*

$$\int_0^\infty w(u)\left(\int_0^u \psi(t)v(t)dt\right)^p du \leq C\int_0^\infty \psi(u)^p\left(\int_0^u v(t)dt\right)^{p-1}\left(\int_u^\infty w(s)ds\right)v(u)du. \tag{2.4}$$

Proof. Let $\int_0^u v(s)ds = V(u)$ and $\int_u^\infty w(u)du = W(u)$. Suppose that $\int_0^b V(u)^{p-1}W(u)v(u)du < \infty$ for some $b > 0$, or the integral on the right-hand side of (2.4) diverges. Then,

$$\begin{aligned} V(a)^p W(a) &\leq V(a)^{p-1}\int_0^a W(u)v(u)du \\ &\leq \int_0^a V(u)^{p-1}W(u)v(u)du \to 0, \quad \textit{as } a \to 0, \end{aligned} \tag{2.5}$$

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i.e., $\lim_{a\to 0} V(a)^p W(a) = 0$. Also, we can suppose that ψ is bounded and compactly supported. Let $F(u) = \int_0^u \psi(t)v(t)dt$. Thus integrating by parts together, (2.5) gives

$$\begin{aligned}\int_0^\infty w(u)F(u)^p du &= p\int_0^\infty W(u)F(u)^{p-1}\psi(u)v(u)du \\ &\leq C\int_0^\infty \psi(u)^p V(u)^{p-1}W(u)v(u)du,\end{aligned}$$

as anticipated. □

3. Some norm inequalities. Let $0 < \alpha < 1$, $1 \leq p$, $\theta < \infty$. The Besov space $B^\alpha_{p,\theta}$ consists all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{B^\alpha_{p,\theta}} = \|f\|_p + \sum_{k=0}^{n}\left(\int_0^\infty [h^{-\alpha}\|\Delta_k(h)f\|_p]^\theta \frac{dh}{h}\right)^{1/\theta} < \infty.$$

The classical different norm embedding theorem is that if $1 \leq p < q < \infty$ and $\alpha > n(1/p - 1/q)$, then for $1 \leq \theta < \infty$,

$$B^\alpha_{p,\theta}(\mathbb{R}^n) \subset B^\beta_{q,\theta}(\mathbb{R}^n), \quad \textit{where } \beta = \alpha - n(1/p - 1/q),$$

and for $f \in B^\alpha_{p,\theta}(\mathbb{R}^n)$,

$$\|f\|_{B^\beta_{q,\theta}} \leq C\|f\|_{B^\alpha_{p,\theta}}$$

[24, Chapter 6].

In this section, we are especially interested in the one-dimensional case of embedding theorem. We shall get different norm inequalities for the Besov spaces defined in some mixed norms which have some connections with embeddings of Sobolev spaces.

Let $\phi \in L^p(\mathbb{R})$ and set

$$\Delta\phi(h)(x) = \phi(x+h) - \phi(x), \quad x \in \mathbb{R},$$

and

$$\omega(f;\delta)_p = \sup_{|h|\leq\delta} \|\Delta(h)f\|_p, \quad \delta \geq 0.$$

Ul'yanov [35] achieved that if $1 \leq p < q < \infty$ and $\phi \in L^p(\mathbb{R})$,

$$\phi^*(t) \leq 2\int_t^\infty s^{-1/p}\omega(\phi;s)_p \frac{ds}{s}, \tag{3.1}$$

$$\|\phi\|_q \leq C\left(\int_0^\infty t^{-q/p}\|\Delta(t)\phi\|_p^q dt\right)^{1/q}, \tag{3.2}$$

and

$$\omega(\phi;\delta)_q \leq C\left(\int_0^\delta t^{-q/p}\|\Delta(t)\phi\|_p^q dt\right)^{1/q}. \tag{3.3}$$

See also [17, 18, 32].

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Let a function f be defined on $\mathbb{R}^n$, $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$ and denote

$$\Delta_k(h)f(x, y) = f(x+h, y) - f(x, y), \ \ h \in \mathbb{R},$$

and

$$\omega_1(f;\delta)_V = \sup_{|h|\le\delta} \|\Delta_1(h)f\|_V, \quad \delta \ge 0.$$

Let $V = V(\mathbb{R}^n)$ be a Banach function space over $\mathbb{R}^n$ (see [5, Chapter 1]) and be translation invariant, that is, $\|\tau_t f\|_V = \|f\|_V$ for all $t \in \mathbb{R}^n$, where $\tau_t f(x) = f(x+t)$. Then, it is easy to see that

$$\omega_1(f;\delta)_V \le \frac{C}{\delta}\int_0^\delta \|\Delta_1(h)f\|_V dh, \tag{3.4}$$

for any $\delta > 0$.

Let $1 \le \theta < \infty$, $0 < \alpha < 1$. Let $V = V(\mathbb{R}^n)(n \ge 2)$ be a translation invariant Banach function space. Denote by $B^\alpha_{\theta;1}(V)$ the class of all functions $f \in V$ such that

$$\|f\|_{B^\alpha_{\theta,1}(V)} = \|f\|_V + \left(\int_0^\infty [h^{-\alpha}\omega_1(f;h)_V]^\theta \frac{dh}{h}\right)^{1/\theta} < \infty.$$

In the light of (3.4) and Hardy's inequality (2.1),

$$\int_0^\infty [h^{-\alpha}\omega_1(f;h)_V]^\theta \frac{dh}{h} \le C\int_0^\infty [h^{-\alpha}\|\Delta_1(h)f\|_V]^\theta \frac{dh}{h}. \tag{3.5}$$

In this section, we write simply $Y[X]$ to represent $Y[X]_1$ (omitting the subindex 1) since the interior norm will be taken only in the first variable.

Recall that $\Lambda^p_{\mathbb{R}^n}(w) = L^p(\mathbb{R}^n)$ if $w = 1$ and thus the next proposition is a generalization of [19, Proposition 3.1] for embedding of Besov spaces.

PROPOSITION 3.1. *Let* $1 \le \theta < \infty$, $1 \le r < p < \infty$ *and* $1/r - 1/p < \alpha < 1$, $\beta = \alpha - 1/r + 1/p$ *and* $w \downarrow$. *Then* $B^\alpha_{\theta,1}(\Lambda^p(w)[L^r]) \subset B^\beta_{\theta,1}(\Lambda^p(w)[L^p])$, *more exactly, for any* $f \in B^\alpha_{\theta,1}(\Lambda^p(w)[L^r])$,

$$\|f\|_{\Lambda^p(w)[L^p]} \le C\|f\|_{B^\alpha_{\theta,1}(\Lambda^p(w)[L^r])} \tag{3.6}$$

and

$$\int_0^\infty h^{-\theta\beta}\|\Delta_1(h)f\|^\theta_{\Lambda^p(w)[L^p]}\frac{dh}{h} \le C\int_0^\infty h^{-\theta\alpha}\|\Delta_1(h)f\|^\theta_{\Lambda^p(w)[L^r]}\frac{dh}{h}. \tag{3.7}$$

Proof. Denote $V = \Lambda^p(w)[L^r]$. Let $f \in B^\alpha_{\theta,1}(V)$. For $y \in \mathbb{R}^{n-1}$, set $f_y(x) = f(x, y)$, $x \in \mathbb{R}$. Due to $1 \le r < p < \infty$, (3.2) leads to

$$\|f_y\|_p^p \le C\int_0^\infty t^{-p/r}\|\Delta(t)f_y\|_r^p dt. \tag{3.8}$$

Since $w \downarrow$, we get [5, Theorem 2.7.5]

$$\|f\|^p_{\Lambda^p(w)[L^p]} = \sup_\rho \int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}|f(x,y)|^p dx\right) w(\rho(y))dy$$

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where the supremum is taken over all measure preserving transformations $\rho : \mathbb{R}^{n-1} \to \mathbb{R}_+$. Thus,

$$\begin{aligned}
\|f\|^p_{\Lambda^p(w)[L^p]} &\leq C \sup_\rho \int_{\mathbb{R}^{n-1}} \int_0^\infty t^{-p/r} \|\Delta(t) f_y\|^p_r dt w(\rho(y)) dy, \ \ by\ (3.8) \\
&= C \sup_\rho \int_0^\infty t^{-p/r} dt \int_{\mathbb{R}^{n-1}} \|\Delta(t) f_y\|^p_r w(\rho(y)) dy \\
&\leq C \int_0^\infty t^{-p/r} dt \int_0^\infty \|\Delta(t) f_y\|^{*p}_r (t) w(t) dt \\
&= C \int_0^\infty t^{-p/r} \|\Delta_1(t) f\|^p_V dt.
\end{aligned}$$

By [5, Chapter 5.4, Theorem 4.6], we obtain

$$\left(\int_0^\infty t^{-p/r} \|\Delta_1(t) f\|^p_V dt \right)^{1/p} \leq C \left[\|f\|_V + \left(\int_0^1 t^{-\alpha\theta} \|\Delta_1(t) f\|^\theta_V \frac{dt}{t} \right)^{1/\theta} \right].$$

Thus, (3.6) holds.

Further, inequality (3.3) implies that ($r < p$)

$$\|\Delta(h) f_y\|^p_p \leq C \int_0^h \|\Delta(t) f_y\|^p_r t^{-p/r} dt.$$

Thus,

$$\begin{aligned}
\|\Delta_1(h) f\|^p_{\Lambda^p(w)[L^p]} &= C \sup_\rho \int_{\mathbb{R}^{n-1}} \|\Delta(h) f_y\|^p_p w(\rho(y)) dy, \ \ w \downarrow \\
&\leq C \sup_\rho \int_0^h t^{-p/r} dt \int_{\mathbb{R}^{n-1}} \|\Delta(t) f_y\|^p_r w(\rho(y)) dy \\
&\leq C \int_0^h t^{-p/r} dt \int_{\mathbb{R}^{n-1}} (\|\Delta(t) f_y\|_r)^{*p}(s) w(s) ds \\
&= C \int_0^h t^{-p/r} \|\Delta_1(t) f\|^p_V dt.
\end{aligned}$$

Therefore,

$$\int_0^\infty h^{-\theta\beta} \|\Delta_1(h) f\|^\theta_{\Lambda^p(w)[L^p]} \frac{dh}{h} \leq C \int_0^\infty h^{-\theta\beta} \left(\int_0^h t^{-p/r} \|\Delta_1(t) f\|^p_V dt \right)^{\theta/p} \frac{dh}{h}.$$

If $\theta \geq p$, applying Proposition 2.1, we get (3.7). Let $\theta < p$. Noting that $\psi(t) = \frac{\omega_1(f;t)_V}{t}$ is quasi-decreasing, we have by Proposition 2.4 and (3.5)

$$\begin{aligned}
\int_0^\infty h^{-\theta\beta} \|\Delta_1(h) f\|^\theta_{\Lambda^p(w)[L^p]} \frac{dh}{h} &\leq C \int_0^\infty h^{-\theta\beta} \left(\int_0^h t^{-p/r} \omega_1(f;t)^p_V dt \right)^{\theta/p} \frac{dh}{h} \\
&\leq C \int_0^\infty h^{-\theta\alpha} \omega_1(f;h)^\theta_V \frac{dh}{h}
\end{aligned}$$

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$$\leq C\int_0^\infty h^{-\theta\alpha}\left(\frac{1}{h}\int_0^h \|\Delta_1(t)f\|_V dt\right)^\theta \frac{dh}{h}$$
$$\leq C\int_0^\infty h^{-\theta\alpha}\|\Delta_1(h)f\|_V^\theta \frac{dh}{h},$$

which is (3.7). □

In the following, we substitute the $\Lambda^p(w)$-norm in (3.6) and (3.7) with $\Lambda^{p,\nu}(w)$ where $w \in B_\nu$. The method will be different from the above for the invalidity of the skill in Proposition 3.1. Indeed, it is notable that there holds the fact (see [5]): if v is a decreasing function on $\mathbb{R}_+$, then

$$\sup \int_{\mathbb{R}^n} |f(x)|v(\rho(x))dx = \int_0^\infty f^*(t)v(t)dt, \tag{3.9}$$

where the supremum is taken over all measure preserving transformations $\rho: \mathbb{R}^n \to \mathbb{R}$, which has been used in the proof of the above proposition. But if v is not decreasing, (3.9) is false. On the other hand, it was induced by Cwikel [10] that if $p \neq \nu$, then there is no inclusive relation between $L^{p,\nu}(\mathbb{R}^2)$ and $L^{p,\nu}(\mathbb{R})[L^{p,\nu}(\mathbb{R})]$. To this end, we introduced the iterated rearrangements and multidimensional Lorentz spaces.

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function. We will denote by $f^*_{x_k}(s, \hat{x}_k)$ the decreasing rearrangement of f with respect to the variable x_k, under fixed variable $\hat{x}_k$, where $x_k \in \mathbb{R}$, $\hat{x}_k \in \mathbb{R}^{n-1}$, $1 \leq k \leq n$. The multivariate decreasing rearrangement of f, first with respect to the first variable x_k and then with respect to $\hat{x}_k$, will be denoted by

$$f^*_{x_k\hat{x}_k}(s, t) = (f^*_{x_k}(s, \cdot))^*_{\hat{x}_k}(t).$$

Specially, denote $f^*_{x_1\hat{x}_1}(s, t) = f^*_2(s, t)$.

Let $p > 0$ and w be a weight on $\mathbb{R}^2_+$ (a nonnegative, locally integrable function on $\mathbb{R}^2_+$, not identically 0). We now say that a measurable function f on $\mathbb{R}^n$ belongs to the multidimensional Lorentz space $\Lambda^p_{x_k\hat{x}_k}(w)$ only if $\|f\|_{\Lambda^p_{x_k\hat{x}_k}(w)} < \infty$, where

$$\|f\|_{\Lambda^p_{x_k\hat{x}_k}(w)} = \left(\int_{\mathbb{R}^2_+} f^{*p}_{x_k\hat{x}_k}(s, t)w(s, t)dsdt\right)^{1/p}.$$

See [6, 4] and [3]. In special, write $\Lambda^p_{x_1\hat{x}_1}(w) = \Lambda^p_2(w)$ if $k = 1$.

As usual, for any $1 \leq p \leq \infty$, $p' = p/(p-1)$. We say that a set $D \subset \mathbb{R}^2_+$ is decreasing (and write $D \in \Delta_d$) if the function χ_D is decreasing in each variable.

PROPOSITION 3.2. *Let $1 \leq \theta < \infty$, $1 \leq \nu \leq p < \infty$, $1 \leq r < p$ and $1/r - 1/p < \alpha < 1$, $\beta = \alpha - 1/r + 1/p$ and $w \in B_p$. For $\theta \geq \nu$, if there exists a function $v: \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$v(h) \leq Ch^{\frac{\nu}{p}-1}, \ \ h > 0; \tag{3.10}$$

$$\left(\int_a^\infty \frac{v(y)^{1-\nu'}}{y^{\nu'}}dy\right)^{1/\nu'}\left(\int_0^a v(y)dy\right)^{1/\nu} \leq C, \quad a > 0; \tag{3.11}$$

$$\left(\int_0^a (v(s)s^{-\frac{\nu}{r}})^{1-\nu'}ds\right)^{1/\nu'}\left(\int_a^\infty \frac{v(s)s^{-\frac{\nu}{r}}}{s^\nu}ds\right)^{1/\nu} \leq C, \quad a > 0; \tag{3.12}$$

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$$\sup_{D\subset\Delta_d}\frac{\int_0^{|D|}W^{\nu/p-1}(x)w(x)dx}{\int_D v(s)W^{\nu/p-1}(t)w(t)dsdt}<\infty, \tag{3.13}$$

then

$$B^{\alpha}_{\theta,1}(\Lambda^{p,\nu}(w)[L^r])\subset B^{\beta}_{\theta,1}(\Lambda^{p,\nu}(w)),$$

more exactly, for any $f\in B^{\alpha}_{\theta,1}(\Lambda^{p,\nu}(w)[L^r])$,

$$\|f\|_{\Lambda^{p,\nu}(w)}\le C\|f\|_{B^{\alpha}_{\theta,1}(\Lambda^{p,\nu}(w)[L^r])} \tag{3.14}$$

and

$$\int_0^\infty h^{-\theta\beta}\|\Delta_1(h)f\|^{\theta}_{\Lambda^{p,\nu}(w)}\frac{dh}{h}\le C\int_0^\infty h^{-\theta\alpha}\|\Delta_1(h)f\|^{\theta}_{\Lambda^{p,\nu}(w)[L^r]}\frac{dh}{h}. \tag{3.15}$$

For $\theta<\nu$, *if the inequalities (3.10)–(3.13) and*

$$\left(\int_0^h v(s)s^{-\frac{\nu}{r}+\nu}ds\right)^{\theta/\nu-1}v(h)h^{-\theta+\frac{\theta}{r}-\frac{\theta}{p}-\frac{\nu}{r}+\nu+1}\le C \tag{$*$}$$

hold, then the results also establish.

Proof. Let $f\in B^{\alpha}_{\theta,1}(\Lambda^{p,\nu}(w)[L^r])$. Set $\phi_h(x,y)=|\Delta_1(h)f(x,y)|$. Fix $s,\ h>0$. For any $t>0$, there is $E=E_{s,t,h}\subset\mathbb{R}^{n-1}$ with $|E|_{n-1}=t$ such that

$$(\phi_h)^*_2(s,t)\le\frac{1}{t}\int_E(\phi_h(\cdot,y))^*(s)dy. \tag{3.16}$$

By (3.1),

$$(\phi_h(\cdot,y))^*(s)\le 2\int_s^\infty\omega(\phi_h(\cdot,y),z)_r\frac{dz}{z^{1+\frac{1}{r}}}. \tag{3.17}$$

Set $g_{z,h}(y)=\omega(\phi_h(\cdot,y);z)_r$. By the property of rearrangement,

$$\frac{1}{t}\int_E g_{z,h}(y)dy\le g^{**}_{z,h}(t). \tag{3.18}$$

Thus, the inequalities (3.16)–(3.18) yield

$$(\phi_h)^*_2(s,t)\le\frac{2}{t}\int_s^\infty\int_E g_{z,h}(y)dy\frac{dz}{z^{1+\frac{1}{r}}}. \tag{3.19}$$

$$\le 2\int_s^\infty g^{**}_{z,h}(t)\frac{dz}{z^{1+\frac{1}{r}}}.$$

Let $\tilde{w}(t)=W^{\nu/p-1}(t)w(t)$. Next, we estimate

$$\|\Delta_1(h)f\|^{\nu}_{\Lambda^{\nu}_2(v\tilde{w})}=\int_{\mathbb{R}^2_+}(\Delta_1(h)f)^{*\nu}_2(s,t)v(s)\tilde{w}(t)dsdt.$$

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Now, by Proposition 2.3 and (3.11),

$$\begin{aligned}
\|\Delta_1(h)f\|^{\nu}_{\Lambda_2^{\nu}(v\tilde{w})} &\le \int_{\mathbb{R}^2}\left(2\int_s^{\infty} g^{**}_{u,h}(t)\frac{dz}{z^{1+\frac{1}{r}}}\right)^{\nu}\tilde{w}(t)v(s)dsdt\\
&= 2^{\nu}\int_0^{\infty}\tilde{w}(t)dt\int_0^{\infty}v(s)\left(\int_s^{\infty}g^{**}_{z,h}(t)\frac{dz}{z^{1+\frac{1}{r}}}\right)^{\nu}ds\\
&\le C\int_0^{\infty}\tilde{w}(t)dt\int_0^{\infty}v(s)\left(\frac{g^{**}_{s,h}(t)}{s^{\frac{1}{r}}}\right)^{\nu}ds\\
&= C\int_0^{\infty}v(s)s^{-\frac{\nu}{r}}ds\int_0^{\infty}\tilde{w}(t)g^{**\nu}_{s,h}(t)dt\\
&= C\int_0^{\infty}v(s)s^{-\frac{\nu}{r}}\|g_{s,h}\|^{\nu}_{\Lambda^{\nu}(\tilde{w})}ds.
\end{aligned}$$

By (3.4), we get

$$g_{s,h}(y)=\omega(\phi_h(\cdot,y),s)_r\le\frac{C}{s}\int_0^s\|\Delta(z)\phi_h(\cdot,y)\|_{L^r}dz.$$

Since $w\in B_p$, it follows that $\tilde{w}\in B_{\nu}$ (see [9, Theorem 1.3.4]) which implies that $\Lambda^{\nu}(\tilde{w})$ is a Banach function space [9]. By Minkowski's inequality (see [30] or [37, Corollary V.1, p. 133]),

$$\begin{aligned}
\|\Delta_1(h)f\|^{\nu}_{\Lambda_2^{\nu}(v\tilde{w})} &\le C\int_0^{\infty}v(s)s^{-\frac{\nu}{r}}\left\|\frac{C}{s}\int_0^s\|\Delta(z)\phi_h(\cdot,y)\|_{L^r}dz\right\|^{\nu}_{\Lambda^{\nu}(\tilde{w})}ds\\
&\le C\int_0^{\infty}v(s)s^{-\frac{\nu}{r}}\left(\frac{C}{s}\int_0^s\|\|\Delta(z)\phi_h(\cdot,y)\|_{L^r}\|_{\Lambda^{\nu}(\tilde{w})}dz\right)^{\nu}ds\\
&= C\int_0^{\infty}v(s)s^{-\frac{\nu}{r}}\left(\frac{C}{s}\int_0^s\|\Delta_1(z)\phi_h\|_V dz\right)^{\nu}ds,
\end{aligned}$$

where $V=\Lambda^{\nu}(\tilde{w})[L^r]$. Using Proposition 2.2 and (3.12), we have

$$\|\Delta_1(h)f\|^{\nu}_{\Lambda_2^{\nu}(v\tilde{w})}\le C\int_0^{\infty}v(s)s^{-\frac{\nu}{r}}\|\Delta_1(s)\phi_h\|^{\nu}_V ds. \tag{3.20}$$

In view of

$$\|\Delta_1(s)\phi_h\|_V\le 2\min(\|\Delta_1(s)f\|_V,\|\Delta_1(h)f\|_V),$$

we get by (3.20)

$$\begin{aligned}
&\int_0^{\infty}h^{-\theta\beta}\|\Delta_1(h)f\|^{\theta}_{\Lambda_2^{\nu}(v\tilde{w})}\frac{dh}{h}\\
&\le\int_0^{\infty}h^{-\theta\beta}\left(\int_0^{\infty}v(s)s^{-\frac{\nu}{r}}\|\Delta_1(s)\phi_h\|^{\nu}_V ds\right)^{\theta/\nu}\frac{dh}{h}\\
&\le C\int_0^{\infty}h^{-\theta\beta}\left(\int_0^{h}v(s)s^{-\frac{\nu}{r}}\|\Delta_1(s)f\|^{\nu}_V ds\right)^{\theta/\nu}\frac{dh}{h}\\
&\quad+C\int_0^{\infty}h^{-\theta\beta}\left(\int_h^{\infty}v(s)s^{-\frac{\nu}{r}}ds\right)^{\theta/\nu}\|\Delta_1(h)f\|^{\theta}_V\frac{dh}{h}\\
&=: C(I_1+I_2).
\end{aligned} \tag{3.21}$$

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First, by (3.10),

$$I_2 \le \int_0^\infty h^{-\theta\alpha}\|\Delta_1(h)f\|_V^\theta \frac{dh}{h}. \tag{3.22}$$

Further, if $\theta > \nu$, by Proposition 2.2 and (3.10),

$$\begin{aligned} I_1 &= \int_0^\infty h^{-\theta\beta}\left(\frac{1}{h}\int_0^h v(s)s^{-\frac{\nu}{r}}\|\Delta_1(s)f\|_V^\nu ds\right)^{\theta/\nu} h^{\theta/\nu}\frac{dh}{h} \\ &\le \int_0^\infty h^{-\theta\beta}\left(v(h)h^{-\frac{\nu}{r}}\|\Delta_1(h)f\|_V^\nu\right)^{\theta/\nu} h^{\theta/\nu}\frac{dh}{h} \\ &\le C\int_0^\infty h^{-\theta\alpha}\|\Delta_1(h)f\|_V^\theta \frac{dh}{h}. \end{aligned} \tag{3.23}$$

If $\theta \le \nu$, we use Proposition 2.4, (3.5) and $(*)$ to get

$$\begin{aligned} I_1 &\le C\int_0^\infty \frac{\omega_1(f;h)_V^\theta}{h^\theta} h^{-\theta\beta}\left(\int_0^h v(s)s^{-\frac{\nu}{r}+\nu}ds\right)^{\theta/\nu-1} v(h)h^{-\frac{\nu}{r}+\nu}dh \\ &\le C\int_0^\infty h^{-\theta\alpha}\|\Delta_1(h)f\|_V^\theta \frac{dh}{h}. \end{aligned} \tag{3.24}$$

In view of (3.13) and [**3**, Theorem 3.2],

$$\Lambda_2^\nu(v(s)\tilde{w}(t)) \subset \Lambda^\nu(\tilde{w}). \tag{3.25}$$

Thus, (3.21)–(3.25) lead to (3.15). The estimate (3.14) follows by analogous arguments, and we omit the details. □

REMARK 3.1. When $w(t) = 1$, there holds that $\Lambda^{p,\nu}(w) = L^{p,\nu}$ and $\Lambda^{p,\nu}(w)[L^r] = L^{p,\nu}[L^r]$. If we take $v(s) = s^{\nu/p-1}$, then equalities (3.13) (see [**36**] and [**3**, Theorem 3.2]), (3.10)–(3.12) and $(*)$ simultaneously hold and thus we get [**19**, Proposition 3.2].

4. Sobolev embedding. In this section, we give Sobolev embedding from the weighted Sobolev–Lorentz spaces to the weighted Besov–Lorentz spaces. For $1 \le p,\ q < \infty$, $k = 1, \dots, n$, u and w two weights on $\mathbb{R}_+$, denote by $V_{q,p,w,u,k}$ the mixed weighted Lorentz space $\Lambda^{q,p}_{\mathbb{R}^{n-1}}(w)[\Lambda^p_{\mathbb{R}}(u)]_k$ obtained by taking first the norm in $\Lambda^p_{\mathbb{R}}(u)$ with respect to the variable x_k, and then the norm in $\Lambda^{q,p}_{\mathbb{R}^{n-1}}(w)$ with respect to $\hat{x}_k$. If w_k, $k = 1, \dots, n$, be weights on $\mathbb{R}^2_+$ and $\vec{w} = (w_1,\ w_2,\ \dots,\ w_n)$, the weighted Lorentz–Sobolev space $W^1_{\Lambda^p(\vec{w})}(\mathbb{R}^n)$ is defined as the class of all functions $f \in \cap_{1\le k\le n}\Lambda^p_{x_k\hat{x}_k}(w_k)$ for which every first-order weak derivative exists and belongs to $\cap_{1\le k\le n}\Lambda^p_{x_k\hat{x}_k}(w_k)$ and define the quasi-norm of f by

$$\|f\|_{W^1_{\Lambda^p(\vec{w})}} = \sum_{1\le k\le n}\|f\|_{\Lambda^p_{x_k\hat{x}_k}(w_k)} + \sum_{1\le i,j\le n}\|D_if\|_{\Lambda^p_{x_j\hat{x}_j}(w_k)}.$$

Obviously, if $w_k(s,t) = 1$, $k = 1, \dots, n$, then $W^1_{\Lambda^p(\vec{w})}(\mathbb{R}^n) = W^1_p(\mathbb{R}^n)$. When $w_k(s,t) = s^{q/p-1}t^{q/p-1}$, $k = 1, \dots, n$, $1 \le p,\ q < \infty$, the space $W^1_{\Lambda^p(\vec{w})}(\mathbb{R}^n)$ has been studied in, e.g., [**6**] and [**19**].

THEOREM 4.1. *Let* $1 < p < q < \infty$, $n \geq 2$, $\alpha = 1 - (n-1)(1/p - 1/q) > 0$. *If* u_k, w_k, $k = 1, \ldots, n$, *are decreasing on* $\mathbb{R}_+$ *and* w_k, $k = 1, \ldots, n$, *satisfies that for all* $r > 0$,

$$\left(\int_0^r w_k(s)ds\right)^{-1} \int_0^r \left(\frac{t}{W_k(t)}\right)^{\frac{p}{n-1}} w_k(t)dt \leq C, \tag{4.1}$$

and for $k = 1, \ldots, n$, *there are* $0 < a_k < 1$, $m_k \in \mathbb{N}$, *such that for all* $r > 0$,

$$\tilde{w_k}(r/m_k) \leq a_k m_k \tilde{w_k}(r), \tag{4.2}$$

where $\tilde{w}_k(r) = W_k^{\frac{p}{q}-1}(r)w_k(r)$, *then for* $f \in W^1_{\Lambda^p(\vec{v})}(\mathbb{R}^n)$, $k = 1, \ldots, n$,

$$\left(\int_0^\infty h^{-\alpha p} \|\Delta_k(h)f\|^p_{V_{q,p,w_k,u_k,k}} \frac{dh}{h}\right)^{1/p} \leq C \sum_{j=1}^n \|D_j f\|_{\Lambda^p_{x_k \hat{x}_k}(u_k w_k)}, \tag{4.3}$$

where $\vec{v}(s, t) = (u_1(s)w_1(t), \ldots, u_n(s)w_n(t))$.

Proof. The proof is inspired by [19]. Without loss of generality, we only prove (4.3) for the case $k = n$. For convenience, write u and w instead of w_n and u_n. Let

$$\psi_j(\hat{x}_n) = \|D_j f(\hat{x}_n, \cdot)\|_{\Lambda^p(u)}$$

and

$$\phi_h(\hat{x}_n) = \|\Delta_n(h)f(\hat{x}_n, \cdot)\|_{\Lambda^p(u)}.$$

Since u is decreasing, we get that

$$\phi_h(\hat{x}_n)^p = \sup_\rho \int_R |\Delta_n(h)f(\hat{x}_n, z)|^p u(\rho(z))dz,$$

where the supremum is taken over all measure preserving transformations $\rho : \mathbb{R} \to \mathbb{R}_+$. Hence, by the fact

$$|\Delta_n(h)f(\hat{x}_n, x_n)| \leq \int_0^h |D_n f(\hat{x}_n, x_n + y)|dy,$$

it follows that

$$\begin{aligned}
\phi_h(\hat{x}_n)^p &\leq \sup_\rho \int_{\mathbb{R}} \left(\int_0^h |D_n f(\hat{x}_n, x_n + y)|dy\right)^p u(\rho(x_n))dx_n \\
&\leq h^{p-1} \sup_\rho \int_0^h dy \int_{\mathbb{R}} |D_n f(\widehat{x}_n, x_n)|^p u(\rho(x_n - y))dx_n, \text{ by Hölder inequality} \\
&\leq h^{p-1} \int_0^h dy \int_{\mathbb{R}} |D_n f(\widehat{x}_n, \cdot)|^{*p}(t)u(t)dt \\
&= h^p \int_{\mathbb{R}} |D_n f(\widehat{x}_n, \cdot)|^{*p}(t)u(t)dt \\
&= h^p \psi_n(\hat{x}_n)^p,
\end{aligned}$$

i.e.,

$$\phi_h^*(t) \leq h\psi_n^*(t). \tag{4.4}$$

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Let us note that, by (4.2) with big r, $w(t) > 0$ if $t > 0$. Thus, W is strictly increasing on $(0, \infty)$. Moreover, integrating (4.2), we get $W^{p/q}(t/m) \leq aW^{p/q}(t)$ if $t > 0$, which leads to $W(\infty) = \infty$ and W is bijective on $(0, \infty) \to (0, \infty)$. Hence

$$\begin{aligned}
K(h) &:= \| \|\Delta_n(h) f(\hat{x}_n, \cdot)\|_{\Lambda^p(u)}\|^p_{\Lambda^{q,p}(w)} \\
&= \int_0^\infty \phi_h^*(t)^p W(t)^{p/q-1} w(t) dt \\
&= \int_{W^{-1}(h^{n-1})}^\infty \phi_h^*(t)^p W(t)^{p/q-1} w(t) dt + \int_0^{W^{-1}(h^{n-1})} \phi_h^*(t)^p W(t)^{p/q-1} w(t) dt \\
&=: K_1(h) + K_2(h).
\end{aligned}$$

Then,

$$\begin{aligned}
J &= \int_0^\infty h^{-\alpha p} \| \|\Delta_n(h) f(\hat{x}_n, \cdot)\|_{\Lambda^p(u)}\|^p_{\Lambda^{q,p}(w)} \frac{dh}{h} \\
&= \int_0^\infty h^{-\alpha p} K(h) \frac{dh}{h} \\
&= \int_0^\infty h^{-\alpha p} (K_1(h) + K_2(h)) \frac{dh}{h} \\
&= \int_0^\infty h^{-\alpha p} K_1(h) \frac{dh}{h} + \int_0^\infty h^{-\alpha p} K_2(h) \frac{dh}{h} \\
&=: J_1 + J_2.
\end{aligned}$$

By (4.4),

$$K_1(h) \leq h^p \int_{W^{-1}(h^{n-1})}^\infty \psi_n^*(t)^p W(t)^{p/q-1} w(t) dt. \tag{4.5}$$

Now by (4.5) and the assumption that w is decreasing,

$$\begin{aligned}
J_1 &\leq \int_0^\infty W(t)^{p/q-1} w(t) \psi_n^*(t)^p dt \int_0^{W(t)^{\frac{1}{n-1}}} h^{-\alpha p + p - 1} dh \\
&= \int_0^\infty \psi_n^*(t)^p w(t) dt \\
&= \sup_\rho \int_{\mathbb{R}^{n-1}} \psi_n(\hat{x}_n)^p w(\rho(\hat{x}_n)) d\hat{x}_n \\
&= \sup_\rho \int_{R^{n-1}} \left(\int_{\mathbb{R}_+} (D_n f)^{*p}_{x_n}(t) u(t) dt \right) w(\rho(\hat{x}_n)) d\hat{x}_n \\
&\leq \int_{\mathbb{R}_+} \sup_\rho \int_{R^{n-1}} (D_n f)^{*p}_{x_n}(t) w(\rho(\hat{x}_n)) d\hat{x}_n u(t) dt \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (D_n f)^{*p}_{x_n \hat{x}_n}(t, s) w(s) u(t) dt
\end{aligned}$$

This holds for all $p > 1$ and $n \geq 2$.

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Next, we estimate J_2. Write $x = (u_1, x_n)$, $u_1 = \hat{x}_n \in \mathbb{R}^{n-1}$. For $u_1 \in \mathbb{R}^{n-1}$ and $t > 0$, denote by $Q_{u_1}(t)$ the cube in $\mathbb{R}^{n-1}$ centered at u_1 with the side length $(2mt)^{1/(n-1)}$, where m from (4.2). Let

$$A_{u_1,t,h} = \{v_1 \in Q_{u_1}(t) : \phi_h(v_1) \le \phi_h^*(mt)\}.$$

Then, $\mathrm{mes}_{n-1} A_{u_1,t,h} \ge mt$. Thus, we have

$$\begin{aligned}\phi_h(u_1) - \phi_h^*(mt) &\le \phi_h(u_1) - \frac{1}{\mathrm{mes}_{n-1} A_{u_1,t,h}} \int_{A_{u_1,t,h}} \phi_h(v_1) dv_1 \\ &\le \frac{1}{mt} \int_{Q_{u_1}(t)} |\phi_h(u_1) - \phi_h(v_1)| dv_1. \end{aligned} \tag{4.6}$$

Note that

$$|\phi_h(u_1) - \phi_h(v_1)| \le C \|f(u_1, \cdot) - f(v_1, \cdot)\|_{\Lambda^p(u)}$$

and

$$|f(u_1, x_n) - f(v_1, x_n)| \le |u_1 - v_1| \sum_{j=1}^{n-1} \int_0^1 |D_j f(u_1 + \tau_1(v_1 - u_1), x_n)| d\tau_1.$$

And, it is easy to see that if $v_1 \in Q_{u_1}(t)$, then $|u_1 - v_1| \le \sqrt{n-1}(mt)^{1/(n-1)}$. So by the fact that $\Lambda^p(u)$ is normable and the Minkowski inequality, for $v_1 \in Q_{u_1}(t)$

$$\begin{aligned} |\phi_h(u_1) - \phi_h(v_1)| &\le c(mt)^{1/(n-1)} \left\| \sum_{j=1}^{n-1} \int_0^1 |D_j f(u_1 + \tau_1(v_1 - u_1), \cdot)| d\tau_1 \right\|_{\Lambda^p(u)} \\ &\le c(mt)^{1/(n-1)} \sum_{j=1}^{n-1} \left\| \int_0^1 |D_j f(u_1 + \tau_1(v_1 - u_1), \cdot)| d\tau_1 \right\|_{\Lambda^p(u)} \\ &\le c(mt)^{1/(n-1)} \sum_{j=1}^{n-1} \int_0^1 \|D_j f(u_1 + \tau_1(v_1 - u_1), \cdot)\|_{\Lambda^p(u_1)} d\tau_1 \\ &= c(mt)^{1/(n-1)} \sum_{j=1}^{n-1} \int_0^1 \psi_j(u_1 + \tau_1(v_1 - u_1)) d\tau_1. \end{aligned}$$

Thus by (4.6)

$$\phi_h(u_1) - \phi_h^*(mt) \le (mt)^{-\frac{n-2}{n-1}} \sum_{j=1}^{n-1} \int_{Q_0(t)} \int_0^1 \psi_j(u_1 + \tau_1 z) d\tau_1 dz.$$

Hence,

$$\begin{aligned} \phi_h^*(t) - \phi_h^*(mt) &\le \frac{1}{t} \sup_{\mathrm{mes}_{n-1} E = t} \int_E [\phi_h(u_1) - \phi_h^*(mt)] du_1 \\ &\le C(mt)^{-\frac{n-2}{n-1}} \sum_{j=1}^{n-1} \sup_{\mathrm{mes}_{n-1} E = t} \int_{Q_0(t)} \int_0^1 \frac{1}{t} \int_E \psi_j(u_1 + \tau_1 z) du_1 d\tau_1 dz \end{aligned}$$

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$$\leq C(mt)^{1/(n-1)} \sum_{j=1}^{n-1} \psi_j^{**}(t). \tag{4.7}$$

Thus, for any $\epsilon > 0$, we have

$$\begin{aligned} J_2(\epsilon)^{1/p} &:= \left(\int_\epsilon^{1/\epsilon} h^{-\alpha p} \int_{W^{-1}(\epsilon^{n-1})}^{W^{-1}(h^{n-1})} \phi_h^*(t)^p W(t)^{p/q-1} w(t) dt \frac{dh}{h} \right)^{1/p} \\ &\leq \left(\int_0^\infty h^{-\alpha p} \int_0^{W^{-1}(h^{n-1})} [\phi_h^*(t) - \phi_h^*(mt)]^p W(t)^{p/q-1} w(t) dt \frac{dh}{h} \right)^{1/p} \\ &\quad + \left(\int_\epsilon^{\frac{1}{\epsilon}} h^{-\alpha p} \int_{W^{-1}(\epsilon^{n-1})}^{W^{-1}(h^{n-1})} [\phi_h^*(mt)]^p W(t)^{p/q-1} w(t) dt \frac{dh}{h} \right)^{1/p} \\ &=: I' + I''(\epsilon). \end{aligned}$$

By (4.7), there holds that

$$\begin{aligned} I' &\leq Cm^{\frac{1}{n-1}} \sum_{j=0}^{n-1} \left(\int_0^\infty h^{-\alpha p} \int_0^{W^{-1}(h^{n-1})} t^{\frac{p}{n-1}} \psi_j^{**p}(t) W(t)^{p/q-1} w(t) dt \frac{dh}{h} \right)^{1/p} \\ &= Cm^{\frac{1}{n-1}} \sum_{j=1}^{n-1} \left(\int_0^\infty t^{\frac{p}{n-1}} \psi_j^{**p}(t) W(t)^{p/q-1} w(t) dt \int_{W(t)^{\frac{1}{n-1}}}^\infty h^{-\alpha p} \frac{dh}{h} \right)^{1/p} \\ &= C \sum_{j=1}^{n-1} \left(\int_0^\infty \psi_j^{**p}(t) \left(\frac{t}{W(t)} \right)^{\frac{p}{n-1}} w(t) dt \right)^{1/p}. \end{aligned}$$

By Hardy's Lemma [5, p. 56] and (4.1), similarly to the proof the evaluation of J_1, we have

$$\begin{aligned} I' &\leq C \sum_{j=1}^{n-1} \left(\int_0^\infty \psi_j^{**p}(t) w(t) dt \right)^{1/p} \\ &\leq C \sum_{j=1}^{n-1} \left(\int_0^\infty \psi_j^{*p}(t) w(t) dt \right)^{1/p}, \quad \text{since } w \in B_p \\ &\leq C \sum_{j=1}^{n-1} \|D_j f\|_{\Lambda^p_{x_n \hat{x}_n}(uw)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} I''(\epsilon) &= \left(\int_\epsilon^{\frac{1}{\epsilon}} h^{-\alpha p} \int_{W^{-1}(\epsilon^{n-1})}^{W^{-1}(h^{n-1})} [\phi_h^*(mt)]^p W(t)^{p/q-1} w(t) dt \frac{dh}{h} \right)^{1/p} \\ &= \left(\int_\epsilon^{\frac{1}{\epsilon}} h^{-\alpha p} \int_{mW^{-1}(\epsilon^{n-1})}^{mW^{-1}(h^{n-1})} \frac{1}{m} [\phi_h^*(t)]^p W\left(\frac{t}{m}\right)^{p/q-1} w\left(\frac{t}{m}\right) dt \frac{dh}{h} \right)^{1/p} \\ &\leq \left(\int_\epsilon^{\frac{1}{\epsilon}} h^{-\alpha p} \int_{W^{-1}(\epsilon^{n-1})}^{W^{-1}(h^{n-1})} \frac{1}{m} [\phi_h^*(t)]^p W\left(\frac{t}{m}\right)^{p/q-1} w\left(\frac{t}{m}\right) dt \frac{dh}{h} \right)^{1/p} \end{aligned}$$

$$+\left(\int_0^\infty h^{-\alpha p}\int_{W^{-1}(h^{n-1})}^\infty \frac{1}{m}[\phi_h^*(t)]^p W\left(\frac{t}{m}\right)^{p/q-1} w\left(\frac{t}{m}\right) dt\frac{dh}{h}\right)^{1/p}$$
$$=: J_2'(\epsilon)+J_1'.$$

Now by (4.4)

$$\begin{aligned} J_1' &\leq \frac{C}{m^{1/p}}\left(\int_0^\infty [\psi_n^*(t)]^p W\left(\frac{t}{m}\right)^{p/q-1} w(\frac{t}{m})W(t)^{1-\frac{p}{q}}dt\right)^{1/p} \\ &\leq \frac{C}{m^{1/p}}\left(\int_0^\infty [\psi_n^*(t)]^p w\left(\frac{t}{m}\right)dt\right)^{1/p}, \quad \text{since } W\in\Delta_2 \\ &\leq \frac{C}{m^{1/p}}\left(\int_0^\infty \left[\psi_n^*\left(\frac{t}{m}\right)\right]^p w\left(\frac{t}{m}\right)dt\right)^{1/p} \\ &= C\|\psi_n\|_{\Lambda^p(w)}. \end{aligned}$$

Similarly to the verification of the estimate of J_1, there holds

$$J_1' \leq C\|D_nf\|_{\Lambda^p_{x_n\hat{x}_n}(uv)}.$$

Due to (4.2), we deduce that

$$\begin{aligned} J_2'(\epsilon) &\leq \left(\int_\epsilon^{1/\epsilon} h^{-\alpha p}\int_{W^{-1}(\epsilon^{n-1})}^{W^{-1}(h^{n-1})} a\phi_h^{*p}(t)W(t)^{p/q-1}w(t)dt\frac{dh}{h}\right)^{1/p} \\ &= a^{1/p}J_2(\epsilon)^{1/p}. \end{aligned}$$

Thus

$$\begin{aligned} J_2(\epsilon)^{1/p} &\leq I'+I''(\epsilon)\leq I'+J_2'(\epsilon)+J_1' \\ &\leq C\sum_{j=1}^{n-1}\|D_jf\|_{\Lambda^p_{x_n\hat{x}_n}(uw)}+a^{1/p}J_2(\epsilon)^{1/p}+C\|D_nf\|_{\Lambda^p_{x_n\hat{x}_n}(uw)}. \end{aligned}$$

Since $0<a<1$, we get

$$J_2(\epsilon)^{1/p}\leq C\sum_{j=1}^n\|D_jf\|_{\Lambda^p_{x_n\hat{x}_n}(uw)}.$$

By letting $\epsilon\to 0$, there holds that

$$J_2^{1/p}\leq C\sum_{j=1}^n\|D_jf\|_{\Lambda^p_{x_n\hat{x}_n}(uw)}.$$

The estimates of J_1 and J_2 give

$$J^{1/p}\leq C(J_1^{1/p}+J_2^{1/p})\leq C\sum_{j=1}^n\|D_jf\|_{\Lambda^p_{x_n\hat{x}_n}(uw)},$$

which implies that (4.3) holds.

□

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REMARK 4.1. (1) If $w(t) = t^{-\alpha} + C_1$; or $w(t) = t^{-\alpha} + C_1 - 1$ if $t \le 1$ and $w(t) = C_1$ if $t > 1$ where $0 \le \alpha < 1$, $C_1 > 0$, then w satisfies all required conditions in Theorem 4.1. Especially, when u and w are all constants, then the result is just [**19**, Theorem 4.2] and thus Theorem 4.1 is an improvement of [**19**, Theorem 4.2] when $p > 1$.

(2) Combining Proposition 3.2 and Theorem 4.1 shows that if $1 < p < q < \infty$, $n \ge 2$, $\alpha = 1 - (n-1)(1/p - 1/q) > 0$, w is decreasing on $\mathbb{R}_+$ and satisfies (4.1) and (4.2), and there exists a nonnegative function v on $\mathbb{R}_+$ such that (3.10)–(3.13) hold, then

$$\sum_{k=1}^{n} \left(\int_0^\infty h^{-\alpha p} \|\Delta_k(h) f\|^p_{\Lambda^{q,p}(w)} \frac{dh}{h} \right)^{1/p} \le C \sum_{k,j=1}^{n} \|D_j f\|_{\Lambda^p_{x_k \hat{x}_k}(w)}. \tag{4.8}$$

In special, if $w = 1$, then (4.8) is just (1.3). Thus, Theorem 4.1 is a refinement of Theorem 1.2 when $p > 1$ and $n \ge 2$.

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