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A STORAGE SYSTEM WITH SPORADIC AND CONTINUOUS CLEARINGS

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We study a cumulative storage system that is totally cleared sporadically at stationary renewal times and whenever a finite-capacity threshold is exceeded. The independent and identically distributed inputs occur at time epochs that also form a stationary renewal process. We determine the distribution of the interoverflow times. Although this distribution is quite intricate when both underlying renewal processes are general, in the special case of Poisson sporadic clearings we obtain a neat formula for its Laplace transform.

1. INTRODUCTION

We consider a storage system that receives inputs of independent and identically distributed (i.i.d.) random sizes at time epochs that form a renewal process. The output is generated by two types of clearing:

1. The system is emptied periodically at random times; we assume that these "sporadic" clearing times form a second renewal process, which is independent of the input times and sizes.
2. The system has a finite capacity, and whenever a freshly arriving input cannot be fully stored, an additional instantaneous clearing takes place.

Hence, the sequence of clearing times is a superposition of the sporadic ones and the "continuous" ones (due to overflows). We study this system in steady state; that is, both renewal processes involved are stationary. The main aim of this article is to derive in closed form the distribution of the length of the period between two successive overflows. As the system is in steady state, the sequence of overflow times is also stationary. It has also a renewal structure if the sporadic clearing times form a Poisson process. In

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general, the renewal property is lost because the future evolution just after an overflow depends on the current residual waiting time until the next sporadic clearing.

In Section 2 we derive the rather intricate interoverflow distribution in the general renewal case in terms of the system primitives: the capacity, and the probability densities of (1) the times between sporadic clearings, (2) the times between inputs, and (3) the input sizes. (We only consider the absolutely continuous case.) Section 3 is devoted to the case of sporadic Poisson clearings, in which the overflow times form a stationary renewal process. In this case, the Laplace transform of the interoverflow distribution takes a neat form, given in Theorem 2.

Stochastic clearing systems (i.e., input/output systems that are periodically emptied by clearing operations) were introduced and investigated in [10–13]. For questions regarding their steady-state behavior, see also [4,15]; related queuing models were studied in [1,3,16]. For production/storage models, we refer to [2,5–9,14]. Most of these articles deal with the derivation of cost functionals and their minimization. For our model, which combines general stationary renewal inputs and sporadic clearings with cumulative jump inputs and continuous clearings, any kind of optimization will be a challenging problem. In this article we only tackle the interoverflow distribution.

2. THE GENERAL CASE

We assume that the sporadic clearing times $0 \leq C_1 < C_2 < \cdots$ and the input times $0 \leq I_1 < I_2 < \cdots$ form stationary renewal processes, independent of each other, with interarrival time densities $p(t)$ and $q(t)$, $t > 0$, respectively. We might then also suppose that the two renewal processes extend to $-\infty$ so that they are the nonnegative points of the two doubly-infinite stationary renewal sequences $\cdots < C_{-2} < C_{-1} < C_0 < 0 \leq C_1 < C_2 < \cdots$ and $\cdots < I_{-2} < I_{-1} < I_0 < 0 \leq I_1 < I_2 < \cdots$ on the entire real line, respectively. The successive inputs are assumed to be i.i.d. positive random variables with common density $r(x)$, $x > 0$, and to be independent of the C_i and the I_i. The capacity of the storage system is denoted by c $(< \infty)$.

For an integrable function $h : [0, \infty) \to \mathbb{R}$, we write $\hat{h}(\alpha) = \int_0^\infty e^{-\alpha x} h(x)\, dx$, $\alpha \geq 0$, for its Laplace transform (LT). We need the LTs $\hat{p}$, $\hat{q}$, and $\hat{r}$. Convolution of two functions is denoted by $*$; we write h^{*n} for the n-fold convolution of h with itself.

Let S_n be the sum of n successive inputs and $R_n(x) = \mathbb{P}(S_n \leq x)$ be its distribution function (having density $r{*}^n$). We also need

$$\begin{aligned} H_n(x) &= \mathbb{P}(S_n \leq x < S_{n+1}) \\ &= R_n(x) - R_{n+1}(x) \\ &= \int_0^c \left(\int_{c-x}^\infty r(y)\, dy \right) r^{*n}(x)\, dx, \qquad n \geq 0, \end{aligned}$$

where $S_0 = 0$.

Our aim is to determine the probability density $f(t)$ of the time T between the first two overflows. Let

$$f_n(t)\,dt = \mathbb{P}\,(T \in dt \text{ and exactly } n \text{ sporadic clearings take place between the first two overflows}).$$

Then

$$f(t) = \sum_{n=0}^{\infty} f_n(t).$$

We now derive formulas for the functions f_n in a series of lemmas, which might be of independent interest.

Let $g_c(t)$ be the conditional density of T given that no sporadic clearings take place between the two overflows. The reason for indicating the dependence of this density on the capacity c is that although $g_c(t)$ itself can only be determined as a convolution series, the double LT $\hat{\hat{g}}(\alpha, \beta) = \int_0^\infty \int_0^\infty e^{-\alpha t - \beta c} g_c(t)\,dt\,dc$ turns out to have a particularly simple form.

LEMMA 1: *The functions* $g_c(t)$ *and* $\hat{\hat{g}}(\alpha, \beta)$ *are given by*

$$g_c(t) = \sum_{n=1}^{\infty} H_{n-1}(c) p^{*n}(t), \qquad t > 0, \tag{2.1}$$

$$\hat{\hat{g}}(\alpha, \beta) = \frac{\hat{p}(\alpha)(1 - \hat{r}(\beta))}{\beta[1 - \hat{p}(\alpha)\hat{r}(\beta)]}, \qquad \alpha, \beta > 0, \tag{2.2}$$

respectively.

PROOF: When no sporadic clearings take place, an interoverflow interval has length t if and only if either (1) the first input in the interval occurs after t time units and is of size greater than c or (2) for some $n \geq 1$, the $(n+1)$st input in the interval occurs at time t, the first n inputs add up to some $x \leq c$, and the first $n+1$ inputs have sum $x + y > c$. By our assumptions, this argument yields

$$g_c(t) = p(t) \int_c^\infty r(x)\,dx + \sum_{n=1}^{\infty} p^{*n+1}(t) \int_0^c \left(\int_{c-x}^\infty r(y)\,dy \right) r^{*n}(x)\,dx, \tag{2.3}$$

and (2.1) is tantamount to (2.3). Multiplying (2.1) by $e^{-\alpha t - \beta c}$ and integrating over t and c yields (2.2) after straightforward calculations. ■

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Remark: Equation (2.2) also holds for β in the complex half-plane Re $\beta > 0$ so that the LT $\hat{g}_c(\alpha)$ of g_c can be determined by Laplace inversion: For any $a > 0$,

$$\hat{g}_c(\alpha) = \frac{\hat{p}(\alpha)}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{(1 - \hat{r}(\beta))\, e^{c\beta}}{\beta[1 - \hat{p}(\alpha)\,\hat{r}(\beta)]}\, d\beta.$$

Let $\rho(s, t)\, ds\, dt$ be the infinitesimal probability that there is an overflow in $[s, s + ds]$ and the first sporadic clearing afterward takes place in $[t, t + dt]$, where $-\infty < s < t < \infty$. By stationarity, $\rho(s, t)$ depends only on $t - s$, and we will determine $\rho(t) = \rho(0, t)$, $t > 0$. For this derivation we need the renewal density

$$u_c(t) = \sum_{n=1}^{\infty} g_c^{*n}(t),$$

associated with g_c, and the function

$$h_c(t) = H_0(c) \int_t^{\infty} p(v)\, dv \int_c^{\infty} r(x)\, dx + \sum_{n=1}^{\infty} H_n(c) \times \int_0^t p^{*n}(v) \left[\int_{t-v}^{\infty} p(w)\, dw \right] dv.$$

Lemma 2:

$$\rho(t) = \int_0^{\infty} q(t + s)[\, h_c(s) + (h_c * u_c)(s)]\, ds. \tag{2.4}$$

Proof: Consider the event that there is an overflow at time 0 and $C_1 \in [t, t + dt]$. Let us collect all of the possibilities contributing to its infinitesimal probability. If the first sporadic clearing after time 0 takes place at time t, the last one before must have occurred at time $-s$ for some $s > 0$. Then there are two cases:

Case 1: The overflow at time 0 is the first one in $(-s, 0]$ and it is due to one of the following:

1. to the first input in $(-s, 0]$, meaning that the last input before time 0 ccurred at some time $-v < -s$ and the one at time 0 is greater than c,
2. to the $(n + 1)st$ input in $(-s, 0]$ for some $n \geq 1$, meaning that there are n inputs in $(-s, 0)$ whose sum does not exceed c but the subsequent input at time 0 leads to an overflow.

Case 2: There is at least one overflow in $(-s, 0)$ so that there are an integer $k \geq 1$ and a $v \in (0, s)$ such that the kth overflow in $(-s, 0)$ occurs at time $-s + v$ and the only overflow in $(-s + v, 0]$ occurs at time 0 (this latter event can then be decomposed as in Case 1).

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In terms of probabilities this decomposition yields

$$\rho(t) = \int_0^\infty q(t+s)\Bigg\{\int_s^\infty p(v)\,dv \int_c^\infty r(x)\,dx \quad + \sum_{n=1}^\infty \int_0^s p^{*n}(v)\left[\int_{s-v}^\infty p(w)\,dw\right]dv$$
$$\times \int_0^c r^{*n}(x)\left[\int_{c-x}^\infty r(y)\,dy\right]dx + \sum_{k=1}^\infty \int_0^s g_c^{*k}(v)\Bigg(\int_{s-v}^\infty p(w)\,dw \int_c^\infty r(x)\,dx$$
$$+ \sum_{n=1}^\infty \int_0^{s-v} p^{*n}(u)\left[\int_{s-v-u}^\infty p(y)\,dy\right]du \int_0^c r^{*n}(x)\left[\int_{c-x}^\infty r(y)\,dy\right]dx\Bigg)dv\Bigg\}ds. \tag{2.5}$$

Equation (2.4) is just a more compact way to write (2.5). ■

Now we condition on $I_1 = 0$ (i.e., the first overflow in $[0, \infty)$ occurs at time 0). Under this condition, let $\sigma(t)\,dt$ be the probability of the event that the first sporadic clearing in $(0, \infty)$ takes place in $[t, t+dt]$. Clearly, the conditional probability $\sigma(t)\,dt$ is proportional to $\rho(t)\,dt$; that is, $\sigma(t) = K\rho(t)$ for some constant K, which can be computed from the normalisation condition

$$K = \left[\int_0^\infty \rho(t)\,dt\right]^{-1}. \tag{2.6}$$

For $t > s > 0$, let $\tau_n(s, t)\,ds\,dt$ be the conditional probability, given $I_1 = 0$, that the nth sporadic clearing in $(0, \infty)$ takes place in $[s, s+ds]$, the next input after this clearing occurs in $[t, t+dt]$, and no overflow has occurred in $(0, t)$. Lemma 3 expresses $\tau_n(s, t)$ in terms of $\tau_{n-1}(\cdot, \cdot)$ and the system primitives and also gives a formula for $\tau_1(s, t)$. Thus, it provides a recursion for $\tau_n(s, t)$.

Lemma 3: *For $n = 2, 3, \ldots$ and $t > s > 0$, we have*

$$\tau_n(s,t) = \int_0^s \tau_{n-1}(s',t)\,q(s-s')\,ds'$$
$$+ R_1(c)\int_0^s p(t-t')\left[\int_0^{t'} \tau_{n-1}(s',t')\,q(s-s')\,ds'\right]dt'$$
$$+ \sum_{k=1}^\infty R_{k+1}(c)\int_0^s p(t-t'')\Bigg(\int_0^{t''} p^{*k}(t''-t')$$
$$\times \left[\int_0^{t'} \tau_{n-1}(s',t')\,q(s-s')\,ds'\right]dt'\Bigg)dt''. \tag{2.7}$$

$\tau_1(s, t)$ is given by

$$\tau_1(s,t) = \sigma(s)\,p(t) + \sigma(s)\sum_{k=1}^\infty R_k(c)\int_0^s p^{*k}(s')\,p(t-s')\,ds'. \tag{2.8}$$

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Proof: Equation (2.8) corresponds to the following partition of the underlying event: The first sporadic clearing in $(0, \infty)$ takes place at time s and either (1) the first input in $(0, \infty)$ occurs at time t or (2) there are $k \geq 1$ inputs in $(0, s)$, none of which causes an overflow, and they are followed by an input at time t.

To see (2.7), note that for the event underlying $\tau_n(s, t)$, there are the following possibilities:

1. The $(n-1)$st sporadic clearing takes place at some time $s' \in (0, s)$ and the nth one takes place $s - s'$ time units later; there is no overflow in $(0, t)$ and the next input after time s' arrives at time t.
2. There are times $0 < s' < t' < s$ such that the following holds: The $(n-1)$st sporadic clearing takes place at s' and the nth one takes place $s - s'$ time units later; there is no overflow in $(0, s')$ and the first input in $[s', \infty)$ arrives at t' and is of size less than or equal to c; the next input after the one at t' arrives $t - t'$ time units later.
3. There are times $0 < s < t' < t'' < s$ and a $k \geq 1$ such that the following holds: The $(n-1)$st regular clearing takes place at s' and the nth one takes place $s - s'$ time units later; there is no overflow in $(0, s')$; there is an input at time t'; the kth input after that at t' occurs at t''; the sum of the input sizes in $[t', t'']$ does not exceed c; the first input after t'' occurs $t - t''$ time units later.

This decomposition yields (2.7). ■

Now we are in a position to determine the interoverflow density f.

Theorem 1: *The density f is given by*

$$f(t) = \sum_{n=0}^{\infty} f_n(t), \tag{2.9}$$

where

$$f_0(t) = g_c(t) \int_t^{\infty} \sigma(s)\, ds \tag{2.10}$$

and the functions f_n, $n \geq 1$, are given by

$$f_n(t) = H_0(c) \int_0^t \tau_n(s,t) \left(\int_{t-s}^{\infty} q(t')\, dt' \right) ds + \sum_{k=1}^{\infty} H_k(c) \int_0^t p^{*k}(t-s') \left[\int_0^{s'} \tau_n(s,s') \int_{t-s}^{\infty} q(s'')\, ds''\, ds \right] ds'. \tag{2.11}$$

The functions g_c, σ, and τ_n have been computed in Lemmas 1–3.

Proof: Clearly, $g_c(t)[\int_t^{\infty} \sigma(s)\, ds]\, dt$ is the probability that following an overflow at time 0, the first sporadic clearing takes place in (t, ∞) and the first overflow in $(0, \infty)$

occurs in $[t, t+dt]$. This is exactly $f_0(t)\,dt$. For $n \geq 1$, the right-hand side of (2.11) times dt is the sum of the probabilities of the following two events:

1. Following an overflow at time 0, the nth sporadic clearing takes place in (s, ∞) for some $s \in (0, t)$ and the next one in (t, ∞); the first input in (s, ∞) occurs in $[t, t+dt]$ and causes an overflow.
2. There are integers $k \geq 1$ and time instants $0 < s < s' < t$ and $s'' > t - s$ such that the following holds: After an overflow at time 0, the nth sporadic clearing takes place in (s, ∞) and the $(n+1)$st takes place in $s + s''$, no overflow occurs in $(0, s)$, the next input after s occurs at s', the kth input thereafter occurs in $[t, t+dt]$, the sum of these $k+1$ inputs is greater than c, and the sum of the k inputs in $[s', t)$ is at most c.

The sum of the probabilities of events 1 and 2 is equal to $f_n(t)\,dt$. ■

3. SPORADIC POISSON CLEARINGS

If the sporadic clearings take place at Poisson times (with intensity λ, say), the overflow times form a renewal process. We now derive the LT of the interoverflow distribution for this case in closed form.

THEOREM 2: *In the Poisson case, the LT of f is given by*

$$\hat{f}(\alpha) = \frac{(1-\hat{p}(\alpha+\lambda))\,\hat{p}(\alpha)\,\hat{g}_c(\alpha+\lambda)}{(1-\hat{p}(\alpha))\,\hat{p}(\alpha+\lambda) + (\hat{p}(\alpha)-\hat{p}(\alpha+\lambda))\hat{g}_c(\alpha+\lambda)}. \quad \mathbf{(3.1)}$$

PROOF: It is assumed that $q(t) = \lambda e^{-\lambda t}$ so that, by the lack of memory of the Poisson process,

$$\sigma(t) = \lambda e^{-\lambda t}. \quad \mathbf{(3.2)}$$

Inserting (3.2) in the (2.7) and (2.8) of Lemma 3 we obtain, for $n = 2, 3, \ldots,$

$$\begin{aligned}\tau_n(s,t) &= \int_0^s \tau_{n-1}(s',t)\lambda e^{-\lambda(s-s')}\,ds' \\ &+ R_1(c)\int_0^s p(t-t')\left[\int_0^{t'} \tau_{n-1}(s',t')\,\lambda e^{-\lambda(s-s')}\,ds'\right]dt' \\ &+ \sum_{k=1}^{\infty} R_{k+1}(c)\int_0^s p(t-t'')\left(\int_0^{t''} p^{*k}(t''-t') \right. \\ &\left.\times\left[\int_0^{t'} \tau_{n-1}(s',t')\lambda e^{-\lambda(s-s')}\,ds'\right]dt'\right)dt'' \end{aligned} \quad \mathbf{(3.3)}$$

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and for $n = 1$,

$$\tau_1(s,\ t) = \lambda e^{-\lambda s}\left(p(t) + \sum_{k=1}^{\infty} R_k(c)\int_0^s p^{*k}(s')\,p(t-s')\,ds'\right). \qquad \textbf{(3.4)}$$

For t, u, $\alpha \in [0, \infty)$, let

$$T_n(t|u) = \int_0^t \tau_n(s,t)\,e^{us}\,ds, \qquad \hat{T}_n\,(\alpha|u) = \int_0^\infty e^{-\alpha t} T_n(t|u)\,dt.$$

Integrating (3.3) with respect to s over $(0, t)$ and using Fubini's theorem we get, for $n = 2, 3, \ldots,$

$$\begin{aligned} T_n(t|0) &= T_{n-1}(t|0) - e^{-\lambda t}T_{n-1}(t|\lambda) \\ &\quad + R_1(c)\left(\int_0^t p(t-t')e^{-\lambda t'}T_{n-1}(t'|\lambda)\,dt'\right. \\ &\quad \left. -e^{-\lambda t}\int_0^t p(t-t')T_{n-1}(t'|\lambda)\,dt'\right) \\ &\quad + \sum_{k=1}^{\infty} R_{k+1}(c)\left[\int_0^t p(t-t'')e^{-\lambda t''}\left(\int_0^{t''} p^{*k}(t''-t')T_{n-1}(t'|\lambda)\,dt'\right)dt''\right. \\ &\quad \left. -e^{-\lambda t}\int_0^t p(t-t'')\left(\int_0^{t''} p^{*k}(t''-t')T_{n-1}(t'|\lambda)\,dt'\right)dt''\right]. \end{aligned} \qquad \textbf{(3.5)}$$

Next, take LTs with respect to t and sum over $n \geq 2$. Rearranging terms it follows easily from (3.5) that

$$\begin{aligned} \hat{T}_1(\alpha|0) &= \left(1 - [\hat{p}(\alpha) - \hat{p}(\alpha+\lambda)]\sum_{k=1}^{\infty} R_k(c)\hat{p}(\alpha+\lambda)^{k-1}\right) \\ &\quad \times \sum_{n=1}^{\infty} \hat{T}_n\,(\alpha+\lambda|\lambda). \end{aligned} \qquad \textbf{(3.6)}$$

According to Theorem 1 and (3.2) we have

$$f_0(t) = e^{-\lambda t} g_c(t) \qquad \textbf{(3.7)}$$

and

$$\begin{aligned} f_n(t) &= e^{-\lambda t}H_0(c)T_n(t|\lambda) + e^{-\lambda t} \\ &\quad \times \sum_{k=1}^{\infty} H_k(c)\int_0^t p^{*k}(t-s')T_n(s'|\lambda)\,ds', \qquad n \geq 1. \end{aligned} \qquad \textbf{(3.8)}$$

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Taking LTs in (3.7) and (3.8) yields

$$\hat{f}_0(\alpha) = \hat{g}_c(\alpha + \lambda) \tag{3.9}$$

and

$$\hat{f}_n(\alpha) = \hat{T}_n(\alpha + \lambda|\lambda)\left(H_0(c) + \sum_{k=1}^{\infty} H_k(c)\,\hat{p}(\alpha + \lambda)^k\right), \qquad n \geq 1. \tag{3.10}$$

By Lemma 1, the second factor on the right-hand side of (3.10) is given by

$$\begin{aligned} H_0(c) + \sum_{k=1}^{\infty} H_k(c)\,\hat{p}(\alpha + \lambda)^k &= \hat{p}(\alpha + \lambda)^{-1} \sum_{k=0}^{\infty} H_k(c)\,\hat{p}(\alpha + \lambda)^{k+1} \\ &= \hat{p}(\alpha + \lambda)^{-1}\hat{g}_c(\alpha + \lambda). \end{aligned} \tag{3.11}$$

Now, summing (3.10) over all $n \geq 1$ and using (3.11) and (3.6) we can conclude that

$$\begin{aligned} \sum_{n=1}^{\infty} \hat{f}_n(\alpha) &= \hat{p}(\alpha + \lambda)^{-1}\hat{g}_c(\alpha + \lambda) \sum_{n=1}^{\infty} \hat{T}_n(\alpha + \lambda|\lambda) \\ &= \hat{p}(\alpha + \lambda)^{-1}\hat{g}_c(\alpha + \lambda)\left[1 - \frac{\hat{p}(\alpha) - \hat{p}(\alpha + \lambda)}{\hat{p}(\alpha + \lambda)} K(\alpha + \lambda)\right]^{-1} \hat{T}_1(\alpha|0), \end{aligned} \tag{3.12}$$

where we have set

$$K(\alpha) = \sum_{k=1}^{\infty} R_k(c)\,\hat{p}(\alpha)^k.$$

There is a simple relation between $K(\alpha)$ and $\hat{g}_c(\alpha)$: We have

$$\begin{aligned} \hat{g}_c(\alpha) &= \sum_{k=1}^{\infty} H_{k-1}(c)\,\hat{p}(\alpha)^k = \sum_{k=1}^{\infty} (R_{k-1}(c) - R_k(c))\,\hat{p}(\alpha)^k \\ &= \hat{p}(\alpha) \sum_{k=1}^{\infty} R_{k-1}(c)\,\hat{p}(\alpha)^{k-1} - K(\alpha) \\ &= \hat{p}(\alpha)[1 + K(\alpha)] - K(\alpha), \end{aligned}$$

so that

$$K(\alpha) = [\hat{p}(\alpha) - \hat{g}_c(\alpha)]/[1 - \hat{p}(\alpha)]. \tag{3.13}$$

We now show that

$$\hat{T}_1(\alpha|0) = [\hat{p}(\alpha) - \hat{p}(\alpha + \lambda)][1 + K(\alpha + \lambda)]. \tag{3.14}$$

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To prove (3.14), recall that, by (2.8), $\hat{T}_1(\alpha|0) = \int_0^\infty e^{-\alpha t}[\int_0^t \tau_1(s,t)\,ds]\,dt$ is the sum $I_1 + I_2$ of the two terms

$$I_1 = \int_0^\infty e^{-\alpha t}\left[\int_0^t \sigma(s)p(t)\,ds\right]dt \tag{3.15}$$

and

$$I_2 = \sum_{k=1}^\infty R_k(c)\int_0^\infty e^{-\alpha t}\left[\int_0^t \lambda e^{-\lambda s}\left(\int_0^s p^{*k}(s')p(t-s')\,ds'\right)ds\right]dt. \tag{3.16}$$

Since $\sigma(s) = \lambda e^{-\lambda s}$, we obtain

$$I_1 = \int_0^\infty e^{-\alpha t}p(t)(1-e^{-\lambda t})\,dt = \widehat{p}(\alpha) - \widehat{p}(\alpha+\lambda). \tag{3.17}$$

Let $U_\alpha(x) = \int_x^\infty e^{-\alpha t}p(t)\ dt$ and $V_{\alpha,k}(x) = e^{-\alpha x}p*^k(x)$. Then $\hat{U}_\alpha(\lambda) = \lambda^{-1}(\hat{p}(\alpha) - \hat{p}(\alpha+\lambda)$ and $\hat{V}_{\alpha,k}(\lambda) = \hat{p}(\alpha+\lambda)^k$. The kth term in the series (3.16) for I_2 can be computed as follows:

$$\begin{aligned}
&\iiint_{0<s'<s<t<\infty} \lambda e^{-\lambda s}e^{-\alpha t}p^{*k}(s')p(t-s')\,dt\,ds'\,ds \\
&= \iint_{0<s'<s<\infty} \lambda e^{-\lambda s}p^{*k}(s')e^{-\alpha s'}U_\alpha(s-s')\,ds\,ds' \\
&= \int_{0<s<\infty} \lambda e^{-\lambda s}(V_{\alpha,k} * G_\alpha)(s)\,ds \\
&= \lambda\hat{V}_{\alpha,k}(\lambda)\hat{U}_\alpha(\lambda) \\
&= [\hat{p}(\alpha) - \hat{p}(\alpha+\lambda)]\hat{p}(\alpha+\lambda)^k.
\end{aligned} \tag{3.18}$$

Combining (3.15)–(3.18) we arrive at (3.14).

Now, we can finally compute $\hat{f}$:

$$\begin{aligned}
\hat{f}(\alpha) &= \hat{f}_0(\alpha) + \sum_{n=1}^\infty \hat{f}_n(\alpha) \\
&= \hat{g}_c(\alpha+\lambda) + \frac{\hat{g}_c(\alpha+\lambda)[\hat{p}(\alpha)-\hat{p}(\alpha+\lambda)][1+K(\alpha+\lambda)]}{\hat{p}(\alpha+\lambda) - [\hat{p}(\alpha)-\hat{p}(\alpha+\lambda)]K(\alpha+\lambda)} \\
&= \frac{\hat{g}_c(\alpha+\lambda)\hat{p}(\alpha)}{\hat{p}(\alpha+\lambda) - [\hat{p}(\alpha)-\hat{p}(\alpha+\lambda)]K(\alpha+\lambda)} \\
&= \hat{g}_c(\alpha+\lambda)\hat{p}(\alpha)\left(\hat{p}(\alpha+\lambda) - [\hat{p}(\alpha)-\hat{p}(\alpha+\lambda)]\frac{\hat{p}(\alpha+\lambda)-\hat{g}_c(\alpha+\lambda)}{1-\hat{p}(\alpha+\lambda)}\right)^{-1} \\
&= \frac{\hat{g}_c(\alpha+\lambda)\hat{p}(\alpha)[1-\hat{p}(\alpha+\lambda)]}{\hat{g}_c(\alpha+\lambda)[\hat{p}(\alpha)-\hat{p}(\alpha+\lambda)] + \hat{p}(\alpha+\lambda)[1-\hat{p}(\alpha)]}.
\end{aligned}$$

We have used (3.9), (3.12), and (3.14) for the second equation and (3.13) for the fourth one. The theorem is proved. ■

Taking derivatives in (3.1) and setting $\alpha = 0$ yields the following:

COROLLARY 1: *The expected time between two overflows is*

$$-\frac{\hat{p}'(0)\,\hat{p}(\lambda)\,[1-\hat{g}_c(\lambda)]}{[1-\hat{p}(\lambda)]\,\hat{g}_c(\lambda)}.$$

References

1. Artalejo, J.R. & Gomez-Corral, A. (1998). Analysis of a stochastic clearing system with repeated attempts. *Stochastic Models* 14: 623–645.
2. Berman, O., Parlar, M., Perry, D., & Posner, M.J.M. (2005). Production/clearing models under continuous and sporadic reviews. *Methodology and Computing in Applied Probability* 7: 203–225.
3. Boxma, O.J., Perry, D., & Stadje, W. (2001). Clearing models for *M*/*G*/1 queues. *Queueing Systems* 38: 287–306.
4. El-Taha, M. (2002). A sample-path condition for the asymptotic uniform distribution of clearing processes. *Optimization* 51: 965–975.
5. Kim, K. & Seila, A.F. (1993). A generalized cost model for stochastic clearing systems. *Computers and Operations Research* 20: 67–82.
6. Liu, B. & Cao, J. (1999). Analysis of a production-inventory system with machine breakdowns and shutdowns. *Computers and Operations Research* 26: 73–91.
7. Moinzadeh, K. & Aggarwal, P. (1997). Analysis of a production/inventory system subject to random disruptions. *Management Science* 43: 1577–1588.
8. Perry, D. & Posner, M.J.M. (2002). Production-inventory models with an unreliable facility operating in a two-state random environment. *Probability in the Engineering and Informational Sciences* 16: 325–338.
9. Perry, D., Stadje, W., & Zacks, S. (2005). Sporadic and continuous clearing policies for a production/inventory system under an *M*/*G* demand process. *Mathematics of Operations Research*, 30: 354–368.
10. Serfozo, R. & Stidham, S. (1978). Semi-stationary clearing processes. *Stochastic Processes and Their Applications* 6: 165–178.
11. Stidham, S. (1974). Stochastic clearing systems. *Stochastic Processes and Their Applications* 2: 85–113.
12. Stidham, S. (1977). Cost models for stochastic clearing systems. *Operations Research* 25: 100–127.
13. Stidham, S. (1986). Clearing systems and (s, S) inventory systems with nonlinear costs and positive lead times. *Operations Research* 34: 276–280.
14. Wang, J., Cao, J., & Liu, B. (2002). Unreliable production-inventory model with a two-phase Erlang demand arrival process. *Computers and Operations Research* 43: 1–13.
15. Whitt, W. (1981). The stationary distribution of a stochastic clearing process. *Operations Research* 29: 294–308.
16. Yang, W.S., Kim, J.D., & Chae, K.C. (2002). Analysis of *M*/*G*/1 stochastic clearing systems. *Stochastic Analysis and Applications* 20: 1083–1100.