Math. Struct. in Comp. Science (1997), *vol.* 7, *pp.* 663–699. Printed in the United Kingdom

Subnets of proof-nets in multiplicative linear logic with MIX

GIANLUIGI BELLIN[†]

Équipe de Logique Mathématique, Université de Paris 7 Denis Diderot, CNRS (URA 753) - UFR de Mathématiques, 2 Place Jussieu - Case 7012, 75251 Paris Cedex 05 (France)

Received 24 May 1995; revised 1 July 1996

This paper studies the properties of the subnets of a proof-net for first-order Multiplicative Linear Logic without propositional constants ($\mathbf{MLL}^-$), extended with the rule of Mix: from $\vdash \Gamma$ and $\vdash \Delta$ infer $\vdash \Gamma, \Delta$. Asperti's correctness criterion and its interpretation in terms of concurrent processes are extended to the first-order case. The notions of *kingdom* and *empire* of a formula are extended from $\mathbf{MLL}^-$ to $\mathbf{MLL}^- + \mathbf{MIX}$. A new proof of the sequentialization theorem is given. As a corollary, a system of proof-nets is given for De Paiva and Hyland's Full Intuitionistic Linear Logic with Mix; this result gives a general method for translating Abramsky-style term assignments into proof-nets, and *vice versa*.

1. Introduction

1.1. *The significance of the Mix rule*

The structural rule of Mix[‡], namely

$$Mix\colon \frac{\Gamma \vdash \Delta \qquad \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}$$

is not accepted in Girard's system of (classical) Linear Logic (**LL**). Nevertheless, the presence of *Mix* is ubiquitous in researches on linear logic: it is satisfied by most models of linear logic, such as the denotational semantics of coherent spaces, the game-theoretic semantics and more. As the example of the game-theoretic semantics shows (Abramsky and Jagadeesan 1994; Hyland and Ong (manuscript)), results are often obtained for **LL** + **MIX** first, and additional efforts are then needed to refine them to the case of **LL**.

† Research supported by EC Individual Fellowship Human Capital and Mobility, contract n. 930142.

‡ The name comes from Girard (1987); the name *Mingle* has been used for a similar rule in relevance logic. The name Mix was used by Gentzen for the following rule:

$$\frac{\Gamma \vdash \Delta, A, \ldots, A \quad A, \ldots, A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}.$$

However, such a variant of Cut cannot be admitted in linear logic, so there is no real danger of confusing Gentzen's use of the term with Girard's.

https://doi.org/10.1017/S0960129597002326 Published online by Cambridge University Press

It could be argued that the rule of *Mix* represents forms of reasoning that are unavoidable in classical logic. Take, for instance, the example by Y. Lafont in the Appendix of Girard *et al.* (1989): given any two proofs $\mathscr{D}_1$ and $\mathscr{D}_2$ of $\vdash A$, let $\mathscr{D}$ be the derivation

$$\mathscr{D}:$$

$$Cut: \dfrac{\dfrac{\begin{matrix}\mathscr{D}_1\\ \vdash A\end{matrix}}{\vdash A, B} \qquad\qquad \dfrac{\begin{matrix}\mathscr{D}_2\\ \vdash A\end{matrix}}{B \vdash A}}{\dfrac{\vdash A, A}{\vdash A}}$$

Lafont correctly argues that, unlike intuitionistic logic, classical logic gives no justification for choosing between two possible reductions of the indicated cut, the first erasing $\mathscr{D}_1$ and yielding $\mathscr{D}_2$, the second erasing $\mathscr{D}_2$ and yielding $\mathscr{D}_1$. Using *Mix*, we have a third possibility, namely, reducing $\mathscr{D}$ to the following derivation

$$\mathscr{D}':$$

$$Mix: \dfrac{\begin{matrix}\mathscr{D}_1\\ \vdash A\end{matrix} \qquad\qquad \begin{matrix}\mathscr{D}_2\\ \vdash A\end{matrix}}{\dfrac{\vdash A, A}{\vdash A}}$$

If cut elimination must preserve the identity of the informal argument formalized by the given proof, common sense indicates that $\mathscr{D}'$ is intuitively very similar to $\mathscr{D}$, while $\mathscr{D}_1$ and $\mathscr{D}_2$ are certainly not identifiable with $\mathscr{D}$. We will not pursue the investigation of *Mix* in classical logic and consider the rule of *Mix* only in the context of linear logic. We will study only the *multiplicative fragment* **MLL** of linear logic; $\mathbf{MLL}^{-}$ denotes the multiplicative fragment *without the propositional constants* **1** and $\bot$.

Notice that several extensions of classical linear logic are possible:

(1) **LL** + **MIX**.
This system has the equivalent axiomatizations:
$\mathbf{LL} \quad + \quad \bot \vdash \mathbf{1}$
or
$\mathbf{LL} \quad + \quad A \otimes B \vdash A \wp B$, for all A and B.

(2) **LL** + **MIX** + the axiom *empty sequent* $\vdash$.
This system has the equivalent axiomatization:
$\mathbf{LL} \quad + \quad \bot \dashv\vdash \mathbf{1}$.

(3) $\mathbf{LL} + \ (A \otimes B) \dashv\vdash (A \wp B)$, for all A and B. We may call this system *'compact closed logic'*.

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(4) **LL** + *unrestricted Weakening*:

$$\frac{\vdash \Gamma}{\vdash \Gamma, A.}$$

This system has the equivalent axiomatization:
LL + $\perp \vdash \mathbf{0}$.
This logic is called *Affine logic* (**AL**).

It is easy to see that the sets of theorems in these systems satisfy

$$(1) \subset (2) \subset (3), \qquad (1) \subset (4), \qquad (2) \not\subset (4),$$

where inclusions are proper. We want to distinguish linear logic and these extensions by their metamathematical and semantical properties.

Affine logic differs considerably from linear logic in its metamathematical properties. For instance, it is known that propositional **AL** is decidable, while propositional linear logic is not. Actually, the idea of a proof-net may have been formulated for the first time for affine logic, which was studied by J. Ketonen and R. Weyhrauch (Ketonen and Weyhrauch 1984) in Stanford in the early 1980s and called *direct logic* (for an improved presentation of a system of proof-nets for multiplicative **AL** and for a discussion of the relations with linear logic, see Bellin and Ketonen (1992)). The proof of the sequentialization theorem for $\mathbf{MLL}^-$ + **MIX** given below goes back to the author's thesis (Bellin 1990) and to the research in Stanford on direct logic. Proof-nets for direct logic were presented as a decision procedure and applied to automated deduction (Ketonen 1984); no mathematical model of cut-elimination was given then. We will not study **AL** in this paper.

The *restricted* Weakening rule

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}$$

and the $\perp$-rule

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

are needed in the system of linear logic if intuitionistic and classical logics are to be interpreted within it. However, Weakening creates some problems in the process of cut-elimination, in particular with respect to the *Church–Rosser property*. As $\perp \vdash ?A$ for any A, the problem is already in the treatment of the $\perp$ rule, as we shall discuss below.

Concerning systems (1), (2) and (3), the denotational semantics of coherent spaces, which gave Girard a main motivation for the creation of **LL**, satisfies the system (2) as well as (1) (at least in its original form (Girard 1987)). Remember that classical **LL** is modelled by $*$-*autonomous* categories (Barr 1979; Barr 1991; Seely 1989) – with additional structure to interpret the exponentials – and that the logic (3) is modelled by *compact closed* categories; logics (1) and (2) can also be modelled by adding suitable morphisms in free $*$-autonomous categories.

An argument in favour of $\mathbf{MLL}^-$ + **MIX** was given by Girard in his fundamental paper: 'One of the arguments for MIX is that, without it, the type of communication considered in proof-nets is very totalitarian: everything communicates with everything, while MIX

https://doi.org/10.1017/S0960129597002326 Published online by Cambridge University Press

could accept more liberal solutions, typically two non-interconnected proof-nets, *etc.*' (Girard 1987, 99–100).

The suggestion that the logic **MLL**$^-$ + **MIX** might be more suitable for the representation of *parallel* logical computations than **MLL**$^-$ itself has been taken up by Asperti (Asperti 1995). He has given convincing evidence of this fact by showing that the verification of correctness for proof-nets in this logic is equivalent to the successful termination of a concurrent game in the style of Petri-nets.

In Girard (1989, Section II.6.), we read:

> 'If one were to accept this rule [*Mix*], then good taste would require to add the void sequent $\vdash$ as an axiom (without weakening, this has no dramatic consequence)'.

If we regard **1** and $\perp$ as weak notions of *truth* and *falsity*, any system allowing the *empty sequent* axiom is *paraconsistent*, in the sense that it allows *local* forms of inconsistency. Even if we reject the interpretation of **1** and $\perp$ as truth values, the meaning of the modalities '**!**' and '**?**' changes drastically if the *empty sequent* is an axiom: indeed by the restricted Weakening we have also $\vdash$ **?**A for all A; in particular, for any theorem A of linear logic both $\vdash$ **!**A and $\vdash (!A)^\perp$ hold. Thus in the system (2) the meaning of **?**A bears little resemblance to that of A.

There are reasons to consider the system (2) other than an interest in paraconsistent logics or good taste. Indeed the system (2) is well behaved with respect to cut-elimination and enjoys the Church–Rosser property. Therefore it could be used to settle the issue of Weakening, if there was a simple method of characterizing the proofs of $\vdash \Gamma$ in (2) that can be transformed into proofs of $\vdash \Gamma$ in linear logic.

In the first paper to be dedicated to the rule of Mix (Fleury and Retoré 1994), A. Fleury and C. Retoré developed the idea of a duality between the rule of Mix and the axiom *empty sequent*. Such a duality is formalized by assigning an integer (*truth-level*) to each sequent with the obvious assignments

$$\vdash^0 A^\perp, A \qquad \vdash^0 \mathbf{1} \qquad \frac{\vdash^m \Gamma, A \qquad \vdash^n \Delta, B}{\vdash^{m+n} \Gamma, \Delta, A \otimes B} \qquad \frac{\vdash^n \Gamma, A, B}{\vdash^n \Gamma, A \wp B},$$

and finally, by letting

$$\vdash^1 \qquad \vdash^1 \perp \qquad \frac{\vdash^m \Gamma \qquad \vdash^n \Delta}{\vdash^{m+n-1} \Gamma, \Delta}.$$

The system of proof-nets for (2) given by Fleury and Retoré (1994) is a natural extension of the system for **MLL**$^-$, and has good metamathematical properties: *e.g.*, it still enjoys the Church–Rosser property.

However, the notion *provable at the zero truth-level* does not coincide with *provable in multiplicative linear logic*, for example, $\vdash^0 \perp \otimes (\mathbf{1} \wp \mathbf{1})$. Some years ago there might have been some hope of finding a feasible algorithm that given a proof in the paraconsistent linear system (2) decides whether or not the proof can be transformed into a proof in linear logic. However, we now know from a result of Lincoln and Winkler (Lincoln and Winkler 1994) that the problem of deciding whether a theorem in the logic (2) is also a theorem of **MLL** is NP-complete; in other words, knowing a proof $\mathscr{D}$ of $\vdash \Gamma$ in the logic

(2) need not reduce the complexity of finding a proof of Γ in **MLL**: *recognizing* a proof in such a representation of **MLL** is as hard as *finding* one.

By contrast, to decide whether a theorem $\vdash \Gamma$ in $\mathbf{MLL}^- + \mathbf{MIX}$ is also a theorem of $\mathbf{MLL}^-$ it is enough to look at Γ, according to the formula (Fleury and Retoré 1994)

$$(*) \qquad \#par + \#conclusions = \#times + 2(\#Mix) + 2;$$

equivalently, if a proof-net with conclusions Γ is given, we may use

$$(**) \qquad \#Mix + \#axiom + 1 = \#par + \#conclusions + \#Cut.$$

Both formulas are easily proved by induction on the derivations in $\mathbf{MLL}^- + \mathbf{MIX}$.

These facts may be taken as evidence that the border between relevance and irrelevance, between consistency and paraconsistency remains marked very strongly within linear logic. This paper 'draws the line' between logics (1) and (2) and is interested in (1) *per se*.

Is the rule of Mix essentially a classical rule, or can it also occur in intuitionistic systems? The multiplicative fragment of *intuitionistic* linear logic may be axiomatized as Gentzen's systems as follows:

(I-1) (**ILL**) the intuitionistic linear consequence relation, axioms, Cut, Exchange, rules for the tensor and linear implication;

(I-2) the classical two-sided linear consequence relation, axioms, Cut, Exchange, rules for the tensor and the par, without negation;

(I-3) (**FILL**, *Full Intuitionistic Linear Logic*) the classical two-sided linear consequence relation, rules for tensor, par and linear implication, but special restrictions must be put on the right rule for implication to guarantee the intuitionistic nature of the connective. It was not immediately obvious how to formulate such restrictions so that the system would enjoy Cut-elimination, as Schellinx (1991) first pointed out.

Thus **FILL** simultaneously embodies features of concurrent logical computations, induced by its connective *par* and the sequential properties of intuitionistic linear implication.

Remember that the systems of intuitionistic linear logic have well-known categorical models: the system (I-1) is modelled by symmetric monoidal closed categories; the system (I-2) is modelled by *weakly distributive* categories (Blute 1993; Blute *et al.* 1996; Cockett and Seely 1992); the system (I-3) also has a categorical model, in fact it was inspired by one of V. de Paiva *Dialectica Categories* (de Paiva 1989a; de Paiva 1989b)).

The rule of Mix can be safely added to the system (I-2); moreover, one of de Paiva's Dialectica categories satisfies the Mix rule. Here we show that Mix can be safely added to (I-3), once the restriction on the right implication rule is correctly formulated. To this purpose, Hyland and de Paiva (1993) uses term assignments to the sequent calculus, which has been refined recently by Bierman. We give an equivalent formulation using a system of proof-nets for **FILL** + **MIX**; since our system admits cut-elimination, it yields an independent verification of the restrictions by term assignments on the intuitionistic implication. This application shows that the rule of Mix need not be excluded if we use linear logic as a framework for the representation of classical *and* intuitionistic logic.

1.2. *Why Proof-Nets?*

Proof-nets are among the most fascinating constructions to have arisen from linear logic. They provide a concise graph-theoretic representation of deductions in fragments and variants of linear logic – principally, first-order **MLL**$^-$ and **MLL**$^-$ + **MIX**. A beautiful part of the subject is the interaction between the *global* correctness conditions, which a proof-structure must satisfy to be a proof-net, and a *local* normalization process. Indeed, a main feature of proof-nets is the *decontextualization* of inferences, which are represented as vertices (*links*) in a graph (*proof-structure*), without distinction between conjunctions and disjunctions. This opens the way to a *concurrent* logical computation (*parallelization of the syntax*). Moreover, each normalization step of proof-nets reduces the size of the data, and the normalization process enjoys the Church–Rosser property.

More precisely, the relation between premises and conclusions of links induce a partial order on proof-structures, which will be called the *structural orientation*. The formulas associated with the premises are subformulas of the formula associated with the conclusion, so the structural orientation is in agreement with the relation of *being a subformula*. In fact, only the axiom (and perhaps Cut) links are needed to define a proof-structure, once a tree of subformulas is given. In this respect, proof-structures are like sequent derivations and unlike natural-deduction derivations. As inferences are decontextualized, the structural orientation is not tree-like, as it is in the sequent calculus: thus one of the functions of a correctness condition is to guarantee the possibility of recovering the tree-like order of a sequential proof.

There is a 'context-forgetting' map $(\ )^-$ from sequent calculus derivations in linear logic to proof-nets, such that $(\mathscr{D})^- = (\mathscr{D}')^-$ if and only if $\mathscr{D}'$ results from $\mathscr{D}$ by successive permutations of inferences. In other words, $(\ )^-$ is a bijection between proof-nets and the equivalence classes of sequent derivations *modulo* permutations of inferences. Given a proof-net $\mathscr{R}$, we have a polynomial time method to obtain a sequent derivation $\mathscr{D}$ such that $(\mathscr{D})^- = \mathscr{R}$ (*sequentialization theorem*). Several correctness conditions have been found. They are directly connected with the game-theoretic semantics of **MLL**$^-$, with coherence theorem in monoidal closed categories, *etc*., and provide tools for the study of normalization in the 'geometry of interaction'. There are tests of correctness that terminate in time at worst *quadratic* on the size of the proof-structure (Gallier (preprint)).

Girard's *no-short-trip* condition (Girard 1987) does not distinguish between correct and incorrect proof-structures for **MLL**$^-$ + **MIX**: this is done by Danos and Regnier's correctness condition (Danos and Regnier 1989), which requires the *acyclicity* of the D-R-graphs on the proof-nets in the case of **MLL**$^-$ + **MIX**, and, additionally, the *connectedness* of such graphs in the case of **MLL**$^-$ (of course, the additional requirement of connectedness may be replaced by counting according to the formulas (*) or (**)). The correctness criterion of Ketonen and Weyhrauch for direct logic also uses acyclicity of *chains*; a chain is just another notation for a path in the D-R-graph.

A significant contribution to proof-nets for **MLL**$^-$ + **MIX** has been given by A. Asperti (Asperti 1995). His criterion appears as the correct generalization to **MLL**$^-$ + **MIX** of Girard's *no-short-trip condition*. While Girard's trips are sequential processes, Asperti's trips are distributed processes. Initially, a token of type $\uparrow$ occurs on each conclusion and

Cut of the proof-structure. They propagate upwards, according to *A-switchings* for the *times* and *cut* links. Whenever both conclusions of an *axiom* are reached by tokens ↑, these are replaced by tokens ↓, which propagate downwards. When tokens ↓ have reached both premises of a *par* link, they are replaced by a token ↓ on the conclusion. When a *times* or *cut* link is reached for the second time, the trip continues with a token ↑ on the premise not yet reached, and so on. The process *terminates successfully* when there are tokens ↓ on all conclusions. A proof structure is correct if for every A-switching, Asperti's trip terminates successfully.

Asperti's trips can also be interpreted in terms of concurrent processes, with formula occurrences as processes. The *activation* or *termination* of a process A is the act of putting a token ↑ or ↓, respectively, on a formula A. A process $A \wp B$ is executed by executing in parallel the processes A and B. The rules of the game on *axioms* and *par* links are *syncronization* requirements between processes. The execution of a process $A \otimes B$ is the execution of A and B as *mutually exclusive* processes, in the order determined by an A-switch. Each A-switching imposes restrictions on the order of the execution of the processes, called *causal dependencies*; Asperti proved that a trip ends in a deadlock if and only if there is a *cyclic causal chain*. This is an interesting process-theoretic interpretation of the condition of acyclicity.

Proof-nets for *first-order* $\mathbf{MLL}^{-}$ were defined by Girard as a straightforward generalization of Danos and Regnier's condition for the propositional case (Girard (preprint)). There are switches on *for all* links, so the conclusion of a *for all* link may be connected either to the premise of the link or to any other formula containing the eigenvariable associated with that link.

One of the contributions of this paper is the extension of Asperti's criterion to the first-order case: when the premise of a *for all* link is activated, its eigenvariable is declared a *global variable*; now the premise of an *exists* link or of a Cut is activated only if all the eigenvariables occurring in it also occur in the list of global variables. Thus the correctness criterion for the first-order case is a natural requirement of synchronization between the activation of processes occurring in *for all* and *exists* links.

The bridge between local and global properties of proof-nets, thus the key to many results in the subject, is the study of the subnets of proof-nets; it is more instructive to study subnets not just as *subgraphs*, but as *subderivations of formulas*: in particular, we consider the *kingdom* $k(A)$ and the *empire* $e(A)$ of an occurrence A in a derivation, *i.e.*, the smallest and the largest subnet having A as a conclusion.

In the system **MLL** with the units **1** and $\bot$, a full decontextualization of inferences would require the introduction of an axiom of the form $\overline{\bot}$. By Lincoln and Winkler's result (Lincoln and Winkler 1994), proof-structures of this kind *underdetermine* a proof: in fact it is easy to construct examples where the same proof-structure corresponds to sequent derivations that are not equivalent *modulo* permutations of inferences. It is therefore necessary to indicate a substructure of a proof-structure where an axiom $\overline{\bot}$ is attached. This is obtained by introducing Weakening boxes (Girard 1987), but then the Church–Rosser property is lost. It suffices to attach the axiom $\overline{\bot}$ to any formula or link in the suitable substructure; it is convenient to choose this area 'as large as possible', the *empire* of $\overline{\bot}$. This idea has been developed and usefully applied in category theory (Blute *et al.*

https://doi.org/10.1017/S0960129597002326 Published online by Cambridge University Press

(to appear)). However, a proof-theorist could argue that such proof-nets are only a small improvement over sequent derivations, as they do not provide a unique representation of equivalence classes of sequent derivations *modulo* permutation of inferences.

In the case of $\mathbf{MLL}^-$ *without Mix* it has been shown by J. van de Wiele and the author (see Bellin and Scott (Theorem 13, 35–45)) that certain of Girard's trips on a proof-net correspond to translations of *intuitionistic* $\mathbf{MLL}^-$ (namely, the multiplicative fragment of **ILL**, see I-1 above) into *classical* $\mathbf{MLL}^-$. More precisely, there is an operation G mapping sequent derivations in *intuitionistic* $\mathbf{MLL}^-$ into sequent derivations in *classical* $\mathbf{MLL}^-$ as follows. In an intuitionistic sequent $\Gamma \vdash A$, the formulas in Γ may be regarded as *inputs* and A as the *output*. The operation G maps a sequent $\Gamma_I \vdash A_O$ to $\vdash \Gamma_I, A_O$, where $G(p_O) = p = G(p_I^\perp)$ and $G(p_I) = p^\perp = G(p_O^\perp)$ and, moreover,

$$G(A \otimes B)_O = A_O \otimes B_O \qquad G(A \otimes B)_I = A_I \wp B_I$$

$$G(A \multimap B)_O = A_I \wp B_O \qquad G(A \multimap B)_I = A_O \otimes B_I.$$

Conversely, we have the following fact: given a *cut-free* proof-net $\mathscr{R}$ with conclusions Γ, A in $\mathbf{MLL}^-$, every trip in the sense of Girard (1987) on a proof-net *reintroduces an input-output orientation* and thus corresponds to a derivation $\mathscr{D}'$ in *intuitionistic* $\mathbf{MLL}^-$ such that $\mathscr{R}$ may be regarded as $(G(\mathscr{D}'))^-$. If $\mathscr{R}$ contains cut-links, only trips that are compatible with the process of cut-elimination (*computationally consistent orientations*) yield such an intutionistic derivation.

This result essentially shows that one can simulate the structure of a *natural deduction derivation* on a proof-net by adding another ordering, the *input-output orientation*, which goes up in the proof-structure from formulas marked *input* to axioms – like in the *elimination part* of a natural deduction path – and down from axioms to formulas marked *output* – like in the *introduction part* of a natural deduction path.

Does an analogous result hold for $\mathbf{MLL}^-$ *with Mix*? The intuitionistic system **ILL** permits the use of Mix only with severe restrictions, which are removed in the systems (I-2) and (I-3), *i.e.*, in **FILL**. Now the translation of the system **FILL** into proof-nets is easy once an adequate restriction on the intutionistic implication rule has been found: the key notion here is that of a *directed chain*, which yields the requirement that for every par link $A_I \wp B_O$, if a directed chain from A_I eventually terminates in X_O, then $X = B$ (*functionality of implication*). On the other hand, the converse result has no analogue: there are proof-nets for $\mathbf{MLL}^-$ that are not in the image of any $\mathbf{FILL}^-$ derivation, *e.g.*, the only subnet with conclusions $p \wp q$, $q^\perp \wp r$, $r^\perp \wp q^\perp$, where functionality of implication fails for every admissible input-output orientation.

The dynamical interpretation of this result is that *every* cut-elimination process in *classical* $\mathbf{MLL}^-$ can be simulated by some cut-elimination process in *intuitionistic* $\mathbf{MLL}^-$; the classical nature of the dynamics of proof-nets emerges only in the fact that the correspondence is many-one. On the other hand, when *Mix* is introduced there are cut-elimination processes in classical $\mathbf{MLL}^- + \mathbf{MIX}$ that have no intuitionistic counterpart in $\mathbf{FILL}^- + \mathbf{MIX}$.

It would be desirable to give these rather technical features of proof-nets a more abstract mathematical presentation. We will not do this here, except for the following

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elementary remarks. Every proof-structure can be associated with a proof in the *compact closed* logic (3), or, in other words, it can be regarded as a morphism in a free compact-closed category. There is a functor F from $*$-autonomous categories to compact closed categories. There are morphisms in a free compact closed category that are not in the image of F, for example, $g \circ f$ where $f : \mathbf{1} \to A \otimes A^*$ and $g : A \otimes A^* \to \bot$. The test of the correctness conditions and the sequentialization algorithm on a proof-structure $\mathcal{R}$ are related to the construction of a morphism f in a free $*$-autonomous category such that $F(f)$ corresponds to $\mathcal{R}$; such a construction does not seem to be functorial.

Also, there is a functor G from symmetric monoidal closed categories to $*$-autonomous categories, which can be described as *forgetting the input-output orientation* of an intuitionistic derivation. Conversely, the result by Bellin and van de Wiele (Bellin and Scott 1994) describes a process of constructing a map in a free symmetric monoidal closed category, given a map in a compact closed category; such a process does not seem to be functorial.

R. Blute made an essential use of proof-nets for $\mathbf{MLL}^- + \mathbf{MIX}$ in his study of coherence in monoidal categories (Blute 1993). Moreover, (two-sided) proof-nets have been used to give categorical models of various extensions of the system (I-2) of weakly distributive categories (Blute 1993; Blute *et al.* 1996); hence proof-nets already play a role in the study of monoidal categories.

In conclusion, in this paper we present the following results:

1 a generalization of the theory of *empires* and *kingdoms* in Bellin and van de Wiele (1995) from $\mathbf{MLL}^-$ to $\mathbf{MLL}^- + \mathbf{MIX}$;
2 a proof of the sequentialization theorem for $\mathbf{MLL}^- + \mathbf{MIX}$;
3 an extension of Asperti's criterion to the first-order case;
4 a system of proof-nets for the multiplicative fragment of de Paiva and Hyland's **FILL** + **MIX**.

The following facts should be noted:

1 The theory of subnets of a proof-net in $\mathbf{MLL}^-$ *with Mix* does not coincide with that for $\mathbf{MLL}^-$ *without Mix* (Bellin and van de Wiele 1995). Indeed, the notion of subnet is trivialized here; instead we need the notion of a *normal* subnet, *i.e.*, a subnet whose sequentialization may be a subderivation of the sequentialization of the whole proof-net. In $\mathbf{MLL}^-$ *without Mix* every subnet is normal.
2 Our argument for the proof of the sequentialization theorem for $\mathbf{MLL}^- + \mathbf{MIX}$ is different from the other existing arguments in that we do not reduce to the case of $\mathbf{MLL}^-$ without Mix. We argue directly about the graph-theoretic configuration of *chains* and about the nesting of kingdoms.
3 The correctness condition for first-order proof-nets in terms of Asperti's games is an efficient characterization, which is more intuitive than the usual graph-theoretic condition of acyclicity using *for all* switches.
4 The specific form of our sequentialization argument for $\mathbf{MLL}^- + \mathbf{MIX}$ easily extends to a proof-net representation of *Full Intuitionistic Linear Logic with Mix*; the method used here for translating sequent calculi with λ-term assignments in the style of Abramsky (1993) into proof-nets, and *vice versa*, may be more generally applicable.

1.3. *Further directions*

A considerable amount of research has taken place on linear logic with *Mix* and other variants in the past decade; proof-nets and other formal tools for these logics are better understood. What directions remain open?

The rule of Mix may find a place in the study of the cut-elimination procedure for classical logic, as indicated above. However, the most interesting developments in the theory of proof-nets will focus on the role of the Exchange rule. *Noncommutative* linear logic, which excludes Exchange, has found applications to computational linguistics. The Exchange rule is studied by the embedding of the proof-graphs in topological spaces. *Braided* proof nets are proof-nets embedded in the space R^3; the embedding of proof-nets in R^2 yields a representation of commutative linear logic in a non-commutative enviromnent with the explicit rule of exchange (Bellin and Fleury 1995). In that context, the techniques of this paper find a more natural presentation: for instance, the correctness criterion for proof-nets in R^2 terminates in *linear time* (Bellin and Fleury 1995). Here, linear logic meets interesting and well-known mathematical objects, such as braids, tangles and knots, and proof-nets are found similar to notations used in physics.

2. Subnets of proof-nets in $\mathbf{MLL}^- + \mathbf{MIX}$

We begin this section with the definitions of the sequent calculus (2.1) and the standard definition of proof-structures and proof-nets. Then we state the sequentialization theorem (2.2) and motivate the definitions of normal subnets, kingdom and empires using the correspondence with subderivations of sequent derivations (2.3). We then present the descriptive notions of a chain and a loop (2.4); with these tools, a characterization is given of kingdoms and empires and of the ordering of the kingdoms (2.5). Finally, we give our proof of the sequentialization theorem, and the characterization of permutations of inferences is obtained (2.6).

2.1. *The language and the sequent calculus*

The first-order language of linear logic is defined in Girard (1987); we consider the first-order $\mathbf{MLL}^-$ (Multiplicative Linear Logic without Constants) fragment. Remember that the operation $(\ .\ )^\perp$ (*linear negation*) applies to atomic formulas only; formulas are built from atoms $p_1, \ldots$ and their negations using the binary connectives $\otimes$ (*times*) and $\wp$ (*par*), and the quantifiers $\forall$ (*for all*) and $\exists$ (*exists*); and negation of non-atomic formulas is defined by $p_i^{\perp\perp} =_d p_i$, $(A \otimes B)^\perp =_d A^\perp \wp B^\perp$, $(A \wp B)^\perp =_d A^\perp \otimes B^\perp$, $(\forall x.A)^\perp =_d \exists x.(A^\perp)$, $(\exists x.A)^\perp =_d \forall x.(A^\perp)$.

The sequent calculus for first-order $\mathbf{MLL}^-$ (Girard 1987) contains *logical axioms*, *cut*, the structural rule of *exchange* and the logical rules for *times*, *par*, *for all* and *exists*.

logical axiom: $\vdash A^\perp, A$ *cut:* $\dfrac{\vdash \Gamma, A^\perp \quad \vdash \Delta, A}{\vdash \Gamma, \Delta}$ *exchange:* $\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$

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$$\textit{times:}\ \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}, \qquad \textit{par:}\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \qquad \textit{for all:}\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x.A} \text{ where } x \notin \Gamma; \qquad \textit{exists:}\ \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x.A}$$

We focus on the extension of $\mathbf{MLL}^{-}$ with the structural rule of *Mix*:

$$\textit{Mix}: \quad \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}.$$

We use the standard terminology for sequent calculi, namely we speak of the *passive*, *active* and *principal formulas* in an inference: *e.g.*, all formula occurrences are passive in a Mix, the cut formulas are active in a Cut, *etc*. Let $\mathscr{I}_1/\mathscr{I}_2$ be a pair of consecutive inferences in a derivation such that the principal formula of $\mathscr{I}_1$ is not active in $\mathscr{I}_2$; observe that we can always permute the order of these inferences, except in the cases indicated in the following table.

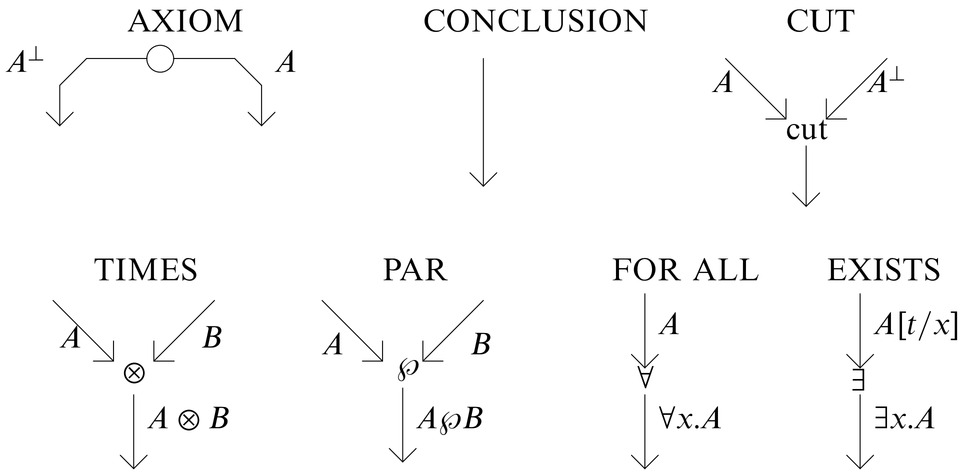
$\mathscr{I}_1$: $\mathscr{I}_2$:	Cut	Mix	$\otimes$	$\wp$	$\forall$	$\exists$
Cut						
Mix						
$\otimes$						
$\wp$	no	no	no			
$\forall$						no
$\exists$						

2.2. *Proof-structures*

In Girard's original formulation (Girard 1987) a proof-structure is a set of formula occurrences and links, where a link is a relation between formula occurrences; this leads to a graphic representation of proof-structures where vertices are associated with formulas and edges with links. More recently, a variant graphic formulation has become usual, and in this, vertices are associated with links and edges with formulas; we will adopt the latter formulation.

Definitions 1.

(i) A *proof-structure* is a graph whose edges are oriented and labelled with occurrences of formulas. (In most cases we will not keep the distiction between edges and their labels in our text.) Vertices are either *conclusions*, with one incident edge, or *links*: with *axioms*, *for all*, *exists* links, of incidence 2, and *cut*, *times* and *par* links, of incidence 3. The arrows pointing at a link are its *premises*, the other incident arrows are its *conclusions*, as indicated in the following table:



(ii) First-order proof-structures must satisfy the following conditions on free and bound variables; for further details and motivations, see Bellin and van de Wiele (1995). Remember that an *eigenvariable* is a free variable that becomes bound in a universal quantification; to each eigenvariable x associate a distinct constant $\underline{x}$.

(a) Each occurrence of a quantifier link uses a distinct bound variable.

(b) If a variable occurs freely in some formula of the structure, the variable is the eigenvariable of exactly one $\forall$-link.

(c) The conclusions of the proof structure are closed formulas.

(d) (*Strictness condition*) No substitution of any number of occurrences of an eigenvariable x with the constant $\underline{x}$ yields a correct proof structure with the same conclusions.

(iii) A Danos-Regnier graph (*D-R-graph*) is the graph resulting from the following transformations:

— for each *par* link select one premise and remove its connection with the link;

— for each *for all* link $\mathscr{L}$ with conclusion $\forall x.A$ select a link $\mathscr{L}'$ whose premise B contains the eigenvariable x; introduce an edge between $\mathscr{L}$ and $\mathscr{L}'$ and at the same time remove the existing connection between the edge A and the link $\mathscr{L}$; if no such B exists, leave the link unchanged.

The set of these choices is called a *switching*; if s is a switching on a proof-structure $\mathscr{R}$, the D-R-graph is written $s\mathscr{R}$. A path ending with the edges A and B in the D-R-graph $s\mathscr{R}$ will be denoted by $path_s(A, B)$.

(iv) A proof-structure $\mathscr{R}$ is a *proof-net* for $\mathbf{MLL}^{-} + \mathbf{MIX}$ if for every switching s the D-R-graph $s\mathscr{R}$ is *acyclic*.

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(v) A *substructure* $\mathscr{R}'$ of a proof-structure $\mathscr{R}$ is a proof-structure together with an embedding $\iota : \mathscr{R}' \to \mathscr{R}$ such that if $A = \iota(A')$, then A' results from A by substitution of eigenvariables with constants (each eigenvariable x being replaced by a distinct constant $\underline{x}$). Obviously, in the propositional case we may let ι be the identity function on a subgraph. Given a formula A in a proof-structure $\mathscr{R}$, which is the conclusion of a link v, we write $st(A)$, or $st(v)$, for *the smallest substructure with A as a conclusion.*

(vi) A *subnet* of a proof-net is a substructure that is a proof-net.

One aim of this paper is to give a new proof of the following theorem.

Theorem 1. There exists a 'context-forgetting' map $(\ .\ )^-$ from sequent derivations in first-order $\mathbf{MLL}^- + \mathbf{MIX}$ to proof-nets with the following properties:

(a) Let $\mathscr{D}$ be a derivation of Γ in the sequent calculus for $\mathbf{MLL}^- + \mathbf{MIX}$; then $(\mathscr{D})^-$ is a proof-net with conclusions Γ.

(b) (Sequentialization) If $\mathscr{R}$ is a proof-net with conclusions Γ for $\mathbf{MLL}^- + \mathbf{MIX}$, there is a sequent calculus derivation $\mathscr{D}$ of Γ such that $\mathscr{R} = (\mathscr{D})^-$.

(c) If $\mathscr{D}$ reduces to $\mathscr{D}'$, then $(\mathscr{D})^-$ reduces to $(\mathscr{D}')^-$.

(d) If $(\mathscr{D})^-$ reduces to $\mathscr{R}'$, there is a $\mathscr{D}'$ such that $\mathscr{D}$ reduces to $\mathscr{D}'$ and $\mathscr{R}' = (\mathscr{D}')^-$.

The proofs of parts (a), (c) and (d) are easy; we will focus on part (b).

2.3. *Permutation of inferences and subnets*

Another aim of this paper is to develop a theory of subnets of proof-nets that yields an answer to the following question: *given an inference $\mathscr{I}$ in a sequent derivation $\mathscr{D}$ in* $\mathbf{MLL}^-$ + $\mathbf{MIX}$*, which inferences $\mathscr{I}'$ of $\mathscr{D}$ can be permuted* above *or* below *$\mathscr{I}$?*

For the system $\mathbf{MLL}^-$ *without Mix* the answer is given by Theorem 2 in Bellin and van de Wiele (1995). The largest and the smallest subnet having A as a conclusion are called the *kingdom kA* and the *empire eA* of A, respectively. Now let $\mathscr{I}$, $\mathscr{I}'$ be inferences in $\mathscr{D}$ and let v, v' be the correponding links in $(\mathscr{D})^-$; suppose v is a *par* or *times* link with premises A, B and conclusion C. Now $\mathscr{I}'$ can be permuted below $\mathscr{I}$ if v' does not occur in kC. To see whether $\mathscr{I}'$ can be permuted above $\mathscr{I}$, we look to see whether v' occurs in $eA \cup eB$ if $\mathscr{I}$ is a *times* rule; if $\mathscr{I}$ is a *par* rule, we look to see whether v' occurs in eC.

We will obtain a similar result for $\mathbf{MLL}^- + \mathbf{MIX}$, but in order to do this we must strengthen the notions of kingdom and empire. In order to see this, consider following derivations:

$$\mathscr{D}_1:\quad \dfrac{\dfrac{\dfrac{\vdash A^\perp, A \qquad \vdash B, B^\perp}{\vdash A^\perp, A, B, B^\perp}\,\text{Mix}}{\vdash A, B, B^\perp, A^\perp}\,\textit{exchanges}}{\vdash A, B, B^\perp \wp A^\perp}\,\wp \qquad\qquad \mathscr{D}_2:\quad \dfrac{\dfrac{\dfrac{\vdash A^\perp, A \qquad \vdash B, B^\perp}{\vdash A^\perp, A \otimes B, B^\perp}\,\otimes}{\vdash A \otimes B, B^\perp, A^\perp}\,\textit{exchanges}}{\vdash A \otimes B, B^\perp \wp A^\perp}\,\wp$$

The 'context forgetting' map sends $\mathscr{D}_1$ and $\mathscr{D}_2$ to the following proof-nets for $\mathbf{MLL}^-$ + $\mathbf{MIX}$:

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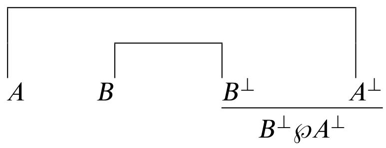
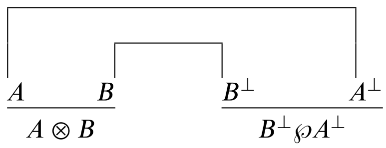
$\mathscr{R}_1 = (\mathscr{D}_1)^- :$



$\mathscr{R}_2 = (\mathscr{D}_2)^- :$



In $\mathbf{MLL}^- + \mathbf{MIX}$, $\mathscr{R}_1$ is the largest subnet of $\mathscr{R}_2$ having A as a conclusion. However, $\mathscr{D}_1$ is *not* a subderivation of $\mathscr{D}_2$ nor of any derivation resulting from $\mathscr{D}_2$ by permutations of inferences; also, we cannot permute the *par* rule above the *times* rule in $\mathscr{D}_2$. Notice that in $\mathbf{MLL}^-$, $\mathscr{R}_1$ is a substructure of $\mathscr{R}_2$, not a subnet; but in $\mathbf{MLL}^- + \mathbf{MIX}$ any *substructure* of a proof-net satisfies the acyclicity condition, hence it is a *subnet*.

Definitions 2.

(i) A *non-logical axiom* is a link with no premise and n conclusions, for some n. We consider proof-structures with non-logical axioms; D-R-graphs for such structures are defined as before. A *proof-net with non-logical axioms* for $\mathbf{MLL}^- + \mathbf{MIX}$ is a proof-structure with non-logical axioms that satisfies the acyclicity condition.

(ii) Let $\mathscr{R}$ be a proof-structure for $\mathbf{MLL}^-$ and let $\mathscr{S}$ be a substructure of $\mathscr{R}$ with conclusions $C_1, \ldots, C_n$. The *complementary substructure* $\overline{\mathscr{S}}$ *of* $\mathscr{S}$ *in* $\mathscr{R}$ consists of all edges and links in $\mathscr{R} \setminus \mathscr{S}$ and, in addition, the edges $C_1, \ldots, C_n$ and a new non-logical axiom $\overline{C_1, \ldots, C_n}$ with these edges as conclusions.

(iii) Let $\mathscr{R}$ be a proof-net for $\mathbf{MLL}^- + \mathbf{MIX}$. A subnet $\mathscr{S}$ of $\mathscr{R}$ is *normal* if the complementary substructure $\overline{\mathscr{S}}$ of $\mathscr{S}$ is a proof-net with a non-logical axiom.

(iv) The *kingdom kA* (or the *empire eA*) of A in $\mathscr{R}$ is the smallest (the largest) *normal* subnet of $\mathscr{R}$ that has A as a conclusion.

Proposition 1. Let $\mathscr{R}$ be a proof-net for $\mathbf{MLL}^- + \mathbf{MIX}$. A subnet $\mathscr{S}$ of $\mathscr{R}$ is normal if and only if the following condition is satisfied: for every X and Y in $\mathscr{S}$ and for every switching s of $\mathscr{R}$, if there is a $path_s(X, Y)$ connecting X and Y in $s\mathscr{R}$, then such a path belongs also to $s\mathscr{S}$ (where we use the same symbol s for the switching of $\mathscr{R}$ and its restriction to $\mathscr{S}$).

Corollary 1. The intersection of two normal subnets is a normal subnet. The union of two normal subnets *need not* be normal.

Proof. The proof is left as an exercise. □

It is not immediately obvious how to prove that given a proof-net $\mathscr{R}$ for $\mathbf{MLL}^- + \mathbf{MIX}$ and a formula occurrence A in $\mathscr{R}$, there exists a subnet that is *normal* and has A as a conclusion. Girard's inductive definition of empires (Bellin and van de Wiele 1995, Proposition 2, Sections 2.3 and 3.3) cannot be used for this purpose in $\mathbf{MLL}^- + \mathbf{MIX}$: in the following example the inductive definition in question applied to $A \wp B$ does not identify a *normal* subnet with $A \wp B$ as a conclusion.

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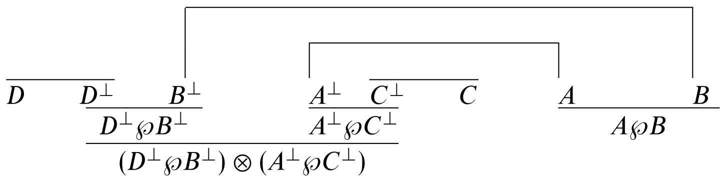

2.4. *Paths, chains and loops*

The definition of a *chain* is just a notational variant for the notion of a *path* in a D-R-graph, which has been used in direct logic (Bellin and Ketonen 1992) and later in the work by Asperti (Asperti 1995). Without introducing switches on the *par* links, a chain between A and B is defined as a path from A to B in the proof-structure that changes direction (with respect to the structural orientation) only at *axioms*, *times* and *cut* links. The notion of a *loop* is fundamental for the study of normal subnets; its geometric properties can be understood best if we consider proof-structures as embedded in a plane, since then a loop is just a particular kind of a 2-cell and a more efficient correctness criterion can be defined in terms of the 2-cells (Bellin and Fleury 1995).

Definitions 3.

(i) The relation between a premise and the conclusion of a link has a transitive closure, which we denote by $\prec$; if $A \prec X$, we say that A is a *hereditary premise* of X or that X is a *hereditary conclusion* of A. Obviously, $\prec$ is also an ordering of links. It may be called the *structural orientation* of the proof-structure.

(ii) In a *first-order* proof-structure the *for all* switches introduce edges between links that are not in the $\prec$ relation. Thus, given a switching s, we consider the order $\prec_s$ of links defined as follows: $v \prec_s u$ if and only if there are *for all* links $w_1, \ldots, w_n$ and links $v_0, \ldots, v_{n+1}$ such that the switching s yields $s(w_i) = v_i$ and, moreover, we have $v = v_0 \preceq v_1,\ w_1 \preceq v_2, \ldots, w_n \preceq v_{n+1} = u$.

(iii) Let $\mathscr{R}$ be a proof-structure and consider $path_s(v, w)$, the path from the links v and w ending with edges A to B in the D-R-graph $s\mathscr{R}$, also written $path_s(A, B)$. Then $path_s(v, w)$ is called a *chain* of the *type* indicated by the following table:

TYPE :	v has A as a	w has B as a
$\lceil v, w \rceil$ or $\lceil A, B \rceil$	premise	premise
$\lceil v, w \rfloor$ or $\lceil A, B \rfloor$	premise	conclusion
$\lfloor v, w \rceil$ or $\lfloor A, B \rceil$	conclusion	premise
$\lfloor v, w \rfloor$ or $\lfloor A, B \rfloor$	conclusion	conclusion.

The notation '$|A, B\rceil$' stands for 'either $\lceil A, B \rceil$ or $\lfloor A, B \rceil$', and, similarly, for $\lceil A, B|$, $|A, B|$. We abbreviate 'a chain γ of type $\lceil A, B \rceil$' by 'a chain $\lceil A - \gamma - B \rceil$'.

(iv) A chain $\lceil A - \gamma - B \rceil$ where A and B are premises of a *par* link will be called a *loop*. The *par* link in question (or its conclusion) is called the *exit* of the loop.

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(v) We define a relation $\ll$ as follows: let $X \ll_0 Y$ if $X \prec Y$ (or $X \prec_s Y$ for some switching s, in the first-order case). Moreover, let $X \ll_1 Y$ if X is in a loop γ with exit Y. Let $\ll$ be the transitive closure of $\ll_0$ and $\ll_1$. Obviously, these relations apply to links as well.

The proofs of the following propositions are left as an easy exercise.

Proposition 2. Let $\mathscr{R}$ be a proof-net. Let A be the conclusion of a link v in $\mathscr{R}$. The *smallest substructure* of $\mathscr{R}$ *with* A *as a conclusion*, denoted by $st(A)$, or $st(v)$, is characterized as the set of links w such that $w \ll_0 v$, together with the edges adjacent to any such w.

Proposition 3. A chain γ of type $\lceil A, B \rceil$ in $\mathscr{R}$ or $\lfloor A, B \rfloor$ has the form

CHAIN OF TYPE $\lceil A, B \rceil$ CHAIN OF TYPE $\lfloor A, B \rfloor$

where the w_i are *Times* or *Cut* links and the v_j are either axioms or links selected by a $\forall$-switch. In a chain of type $\lfloor A, B \rfloor$ we have $w_i \gg_0 v_i \ll_0 w_{i+1}$ for $i \leqslant n$, and, moreover, $A \ll_0 w_1$, $B \ll_0 w_{n+1}$. Similar facts hold for chains of other types.

We call the links w_i *lower links* of the chain γ; A, B and the conclusions of the links $w_1, \ldots, w_n$ are called the *lower members* of γ.

Proposition 4. Let $\mathscr{R}$ be a proof-net, let u be any link in $\mathscr{R}$, let $\gamma_1 = \text{path}_{s'}(v, u)$ and $\gamma_2 = \text{path}_{s''}(u, w)$ be incident to u by different edges. Then one of the following is the case:

1 u is a *par* or *for all* link and the chains have types $\lfloor v - \gamma_1 - u \rceil$, $\lceil u - \gamma_2 - w \rfloor$;
2 otherwise, if $\gamma_1 \cap \gamma_2 = \{u\}$, there exists a chain $\gamma = path_s(v, w)$, for some s, the *concatenation* of γ_1 and γ_2, written $\gamma = \gamma_1 * \gamma_2$;
3 otherwise, there exists a *par* or *for all* link v' such that

$$\gamma_1 = \lfloor v - \gamma_1^1 - v' \rfloor * \lceil v' - \gamma_1^2 - u \rfloor, \qquad \gamma_2 = \lfloor u - \gamma_2^1 - v' \rceil * \lfloor v' - \gamma_2^2 - w \rfloor$$

and $\gamma_1^2 * \gamma_2^1$ is a loop with exit v'.

2.5. *Kingdoms and their ordering*

In the case of **MLL**$^-$ without Mix the fact that the relation $\ll$ is an ordering follows easily from a simple fact about the nesting of empires and kingdom (Bellin and van de Wiele 1995, Lemma 3). But such a nesting no longer holds in **MLL**$^-$ + **MIX**, and an explicit graph-theoretic analysis is needed to prove the property of ordering. The definition of kingdom is a direct generalization of the one for **MLL**$^-$ without Mix (Bellin and van de Wiele 1995, Proposition 3); the definition of empire is due to Asperti (1995).

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Lemma 1. If $\mathscr{R}$ a proof-net for $\mathbf{MLL}^- + \mathbf{MIX}$ and $A \ll B$, there exist chains γ of type $|A, B\rceil$. Moreover, we may assume that either A is a lower member of γ or γ is of type $\lfloor A, B\rceil$.

Proof. The argument is by induction on $\ll$. For the base case, let γ_0 be any loop with exit v_0. If $A \in \gamma_0$ or if $A \ll_0 u$ with u in γ_0 and $v_0 \ll_0 B$, then, by choosing suitable switches and using a part of γ_0, we find a chain $|A - \gamma - B\rceil$. Such a chain will be of type $\lfloor A, B\rceil$, unless A is a lower member of γ_0, in which case γ is of type $\lceil A, B\rceil$.

For $i = 0, \ldots, n$, let γ_i be a loop with exit v_i, and let u_i be a link in γ_i such that

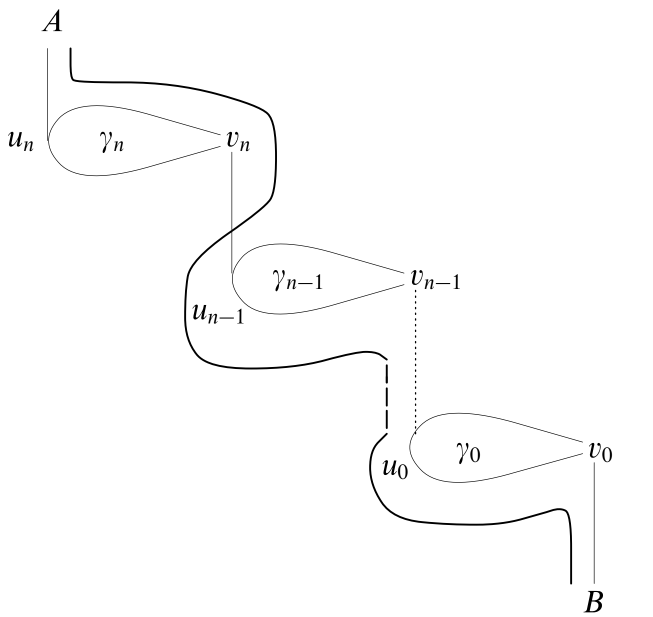
$$A \ll_0 u_n \ll_1 v_n \quad \ll_0 \quad u_{n-1} \ll_1 v_{n-1} \quad \ll_0 \ldots u_0 \quad \ll_1 v_0 \ll_0 B.$$

By the induction hypothesis, we have a chain $\lfloor v_n - \gamma - B\rceil$. We show that $\gamma_n \cap \gamma = \{v_n\}$. Suppose this is not true. Starting from v_n, follow γ and let u be the first link in γ, different from v_n, such that $u \in \gamma \cap \gamma_n$.

If Case (1) of Proposition 4 does not apply to u, we are in Case (2) and there are subchains $\overline{\gamma_n}$ of γ_n and $\overline{\gamma}$ of γ whose concatenation $\lfloor v_n - \overline{\gamma} * \overline{\gamma_n} - v_n\rceil$ is cyclic; this is impossible in a proof-net.

Therefore u is a *par* or *for all* link and the subchain $\overline{\gamma}$ of γ from v_n to u is of type $\lfloor v_n, u\rceil$. Since γ_n is a loop, there is a lower link $w \in \gamma_n$ such that $u \ll_0 w$, and therefore we can find a subchain $\overline{\gamma_n}$ of γ_n of type $\lfloor u, v_n\rceil$. But $\overline{\gamma_n}$ and $\overline{\gamma}$ intersect only at $\{v_n, u\}$; thus $\lfloor v_n - \overline{\gamma} * \overline{\gamma_n} - v_n\rceil$ is cyclic, which is again a contradiction.

Hence $\gamma_n \cap \gamma = \{v_n\}$. The argument for the base case yields a subchain γ' of γ_n of type $|A, v_n\rceil$ and the result is given by $\gamma' * \gamma$. □



Corollary 2. (Kingdom Ordering) In a proof-net for $\mathbf{MLL}^- + \mathbf{MIX}$ the relation $\ll$ is an ordering.

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The name of the Corollary is justified by the following characterization of kingdoms.

Lemma 2. Let $\mathscr{R}$ be a proof-net for **MLL**$^-$ + **MIX** and let A be a formula occurrence in $\mathscr{R}$. Then kA, the smallest normal subnet having A as a conclusion, exists and is characterized by the following equivalent conditions:

(a) the smallest set closed under the induction conditions

(0) $A \in kA$.

(i) If $v = \overline{X \quad X^\perp}$, then $k(v) = kX = kX^\perp$.

(ii) If $v = \dfrac{X \quad Y}{X \otimes Y}$, then $k(v) = k(X \otimes Y) = kX \cup kY \cup \{X \otimes Y\}$,

and similarly for $v = \dfrac{A[t/x]}{\exists x.A}$.

(iii) If $v = \dfrac{X \quad Y}{X \wp Y}$, then

$$k(v) \;=\; kX \wp Y \;=\; \bigcup_s \bigcup_{Z \in path_s(X,Y)} kZ \cup \{X \wp Y\}$$

where s ranges over all switchings of $\mathscr{R}$,

$$= \bigcup_{\lceil X - \gamma - Y \rceil} \bigcup_{Z \in \gamma} kZ \cup \{X \wp Y\},$$

where γ ranges over loops,

and similarly for $v = \dfrac{A}{\forall x.A}$.

(b) $\bigcup_{X \ll A} st(X) \;\cup\; st(A)$.

Proof. It is easy to see, by induction on $\ll$, that Conditions (a) and (b) define the same set, call it kA: indeed

$$\begin{aligned} kA &= st(A) \;\cup \bigcup_{X \wp Y \leq A} k(X \wp Y) \\ &= st(A) \;\cup \bigcup_{X \wp Y \leq A} \; \bigcup_{Z \in \lceil X - \gamma - Y \rceil} kZ \\ &\quad \text{where } \gamma \text{ ranges over loops} \\ &= st(A) \;\cup \bigcup_{X \ll_1 A} kX, \end{aligned}$$

from which the result follows by the induction hypothesis. Now the set (b) is clearly a substructure of $\mathscr{R}$, hence (in **MLL**$^-$ + **MIX**) it is a subnet of $\mathscr{R}$. The fact that kA has A as a conclusion follows from the fact that $\ll$ is an order. It remains to show that kA is a normal subnet.

Let V and Z occur in kA and let $\gamma = path_s(V, Z)$ be any chain such that $\gamma \not\subset kA$. Starting from V, follow γ and let U be the first element such that U is in kA but the

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next element U' is not. Similarly, continuing from U' along γ, let W' be the first element such that W' does not belong to kA, but the next element W does. By Lemma 1, we have chains $\lceil A - \gamma_U - U \rceil$ and $\lceil A - \gamma_W - W \rceil$; we also have a subchain $\lfloor U' - \overline{\gamma} - W' \rfloor$ of γ, and $\overline{\gamma}$ is disjoint from γ_U and γ_W. By Proposition 4, the concatenation $\gamma_U * \overline{\gamma} * \gamma_W$ yields a loop with exit in kA; but then $\overline{\gamma} \subset kA$ and this is a contradiction. Hence kA is a normal subnet with A as a conclusion, indeed the smallest subnet with these properties, since Conditions (a) must be satisfied by any normal subnet containing A. □

Lemma 3. Let $\mathscr{R}$ be a proof-net for $\mathbf{MLL}^- + \mathbf{MIX}$ and let A be a formula occurrence in $\mathscr{R}$. Then eA (the largest normal subnet having A as a conclusion) exists and is characterized by the condition

(a) $$\{X : \text{ there is no chain } |X, \ldots, A\rfloor \text{ in } \mathscr{R}\}$$

Proof. Normal subnets with A as a conclusion exist, by Lemma 2. If there is a chain $|X, \ldots, A\rfloor$ and $\mathscr{S}$ is a normal subnet containing X and A, then A cannot be a conclusion of $\mathscr{S}$. Hence every normal subnet with A as a conclusion is included in the set (a).

If a formula occurrence X is in the set (a), all the hereditary premises of X and the axioms above them are in the set (a). Therefore the set (a) is a substructure of $\mathscr{R}$, hence (in $\mathbf{MLL}^- + \mathbf{MIX}$) it is a subnet. To see that the set (a) is normal, let X and Y be distinct formula occurrences in $\mathscr{R}$ and let $|X - \gamma - Y|$ be a chain such that some link w in γ is not in the set (a). This means that there is a chain γ' of type $|w, A\rfloor$. We may assume that $\gamma \cap \gamma' = \{w\}$ (otherwise we take a subchain of γ'), and that γ is the concatenation

$$\gamma = |X - \gamma_X - w\rceil * \lfloor w - \gamma_Y - Y|.$$

If γ' has type $\lfloor w, A\rfloor$, then $\gamma_X * \gamma'$ is a chain of type $|X, A\rfloor$. If γ' has type $\lceil w, A\rfloor$, then $\gamma_Y * \gamma'$ is a chain of type $|Y, A\rfloor$. In both cases we contradict the fact that X and Y belong to the set (a). □

2.6. *The sequentialization theorem*

As pointed out above, the structure of kingdoms and empires in $\mathbf{MLL}^- + \mathbf{MIX}$ is different from that in $\mathbf{MLL}^-$ without *Mix*; as a consequence, the standard proof of the sequentialization theorem for $\mathbf{MLL}^-$ does not carry through. Crucial to our proof is the *ordering of the kingdoms* and a direct graph-theoretic analysis, which in substance was given in the author's thesis (Bellin 1990). The general structure of the argument differs from those in Fleury and Retoré (1994) and Asperti (1994); an original feature of our proof is the fact that we do not reduce the problem to the case of $\mathbf{MLL}^-$ without Mix.

Theorem 1.b. If $\mathscr{R}$ is a proof-net for $\mathbf{MLL}^- + \mathbf{MIX}$ with conclusions Γ, then there is a sequent calculus derivation $\mathscr{D}$ of Γ such that $\mathscr{R} = (\mathscr{D})^-$.

Proof. By induction on the size of $\mathscr{R}$. The following case is trivial:

Case 1.

1 $\mathscr{R}$ is an axiom;

2 $\mathscr{R}$ consists of two proof structures without axiomatic connections with each others: just apply the induction hypothesis to them and then use the rule of Mix;

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3 one of the lowermost links of $\mathscr{R}$ is a *par*, *for all* link or an *exists* link whose premise contains no eigenvariable: just remove such a link, apply the induction hypothesis and use the *par*, *for all* or *exists* rule of the sequent calculus.

Case 2. Otherwise, we may assume that among the lowermost links of $\mathscr{R}$ there are Cut, *times* or *exists* links, but no *par* or *for all* links. Consider a *times* (or Cut) link

$$v : \frac{A \quad B}{A \otimes B}, \qquad A \otimes B \in \Gamma$$

maximal with respect to $\ll$ (the case of an *exists* link is entirely similar).

Subcase 2.1. The link v is splitting, *i.e.*, its removal yields two disconnected proof structures; apply the induction hypothesis to them and then use the *Times* rule of the sequent calculus.

Unlike the case of $\mathbf{MLL}^-$, in $\mathbf{MLL}^- + \mathbf{MIX}$ a link maximal with respect to $\ll$ need not be splitting. For instance, there may be a sequence of chains of the form

$$\lceil A - \gamma_A - C_0 \rceil,\ \lceil D_0 - \gamma_1 - C_1 \rceil,\ \ldots,\ \lceil D_{n-1} - \gamma_n - C_n \rceil,\ \lceil D_n - \gamma_B - B \rceil$$

with links

$$\mathscr{L}'_i : \frac{C_i \quad D_i}{C_i \wp D_i}$$

for $i \leqslant n$, where $\gamma_1, \ldots, \gamma_n$ are in $e(A) \cap e(B)$; clearly in this case we cannot split $\mathscr{R}$ by removing the link $\mathscr{L}_0$ with conclusion $A \otimes B$. In Bellin (1990) this situation was described as '$A \otimes B$ *is inside a maze*'. We claim that in any case we can find a splitting link $\mathscr{L}^*$.

Subcase 2.2. The given link v is not splitting. Let

$$\mathscr{G}_A = \bigcup \{\gamma : \gamma \text{ is a chain of type } \lceil A, X | \text{ for some } X\}$$

$$\mathscr{G}_B = \bigcup \{\gamma : \gamma \text{ is a chain of type } \lceil B, Y | \text{ for some } Y\}.$$

Notice that if $\gamma \in \mathscr{G}_A$ and $\gamma' \in \mathscr{G}_B$, then $\gamma \cap \gamma' = \varnothing$; otherwise, follow γ starting from A and let u be the first link in $\gamma \cap \gamma'$. If u does not satisfy Case (1) of Proposition 4, then we can obtain a cyclic chain; if u does satisfy Case (1), then $A \otimes B$ cannot be maximal with respect to $\ll$.

Furthermore, notice that any chain $\lceil B, X|$ can be extended to a chain $\lfloor A, X|$ including the *times* link v, and conversely, every chain $\lfloor A, X|$ can be reduced to a chain $\lceil B, \ldots, X|$. Since

$$eA \cap eB = \{X : \text{ there is no chain of type } \lfloor A, X| \text{ nor of type } \lfloor B, X|\}$$

and $\mathscr{R}$ is a proof-net, it follows that the following is a partition of $\mathscr{R}$

$$\mathscr{R} = \mathscr{G}_A \cup \mathscr{G}_B \cup (eA \cap eB) \cup \{A \otimes B\}.$$

Since by the hypothesis of the subcase, $\mathscr{R}$ cannot be decomposed in two disconnected proof-structures, $eA \cap eB$ is nonempty. Now suppose we have a chain $\lceil A - \gamma_A - u \rceil$ and a chain $\lceil u - \gamma_1 - z|$ such that $\gamma_1 \subset e(A) \cap e(B)$, and u satisfies Case (1) of Proposition 4, for example, u is a *par* link with conclusion $C_0 \wp D_0$. Let v_0 be the lowermost link of the proof-net such that $u \prec v_0$: by the assumption of the case, v_0 is a *times*, *cut* or *exists* link.

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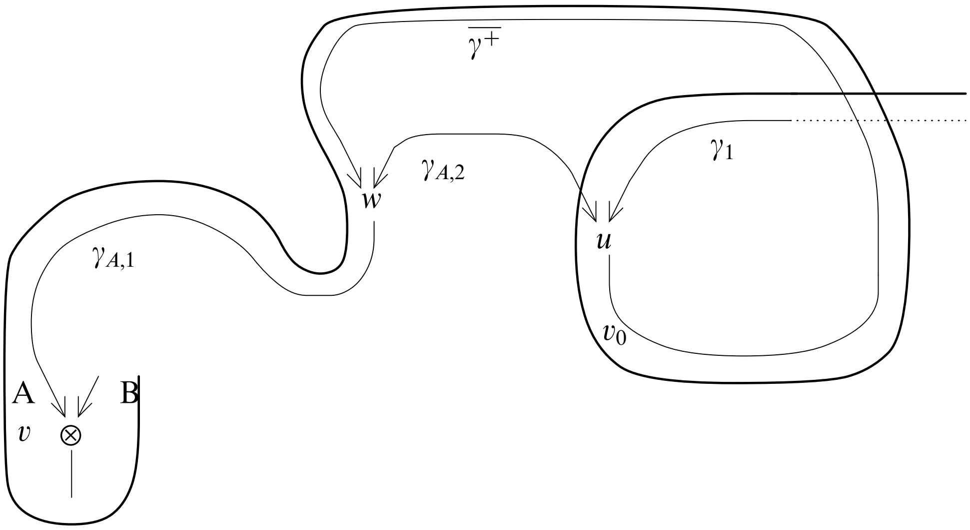
Subcase 2.2.1. If v_0 is also maximal with respect to $\ll$, then we repeat the argument of Case 2, applied to v_0. Notice that the chain γ_A is extended to a chain $\lceil v - \gamma' - v_0 \rceil$.

Subcase 2.2.2. If v_0 is not maximal with respect to $\ll$, then it belongs to a loop with exit u_1. Consider the lowermost link v_1 of the proof-structure such that $u_1 \prec v_1$: by the assumption of the case, v_1 is a *times*, *cut* or *exists* link. Since v_0 is in a loop, clearly there is a chain $\lfloor u - \gamma^+ - v_1 \rceil$. *We claim that* $\gamma_A \cap \gamma^+ = \{u\}$*, so the chain* $\gamma_A * \gamma^+$ *properly extends* γ_A.

Suppose the claim fails. Following γ_A starting from u, let w be the first link different from u such that $w \in \gamma_A \cap \gamma^+$. By Proposition 4, w is the exit of a loop; such a loop is the concatenation of a subchain of γ_A and of a subchain of γ^+. More precisely, γ_A splits as $\gamma_{A,1} * \gamma_{A,2}$, with $\lceil w - \gamma_{A,2} - u \rceil$ and the loop in question is $\gamma_{A,2} * \overline{\gamma^+}$, where $\lfloor u - \overline{\gamma^+} - w \rceil$ is a subchain of γ^+. It follows that $\gamma_{A,1}$ is of type $\lceil v, w \rfloor$. *But then the concatenation*

$$\lceil v - \gamma_{A,1} - w \rfloor * \lceil w - \overline{\gamma^+} - u \rfloor * \lceil u - \gamma_1 - z |$$

belongs to $\mathscr{G}_A$ *and this implies that* γ_1 *is not in* $e(B)$.



The claim is proved. Since in passing from v to v_0 and to v_1 we extend the chain starting from A, and since $\mathscr{R}$ is finite, the process must eventually terminate in a link v^* that is maximal with respect to $\ll$ and 'not in a maze'. □

Theorem 2. (Permutability of Inferences)

1 Let $\mathscr{D}$ and $\mathscr{D}'$ be a pair of derivations of the same sequent $\vdash \Gamma$ in propositional **MLL**$^-$ + **MIX**. Then $(\mathscr{D})^- = (\mathscr{D}')^-$ if and only if there exists a sequence of derivations $\mathscr{D} = \mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_n = \mathscr{D}'$ such that $\mathscr{D}_i$ and $\mathscr{D}_{i+1}$ differ only for a permutation of two consecutive inferences.

2 Let $\mathscr{R}$ be a proof-net and let A be a formula occurrence in $\mathscr{R}$. Then there exists a derivation $\mathscr{D}$ with $(\mathscr{D})^- = \mathscr{R}$ and a subderivation $\mathscr{B}$ of $\mathscr{D}$ such that $(\mathscr{B})^- = eA$. A similar statement holds for kA.

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Proof. Part (ii) is an immediate consequence of the definition of *normal subnet* and of the sequentialization theorem (generalized to proof-nets with non-logical axioms). Part (i) is obvious in the *if* case. The *only if* case is proved by induction on the size of the proof-net $\mathscr{R}$ such that $(\mathscr{D})^- = \mathscr{R} = (\mathscr{D}')^-$.

First notice if $\mathscr{B}$ is any subderivation of $\mathscr{D}$, then $(\mathscr{B})^-$ is a *normal* subnet of $\mathscr{R}$; hence given a chain $\lceil A - \gamma - B \rceil$ in $\mathscr{R}$, if $A, B \in (\mathscr{B})^-$ then $\gamma \subset (\mathscr{B})^-$ also.

Now consider a branch of $\mathscr{D}$ and let $\mathscr{I}_0$ be the last inference from the bottom up where $\mathscr{D}$ agrees with $\mathscr{D}'$. If $\mathscr{I}_0$ is an axiom, then $\mathscr{D}$ and $\mathscr{D}'$ entirely agree in the order of the inferences in this branch. Otherwise, let $\mathscr{I}$, $\mathscr{I}'$ be the inferences immediately above $\mathscr{I}_0$ in $\mathscr{D}$ and $\mathscr{D}'$, respectively, and let $\mathscr{B}$ and $\mathscr{B}'$ be the subderivations of $\mathscr{D}$ and $\mathscr{D}'$ ending with $\mathscr{I}$ and $\mathscr{I}'$, respectively.

Case 1. Both $\mathscr{I}$ and $\mathscr{I}'$ are Mix rules. Thus $\mathscr{B}$ and $\mathscr{B}'$ have the forms

$$\mathscr{I} : \frac{\begin{matrix}\mathscr{B}_1 \\ \vdots \\ \vdash \Gamma_1, \Gamma_2\end{matrix} \quad \begin{matrix}\mathscr{B}_2 \\ \vdots \\ \vdash \Gamma_3, \Gamma_4\end{matrix}}{\vdash \Gamma} \qquad \mathscr{I}' : \frac{\begin{matrix}\mathscr{B}'_1 \\ \vdots \\ \vdash \Gamma_1, \Gamma_3\end{matrix} \quad \begin{matrix}\mathscr{B}'_2 \\ \vdots \\ \vdash \Gamma_2, \Gamma_4\end{matrix}}{\vdash \Gamma}$$

respectively, where $\Gamma = \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ and one of the Γ_i may be empty. Consider the subnets

$$\begin{matrix}\mathscr{S}_1 \\ \Gamma_1\end{matrix} = (\mathscr{B}_1)^- \cap (\mathscr{B}'_1)^- \qquad \text{and} \qquad \begin{matrix}\mathscr{S}_2 \\ \Gamma_2\end{matrix} = (\mathscr{B}_1)^- \cap (\mathscr{B}'_2)^-.$$

We claim that $\mathscr{S}_1$ and $\mathscr{S}_2$ are disjoint. In fact, in $\mathscr{R}$ there can be no axiom link connecting hereditary premises of Γ_1 and Γ_2 nor any chain with Cuts as lower members connecting Γ_1 and Γ_2. This is because $(\mathscr{B}')^-$ is a normal subnet of $\mathscr{R}$ in which the hereditary premises of Γ_1 and Γ_2 belong to two separated proof-nets $(\mathscr{B}'_1)^-$ and $(\mathscr{B}'_2)^-$. By the same argument, we have two disjoint subnets $\mathscr{S}_3$ and $\mathscr{S}_4$ with conclusions Γ_3 and Γ_4, respectively.

By the sequentialization theorem, we have derivations $\mathscr{D}_1, \ldots, \mathscr{D}_4$ such that $\mathscr{S}_i = (\mathscr{D}_i)^-$ and

$$\mathscr{D}_{1,2} \quad \mathscr{M}_1 : \frac{\begin{matrix}\mathscr{D}_1 \\ \vdots \\ \vdash \Gamma_1\end{matrix} \quad \begin{matrix}\mathscr{D}_2 \\ \vdots \\ \vdash \Gamma_2\end{matrix}}{\vdash \Gamma_1, \Gamma_2} \qquad \mathscr{D}_{3,4} \quad \mathscr{M}_2 : \frac{\begin{matrix}\mathscr{D}_3 \\ \vdots \\ \vdash \Gamma_3\end{matrix} \quad \begin{matrix}\mathscr{D}_4 \\ \vdots \\ \vdash \Gamma_4\end{matrix}}{\vdash \Gamma_3, \Gamma_4}$$

are derivations ending with Mixes; by the induction hypothesis, $\mathscr{D}_{1,2}$ can be obtained from $\mathscr{B}_1$, and $\mathscr{D}_{3,4}$ can be obtained from $\mathscr{B}_2$ by successive permutations of inferences.

If we repeat exactly the same argument for $\mathscr{B}'$, we obtain derivations

$$\mathscr{D}_{1,3} \quad \mathscr{M}_3 : \frac{\begin{matrix}\mathscr{D}_1 \\ \vdots \\ \vdash \Gamma_1\end{matrix} \quad \begin{matrix}\mathscr{D}_3 \\ \vdots \\ \vdash \Gamma_3\end{matrix}}{\vdash \Gamma_1, \Gamma_3} \qquad \mathscr{D}_{2,4} \quad \mathscr{M}_4 : \frac{\begin{matrix}\mathscr{D}_2 \\ \vdots \\ \vdash \Gamma_2\end{matrix} \quad \begin{matrix}\mathscr{D}_4 \\ \vdots \\ \vdash \Gamma_4\end{matrix}}{\vdash \Gamma_2, \Gamma_4}$$

such that $\mathscr{D}_{1,3}$ can be obtained from $\mathscr{B}'_1$, and $\mathscr{D}_{2,4}$ can be obtained from $\mathscr{B}'_2$ by successive permutations of inferences. Now it is evident that by successive permutations of the Mixes

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$\mathscr{M}_3$ and $\mathscr{M}_4$ with $\mathscr{I}'$ we identify

$$\mathscr{I} : \frac{\mathscr{D}_{1,2} \quad \mathscr{D}_{3,4}}{\vdash \Gamma} \quad \text{and} \quad \mathscr{I}' : \frac{\mathscr{D}_{1,3} \quad \mathscr{D}_{2,4}}{\vdash \Gamma}.$$

Case 2. One of $\mathscr{I}$, $\mathscr{I}'$ is not a Mix, say $\mathscr{I}$. Let $\mathscr{I}$ have principal formula A and let $\mathscr{I}'_A$ be the inference of $\mathscr{D}'$ corresponding to the same link of $\mathscr{R}$ as $\mathscr{I}$; such an $\mathscr{I}'_A$ exists, since $(\mathscr{D})^- = (\mathscr{D}')^-$.

Moreover, let $\mathscr{I}'_1, \ldots, \mathscr{I}'_k$ be the inferences that occur in $\mathscr{D}'$ between $\mathscr{I}'_A$ and $\mathscr{I}_0$ (proceeding downwards).

If $A = A_1 \wp A_2$, the *par* rule $\mathscr{I}'_A$ clearly can be permuted below $\mathscr{I}'_1, \ldots, \mathscr{I}'_k$, as required.

If $A = A_1 \otimes A_2$, then $\mathscr{I}'_A$ can always be permuted below any $\mathscr{I}'_i$, unless $\mathscr{I}'_i$ is a *par* rule, say, with principal formula B. Let $\mathscr{I}_B$ be the inference of $\mathscr{D}$ corresponding to the same link of $\mathscr{R}$ as $\mathscr{I}'_i$. Now $\mathscr{I}_B$ occurs *above* the inference $\mathscr{I}_0$ by our assumption that $\mathscr{D}$ and $\mathscr{D}'$ agree in the given branch up to $\mathscr{I}_0$. Hence the subderivation $\mathscr{D}_B$ of $\mathscr{D}$ ending with $\mathscr{I}_B$ is a proper subderivation of $\mathscr{B}$ and does not contain $\mathscr{I}$, and hence $A \notin k(B)$, since $A \notin (\mathscr{D}_B)^-$.

Let $\mathscr{D}'_B$ be the subderivation of $\mathscr{D}'$ ending with $\mathscr{I}'_i$. By Part (ii) of the theorem and induction hypothesis, $\mathscr{D}'_B$ can be transformed by successive permutations of inferences into a derivation $\mathscr{D}^*$ where the inference with principal formula B occurs *above* that with principal formula A.

If $A = cut$ or if $A = \forall x.A_1$ the argument is similar. By repeating the argument, we eventually permute A below $\mathscr{I}'_k$, as required. □

3. Asperti's concurrent processes

We consider a variant of Asperti's token game (Asperti 1995). The original formulation by Asperti is in terms of Petri Nets; we speak informally of trips of tokens in a proof-structure and regard this condition as the correct generalization of Girard's *no-short-trip* condition to the case of $\mathbf{MLL}^- + \mathbf{MIX}$.

There are tokens of type $\uparrow$ and $\downarrow$. There is a Left or Right switch on each *times* link (*Asperti's switching*). Given a multiplicative proof-structure $\mathscr{R}$, in the *initial position* we have a token of type $\uparrow$ on each conclusion of $\mathscr{R}$. The game *succeeds* if it reaches the *terminal position* where there is a token of type $\downarrow$ on each conclusion of $\mathscr{R}$. The permissible movements of the tokens are those in accordance with the following instructions:

— case of an *axiom* link $\overline{A \quad A^\perp}$:

$$\text{from a pair } \uparrow A, \uparrow A^\perp \text{ go to the pair } \downarrow A, \downarrow A^\perp$$

— case of a *par* link $\dfrac{A \quad B}{A \wp B}$:

$$\text{from } \uparrow A \wp B \text{ go to the pair } \uparrow A, \uparrow B;$$

$$\text{from the pair } \downarrow A, \downarrow B, \text{ go to } \downarrow A \wp B;$$

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— case of a *times* link $\dfrac{A \quad B}{A \otimes B}$:

	Right Switch	Left Switch
(1)	from $\uparrow A \otimes B$, go to $\uparrow B$	from $\uparrow A \otimes B$, go to $\uparrow A$
(2)	from $\downarrow B$, go to $\uparrow A$	from $\downarrow A$, go to $\uparrow B$
(2)	from $\downarrow A$, go to $\downarrow A \otimes B$	from $\downarrow B$, go to $\downarrow A \otimes B$

The case of *cut* is identical to that of a *times* link.

Definitions 4.

(i) A *deadlock* for a given switching is a position of the tokens that is reachable from the initial position from which the game cannot successfully terminate.

(ii) Given a proof-structure and a switching for the Asperti game, a *causal path* or *causal chain* is a path of $n+1$ edges together with n transitions such that the transition t_i takes a token from the edge e_{i-1} and puts a token in the edge e_i. A causal path is *cyclic* if the edges A_1 and A_{n+1} coincide. A causal path where the first transition is of the form $\uparrow A$ and the last is $\uparrow B$ is said *of type* $\uparrow A, \uparrow B$, and similarly for the types $\uparrow A, \downarrow B$, and so on.

Asperti's Theorem. A propositional proof-structure is a proof-net if and only if for every A-switching there is no deadlock in Asperti's game.

The following facts are needed to prove Asperti's theorem. Given a proof-structure and a switching for the Asperti game, let M_0 and M_T denote the initial and terminal successful position, respectively.

Proposition A-1.

(i) In any computation $M_0 \Rightarrow M'$ every transition can be fired at most once.

(ii) We cannot have infinite computations starting from M_0.

(iii) In any computation $M_0 \Rightarrow M_T$ every transition is fired exactly once.

Proof. The proof is left as an exercise (see Asperti (1995, 3.13, 3.15, 3.16)). □

Proposition A-2. In some computation $M_0 \Rightarrow M'$ there is a deadlock if and only if there is a cyclic causal path if and only if in $\mathscr{R}$ there is a cyclic chain.

Proof. The proof is left as an exercise (see Asperti (1995, Theorem 3.24)). □

Finally, note that an Asperti game is reversible: the *dual* of a given transition is obtained by changing the kind of the tokens, by choosing the opposite switch in the case of a *times* link and by performing the transition in the reverse order (note that the dual of a transition is a transition). The *dual* of an Asperti game is obtained by performing the dual transitions in the reverse order.

Proposition A-3. The dual of an Asperti game is an Asperti game.

Proof. The proof is left as an exercise (see Asperti (1995, 3.20)). □

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In the process interpretation, a causal path $\uparrow A, \uparrow B$ for a certain A-switching s means that under the restrictions on the order of the execution of the processes, induced by s, the activation of A must precede the activation of B or, in other terms, there is a *causal dependency* between the activations of those processes. The proofs of propositions A-2 and A-3 yield the following

Corollary 3. Chains are related to causal dependencies between processes as follows:

— the termination of A [B] may depend on the activation of B [A] if and only if there exists a chain of type $|A, B|$.
— the activation of A [B] may depend on the termination of B [A] if and only if there exists a chain of type $\lfloor A, B \rfloor$;
— the activation of A may depend on the activation of B if and only if there exists a chain of type $\lfloor A, B|$; dually,
— the termination of A may depend of the termination of B if and only if there exists a chain of type $|A, B\rfloor$.

Also the *empire* of a formula A in a proof-net has the following characterization (see Asperti (1995, the Remark after Proposition 4.10)):

$$eA \quad = \quad \{X : \text{ the activation of } A \text{ cannot depend on the activation of } X\}.$$

3.1. *Asperti's correctness condition, first-order case*

The main technical idea in Girard's treatment of quantifiers (Girard (preprint)) is to define D-R-graphs so that the conclusion of a *for all* link may be connected either to the premise of the link or to any other formula containing the eigenvariable associated with that link. Given the characterization of the empires in Lemma 3, the requirement that such a D-R-graph should be acyclic implies that an eigenvariable associated with a *for all* link cannot occur outside the *empire* of the premise of such a link. The refinement in Bellin and van de Wiele (1995) requiring the strictest possible use of eigenvariables, implies that an eigenvariable cannot occur outside the *kingdom* of the premise of its associated link.

We extend Asperti's characterization of proof-nets to the first-order case[†]. The restriction on the eigenvariables is interpreted here as a synchronization requirement. The resulting correctness condition seems more intuitive than the official one using *for all* switchings.

To a proof-structure we associate a list of *global variables* that is empty at the beginning of the verification of the correctness condition. Asperti's games are defined as before, with the addition of the following cases:

— case of a *for all* link $\dfrac{A}{\forall x.A}$:

$$\text{from } \uparrow\forall x.A \text{ go to } \uparrow A;$$

[†] This section results from a discussion with A. Fleury.

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and add x to the list of *global variables*;

$$\text{from } \downarrow A, \text{ go to } \downarrow \forall x.A;$$

— case of an *exists* link $\dfrac{A[t/x]}{\exists x.A}$:

$$\text{from } \uparrow \exists x.A \text{ go to } \uparrow A[t/x]$$

if every eigenvariable occurring in t occurs already in the list of *global variables*;

$$\text{from } \downarrow A[t/x], \text{ go to } \downarrow \exists x.A.$$

— case of a *cut* link $\dfrac{A \qquad A^{\perp}}{cut}$:
as in the case of a *times* link, except that the transition from $\uparrow cut$ is not activated unless every eigenvariable occurring in A already occurs in the list of *global variables*.

Theorem 3. A first-order proof-structure $\mathscr{R}$ is a proof-net if and only if for every A-switching there is no deadlock in Asperti's game for $\mathscr{R}$.

Proof. (Sketch) The restriction on the *exists* case, *i.e.*, the synchronization of an *exists* process with a *for all* process, may be regarded as a transition rule

$$\text{from } \uparrow \forall x.A \text{ and } \uparrow \exists y.B \text{ to } \uparrow B[t/y].$$

Clearly this corresponds to the chain $\lfloor B[t/y], \forall x.A \rceil$, in the sense that if v and w are the *for all* and *exists* links in question, the transition from $\uparrow \forall x.A$ to $\uparrow B[t/y]$ may be regarded as passing through the edge between v and w in the D-R-graph determined by a switch for $\forall x.A$. A similar remark applies to the restriction on the *cut* process.

As in the propositional case, we must prove Proposition A-2. Using the previous paragraph, it can be shown that if there is a deadlock, there is a cyclic causal path, and this may be regarded as a cyclic chain. Conversely, if there is a cyclic chain, first apply Proposition 5 below and consider a cyclic chain where the conclusions of all *for all* links are connected to an *exists* or *cut* link. Now consider a D-R-switching s yielding a chain γ of type $\lfloor v, w \rfloor$, where v has $\forall x.A$ as a conclusion and w has $B[t/y]$ as a premise, which becomes cyclic when we add the edge induced by the switch $s(v) = w$, so the eigenvariable x must occur in t. By induction on γ, we find an A-switching that determines a causal path from $\uparrow B[t/y]$ to $\uparrow \forall x.A$, that is, the activation of $\forall x.A$ depends on the activation of $B[t/y]$. But the transition from $\uparrow \exists y.B$ to $\uparrow B[t/y]$, where t contains x, is permissible only if x is already declared a global variable, and this requires that $\uparrow \forall x.A$ has already been reached: thus the activation of $B[t/y]$ depends on the activation of $\forall x.A$ and this is a deadlock. □

Proposition 5. Let $\mathscr{R}$ be a first-order proof-structure. If in $\mathscr{R}$ there is a cyclic chain, we can find a cyclic chain where every *for all* switch selects either the premise of its link or an *exists* link or a Cut.

Proof. Suppose $s\mathscr{R}$ contains a cycle γ, a link v with conclusion $\forall x.A$ is in γ and $s(v) = u$. If u is neither an *exists* link nor a Cut, its conclusion C still contains the eigenvariable x. Hence C cannot be a conclusion of $\mathscr{R}$ and it must be a premise of a link w.

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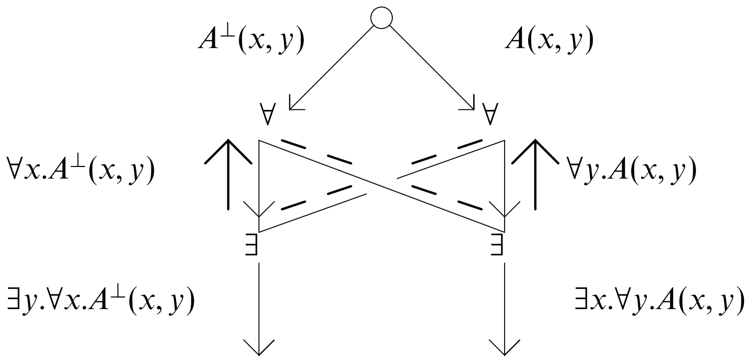
Let s' be a switching which is like s, except that $s'(v) = w$. If w is also in γ, then $s'(\mathscr{R})$ still contains a cycle. If w does not belong to γ, then $s'(\mathscr{R})$ still contains a cycle, unless w is a *par* or *for all* link and the switching $s'(w) = s(w)$ does not choose C. In the latter case, take a switching s'' that is like s' except that $s''(w) = C$. Since the choice $s(w)$ does not determine γ, it is clear that $s''\mathscr{R}$ again contains a cycle.

Repeating this process, we obtain a switching s^* with a cyclic $s^*(\mathscr{R})$ where $s^*(v)$ is either an *exists* link or a Cut, as required. □

Example. The solid arrows indicate the mutual causal dependency.



4. Full intuitionistic linear logic

In previous work by G. Bellin and J. van de Wiele (Bellin and Scott 1994, Section 5.4) it has been shown that each sequent derivation of multiplicative *intuitionistic linear logic* **ILL** can be represented as a classical proof-net of $\mathbf{MLL}^-$ together with an *Input-Output orientation*; conversely, each proof-net for $\mathbf{MLL}^-$ corresponds to a *set* of sequent derivations in **ILL**, where each translation from $\mathbf{MLL}^-$ to **ILL** is determined by an I-O orientation satisfying certain conditions. Moreover, suitable orientations are related to Girard's trips (or D-R-graphs). In particular, in the case of a cut-free proof-net each D-R-graph determines a suitable orientation.

In this section we show how to extend this result to $\mathbf{MLL}^- + \mathbf{MIX}$†. It turns out that the right intuitionistic system for this purpose is *Full Intuitionistic Linear Logic* **FILL** introduced by M. Hyland and V. de Paiva (1993). We will not discuss the considerations of categorical logic that motivate Hyland and de Paiva's work. We consider only the multiplicative fragment of **FILL**.

The language of this fragment has the connectives $\otimes$ (*times*), $\wp$ (*par*), $\multimap$ (*linear implication*) and their units, the propositional constants $\perp$ and $\mathbf{1}$ for falsity and truth. We use the same symbols as in classical linear logic, although the meaning is obviously different. Linear negation is defined as $A^\perp =_{df} A \multimap \perp$.

The *consequence relation* of **FILL** is *classical linear* ; thus sequents have the form $\Gamma \vdash \Delta$, where Δ may contain several occurrences of formulas.

† The question as to what fragment of intuitionistic linear logic could be represented in $\mathbf{MLL}^- + \mathbf{MIX}$ by an extension of the *I-O-translations* for $\mathbf{MLL}^-$ was asked by P. Scott to the author in private communication.

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4.1. *Term calculus for multiplicative FILL*

We give here the basic definitions of de Paiva and Hyland's term calculus.

Definitions 5.

(i) Given a set X of variables and constant terms $\circ$, $-$, define the set $\mathscr{P}_X$ of *patterns* with variables in X by the inductive clauses

$$x \otimes y \in \mathscr{P}_{\{x,y\}} \qquad x- \in \mathscr{P}_{\{x\}} \qquad -y \in \mathscr{P}_{\{x\}}$$

Then define the set $\mathscr{T}_X$ of *linear terms* with variables in X inductively as follows:

— $\circ \in \mathscr{T}_{\{\}}$, $- \in \mathscr{T}_{\{\}}$;

— $x \in \mathscr{T}_{\{x\}}$;

— $t \in \mathscr{T}_X$, $u \in \mathscr{T}_Y$, $X \cap Y = \varnothing$ implies $tu \in \mathscr{T}_{X\cup Y}$;

— $t \in \mathscr{T}_{X\cup\{x\}}$, $x \notin \mathscr{T}_X$ implies $\lambda x.t \in \mathscr{T}_X$;

— $t \in \mathscr{T}_X$, $u \in \mathscr{T}_Y$, $X \cap Y = \varnothing$ implies $t \otimes u, t \wp u \in \mathscr{T}_{X\cup Y}$;

— $t \in \mathscr{T}_X$, $p \in \mathscr{P}_Y$, $e \in \mathscr{T}_{Y\cup Z}$, $X \cap Z = \varnothing$, $Y \cap Z = \varnothing$ implies $\texttt{let}\ t\ \texttt{be}\ p\ \texttt{in}\ e \in \mathscr{T}_{X\cup Z}$.

(ii) The sequent calculus rules with the associated term assignment are as follows. We use $\overline{x}$, $\overline{y}$, $\overline{z}$, $\overline{v}$ $\overline{f}$ for sequences of variables, $\overline{r}$, $\overline{s}$, $\overline{t}$, $\overline{u}$ for sequences of terms. If $\overline{t}$ is the sequence of terms $t_1, \ldots, t_n$, then $\overline{t}[u/x]$ is the sequence $t_1[u/x], \ldots, t_n[u/x]$.
If $\overline{x}$ and $\overline{y}$ are the variables occurring in the premises of a two-premised sequent rule, it is understood that no variable in $\overline{x}$ occurs in $\overline{y}$ and *vice versa.*

Identity

$$\textit{Axiom}: \quad x : A \vdash x : A \qquad \textit{Cut}: \ \frac{\overline{x} : \Gamma \vdash \overline{u} : \Delta, t : A \qquad x : A,\ \overline{y} : \Pi \vdash \overline{f} : \Lambda}{\overline{x} : \Gamma, \overline{y} : \Pi \vdash \overline{u} : \Delta, \overline{f}[t/x] : \Lambda}$$

Times

$$\otimes - R : \ \frac{\overline{x} : \Gamma \vdash \overline{r} : \Delta, t : A \qquad \overline{y} : \Pi \vdash \overline{s} : \Lambda, u : B}{\overline{x} : \Gamma, \overline{y} : \Pi \vdash \overline{r} : \Delta, \overline{s} : \Lambda, t \otimes u : A \otimes B}$$

$$\otimes - L : \ \frac{\overline{v} : \Gamma, x : A, y : B \vdash \overline{t} : \Delta}{\overline{v} : \Gamma, z : A \otimes B \vdash \overline{t'} : \Delta}$$

where for each $t'_i \in \overline{t'}$ we have

$$t'_i = \begin{cases} \texttt{let}\ z\ \texttt{be}\ x \otimes y\ \texttt{in}\ t_i, & \text{if } x \text{ or } y \text{ occurs in } t_i; \\ t_i, & \text{otherwise.} \end{cases}$$

Par

$$\wp - R : \ \frac{\overline{x} : \Gamma \vdash x : A, y : B, \overline{u} : \Delta}{\overline{x} : \Gamma, \vdash x \wp y : A \wp B, \overline{u} : \Delta}$$

$$\wp - L : \ \frac{\overline{x} : \Gamma, x : A \vdash \overline{r} : \Delta \qquad \overline{y} : \Pi, y : B \vdash \overline{s} : \Lambda}{\overline{x} : \Gamma, \overline{y} : \Pi, z : A \wp B \vdash \overline{r'} : \Delta, \overline{s'} : \Lambda}$$

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where for each $r'_i \in \overline{r'}$ we have

$$r'_i = \begin{cases} \texttt{let } z \texttt{ be } x- \texttt{ in } r_i, & \text{if } x \text{ occurs in } r_i; \\ r_i, & \text{otherwise.} \end{cases}$$

and for each $s'_j \in \overline{s'}$ we have

$$s'_j = \begin{cases} \texttt{let } z \texttt{ be } -y \texttt{ in } r_i, & \text{if } y \text{ occurs in } s_j; \\ s_j, & \text{otherwise.} \end{cases}$$

Linear Implication

$$\multimap - R : \frac{\overline{x} : \Gamma, x : A \vdash t : B, \overline{u} : \Delta}{\overline{x} : \Gamma \vdash \lambda x.t : A \multimap B, \overline{u} : \Delta} \text{ where } x \text{ does not occur in } \overline{u}.$$

$$\multimap - L : \frac{\overline{x} : \Gamma \vdash \overline{r} : \Delta, t : A \qquad x : B, \overline{y} : \Pi \vdash \overline{s} : \Lambda}{\overline{x} : \Gamma, f : A \multimap B, \overline{y} : \Pi \vdash \overline{r} : \Delta, \overline{s}[f(a)/x] : \Lambda}$$

Structural Rules

The term assignments for the rules Mix, Exchange Left and Right are straightforward. For instance, in the case of Mix we have

$$Mix : \frac{\overline{x} : \Gamma \vdash \overline{t} : \Delta \qquad \overline{y} : \Pi \vdash \overline{u} : \Lambda}{\overline{x} : \Gamma, \overline{y} : \Pi \vdash \overline{t} : \Delta, \overline{u} : \Lambda}$$

where $\overline{x} \cap \overline{y} = \emptyset$, as indicated above.

Multiplicative Propositional Constants

$$Axiom : \ \vdash \circ : \mathbf{1} \qquad \mathbf{1} - L : \frac{\overline{x} : \Gamma \vdash \overline{u} : \Delta}{- : \mathbf{1}, \overline{x} : \Gamma \vdash \overline{u} : \Delta}$$

$$Axiom : \ x : \bot \vdash \qquad \bot - R : \frac{\overline{x} : \Gamma \vdash \overline{u} : \Delta}{\overline{x} : \Gamma \vdash \overline{u} : \Delta, - : \bot}$$

4.2. *Proof-nets with orientations*

The consideration of the units is essential in the logic **FILL**, although it makes sense to consider the subsystem **FILL**$^-$ with the axioms **1**-right and $\bot$-left, but without the rules **1**-left and $\bot$-right. To represent **FILL** + **MIX** in **MLL** + **MIX** *with the propositional constants* **1** and $\bot$, we consider proof-nets with links of the form

$$\mathbf{1} - axiom\colon \overline{\mathbf{1}} \qquad \bot - axiom\colon \overline{\bot}$$

where the $\bot$-axioms are *attached* to other links $\overline{\bot} \rightharpoonup v$ in correspondence with a Weakening:

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}.$$

An attachment induces an edge in every D-R-graph. We have the reduction:

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1-*Reductions*

$$\underline{\mathbf{1} \quad \perp} \rightarrow \begin{array}{c} X \\ \vdots \end{array} \quad \text{reduces to} \quad \begin{array}{c} X \\ \vdots \end{array}$$

For the reasons given in the Introduction, the theory of proof-nets for such a system is not fully satisfactory; however, almost all the basic results hold *modulo* a given choice of attachments. In a sequent derivation we can permute the $\perp$-rule downwards (unless its principal formula becomes active in another inference); this corresponds to attaching the $\perp$-axiom *as low as possible* and gives a sort of 'normal form' for the attachment. On the other hand, we can always permute the $\perp$-rule upwards, but in the case of a *times* or *mix* inference the choice of the branch is arbitrary; thus the notion of *kingdom* of a $\perp$-axiom is not well defined.

Definitions 6.

(i) Given such a proof-net $\mathscr{S}$, an orientation is a map $\delta : \mathscr{S} \rightarrow \{O, I\}$ satisfying the following restrictions:

$$\text{axiom:} \quad (0)\ \overline{O \quad I} \qquad\qquad \overline{I \quad O}$$

$$\text{tensor:} \quad (1)\ \frac{O \quad I}{I} \quad (2)\ \frac{I \quad O}{I} \quad (3)\ \frac{O \quad O}{O} \quad (4)\ \frac{I \quad I}{I}$$

$$\text{par:} \quad (5)\ \frac{I \quad O}{O} \quad (6)\ \frac{O \quad I}{O} \quad (7)\ \frac{I \quad I}{I} \quad (8)\ \frac{O \quad O}{O}$$

For the units we have all possibilities $\delta(\mathbf{1}) = I, O$, $\delta(\perp) = I, O$. We write A_I, A_O for $\delta(A) = I$, $\delta(A) = O$, respectively.

(ii) An orientation $\delta : \mathscr{S} \rightarrow \{O, I\}$ has *a deadlock* if it makes the assignments $\underline{A_I \quad A_I^\perp}$ or $\underline{A_O \quad A_O^\perp}$ to some cut link, and is *deadlock-free* otherwise.

(iii) An orientation $\delta : \mathscr{S} \rightarrow \{O, I\}$ is *computationally consistent* if no sequence of cut reductions yields an orientation with a deadlock.

Remark. If a proof-net $\mathscr{S}$ reduces to $\mathscr{S}'$ by a cut-reduction, an orientation $\delta : \mathscr{S} \rightarrow \{O, I\}$ when restricted to $\mathscr{S}'$ is still an orientation. It is easy to see (*e.g.*, when the links immediately above a cut have orientations (2) and (5) above) that $\delta : \mathscr{S} \rightarrow \{O, I\}$ may be deadlock-free, but not $\delta : \mathscr{S}' \rightarrow \{O, I\}$.

We can extend the map $(\ )^-$ of Theorem 1 so that, given a sequent derivation $\mathscr{D}$ in multiplicative **FILL** + **MIX**, we obtain a proof-net with orientation $(\mathscr{D})^- = \delta : \mathscr{R} \rightarrow \{O, I\}$. The only question is: *what condition should correspond to the restriction on the implication introduction rule?*

Definitions 7. Let $\delta : \mathscr{R} \rightarrow \{I, O\}$ be a proof-net for **MLL**$^-$ + **MIX** with an orientation.

(i) A chain γ is *directed* if it does not pass through an attachment and for every link $\mathscr{L}$ that occurs in γ the following hold:

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(a) if $\mathscr{L}$ is a lower link (thus a *times* or cut link), it has the orientation

$$(1)\quad \frac{O \quad I}{I} \quad \text{or} \quad (2)\quad \frac{I \quad O}{I};$$

(b) otherwise, the two formula occurrences in $\mathscr{L}$ that belong to γ have the same orientation.

(ii) A *par* link with orientation

$$(5)\quad \frac{I \quad O}{O} \quad \text{or} \quad (6)\quad \frac{O \quad I}{O}$$

will be called an *implication*. Let $\mathscr{L}$ be

$$\frac{A_I \quad B_O}{(A \wp B)_O}:$$

we say that the orientation $\delta : \mathscr{R} \to \{I, O\}$ *makes the implication* $\mathscr{L}$ *functional* if for every directed chain of type $\lceil A_I, C_O \rceil$ where C is a door of $e(A)$ we have that C is precisely the formula occurrence B.

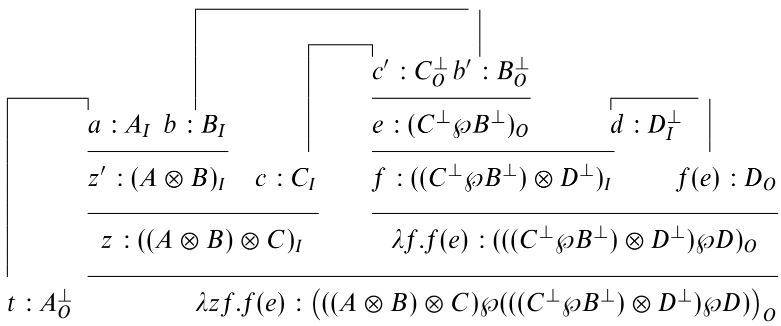
(iii) We say that $\delta : \mathscr{R} \to \{I, O\}$ is a *proof-net for* multiplicative **FILL + MIX** if $\mathscr{R}$ is a proof-net for **MLL + MIX** and δ is a computationally consistent orientation that makes all implications functional.

Remark. It is easy to see that if γ is a directed chain of type $\lceil A_I, B_O \rceil$ and

$$\mathscr{L} : \frac{C_O \quad D_I}{(C \otimes D)_I}$$

is a lower link of the chain, then γ may only result from subchains $\lceil A_I, C_O \rceil$ and $\lceil D_I, B_O \rceil$ connected by $\mathscr{L}$ (*cf.* Proposition 6 below).

Example.



(The term assignment arises from the attempted translation into **FILL**.) Of the two implications in the example, the higher one is functional, since there is only one directed chain $\lceil ((C^\perp \wp B^\perp) \otimes D^\perp)_I, D_O \rceil$ between its premises, but the lower one is not, because of the directed chain $\lceil ((A \otimes B) \otimes C)_I, A_O^\perp \rceil$. Indeed, letting

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$t = \texttt{let}\ z\ \texttt{be}\ z'\wp -\ \texttt{in}\ (\texttt{let}\ z'\ \texttt{be}\ a\wp -\ \texttt{in}\ a)$,
$b' = \texttt{let}\ z\ \texttt{be}\ z'\wp -\ \texttt{in}\ (\texttt{let}\ z'\ \texttt{be}\ -\wp b\ \texttt{in}\ b)$,
$c' = \texttt{let}\ z\ \texttt{be}\ -\wp c\ \texttt{in}\ c$
$e = c'\wp b'$
we have that

$$z : (A^{\perp}\wp B^{\perp})\wp C^{\perp} \vdash t : A^{\perp}, \lambda f.f(e) : ((C^{\perp}\wp D^{\perp}) \multimap D) \multimap D$$

is provable in multiplicative **FILL**, but

$$\vdash t : A^{\perp}, \lambda zf.f(e) : (A^{\perp}\wp B^{\perp})\wp C^{\perp} \multimap ((C^{\perp}\wp D^{\perp}) \multimap D) \multimap D$$

is not. The general case is given by the following result.

Lemma 4. Let $\mathscr{D}$ be a derivation in multiplicative **FILL** + **MIX** *of* $\overline{v} : \Gamma \vdash \overline{t} : \Delta$ and let $(\mathscr{D})^{-} = \delta : \mathscr{R} \to \{I, O\}$. For each variable $v_i : C_i$ in $\overline{v}$ and every term $t_j : D_j$ in $\overline{t}$, v_i occurs in t_j if and only if there exists a directed chain $\lceil C_i, D_j \rceil$ in $(\mathscr{D})^{-}$.

Proof. The proof is by induction on $\mathscr{D}$. If $\mathscr{D}$ is an axiom the result is clear. If the last inference of $\mathscr{D}$ is Mix, the result is immediate from the induction hypothesis and the fact that the variables occurring in different branches of a derivation are distinct. If the last inference is $\perp$-right or **1**-left, the term assignments to the passive formulas are unchanged and the directed chains in $\mathscr{D}^{-}$ do not propagate through the new attachment. If the last inference of $\mathscr{D}$ is *Times* Left or *Par* Left, the variable $z : A \circ B$ assigned to the principal formula occurs in the term $t' : D$ in the succedent of the conclusion if and only if one of the variables $x : A$ or $y : B$ occurs in $t : D$ in the succedent of the premise. By the induction hypothesis this is the case if and only if there is a directed chain of type $\lceil A, D \rceil$ or $\lceil B, D \rceil$ in $(\mathscr{D})^{-}$ if and only if there is a directed chain of type $\lceil A \circ B, D \rceil$. The cases when the last inference of $\mathscr{D}$ is *Times* Right or *Par* Right or *Linear Implication* Right are similar.

Now suppose the last inference of $\mathscr{D}$ is *Linear Implication* Left. If the variable v and the term t are both assigned to *passive formulas* and have immediate ancestors in the same branch of the derivation, the result is immediate from the induction hypothesis.

For any passive formula L in the *succedent* of the conclusion, the variable $f : A \multimap B$ occurs in $w : L$ if and only if $w = s[f(t)/x]$ if and only if there is a directed chain $\lceil B_I, L_O \rceil$ if and only if there is a directed chain $\lceil (A \otimes B)_I, L_O \rceil$.

For any passive formula C in the antecedent of the *left* premise and any passive formula L in the succedent of the *right* premise, the variable $v : C$ occurs in the term $w : L$ if and only if w is $s[f(t)/x]$ and v occurs in t if and only if there are chains $\lceil C_I, A_O \rceil$ and $\lceil B_I, L_O \rceil$ connected by the link

$$\frac{A_O \quad B_I}{(A \otimes B)_I}.$$

Finally, for any passive formula P in the antecedent of the *right* premise and any passive formula D in the succedent of the *left* premise, the variable $y : P$ does not occur in the term $r : D$ and any chain $\lceil P, D \rceil$ cannot be directed, since it must consist of two

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disjoint subchains of types $\lceil P_I, B_I \rceil$ and $\lceil A_O, D_O \rceil$ connected by the link

$$\frac{A_O \quad B_I}{(A \otimes B)_I},$$

and by the above Remark this is impossible in a directed chain. The case of Cut is similar. □

We also need the following fact.

Proposition 6. Let $\mathscr{R}$ be a proof-net for **MLL** + **MIX** and let $\delta : \mathscr{R} \to \{I, O\}$ be an orientation satisfying (0)–(8) above and let C be a conclusion of $\mathscr{R}$ such that $\delta(C) = I$. Every maximal directed chain starting from C_I has one of the forms

$$(o)\ \lceil C_I, \dots, D_O \rceil \qquad \text{or} \qquad (i)\ \lceil C_I, \dots, \mathbf{1}_I \rfloor$$

where D is a conclusion of $\mathscr{R}$ and $\overline{\mathbf{1}}$ is an axiom.

Proof. A directed chain $path_s$ is obtained by determining the switching s according to the orientation as follows.

1 Start from C_I and proceed upwards.
2 Going up, always remain within formulas marked I, fixing the switching, if necessary, so that from a conclusion marked I the path reaches a premise marked I. The step is determined in Cases (1) and (2) (*times*), and an arbitrary choice is made in Cases (4) (*times*) and (7) (*par* links).
3 At an axiom or cut, change the direction.
4 Going down, remain within formulas marked O whenever possible; namely, proceed from a premise marked O to the conclusion marked O in Cases (3) (*times* links) (5) and (6) and (8) (*par*) fixing the switching accordingly.
5 If going down you reach a *times* link with conclusion marked I (Cases (1) and (2)), then from the premise marked O proceed up to the premise marked I and continue as in Step 2.

Since every path is acyclic, the process terminates, either (i) going downwards at a conclusion D_O or (ii) going upwards at a link $\overline{\mathbf{1}_I}$, as claimed. □

The proof of the sequentialization theorem for **FILL**$^-$ + **MIX** is essentially the same as the proof on Theorem 1 for **MLL**$^-$ + **MIX**. It is in the treatment of the axioms $\bot \vdash$ of **FILL** that the specific graph theoretic analysis contained in the proof becomes necessary.

Theorem 4. There exists a 'context forgetting' map $(\ )^-$ from sequent derivations in first-order *multiplicative* **FILL** + **MIX** to proof-nets for **FILL** + **MIX** with the following properties:

(a) Let $\mathscr{D}$ be a derivation of $\overline{x} : \Gamma \vdash \overline{t} : \Delta$, then $(\mathscr{D})^-$ is a proof-net with conclusions $\vdash \Gamma_I, \Delta_O$.
(b) If $\delta : \mathscr{R} \to \{I, O\}$ is a proof-net net for **FILL** + **MIX** with conclusions $\vdash \Gamma_I, \Delta_O$, then there is a sequent calculus derivation $\mathscr{D}$ of $\overline{x} : \Gamma \vdash \overline{t} : \Delta$ such that $(\mathscr{D})^- = \delta : \mathscr{R} \to \{I, O\}$.
(c) If $\mathscr{D}$ reduces to $\mathscr{D}'$, then $(\mathscr{D})^-$ reduces to $(\mathscr{D}')^-$.

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(d) If $(\mathscr{D})^{-} = \delta : \mathscr{R} \to \{I, O\}$ and $\mathscr{R}$ reduces to $\mathscr{R}'$, then there is a $\mathscr{D}'$ such that $\mathscr{D}$ reduces to $\mathscr{D}'$ and $(\mathscr{D}') = \delta : \mathscr{R}' \to \{I, O\}$.

Proof. To prove (a) use Lemma 4.
(b) If $\mathscr{R}$ consists of an axiom, of disconnected structures, and if a terminal $\bot$-axiom is attached to some link in $\mathscr{R}$ or if $\mathscr{R}$ ends with a *par* link (Case 1), then the argument is easy; when a terminal *par* link is an implication we use Lemma 4. When no terminal link is a *par* link (Case 2) the algorithm in our proof of Theorem 1 selects a splitting *times* link v: we need to show that the two splitting substructures still satisfy the condition that all implications are functional.

This is immediate if v has orientation

$$(3)\quad \frac{O \quad O}{O} \quad \text{or} \quad (4)\quad \frac{I \quad I}{I}.$$

Now suppose the orientation of v is of type (1) or (2), say

$$v : \frac{A_O \quad B_I}{(A \otimes B)_I},$$

and let $\mathscr{S}_A$ and $\mathscr{S}_B$ be the splitting substructures. Notice that since v is maximal with respect to $\ll$, for every implication

$$w : \frac{C_I \quad D_O}{(C \wp D)_O}$$

and every directed chain γ of type $\lceil C_I, D_O \rceil$, the link v cannot be a lower link of γ; however, *v may be a lower link of a maximal directed chain of type* $\lceil C_I, \mathbf{1} \rfloor$. If every maximal directed chain γ starting from B_I is of type $\lceil B_I, \mathbf{1}_I \rfloor$, then it may very well be the case that an implication

$$w : \frac{C_I \quad D_O}{(C \wp D)_O}$$

is no longer functional in $\mathscr{S}_A$, while it was functional in $\mathscr{R}$ if some directed chain γ' from C_I passes through v and $v \in e(w)$. To conclude the proof it is enough to refine our argument for Theorem 1 as follows.

Suppose v is splitting as in Subcase 2.1. If in $\mathscr{S}_A$ all implications are functional, we are done. Otherwise, we have a chain γ' as above, and since $C \wp D$ is not a conclusion of $\mathscr{R}$, we consider the lowermost link

$$v_0 : \frac{E \quad F}{E \otimes F}$$

occurring below $C \wp D$. If v_0 was v, then $w \prec v$, and hence we cannot have $v \in e(w)$, so we may suppose that v_0 is different from v. Notice that we have extended γ' to a longer chain (not necessarily directed) passing through v and v_0.

We now proceed as in Subcase 2.2 of our proof of Theorem 1.

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Subcase 2.2.1. If v_0 is maximal with respect to $\ll$, then we test whether v_0 is splitting and whether removing v_0 preserves the functionality of all implications, as in the case of v; if this is the case we are done.

Subcase 2.2.2. Otherwise, we find another terminal *times* of Cut link v_1 and a chain γ'' properly extending γ' and passing through v_0. In this case, we repeat the argument with v_1 in place of v_0.

And so on. Since $\mathscr{R}$ is finite, we must eventually find a splitting link such that the resulting structures have functional implications. This proves (b).

The proof of (c) is routine.

Finally, to prove (d), we must show that our correctness condition for **FILL** is preserved under cut-elimination, *i.e.*, that if a proof-net $\delta : \mathscr{R} \to \{I, O\}$ reduces to $\delta : \mathscr{R}' \to \{I, O\}$, then all implications in $\delta : \mathscr{R}' \to \{I, O\}$ are functional. This is immediate for Cut reductions with the following orientation:

$$\dfrac{\dfrac{I \quad I}{I} \qquad \dfrac{O \quad O}{O}}{cut}$$

reduces to

$$\dfrac{I \quad O}{cut} \qquad \dfrac{I \quad O}{cut}$$

In the remaining cases, *e.g.* in

$$\dfrac{\dfrac{A_I^\perp \quad B_O}{(A^\perp \wp B)_O} \qquad \dfrac{A_O \quad B_I^\perp}{(A \otimes B^\perp)_I}}{cut}$$

reduces to

$$\dfrac{A_I^\perp \quad A_O}{cut_1} \qquad \dfrac{B_O \quad B_I^\perp}{cut_2},$$

we argue thus. Let

$$\dfrac{C_I \quad D_O}{(C \wp D)_O}$$

be an implication in $\delta : \mathscr{R}' \to \{I, O\}$ and let γ be a directed chain of type $\lceil C_I, X_O \rceil$.

Suppose γ reaches cut_2 first. Notice that γ cannot pass through cut_1, since otherwise, by the above Remark, γ would be the concatenation

$$\lceil C_I - \gamma_1 - B_O \rceil * \lceil B_I^\perp - \gamma_2 - A_O \rceil * \lceil A_I^\perp - \gamma_3 - X_O \rceil$$

and γ_2 would yield a cyclic chain in $\mathscr{R}$. Therefore γ results from a directed chain $\gamma' = \lceil C_I - \gamma_1' - (A^\perp \wp B)_O \rceil * \lceil (A \otimes B^\perp)_I - \gamma_2' - X_O \rceil$ in $\delta : \mathscr{R} \to \{I, O\}$, where all implications are functional, so $X = D$.

Now suppose γ reaches cut_1 first. Then γ must also reach cut_2: indeed in $\delta : \mathscr{R} \to \{I, O\}$ the implication

$$\dfrac{A_I^\perp \quad B_O}{(A^\perp \wp B)_O}$$

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is functional, and so in any directed chain of type $\lceil A_I^\perp, Y_O \rceil$ we have $Y = B$. Therefore γ has the form

$$\lceil C_I - \gamma_1 - A_O \rceil * \lceil A_I^\perp - \gamma_2 - B_O \rceil * \lceil B_I^\perp - \gamma_3 - X_O \rceil.$$

But then

$$\gamma' = \lceil C_I - \gamma_1 - A_O \rceil * \lceil B_I^\perp - \gamma_3 - X_O \rceil$$

is a directed chain in $\delta : \mathscr{R} \to \{I, O\}$, where all implications are functional, so $X = D$. □

References

Abramsky, S. (1993) Computational Interpretations of Linear Logic. *Theoretical Computer Science* **111** 3–57.

Abramsky, S. and Jagadeesan, R. (1994) Games and Full Completeness Theorem for Multiplicative Linear Logic. *J. Symbolic Logic* **59** (2) 543–574.

Asperti, A. (1991) A linguistic approach to deadlock. Rapport de Recherche du Laboratoire d'Informatique de l'Ecole Normale Supérieure de Paris, LIENS-91-15.

Asperti, A. (1995) Causal Dependencies in Multiplicative Linear Logic with MIX. *Mathematical Structures in Computer Science* **5** 351–380.

Barr, M. (1979) ∗-Autonomous categories. *Springer-Verlag Lecture Notes in Mathematics* **752**.

Barr, M. (1991) ∗-Autonomous categories and linear logic. *Mathematical Structures in Computer Science* **1** 159–178.

Bellin, G. (1990) *Mechanizing Proof Theory: Resource-Aware Logics and Proof-Transformations to Extract Implicit Information*, Ph.D. thesis, Stanford University. (Available as: Report CST-80-91, June, Dept. of Computer Science, University of Edinburgh.)

Bellin, G. and Fleury, A. (1995) Braided Proof-Nets for Multiplicative Linear Logic with Mix. To appear in the *Archive for Mathematical Logic*, special number devoted to papers from the Logic Colloquium 1995.

Bellin, G. and Ketonen, J. (1992) A Decision Procedure Revisited: Notes on Direct Logic, Linear Logic and its Implementation. *Theoretical Computer Science* **95** (1) 115–142.

Bellin, G. and Scott, P. J. (1994) Selected papers of the meeting on Mathematical Foundations of Programming Semantics (MFPS 92), Part 1 (Oxford 1992). *Theoretical Computer Science* **135** (1) 11–65.

Bellin, G. and van de Wiele, J. (1995) Subnets of Proof-nets in **MLL**$^-$. In: Girard J-Y., Lafont, Y. and Regnier, L. (eds.) Advances in Linear Logic. *London Mathematical Society Lecture Note Series* **222**, Cambridge University Press 249–270.

Blute, R. F. (1993) Linear Logic, Coherence and Dinaturality. *Theoretical Computer Science* **115** (1) 3–41.

Blute, R. F., Cockett, J. R. B. and Seely, R. A. G. (1996) **!** and **?**, Storage as tensorial strength. *Mathematical Structures in Computer Science* **6** 313–351.

Blute, R. F., Cockett, J. R. B., Seely, R. A. G. and Trimble, T. H. (to appear) Natural Deduction and Coherence for Weakly Distributive Categories. *Journal of Pure and Applied Algebra*.

Cockett, J. R. B. and Seely, R. A. G. (1992) Weakly distributive categories. In: Fourman, Johnstone and Pitts (eds.) Applications of Categories to Computer Science. *London Mathematical Society Lecture Note Series* **177** 45–65. (The full paper is to appear in the *Journal of Pure and Applied Logic*.)

Danos, V. (1990) *La logique linéaire appliquée à l'étude de divers processus de normalisation et principalement du λ-calcul*, Thèse de doctorat, U. Paris VII.

Danos, V. and Regnier, L. (1989) The Structure of Multiplicatives. *Arch. Math. Logic* **28** 181–203.

de Paiva, V. C. V. (1989a) The Dialectica Categories. In: Gray, J. and Scedrov, A. (eds.) Categories in Computer Science and Logic (Boulder, CO, 1987). *Contemporary Mathematics* **92**, American Mathematical Society 47–72.

de Paiva, V. C. V. (1989b) A Dialectica-like Model of Linear Logic. In: Pitts, D. H. (ed.) Category Theory and Computer Science, Manchester, UK, September 1989. *Springer-Verlag Lecture Notes in Computer Science* **389** 341–356.

Fleury, A. and Retoré, C. (1994) The Mix Rule. *Mathematical Structures in Computer Science* **4** 273–85.

Gallier, J. (preprint) Constructive Logics. Part II: Linear Logic and Proof Nets. Preprint, Dept. of Computer and Information Science, University of Pennsylvania, 200 South 33rd St., Philadelphia, PA 19104, USA (Email: `jean@saul.cis.upenn.edu`).

Girard, J-Y. (1987) Linear Logic. *Theoretical Computer Science* **50** 1–102.

Girard, J-Y. (1989) Towards a Geometry of Interaction. In: Gray, J. and Scedrov, A. (eds.) Categories in Computer Science and Logic (Boulder, CO, 1987). *Contemporary Mathematics* **92**, American Mathematical Society 69–108.

Girard, J-Y. (preprint) Quantifiers in linear logic II. Preprint, Équipe de Logique Mathématique, Université Paris VII, Tour 45-55. 5^{e} étage, 2, Place Jussieu, 75251 PARIS Cedex 05. France.

Girard, J-Y., Lafont, Y. and Taylor, P. (1989) *Proofs and Types*, Cambridge Tracts in Theoretical Computer Science **7**, Cambridge University Press.

Hyland, J. M. and de Paiva, V. C. V. (1993) Full Intuitionistic Linear Logic. *Annals of Pure and Applied Logic* **64** 273–91.

Hyland, M. and Ong, L. (manuscript) Fair Games and Full Completeness for Multiplicative Logic without the rule of Mix – Computer Laboratory and Department of Pure Mathematics and Mathematical Statistics, University of Cambridge.

Ketonen, J. (1984) EKL - A Mathematically Oriented Proof Checker. In: Shostak (ed.) Proceedings of the Seventh Conference on Automated Deduction. *Springer-Verlag Lecture Notes in Computer Science* **170** 65–79.

Ketonen, J. and Weyhrauch, R. (1984) A Decidable Fragment of Predicate Calculus. *Theoretical Computer Science* **32** 297–307.

Lincoln, P. and Winkler, T. (1994) Constant-only multiplicative linear logic is NP-complete. Selected papers of the meeting on Mathematical Foundations of Programming Semantics (MFPS 92), Part 1 (Oxford 1992). *Theoretical Computer Science* **135** (1) 155–169.

Retoré, C. (1993a) *Réseaux et Séquents Ordonnées*, Ph.D. thesis, Université Paris VII, Mathématiques.

Retoré, C. (1993b) Graph Theory from Linear Logic: Agregates. Preprint **47**, Équipe de Logique Mathématique, Université Paris VII.

Schellinx, H. (1991) Some syntactical observations on linear logic. *Journal of Logic and Computation* **1** (4) 537–559.

Seely, R. A. G. (1989) Linear logic, $*$-autonomous categories and cofree coalgebras. In: Gray, J. and Scedrov, A. (eds.) Categories in Computer Science and Logic (Boulder, CO, 1987). *Contemporary Mathematics* **92**, American Mathematical Society 371–382.