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AN APPLICATION OF SCHUR'S ALGORITHM TO VARIABILITY REGIONS OF CERTAIN ANALYTIC FUNCTIONS II

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Abstract

We extend our study of variability regions, Ali *et al.* ['An application of Schur algorithm to variability regions of certain analytic functions–I', *Comput. Methods Funct. Theory*, to appear] from convex domains to starlike domains. Let $\mathcal{CV}(\Omega)$ be the class of analytic functions f in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ satisfying $1 + zf''(z)/f'(z) \in \Omega$. As an application of the main result, we determine the variability region of $\log f'(z_0)$ when f ranges over $\mathcal{CV}(\Omega)$. By choosing a particular Ω, we obtain the precise variability regions of $\log f'(z_0)$ for some well-known subclasses of analytic and univalent functions.

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1. Introduction

Let $\mathbb{C}$ be the complex plane. For $c \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(c, r) := \{z \in \mathbb{C} : |z - c| < r\}$ and $\overline{\mathbb{D}}(c, r) := \{z \in \mathbb{C} : |z - c| \le r\}$. In particular, we denote the unit disk by $\mathbb{D} := \mathbb{D}(0, 1)$. Let $\mathcal{A}(\mathbb{D})$ be the class of analytic functions in the unit disk $\mathbb{D}$ endowed with the topology of uniform convergence on every compact subset of $\mathbb{D}$. Denote by $\mathcal{A}_0$ functions f in $\mathcal{A}(\mathbb{D})$ normalised by $f(0) = f'(0) - 1 = 0$. Further, let $\mathcal{S}$ denote the standard subclass of $\mathcal{A}_0$ of normalised univalent functions in $\mathbb{D}$. A function f in $\mathcal{A}_0$ is called starlike (respectively convex) if f is univalent and $f(\mathbb{D})$ is starlike with respect to 0 (respectively convex). Let $\mathcal{S}^*$ and $\mathcal{CV}$ denote the classes of starlike and convex functions, respectively. It is well known that a function $f \in \mathcal{A}_0$ is in $\mathcal{S}^*$ if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$ and in $\mathcal{CV}$ if and only if $\operatorname{Re}\{zf''(z)/f'(z)\} + 1 > 0$ for $z \in \mathbb{D}$.

Let $\mathcal{F}$ be a subclass of $\mathcal{A}(\mathbb{D})$ and $z_0 \in \mathbb{D}$. The upper and lower estimates,

$$M_1 \le |f'(z_0)| \le M_2 \quad \text{and} \quad m_1 \le \operatorname{Arg} f'(z_0) \le m_2 \quad \text{for all } f \in \mathcal{F},$$

where the M_j and m_j are nonnegative constants, are respectively called distortion and rotation theorems at z_0 for $\mathcal{F}$. These estimates deal only with the absolute value or

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argument of $f'(z_0)$. If one wants to study the complex value $f'(z_0)$ itself, it is necessary to consider the variability region of $f'(z_0)$ when f ranges over $\mathcal{F}$, that is, to consider the set $\{f'(z_0) : f \in \mathcal{F}\}$. For example [4, Ch. 2, Exercises 10, 11 and 13],

$$\{\log f'(z_0) : f \in \mathcal{CV}\} = \left\{\log \frac{1}{(1-z)^2} : |z| \le |z_0|\right\}.$$

For $f \in \mathcal{CV}$, an easy consequence of Schwarz's lemma is that $|f''(0)| \le 2$. For fixed $z_0 \in \mathbb{D}$ and $\lambda \in \overline{\mathbb{D}}$, Gronwall [7] obtained the sharp lower and upper estimates for $|f'(z_0)|$ when $f \in \mathcal{CV}$ satisfies the additional condition $f''(0) = 2\lambda$ (see also [5]). Let

$$\widetilde{V}(z_0, \lambda) = \{\log f'(z_0) : f \in \mathcal{CV} \text{ and } f''(0) = 2\lambda\}.$$

If $|\lambda| = 1$, then, by Schwarz's lemma, for $f \in \mathcal{CV}$ the condition $f''(0) = 2\lambda$ forces $f(z) \equiv z/(1-\lambda z)$ and hence $\widetilde{V}(z_0, \lambda) = \{\log 1/(1-\lambda z_0)^2\}$. Since $\widetilde{V}(e^{-i\theta} z_0, e^{i\theta}\lambda) = \widetilde{V}(z_0, \lambda)$ for all $\theta \in \mathbb{R}$, without loss of generality we may assume that $0 \le \lambda < 1$. In 2006, Yanagihara [20] obtained the following extension of Gronwall's result.

THEOREM 1.1. *For any $z_0 \in \mathbb{D}\backslash\{0\}$ and $0 \le \lambda < 1$, the set $\widetilde{V}(z_0, \lambda)$ is a convex closed Jordan domain surrounded by the curve*

$$(-\pi, \pi] \ni \theta \mapsto -\left(1 - \frac{\lambda \cos(\theta/2)}{\sqrt{1-\lambda^2 \sin^2(\theta/2)}}\right) \log\left\{1 - \frac{e^{i\theta/2} z_0}{i\lambda \sin(\theta/2) - \sqrt{1-\lambda^2 \sin^2(\theta/2)}}\right\}$$
$$- \left(1 + \frac{\lambda \cos(\theta/2)}{\sqrt{1-\lambda^2 \sin^2(\theta/2)}}\right) \log\left\{1 - \frac{e^{i\theta/2} z_0}{i\lambda \sin(\theta/2) + \sqrt{1-\lambda^2 \sin^2(\theta/2)}}\right\}.$$

Theorem 1.1 can be equivalently written as follows.

THEOREM 1.2. *Let $\mathbb{H} = \{w \in \mathbb{C} : \mathrm{Re}\, w > 0\}$. For any $z_0 \in \mathbb{D}\backslash\{0\}$ and $0 \le \lambda < 1$, the variability region*

$$\left\{\int_0^{z_0} \frac{g(\zeta) - g(0)}{\zeta}\, d\zeta : g \in \mathcal{A}(\mathbb{D}) \text{ with } g(0) = 1, g'(0) = 2\lambda, g(\mathbb{D}) \subset \mathbb{H}\right\}$$

coincides with the convex closed Jordan domain defined in Theorem 1.1.

Theorem 1.1 is a direct consequence of Theorem 1.2 with $g(z) = 1 + zf''(z)/f'(z)$. For similar results, we refer to [11, 13, 18, 19, 21] and the references therein.

Recently, the present authors [2] extended Theorem 1.2 to the most general setting.

Let Ω be a simply connected domain in $\mathbb{C}$ with $\Omega \ne \mathbb{C}$ and let P be a conformal map of $\mathbb{D}$ onto Ω. Let $\mathcal{F}_\Omega$ be the class of analytic functions g in $\mathbb{D}$ with $g(\mathbb{D}) \subset \Omega$. Then the map $P^{-1} \circ g$ maps $\mathbb{D}$ into $\mathbb{D}$. For $c = (c_0, c_1, \ldots, c_n) \in \mathbb{C}^{n+1}$, let

$$\mathcal{F}_\Omega(c) = \{g \in \mathcal{F}_\Omega : (P^{-1} \circ g)(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots \in \mathbb{D}\}.$$

Let $H^\infty(\mathbb{D})$ be the Banach space of analytic functions f in $\mathbb{D}$ with the norm defined by $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, and $H_1^\infty(\mathbb{D}) = \{\omega \in H^\infty(\mathbb{D}) : \|\omega\|_\infty \le 1\}$ be the closed unit ball of

$H^\infty(\mathbb{D})$. The coefficient body $C(n)$ defined by

$$C(n) = \{c = (c_0, c_1, \dots, c_n) \in \mathbb{C}^{n+1} : \text{there exists } \omega \in H_1^\infty(\mathbb{D}) \text{ such that } \omega(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots\}$$

is a compact and convex subset of $\mathbb{C}^{n+1}$. The coefficient body $C(n)$ has been completely characterised by Schur [15, 16]. For a detailed treatment, we refer to [6, Ch. I] and [3, Ch. 1].

We call $c = (c_0, \dots, c_n)$ the Carathéodory data of length $n + 1$. For given Carathéodory data $c = (c_0, \dots, c_n) \in \mathbb{C}^{n+1}$, the Schur parameter $\gamma = (\gamma_0, \dots, \gamma_k)$, $k = 0, 1, \dots, n$, is defined as follows.

For $j = 0, 1, \dots,$ define recursively $c^{(j)} = (c_0^{(j)}, c_1^{(j)}, \dots, c_{n-j}^{(j)})$ and $\gamma_j = c_0^{(j)}$ by

$$c_0^{(j)} = \frac{c_1^{(j-1)}}{1 - |\gamma_{j-1}|^2}, \quad c_p^{(j)} = \frac{c_{p+1}^{(j-1)} + \overline{\gamma_{j-1}} \sum_{\ell=1}^{p} c_{p-\ell}^{(j)} c_\ell^{(j-1)}}{1 - |\gamma_{j-1}|^2} \quad (1 \le p \le n - j)$$

with $c^{(0)} = c = (c_0, \dots, c_n)$. In the jth step ($j = 0, 1, \dots$), if $|\gamma_j| > 1$, then we put $k = j$ and $\gamma = (\gamma_0, \dots, \gamma_j)$; if $|\gamma_j| = 1$, then we put $k = n$ and, for $p = j + 1, \dots, n$, we take

$$\gamma_p = \begin{cases} \infty & \text{if } c_{p-j}^{(j)} \neq 0, \\ 0 & \text{if } c_{p-j}^{(j)} = 0; \end{cases}$$

if $|\gamma_j| < 1$, then we proceed to the $(j + 1)$th step. Applying this procedure recursively, we obtain the Schur parameter $\gamma = (\gamma_0, \dots, \gamma_k)$, $k = 0, \dots, n$, of $c = (c_0, \dots, c_n)$.

When $|\gamma_0| < 1, \dots, |\gamma_n| < 1$, each of $c = (c_0, \dots, c_n) = c^{(0)}$ and $\gamma = (\gamma_0, \dots, \gamma_n)$ is uniquely determined by the other. For an explicit representation of γ in terms of c, we refer to [15, 16]. For given $c = (c_0, \dots, c_n) \in \mathbb{C}^{n+1}$, Schur [15, 16] proved that $c \in \operatorname{Int} C(n)$, $c \in \partial C(n)$ and $c \notin C(n)$ are respectively equivalent to the conditions:

- **(C1)** $k = n$ and $|\gamma_i| < 1$ for $i = 1, 2, \dots, n$;
- **(C2)** $k = n$ and $|\gamma_0| < 1, \dots, |\gamma_{i-1}| < 1$, $|\gamma_i| = 1$, $\gamma_{i+1} = \cdots = \gamma_n = 0$ for some i with $i = 0, \dots, n$; and
- **(C3)** neither **(C1)** nor **(C2)** holds.

For $c \in \operatorname{Int} C(n)$, the Schur parameter can be computed as follows. Let $\omega \in H_1^\infty(\mathbb{D})$ be such that $\omega(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots$. Define

$$\omega_0(z) = \omega(z) \quad \text{and} \quad \omega_k(z) = \frac{\omega_{k-1}(z) - \omega_{k-1}(0)}{z(1 - \overline{\omega_{k-1}(0)}\omega_{k-1}(z))} \quad (k = 1, 2, \dots, n).$$

Then $\gamma_p = \omega_p(0)$ and $\omega_p(z) = c_0^{(p)} + c_1^{(p)} z + \cdots + c_{n-p}^{(p)} z^{n-p} + \cdots$ for $p = 0, 1, \dots, n$. For a detailed proof, we refer to [6, Ch. 1].

For $a \in \mathbb{D}$, define $\sigma_a \in \operatorname{Aut}(\mathbb{D})$ by

$$\sigma_a(z) = \frac{z + a}{1 + \bar{a}z}, \quad z \in \mathbb{D}.$$

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For $\varepsilon \in \overline{\mathbb{D}}$ and the Schur parameter $\gamma = (\gamma_0, \ldots, \gamma_n)$ of $c \in \operatorname{Int} C(n)$, let

$$\begin{aligned} \omega_{\gamma,\varepsilon}(z) &= \sigma_{\gamma_0}(z\sigma_{\gamma_1}(\cdots z\sigma_{\gamma_n}(\varepsilon z)\cdots)), \quad z \in \mathbb{D}, \\ Q_{\gamma,j}(z,\varepsilon) &= \int_0^z \zeta^j\{P(\omega_{\gamma,\varepsilon}(\zeta)) - P(c_0)\}\, d\zeta, \quad z \in \mathbb{D} \text{ and } \varepsilon \in \overline{\mathbb{D}}. \end{aligned} \tag{1.1}$$

Then $\omega_{\gamma,\varepsilon} \in H_1^\infty(\mathbb{D})$ with Carathéodory data c, that is, $\omega_{\gamma,\varepsilon}(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots$. By using the Schur algorithm, the present authors [2] obtained the following general result for the region of variability.

Theorem 1.3 [2]. *Let $n \in \mathbb{N} \cup \{0\}$, $j \in \{-1, 0, 1, 2, \ldots\}$ and $c = (c_0, \ldots, c_n) \in \mathbb{C}^{n+1}$ be Carathéodory data. Let Ω be a convex domain in $\mathbb{C}$ with $\Omega \neq \mathbb{C}$ and P be a conformal map of $\mathbb{D}$ onto Ω. For each fixed $z_0 \in \mathbb{D}\backslash\{0\}$, let*

$$V_\Omega^j(z_0, c) = \left\{ \int_0^{z_0} \zeta^j (g(\zeta) - g(0))\, d\zeta : g \in \mathcal{F}_\Omega(c) \right\}.$$

(i) *If $c = (c_0, \ldots, c_n) \in \operatorname{Int} C(n)$ and $\gamma = (\gamma_0, \ldots, \gamma_n)$ is the Schur parameter of c, then $Q_{\gamma,j}(z_0, \varepsilon)$ defined by (1.1) is a convex univalent function of $\varepsilon \in \overline{\mathbb{D}}$ and*

$$V_\Omega^j(z_0, c) = Q_{\gamma,j}(z_0, \overline{\mathbb{D}}) := \{Q_{\gamma,j}(z_0, \varepsilon) : \varepsilon \in \overline{\mathbb{D}}\}.$$

Furthermore,

$$\int_0^{z_0} \zeta^j\{g(\zeta) - g(0)\}\, d\zeta = Q_{\gamma,j}(z_0, \varepsilon)$$

for some $g \in \mathcal{F}_\Omega(c)$ and $\varepsilon \in \partial\mathbb{D}$ if and only if $g(z) \equiv P(\omega_{\gamma,\varepsilon}(z))$.

(ii) *If $c \in \partial C(n)$ and $\gamma = (\gamma_0, \ldots, \gamma_i, 0, \ldots, 0)$ is the Schur parameter of c, then $V_\Omega^j(z_0, c)$ reduces to a set consisting of a single point w_0, where*

$$w_0 = \int_0^{z_0} \zeta^j\{P(\sigma_{\gamma_0}(\zeta\sigma_{\gamma_1}(\cdots\zeta\sigma_{\gamma_{i-1}}(\gamma_i\zeta)\cdots))) - P(c_0)\}\, d\zeta.$$

(iii) *If $c \notin C(n)$, then $V_\Omega^j(z_0, c) = \emptyset$.*

In the present article, we first show that in the case $n = 0$, $j = -1$ and $c = 0$, the conclusion of Theorem 1.3 holds when one weakens the assumption that Ω is convex to Ω is starlike with respect to $P(0)$ (Theorem 2.1). We then present several applications of Theorems 1.3 and 2.1 to obtain the precise variability region for several well-known subclasses of analytic and univalent functions. We also obtain certain subordination results.

2. Main results

Before we state our first result, let us recall the definition of subordination. For two analytic functions f and g in $\mathbb{D}$, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. If g is univalent in $\mathbb{D}$, the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

THEOREM 2.1. *Let $b \in \mathbb{C}$, $z_0 \in \mathbb{D}\backslash\{0\}$ and Ω be a starlike domain with respect to b satisfying $\Omega \neq \mathbb{C}$. Let P be a conformal map of $\mathbb{D}$ onto Ω with $P(0) = b$. Then the region of variability*

$$V_\Omega^{-1}(z_0, 0) = \left\{ \int_0^{z_0} \frac{g(\zeta) - b}{\zeta}\, d\zeta : g \in \mathcal{F}_\Omega, g(0) = b \right\}$$

is a convex closed Jordan domain that coincides with the set $K(\overline{\mathbb{D}}(0, |z_0|))$, where $K(z) = \int_0^z \zeta^{-1}(P(\zeta) - b)\, d\zeta$. Furthermore, for $|\varepsilon| = 1$ and $g \in \mathcal{F}_\Omega$ with $g(0) = b$, the relation $\int_0^{z_0} \zeta^{-1}(g(\zeta) - b)\, d\zeta = K(\varepsilon z_0)$ holds if and only if $g(z) \equiv P(\varepsilon z)$.

PROOF. Let $g \in \mathcal{A}(\mathbb{D})$ be such that $g(0) = b$ and $g(\mathbb{D}) \subset \Omega$. Then g is subordinate to P. By using a result of Suffridge [17], we may conclude that

$$\int_0^z \frac{g(\zeta) - b}{\zeta}\, d\zeta \prec K(z) := \int_0^z \frac{P(\zeta) - b}{\zeta}\, d\zeta.$$

Thus, there exists $\omega \in H_1^\infty(\mathbb{D})$ with $\omega(0) = 0$ and $\int_0^z \zeta^{-1}\{g(\zeta) - b\}\, d\zeta = K(\omega(z))$ and so

$$V_\Omega^{-1}(z_0, 0) \subset \{K(\omega(z_0)) : \omega \in H_1^\infty(\mathbb{D}) \text{ and } \omega(0) = 0\} = K(\overline{\mathbb{D}}(0, |z_0|)).$$

For $\varepsilon \in \overline{\mathbb{D}}$, let $g_\varepsilon(z) = P(\varepsilon z)$. Then $g_\varepsilon(0) = P(0) = b$ and $g_\varepsilon(\mathbb{D}) = P(\mathbb{D}) = \Omega$. Therefore,

$$K(\varepsilon z_0) = \int_0^{\varepsilon z_0} \frac{P(\zeta) - b}{\zeta}\, d\zeta = \int_0^{z_0} \frac{g_\varepsilon(\zeta) - b}{\zeta}\, d\zeta \in V_\Omega^{-1}(z_0, 0)$$

and hence $K(\overline{\mathbb{D}}(0, |z_0|)) \subset V_\Omega^{-1}(z_0, 0)$.

We now deal with the uniqueness. Suppose that

$$\int_0^{z_0} \frac{g(\zeta) - b}{\zeta}\, d\zeta = K(\varepsilon z_0) \tag{2.1}$$

for some g with $g(0) = b$ and $g(\mathbb{D}) \subset \Omega$ and $|\varepsilon| = 1$. Then there exists $\omega \in H_1^\infty(\mathbb{D})$ with $\omega(0) = 0$ such that $\int_0^z \zeta^{-1}\{g(\zeta) - b\}\, d\zeta = K(\omega(z))$. From (2.1), $K(\omega(z_0)) = K(\varepsilon z_0)$. Since K is a convex univalent function, $\omega(z_0) = \varepsilon z_0$. It follows from Schwarz's lemma that $\omega(z) \equiv \varepsilon z$. Consequently, $g(z) \equiv P(\varepsilon z)$. □

2.1. The class $\mathcal{CV}(\Omega)$. Suppose that Ω is a simply connected domain with $1 \in \Omega$. Define

$$\mathcal{CV}(\Omega) = \left\{ f \in \mathcal{A}_0(\mathbb{D}) : 1 + z\frac{f''(z)}{f'(z)} \in \Omega \text{ for all } z \in \mathbb{D} \right\}.$$

Let P be the conformal map of $\mathbb{D}$ onto Ω with $P(0) = 1$. Then $1 + zf''(z)/f'(z) \prec P$ for each $f \in \mathcal{CV}(\Omega)$. For $\alpha \in \mathbb{R}$, let $\mathbb{H}_\alpha := \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$ and $\mathbb{H}_0 = \mathbb{H}$. If $\Omega = \mathbb{H}$ and $P(z) = (1 + z)/(1 - z)$, then $\mathcal{CV}(\mathbb{H}) = \mathcal{CV}$ is the well-known class of normalised convex functions in $\mathbb{D}$. If $\Omega \subset \mathbb{H}$, then $\mathcal{CV}(\Omega)$ is a subclass of $\mathcal{CV}$. For $0 \leq \alpha < 1$, $\mathcal{CV}(\alpha) := \mathcal{CV}(\mathbb{H}_\alpha)$ is the class of convex functions of order α. In this case, we have $P(z) = \{1 + (1 - 2\alpha)z\}/(1 - z)$. If $0 < \beta \leq 1$, then $\mathcal{CV}_\beta := \mathcal{CV}(\{w \in \mathbb{C} : |\arg w| < \pi\beta/2\})$ is the class of strongly convex functions of order β and $P(z) = \{(1 + z)/(1 - z)\}^\beta$.

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As an application of Theorem 2.1, we determine the variability region of $\log f'(z_0)$ when f ranges over $\mathcal{CV}(\Omega)$.

Theorem 2.2. *Let Ω be a starlike domain with respect to 1 and P be a conformal map of $\mathbb{D}$ onto Ω with $P(0) = 1$. Then, for each fixed $z_0 \in \mathbb{D}\backslash\{0\}$, the region of variability*

$$V_{\mathcal{CV}(\Omega)}(z_0) := \{\log f'(z_0) : f \in \mathcal{CV}(\Omega)\}$$

is a convex closed Jordan domain that coincides with the set $K(\overline{\mathbb{D}}(0, |z_0|))$, where $K(z) = \int_0^z \zeta^{-1}(P(\zeta) - 1)\, d\zeta$ is a convex univalent function in $\mathbb{D}$. Furthermore, $\log f'(z_0) = K(\varepsilon z_0)$ for some ε with $|\varepsilon| = 1$ and $f \in \mathcal{CV}(\Omega)$ if and only if $f(z) = \varepsilon^{-1}F(\varepsilon z)$, where $F(z) = \int_0^z e^{K(\zeta)}\, d\zeta$.

Proof. Let $c = 0 \in \mathbb{C}^1$ be given Carathéodory data of length one. In that case, $\mathcal{F}_\Omega(0) = \{g \in \mathcal{A}(\mathbb{D}) : g(\mathbb{D}) \subset \Omega \text{ and } (P^{-1} \circ g)(0) = 0\}$. It is easy to see that the map

$$\mathcal{CV}(\Omega) \ni f \mapsto g(z) = 1 + z\frac{f''(z)}{f'(z)} \in \mathcal{F}_\Omega(0)$$

is bijective. Indeed, since $g(z) = 1 + zf''(z)/f'(z)$ is analytic in $\mathbb{D}$, $f'(z)$ does not have zeros in $\mathbb{D}$ and so

$$\log f'(z) = \int_0^z \zeta^{-1}(g(\zeta) - 1)\, d\zeta, \tag{2.2}$$

where $\log f'$ is a single-valued branch of the logarithm of f' with $\log f'(0) = 0$. The conclusions now follow from Theorem 2.1 and (2.2). □

As an application of Theorem 1.3, we determine the variability region of $\log f'(z_0)$ when f ranges over $\mathcal{CV}(\Omega)$ with the conditions $f''(0) = 2\lambda$ and $f'''(0) = 6\mu$. Here $z_0 \in \mathbb{D}\backslash\{0\}$; $\lambda, \mu \in \mathbb{C}$ are arbitrarily preassigned values. By letting Ω be one of the particular domains mentioned above, we can determine variability regions of $\log f'(z_0)$ for various subclasses of $\mathcal{CV}$.

Let Ω be a simply connected domain with $\Omega \neq \mathbb{C}$ and P be a conformal map of $\mathbb{D}$ onto Ω with $P(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots$. Let g be an analytic function in $\mathbb{D}$ with $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ satisfying $g(\mathbb{D}) \subset \Omega$. For simplicity, we assume that $P(0) = g(0)$, that is, $\alpha_0 = b_0$. Let

$$\omega(z) = (P^{-1} \circ g)(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \quad z \in \mathbb{D}.$$

Then

$$c_0 = 0, \quad c_1 = \frac{b_1}{\alpha_1}, \quad c_2 = \frac{\alpha_1^2 b_2 - \alpha_2 b_1^2}{\alpha_1^3}. \tag{2.3}$$

By Schwarz's lemma, $|b_1| \le |\alpha_1|$ with equality if and only if $g(z) = P(\varepsilon z)$ for some $\varepsilon \in \partial\mathbb{D}$. Let $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ be the Schur parameter of the Carathéodory data $c = (0, c_1, c_2)$. Then $\gamma_0 = \omega(0) = c_0 = 0$ and

$$\omega_1(z) = \frac{\omega(z)}{z}, \quad \gamma_1 = \omega_1(0), \quad \omega_2(z) = \frac{\omega_1(z) - \gamma_1}{z(1 - \overline{\gamma_1}\omega_1(z))}, \quad \gamma_2 = \omega_2(0).$$

A simple computation shows that

$$\gamma_0 = 0, \quad \gamma_1 = c_1 = \frac{b_1}{\alpha_1}, \quad \gamma_2 = \frac{c_2}{1-|c_1|^2} = \frac{\overline{\alpha_1}(\alpha_1^2 b_2 - \alpha_2 b_1^2)}{\alpha_1^2(|\alpha_1|^2 - |b_1|^2)}. \tag{2.4}$$

For $f \in \mathcal{CV}(\Omega)$ and $k \in \mathbb{N}$, let $a_k(f) = f^{(k)}(0)/k!$. Also let $g(z) = 1 + zf''(z)/f'(z) = 1 + b_1 z + b_2 z^2 + \cdots$. Then

$$b_1 = 2a_2(f) \quad \text{and} \quad b_2 = 6a_3(f) - 4a_2(f)^2. \tag{2.5}$$

From (2.4) and (2.5),

$$\gamma_0 = 0, \quad \gamma_1 = \frac{2a_2(f)}{\alpha_1}, \quad \gamma_2 = \frac{2\overline{\alpha_1}\{3\alpha_1^2 a_3(f) - 2(\alpha_1^2 + \alpha_2)a_2(f)^2\}}{\alpha_1^2(|\alpha_1|^2 - 4|a_2(f)|^2)}.$$

Let $\mathcal{A}(2,\Omega) = \{a_2(f) : f \in \mathcal{CV}(\Omega)\}$. By Schwarz's lemma, $\mathcal{A}(2,\Omega) = \overline{\mathbb{D}}(0, |\alpha_1|/2)$. For $f \in \mathcal{CV}(\Omega)$ and $\lambda \in \partial\mathcal{A}(2,\Omega)$, we have $a_2(f) = \lambda$ if and only if $f(z) \equiv \gamma_1^{-1}F(\gamma_1 z)$, where $\gamma_1 = 2\lambda/\alpha_1$. By applying Theorem 1.3 with $n = 1$ and $j = -1$, we obtain the following generalisation of Theorem 1.1.

THEOREM 2.3. *Let Ω be a convex domain with $1 \in \Omega$ and P be a conformal map of $\mathbb{D}$ onto Ω with $P(z) = 1 + \alpha_1 z + \cdots$. For $\lambda \in \mathbb{C}$ with $|\lambda| \le |\alpha_1|/2$ and $z_0 \in \mathbb{D}\backslash\{0\}$, consider the variability region*

$$V_{\mathcal{CV}(\Omega)}(z_0, \lambda) := \{\log f'(z_0) : f \in \mathcal{CV}(\Omega) \text{ with } a_2(f) = \lambda\}.$$

(i) *If $|\lambda| = |\alpha_1|/2$, then $V_{\mathcal{CV}(\Omega)}(z_0,\lambda)$ reduces to a set consisting of a single point w_0, where $w_0 = \int_0^{z_0} \zeta^{-1}\{P(\gamma_1\zeta) - 1\}\,d\zeta$ with $\gamma_1 = 2\lambda/\alpha_1$.*

(ii) *If $|\lambda| < |\alpha_1|/2$, then $V_{\mathcal{CV}(\Omega)}(z_0,\lambda) = Q_{\gamma_1}(z_0, \overline{\mathbb{D}})$, where $\gamma_1 = 2\lambda/\alpha_1$ and*

$$Q_{\gamma_1}(z_0, \varepsilon) = \int_0^{z_0} \zeta^{-1}\Big\{P\Big(\zeta\frac{\varepsilon\zeta + \gamma_1}{1 + \overline{\gamma_1}\varepsilon\zeta}\Big) - 1\Big\}\,d\zeta$$

is a convex, univalent and analytic function of $\varepsilon \in \overline{\mathbb{D}}$. Furthermore, $\log f'(z_0) = Q_{\gamma_1}(z_0, \varepsilon)$ for some $\varepsilon \in \partial\mathbb{D}$ and $f \in \mathcal{CV}(\Omega)$ with $a_2(f) = \lambda$ if and only if

$$f(z) = \int_0^z e^{Q_{\gamma_1}(\zeta,\varepsilon)}\,d\zeta, \quad z \in \mathbb{D}.$$

Next let $\mathcal{A}(3,\Omega) = \{(a_2(f), a_3(f)) \in \mathbb{C}^2 : f \in \mathcal{CV}(\Omega)\}$ and, for $\lambda, \mu \in \mathbb{C}$, let $\gamma_1 := \gamma_1(\lambda,\mu)$ and $\gamma_2 := \gamma_2(\lambda,\mu)$ be given by

$$\gamma_1 = \frac{2\lambda}{\alpha_1} \tag{2.6}$$

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and

$$\gamma_2 = \begin{cases} \dfrac{2\overline{\alpha_1}\{3\alpha_1^2\mu - 2(\alpha_1^2+\alpha_2)\lambda^2\}}{\alpha_1^2(|\alpha_1|^2 - 4|\lambda|^2)} & \text{if } |\gamma_1| < 1, \\ 0 & \text{if } |\gamma_1| = 1 \text{ and } 3\alpha_1^2\mu = 2(\alpha_1^2+\alpha_2)\lambda^2, \\ \infty & \text{if } |\gamma_1| = 1 \text{ and } 3\alpha_1^2\mu \neq 2(\alpha_1^2+\alpha_2)\lambda^2. \end{cases} \tag{2.7}$$

Then $(\lambda,\mu) \in \mathcal{A}(3,\Omega)$ if and only if one of the following conditions holds:

(a) $|\gamma_1(\lambda,\mu)| = 1$ and $\gamma_2(\lambda,\mu) = 0$;
(b) $|\gamma_1(\lambda,\mu)| < 1$ and $|\gamma_2(\lambda,\mu)| = 1$;
(c) $|\gamma_1(\lambda,\mu)| < 1$ and $|\gamma_2(\lambda,\mu)| < 1$.

In case (a), for $f \in \mathcal{CV}(\Omega)$, $(a_2(f), a_3(f)) = (\lambda,\mu)$ if and only if $g(z) = P(\gamma_1 z)$, that is, $f(z) = \gamma_1 F(\gamma_1 z)$, where $\gamma_1 = \gamma_1(\lambda,\mu)$. Similarly, in case (b), for $f \in \mathcal{CV}(\Omega)$, $(a_2(f), a_3(f)) = (\lambda,\mu)$ if and only if $g(z) = P(z\sigma_{\gamma_1}(\gamma_2 z))$, that is,

$$f(z) = \int_0^z \exp\left[\int_0^{\zeta_1} \zeta_2^{-1}\{P(\zeta_2\sigma_{\gamma_1}(\gamma_2\zeta_2)) - 1\}\,d\zeta_2\right] d\zeta_1.$$

We note that $(\lambda,\mu) \in \partial\mathcal{A}(3,\Omega)$ if and only if either (a) or (b) holds.

Suppose that (c) holds, that is, $(\lambda,\mu) \in \operatorname{Int}\mathcal{A}(3,\Omega)$. Then, for $f \in \mathcal{CV}(\Omega)$, $(a_2(f), a_3(f)) = (\lambda,\mu)$ if and only if there exists $\omega^* \in H_1^\infty(\mathbb{D})$ such that

$$g(z) = 1 + \frac{zf''(z)}{f'(z)} = P(z\sigma_{\gamma_1}(z\sigma_{\gamma_2}(z\omega^*(z)))).$$

Let

$$Q_{\gamma_1,\gamma_2}(z,\varepsilon) = \int_0^z \zeta^{-1}\{P(\zeta\sigma_{\gamma_1}(\zeta\sigma_{\gamma_2}(\varepsilon\zeta))) - 1\}\,d\zeta, \quad z \in \mathbb{D} \text{ and } \varepsilon \in \overline{\mathbb{D}}. \tag{2.8}$$

Then, for any fixed $\varepsilon \in \overline{\mathbb{D}}$, $Q_{\gamma_1,\gamma_2}(z,\varepsilon)$ is an analytic function of $z \in \mathbb{D}$ and, for each fixed $z \in \mathbb{D}$, $Q_{\gamma_1,\gamma_2}(z,\varepsilon)$ is an analytic function of $\varepsilon \in \overline{\mathbb{D}}$. Theorem 1.3 leads to the following result.

THEOREM 2.4. *Let Ω be a convex domain with $1 \in \Omega$ and P be a conformal map of $\mathbb{D}$ onto Ω with $P(z) = 1 + \alpha_1 z + \cdots$. Let $(\lambda,\mu) \in \mathbb{C}^2$ and $\gamma_1 = \gamma_1(\lambda,\mu)$ and $\gamma_2 = \gamma_2(\lambda,\mu)$ be defined by (2.6) and (2.7), respectively. For $z_0 \in \mathbb{D}\backslash\{0\}$, consider the variability region*

$$V_{\mathcal{CV}(\Omega)}(z_0,\lambda,\mu) := \{\log f'(z_0) : f \in \mathcal{CV}(\Omega) \text{ with } (a_2(f), a_3(f)) = (\lambda,\mu)\}.$$

(i) *If $|\gamma_1(\lambda,\mu)| = 1$ and $|\gamma_2(\lambda,\mu)| = 0$, then $V_{\mathcal{CV}(\Omega)}(z_0,\lambda,\mu)$ reduces to a set consisting of a single point w_0, where $w_0 = \int_0^{z_0} \zeta^{-1}\{P(\gamma_1\zeta) - 1\}\,d\zeta$.*
(ii) *If $|\gamma_1(\lambda,\mu)| < 1$ and $|\gamma_2(\lambda,\mu)| = 1$, then $V_{\mathcal{CV}(\Omega)}(z_0,\lambda,\mu)$ reduces to a set consisting of a single point w_0, where $w_0 = \int_0^{z_0} \zeta^{-1}\{P(\zeta\sigma_{\gamma_1}(\gamma_2\zeta)) - 1\}\,d\zeta$.*

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(iii) *If* $|\gamma_1(\lambda,\mu)| < 1$ *and* $|\gamma_2(\lambda,\mu)| < 1$, *that is,* $(\lambda,\mu) \in \mathrm{Int}\,\mathcal{A}(3,\Omega)$, *then* $Q_{\gamma_1,\gamma_2}(z_0,\varepsilon)$ *defined by (2.8) is a convex, univalent and analytic function of* $\varepsilon \in \overline{\mathbb{D}}$ *and*

$$V_{\mathcal{CV}(\Omega)}(z_0,\lambda,\mu) = Q_{\gamma_1,\gamma_2}(z_0,\overline{\mathbb{D}}).$$

Furthermore, $\log f'(z_0) = Q_{\gamma_1,\gamma_2}(z_0,\varepsilon)$ *for some* ε *with* $|\varepsilon| = 1$ *and* $f \in \mathcal{CV}(\Omega)$ *with* $(a_2(f), a_3(f)) = (\lambda,\mu)$ *if and only if*

$$f(z) = \int_0^z \exp\Big[\int_0^{\zeta_1} \zeta_2^{-1}\{P(z\sigma_{\gamma_1}(z\sigma_{\gamma_2}(\varepsilon\zeta_2))) - 1\}\,d\zeta_2\Big]\,d\zeta_1.$$

REMARK 2.5. For a simply connected domain Ω with $1 \in \Omega$, define

$$\mathcal{S}^*(\Omega) = \Big\{f \in \mathcal{A}_0(\mathbb{D}) : z\frac{f'(z)}{f(z)} \in \Omega \text{ for all } z \in \mathbb{D}\Big\}.$$

Then $f \in \mathcal{CV}(\Omega)$ if and only if $zf'(z) \in \mathcal{S}^*(\Omega)$. Thus, we can easily translate the theorems of this section to results about variability regions of $\log\{f(z_0)/z_0\}$ when f ranges over $\mathcal{S}^*(\Omega)$ with or without the conditions $f''(0) = \lambda$ and $f'''(0) = \mu$.

2.2. Uniformly convex functions. For $0 \le k < \infty$, the class k-$\mathcal{UCV}$ of k-uniformly convex functions is $\mathcal{CV}(\Omega_k)$, where $\Omega_k := \{w \in \mathbb{C} : \mathrm{Re}\,w > k|w-1|\}$. Here Ω_k is a convex domain containing 1, bounded by a conic section. The conformal map P_k that maps the unit disk $\mathbb{D}$ conformally onto Ω_k is given by

$$P_k = \begin{cases} \dfrac{1}{1-k^2}\cosh\Big(A\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\Big) - \dfrac{k^2}{1-k^2} & \text{for } 0 \le k < 1,\\[2mm] 1 + \dfrac{2}{\pi^2}\Big(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\Big)^2 & \text{for } k = 1,\\[2mm] \dfrac{1}{k^2-1}\sin\Big(\dfrac{\pi}{2K(x)}\displaystyle\int_0^{u(z)/\sqrt{x}}\dfrac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}\Big) + \dfrac{k^2}{k^2-1} & \text{for } 1 < k < \infty,\end{cases}$$

where $A = (2/\pi)\arccos k$, $u(z) = (z-\sqrt{x})/(1-\sqrt{x}z)$ and $K(x)$ is the elliptic integral defined by

$$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}, \quad x \in (0,1).$$

For more details concerning uniformly convex functions, we refer to [10, 14]. When $k = 0$, the class 0-$\mathcal{UCV}$ is essentially the same as $\mathcal{CV}$. Let $P_k(z) = 1 + \alpha_{k1}z + \alpha_{k2}z^2 + \cdots$. Then it is a simple exercise to see that

$$\alpha_{k1} = \begin{cases} \dfrac{2A^2}{1-k^2} & \text{for } 0 \le k < 1,\\[2mm] 8/\pi^2 & \text{for } k = 1,\\[2mm] \dfrac{\pi^2}{4(k^2-1)K^2(x)(1+x)\sqrt{x}} & \text{for } 1 < k < \infty.\end{cases}$$

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Let $f \in k\text{-}\mathcal{UCV}$ be of the form $f(z) = z + a_2 z + a_3 z^2 + \cdots$ and $g(z) = 1 + zf''(z)/f'(z)$. Then, from (2.3) and (2.5), we obtain $|a_2| \le \alpha_{k1}/2$. For $z_0 \in \mathbb{D}\backslash\{0\}$ and $|\lambda| \le \alpha_{k1}/2$, consider the region of variability

$$V_{k\text{-}\mathcal{UCV}}(z_0, \lambda) = \{\log f'(z_0) : f \in k\text{-}\mathcal{UCV} \text{ with } a_2(f) = \lambda\}.$$

The following corollary is a simple consequence of Theorem 2.3.

COROLLARY 2.6. *Let $z_0 \in \mathbb{D}\backslash\{0\}$ and $\lambda \in \mathbb{C}$ with $|\lambda| \le \alpha_{1k}/2$. Let $\gamma_1 = 2\lambda/\alpha_{1k}$.*

(i) *If $|\gamma_1| = 1$, then $V_{k\text{-}\mathcal{UCV}}(z_0, \lambda) = \{w_0\}$, where $w_0 = \int_0^{z_0} \zeta^{-1}\{P_k(\gamma_1\zeta) - 1\}\, d\zeta$.*

(ii) *If $|\gamma_1| < 1$, then $V_{k\text{-}\mathcal{UCV}}(z_0, \lambda) = Q_{\gamma_1}(z_0, \overline{\mathbb{D}})$, where*

$$Q_{\gamma_1}(z_0, \varepsilon) = \int_0^{z_0} \zeta^{-1}\Big\{P_k\Big(\zeta\frac{\varepsilon\zeta + \gamma_1}{1 + \overline{\gamma_1}\varepsilon\zeta}\Big) - 1\Big\}\, d\zeta$$

is a convex, univalent and analytic function of $\varepsilon \in \overline{\mathbb{D}}$. Furthermore,

$$\log f'(z_0) = Q_{\gamma_1}(z_0, \varepsilon)$$

for some $\varepsilon \in \partial\mathbb{D}$ and $f \in k\text{-}\mathcal{UCV}$ with $a_2(f) = \lambda$ if and only if

$$f(z) = \int_0^z e^{Q_{\gamma_1}(\zeta, \varepsilon)}\, d\zeta, \quad z \in \mathbb{D}.$$

2.3. Janowski starlike and convex functions. For $A, B \in \mathbb{C}$ with $|B| \le 1$ and $A \neq B$, let $P_{A,B}(z) := (1 + Az)/(1 + Bz)$. Then $P_{A,B}$ is a conformal map of $\mathbb{D}$ onto a convex domain $\Omega_{A,B}$. In this case, the classes $\mathcal{S}^*(\Omega_{A,B})$ and $\mathcal{CV}(\Omega_{A,B})$ reduce to

$$\mathcal{S}^*(A, B) := \Big\{f \in \mathcal{A}_0 : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}\Big\}$$

and

$$\mathcal{CV}(A, B) := \Big\{f \in \mathcal{A}_0 : \frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 + Az}{1 + Bz}\Big\},$$

respectively. Since $P_{A,B}(\mathbb{D}) = P_{-A,-B}(\mathbb{D})$, without loss of generality we may assume that $A \in \mathbb{C}$ with $-1 \le B \le 0$ and $A \neq B$. It is important to note that functions in $\mathcal{S}^*(A, B)$ with $A \in \mathbb{C}$, $-1 \le B \le 0$ and $A \neq B$ are not in general univalent. For $-1 \le B < A \le 1$, it is easy to see that $\Omega_{A,B} \subset \mathbb{H}$ and so $\mathcal{S}^*(A, B) \subset \mathcal{S}^*$. A similar result holds for $\mathcal{CV}(A, B)$. For $-1 \le B < A \le 1$, the class $\mathcal{S}^*(A, B)$ was first introduced and investigated by Janowski [9].

Note that $P_{A,B}(z) := (1 + Az)/(1 + Bz) = 1 + (A - B)z + \cdots$. For $f \in \mathcal{CV}(A, B)$, from (2.3) and (2.5) we immediately obtain $|a_2(f)| \le |A - B|/2$. For $z_0 \in \mathbb{D}\backslash\{0\}$ and $|\lambda| \le |A - B|/2$, consider

$$V_{\mathcal{CV}(A,B)}(z_0) := \{\log f'(z_0) : f \in \mathcal{CV}(A, B)\},$$
$$V_{\mathcal{CV}(A,B)}(z_0, \lambda) := \{\log f'(z_0) : f \in \mathcal{CV}(A, B) \text{ with } a_2(f) = \lambda\}.$$

The following corollary is a simple consequence of Theorems 2.2 and 2.3.

COROLLARY 2.7. *Let $z_0 \in \mathbb{D}\backslash\{0\}$ be fixed and $\lambda \in \mathbb{C}$ be such that $|\lambda| \le |A - B|/2$. Also, let $\gamma_1 = 2\lambda/(A - B)$.*

(i) *The region of variability $V_{\mathcal{CV}(A,B)}(z_0)$ is a convex, closed Jordan domain and coincides with the set $K(\overline{\mathbb{D}}(0, |z_0|))$, where*

$$K(z) = \int_0^z \frac{A - B}{1 + B\zeta}\, d\zeta$$

is a convex univalent function in $\mathbb{D}$. Furthermore, $\log f'(z_0) = K(\varepsilon z_0)$ for some ε with $|\varepsilon| = 1$ and $f \in \mathcal{CV}(A, B)$ if and only if $f(z) = \varepsilon^{-1}F(\varepsilon z)$, where $F(z) = \int_0^z e^{K(\zeta)}\, d\zeta$.

(ii) *If $|\gamma_1| = 1$, then $V_{\mathcal{CV}(A,B)}(z_0, \lambda) = \{w_0\}$, where $w_0 = \int_0^{z_0} \zeta^{-1}\{P(\gamma_1\zeta) - 1\}\, d\zeta$.*

(iii) *If $|\gamma_1| < 1$, then $V_{\mathcal{CV}(A,B)}(z_0, \lambda) = Q_{\gamma_1}(z_0, \overline{\mathbb{D}})$, where*

$$Q_{\gamma_1}(z_0, \varepsilon) = \int_0^{z_0} \frac{(A - B)\sigma_{\gamma_1}(\varepsilon\zeta)}{1 + B\zeta\sigma_{\gamma_1}(\varepsilon\zeta)}\, d\zeta$$

is a convex, univalent and analytic function of $\varepsilon \in \overline{\mathbb{D}}$. Furthermore,

$$\log f'(z_0) = Q_{\gamma_1}(z_0, \varepsilon)$$

for some ε with $\varepsilon \in \partial\mathbb{D}$ and $f \in \mathcal{CV}(\Omega)$ with $a_2(f) = \lambda$ if and only if

$$f(z) = \int_0^z e^{Q_{\gamma_1}(\zeta,\varepsilon)}\, d\zeta, \quad z \in \mathbb{D}.$$

REMARK 2.8. The region of variability $V_{\mathcal{CV}(A,B)}(z_0, \lambda)$ for the class $\mathcal{CV}(A, B)$ was first obtained by Ul-Haq [18] for $-1 \le B < 0$ and $A > B$. Although Ul-Haq considered the problem for $A \in \mathbb{C}$, $0 < B \le 1$ and $A \ne B$, the computation is valid only for $-1 \le B < 0$ and $A > B$. We also note that the Herglotz representation [18, formula (2)] for functions in $\mathcal{CV}(A, B)$ is not valid when $-1 < B < 0$.

In particular, for $A = e^{-2i\alpha}$ with $\alpha \in (-\pi/2, \pi/2)$ and $B = -1$, the class $\mathcal{CV}(A, B)$ reduces to the class of functions that satisfy $\mathrm{Re}\,\{e^{i\alpha}(1 + zf''(z)/f'(z))\} > 0$ for $z \in \mathbb{D}$. The functions in this class, denoted by $\mathcal{S}_\alpha$, are known as Robertson functions. If we choose $A = e^{-2i\alpha}$ with $\alpha \in (-\pi/2, \pi/2)$ and $B = -1$ in Corollary 2.7, then we obtain the result obtained in [13].

For $A = 1 - 2\alpha$ with $-1/2 \le \alpha < 1$ and $B = -1$, the class $\mathcal{CV}(A, B)$ reduces to the class of functions f satisfying $\mathrm{Re}\,(1 + zf''(z)/f'(z)) > \alpha$ for $z \in \mathbb{D}$. This is the class $\mathcal{CV}(\alpha)$ of convex functions of order α. For $0 \le \alpha < 1$, $\mathcal{CV}(\alpha) \subset \mathcal{CV}$. On the other hand, for $-1/2 \le \alpha < 0$, functions in $\mathcal{CV}(\alpha)$ are convex functions in some direction (see [12]). If we choose $A = 1 - 2\alpha$ with $-1/2 \le \alpha < 1$ and $B = -1$ in Corollary 2.7, then we obtain the precise region of variability $V_{\mathcal{CV}(\alpha)}(z_0) := \{\log f'(z_0) : f \in \mathcal{CV}(\alpha)\}$ and $V_{\mathcal{CV}(\alpha)}(z_0, \lambda) := \{\log f'(z_0) : f \in \mathcal{CV}(\alpha) \text{ and } a_2(f) = \lambda\}$, which gives a generalisation of Theorem 1.1. In particular, if we choose $A = 2$ and $B = -1$ in Corollary 2.7, then we obtain the result obtained by Ponnusamy and Vasudevarao [11, Theorem 2.6]. Similarly, for $A = -2$ and $B = -1$, the class $\mathcal{CV}(A, B)$

reduces to the class of functions f that satisfy $\operatorname{Re}(1 + zf''(z)/f'(z)) < 3/2$ for $z \in \mathbb{D}$. Functions in the class $\mathcal{CV}(-2, -1)$ are starlike, but not necessarily convex [1]. If we choose $A = -2$ and $B = -1$ in Corollary 2.7, then we obtain the result in [11, Theorem 2.8].

Since $f \in \mathcal{CV}(A, B)$ if and only if $zf'(z) \in \mathcal{S}^*(A, B)$, we can easily translate the above results about variability regions of $\log\{f(z_0)/z_0\}$ when f ranges over $\mathcal{S}^*(A, B)$ with or without the condition $f''(0) = 2\lambda$.

3. Concluding remark

Theorem 2.1 demonstrates that our results are closely related to the concept of subordination. Our assumption that $g \in \mathcal{F}_\Omega(c)$ in Theorem 1.3 can be rewritten as $g \prec P$ when $c_0 = 0$. In this case, $P^{-1}(g(z)) = c_1 z + \cdots + c_n z^n + \cdots$. However, apart from a few exceptional cases, we cannot express our conclusions in terms of subordination relations. Let $c = (c_0, \ldots, c_{n-1}) = (0, \ldots, 0) \in \mathbb{C}^n$. Then the Schur parameter for c is given by $\gamma = (\gamma_0, \ldots, \gamma_{n-1}) = (0, \ldots, 0)$. For this particular choice of c, the function $Q_{\gamma,j}$ defined by (1.1) becomes

$$Q_{\gamma,j}(z, \varepsilon) = \int_0^z \zeta^j \{P(\varepsilon\zeta^n) - 1\}\, d\zeta.$$

Let

$$H(z) = \frac{j+1}{z^{(j+1)/n}} \int_0^{z^{1/n}} \zeta^j \{P(\zeta^n) - 1\}\, d\zeta.$$

Then

$$\frac{j+1}{z^{j+1}} Q_{\gamma,j}(z, \varepsilon) = H(\varepsilon z^n).$$

By Theorem 1.3, for each fixed $z \in \mathbb{D}\backslash\{0\}$, $Q_{\gamma,j}(z, \varepsilon)$ is a convex univalent function of $\varepsilon \in \overline{\mathbb{D}}$ and $H(\varepsilon z^n)$ is also a convex univalent function of $\varepsilon \in \overline{\mathbb{D}}$. Letting $z \to 1$ in $\mathbb{D}$ shows that $H(\varepsilon)$ is also convex univalent in $\mathbb{D}$. Let $g \in \mathcal{F}_\Omega$ with $g'(0) = \cdots = g^{(n-1)}(0) = 0$. It follows from Theorem 1.3 that for any $z \in \mathbb{D}\backslash\{0\}$, there exists $\varepsilon \in \overline{\mathbb{D}}$ satisfying

$$\int_0^z \zeta^j \{g(\zeta) - 1\}\, d\zeta = Q_{\gamma,j}(z, \varepsilon).$$

Thus, for all $z \in \mathbb{D}$,

$$\frac{j+1}{z^{j+1}} \int_0^z \zeta^j \{g(\zeta) - 1\}\, d\zeta = \frac{j+1}{z^{j+1}} Q_{\gamma,j}(z, \varepsilon) = H(\varepsilon z^n) \subset H(\mathbb{D}).$$

Consequently, in view of the univalence of H, we obtain the subordination relation

$$\frac{j+1}{z^{j+1}} \int_0^z \zeta^j \{g(\zeta) - 1\}\, d\zeta \prec H(z).$$

This was previously proved by Hallenbeck and Ruscheweyh [8] when $\operatorname{Re} j \geq -1$ with $j \neq -1$.

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