Can. J. Math. Vol. 47 (6), 1995 pp. 1148–1176

PRIME SEGMENTS OF SKEW FIELDS

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ABSTRACT. An additive subgroup P of a skew field F is called a *prime* of F if P does not contain the identity, but if the product xy of two elements x and y in F is contained in P, then x or y is in P. A prime segment of F is given by two neighbouring primes $P_1 \supset P_2$; such a segment is invariant, simple, or exceptional depending on whether $A(P_1) = \{a \in P_1 \mid P_1 a P_1 \subset P_1\}$ equals P_1, P_2 or lies properly between P_1 and P_2. The set $T(F)$ of all primes of F together with the containment relation is a tree if $|T(F)|$ is finite, and $1 < |T(F)| < \infty$ is possible if F is not commutative. In this paper we construct skew fields with prescribed types of sequences of prime segments as skew fields F of fractions of group rings of certain right ordered groups. In particular, groups G of affine transformations on ordered vector spaces V are considered, and the relationship between properties of Dedekind cuts of V, certain right orders on G, and chains of prime segments of F is investigated. A general result in Section 4 describing the possible orders on vector spaces over ordered fields may be of independent interest.

1. **Introduction.** A subring B of a skew field F is called a *valuation ring* of F if $x \in F \setminus B$ implies $x^{-1} \in B$. The pair (F, B) is then called a *valued skew field*. For any two right (left) ideals I_1, I_2 of B we have $I_1 \subseteq I_2$ or $I_2 \subset I_1$, *i.e.* B is a chain domain with F as skew field of quotients.

Two distinct completely prime ideals $P_2 \subset P_1$ of B are called a *prime segment* of B if no further completely prime ideal of B lies between P_1 and P_2. Prime segments can also be defined directly for F without the introduction of B by considering certain subsets P of F. These primes P of F are in one-to-one correspondence with the valuation subrings B of F such that $P = \mathcal{J}(B)$, the maximal ideal of B.

A prime segment $P_2 \subset P_1 \subset B$ of a skew field F falls into one of three categories: It is *invariant* if the ring $\bar{B} = B_{P_1}/P_2$, the factor ring of the localization B_{P_1} of B on P_1 modulo P_2, is an invariant ring, *i.e.* all its one-sided ideals are two-sided and the ordered semigroup of non-zero principal ideals of this ring can then be embedded into the ordered group $(\mathbb{R}, +)$ of real numbers with addition as operation. If $\bar{B}$ is a nearly simple ring with (0) and $\mathcal{J}(\bar{B})$ as its only proper ideals we say that the segment $P_2 \subset P_1$ is *simple*. Finally, the prime segment is called *exceptional* if there exists a prime ideal Q of B between P_1 and P_2 which is not completely prime. There are then no further ideals between P_1 and Q and $\cap Q^n = P_2$ (see [BBT] Theorem 6.2 and [BT 1]).

These cases are characterized directly for primes of F in Theorem 2.6 without the use of valuation rings.

The first author is supported in part by NSERC.
Received by the editors February 15, 1994.
AMS subject classification: Primary: 16W60, 16S35, 16L30; secondary: 16W80, 20F60.

It follows from [G] (see also [BG]) that skew fields F finite dimensional over their centers have invariant prime segments only. Valuation rings with invariant prime segments only are called *locally invariant*. Such rings are necessarily invariant in the rank one case, but not otherwise (see also [BT]).

The existence of prime ideals in valuation rings which are not completely prime has been an open question for some time (see for example: [P], [O], [BT1], [D2], [S], [S1], [S2]). However, N. I. Dubrovin has announced that he was able to overcome the difficulties he encountered in [D2] and that he has constructed skew fields with exceptional prime segments.

In Section 2 we discuss $T(F)$, the partially ordered set of all primes of a skew field F, see Lemma 2.4, and characterize prime segments of F. Unlike the commutative case, it is possible to have $1 < |T(F)| < \infty$.

In order to construct valued skew fields (F,B) such that B has a prescribed sequence of segments which are either invariant or simple, see Example 3.10, we discuss two construction methods for F as a skew field of fractions of group rings over right ordered groups in Theorems 3.2 and 3.7.

The principal one-sided ideals $\neq (0)$ of the valuation ring B are parameterized by the elements of the generalized positive cone P of a right ordered group and the correspondence between ideals of B and ideals of P is worked out in Lemma 3.5, Theorem 3.6 and Corollary 3.8.

The right *cofinality type* r-cofin(P_1,P_2) of a prime segment $P_2 \subset P_1$ of a valuation ring B of F is defined as the cofinality type of the ordered set $(M,\leq)$ of right ideals I, $I \neq P_2$, between P_1 and P_2 with $I \leq I'$ if $I \supseteq I'$. Hence, r-cofin$(P_1,P_2) = \omega_\alpha$ is the minimal ordinal β such that there exists a well ordered subset N of M of order type β with N cofinal in M, *i.e.* for every $m \in M$ there exists $n \in N$ with $m \leq n$. The left cofinality type ℓ-cofin(P_1,P_2) of $P_2 \subset P_1$ is defined similarly. Finally, one can also define the right cofinality type, r-cofin(B), of B as the cofinality type of the ordered set of all non-zero right ideals of B or equivalently of $W_r = \{aB \mid 0 \neq a \in B\}$ with $aB \geq bB$ if $aB \subseteq bB$, and ℓ-cofin(B) is defined similarly. It follows from the above remarks that for an invariant or exceptional prime segment of a valuation ring B the right and left cofinality types are both equal to ω_0. On the other hand we construct in Sections 3, 4 and 5 examples of valuation rings with simple prime segments of arbitrary cofinality types ω_α. However, we were not able to decide whether the right and left cofinality types always agree for any segment or any valuation ring. A valuation ring B with r-cofin$(B) \neq \ell$-cofin(B) must certainly satisfy the following condition

$(*)$ $\bigcap_{I\neq(0)} I \neq (0)$, where the intersection is taken over all non-zero ideals I of B.

Valuation rings B with $(*)$ are called *finally simple* and it follows that $P_{\min} = \bigcap I$, $I \neq (0)$, is a completely prime ideal of B with $(0) \subset P_{\min}$ a simple prime segment whose cofinality types agree with the cofinality types of B.

We recall that for a valued skew field (F,B) the set $T = \{aBb \mid a,b \in B\backslash\{0\}\}$ defines a neighborhood basis of 0 for the coarsest ring topology $\mathcal{T}_B$ defined on F such that B is open, see [S1] and [HA]. We call $\mathcal{T}_B$ the *valuation topology* of (F,B). This is a skew field topology (see [S1]). With the help of Cauchy filters one obtains the completion $\hat{F}$ and $\hat{B}$ of F and B respectively. The valuation topology of (F, B) can be defined using the set of non-zero two-sided ideals of B as neighborhood basis of 0 if and only if the condition $(*)$ above does not hold. In this case $\mathcal{T}_B$ is a V-topology and $(\hat{F}, \hat{B})$ is a valued skew field, see [K] and [M2].

However, Liepold in [L] proved on the one hand that $\hat{F}$ is not a skew field if $(*)$ holds for (F, B) and if $r\text{-cofin}(B) = \ell\text{-cofin}(B) = \omega_0$, but on the other hand that for every cofinality type $\omega_\alpha > \omega_0$ there exists an example of a valued skew field (F, B) with $(*)$ and $r\text{-cofin}(B) = \ell\text{-cofin}(B) = \omega_\alpha$ and $\hat{F}$ is a skew field.

In Section 4 of this paper we construct particular right ordered groups G for which Theorem 3.2 is applicable. These groups are groups of affine transformations on ordered K-vector spaces V and the right order is defined using Dedekind cuts $\mathcal{C} = (U, O)$ of V. After obtaining a very general result—Theorem 4.1—about ordered vector spaces, the connection between properties of the ordered field K, of V and of $\mathcal{C}$ on the one hand and properties of the resulting right ordered group G and the valued skew field (F, B) on the other hand is explored; see Theorem 4.4 for conditions for (F, B) to be finally simple, Theorem 4.8 for conditions for (F, B) to have simple segments only, and Corollary 4.9 describes the nearly simple case. The general result in Corollary 4.7 shows that for the valued skew fields so constructed there exists a completely prime ideal D of B such that a prime segment $P_1 \supset P_2$ of B is simple if $D \supseteq P_1$ and it is invariant if $P_2 \supseteq D$. Finally, the cofinality types of the simple segments are all equal to the cofinality type of the ordered field K. These results are illustrated by various examples in Section 5.

2. **Primes and prime segments.** Let F be a skew field. A subring B of F is called a *valuation* (or *total*) subring of F if $x \in F \setminus B$ implies $x^{-1} \in B$. We call the pair (F, B) a *valued skew field*. Since for any $a \neq 0 \neq b$ in F we have $a^{-1}b \in B$ or $b^{-1}a \in B$ we have $bB \subseteq aB$ or $aB \subset bB$ and B is a right (and similarly a left) chain domain. We denote with $\mathcal{J}(B)$ the maximal ideal of B.

We consider the chain of completely prime ideals of B and we say that two completely prime ideals $P_1 \supset P_2 \supseteq (0)$ in B define a *prime segment* if no further completely prime ideal of B exists between P_1 and P_2.

THEOREM 2.1. *Let (F, B) be a valued skew field and $P_1 \supset P_2$ be a prime segment of B. Then exactly one of the following possibilities occurs:*

(i) $aP_1 = P_1a$ for any $a \in P_1 \setminus P_2$;
(ii) There are no further two-sided ideals of B between P_1 and P_2;
(iii) There exists a prime ideal Q of B with $P_1 \supset Q \supset P_2$ and no further two-sided ideal exists between P_1 and Q.

DEFINITION 2.2. A prime segment $P_1 \supset P_2$ of B is called *invariant* or of type ***i*** if condition (i) in 2.1 holds, it is called *simple* or of type **s** if condition (ii) in 2.1 holds and it is called *exceptional* or of type **x** if condition (iii) in 2.1 holds.

REMARK. N. I. Dubrovin has announced that he has recently (*i.e.* 1992) constructed skew fields with exceptional primes.

PROOF OF THEOREM 2.1. Assume that the condition (i) is not satisfied and hence there exists an element z in $P_1 \setminus P_2$ with $zP_1 \neq P_1z$. This implies that either $P_1 \not\subseteq z^{-1}P_1z$ or $P_1 \not\subseteq zP_1z^{-1}$, since otherwise $zP_1 \subseteq P_1z \subseteq zP_1$. Therefore, there exists an element $x \in P_1$ with $x \notin z^{-1}P_1z$ or $x \notin zP_1z^{-1}$. We claim that there can not exist an ideal I of B with $x \in I$ and $\bigcap I^n \subseteq P_2$ with $n \in \mathbb{N}$. Assume to the contrary that I is such an ideal. Since $z \in P_1 \setminus P_2$ and B is a chain ring, we have $P_2 \subset z^{-1}P_1$, $P_2 \subset Bz$, $P_2 \subset P_1z^{-1}$, $P_2 \subset zB$ and there exists an $n \in \mathbb{N}$ with $I^n \subseteq z^{-1}P_1 \cap Bz \cap P_1z^{-1} \cap zB$. Therefore, $x^{2n} \in I^{2n} \subseteq z^{-1}P_1Bz = z^{-1}P_1z$ and similarly $x^{2n} \in zP_1z^{-1}$, and we obtain $(zxz^{-1})^{2n} \in P_1$ which implies $zxz^{-1} \in B$ and $zxz^{-1} \in P_1$. Similarly, $z^{-1}xz \in P_1$. However, this contradicts the assumption that $x \notin z^{-1}P_1z$ or $x \notin zP_1z^{-1}$ and the above claim is proved.

From this we conclude that P_1 is idempotent, since otherwise $x \in I := P_1$ and $\bigcap P_1^n = P_2$ leads to a contradiction. This follows since (by [BBT]) for any ideal I of B the intersection $\bigcap I^n$ is completely prime and therefore for $P_1 \subset I \subseteq P_2$ we have $\bigcap I^n = P_2$.

Next we consider the ideal I_0 of B maximal with the property of not containing x. It follows that $I_0 \subset P_1$. There can be no two-sided ideal I in B with $I_0 \subset I \subset P_1$, since otherwise $x \in I$ and $\bigcap I^n = P_2$, contradicting the above result. We conclude that only the following possibilities remain: $I_0 = P_2$ and the segment $P_1 \supset P_2$ is simple or $P_1 \supset I_0 \supset P_2$ with $P_1^2 = P_1$ and there are no further two-sided ideals between P_1 and I_0. In this case we obtain from $A \supset I_0 \subset C$ for two-sided ideals A and C of B that $A \supseteq P_1 \subseteq C$ and $AC \supseteq P_1 \supset I_0$, *i.e.* $I_0 = Q$ is prime and the segment $P_1 \supset P_2$ is exceptional. ∎

We mention that valued skew fields (F, B) can also be defined through a valuation map v from F onto a totally ordered set with maximal element ∞, such that

$$v(a) = \infty \quad \text{if and only if } a = 0, v(a+b) \geq \min\{v(a), v(b)\} \quad \text{and}$$
$$v(a) \geq v(b) \text{ implies } v(ca) \geq v(cb) \quad \text{for all } a, b, c \text{ in } F.$$

The corresponding valuation subring B of F is then equal to $B_v = \{a \in F \mid v(a) \geq v(1)\}$.

In the next result we will show that a valuation subring B of a skew field F is defined through its maximal ideal $P = \mathcal{J}(B)$ considered as a subset of F which satisfies certain properties.

THEOREM 2.3. *A nonempty subset P of the skew field F is the maximal ideal of a valuation subring B of F if and only if the following conditions hold:*

(i) $1 \notin P$;

(ii) $a, b \in P$ *implies* $a - b \in P$;

(iii) $x, y \in F$, $xy \in P$, $x \notin P$ *imply* $y \in P$.

In that case, $B = \{x \in F^* \mid x^{-1} \notin P\} \cup \{0\}$ *and we say that P is a prime of F.*

PROOF. If B is a valuation subring of F with maximal ideal P, then $0 \neq x \in B$ implies either $x \in P$, hence $x^{-1} \notin P$, or $x \in B \setminus P$ and $x^{-1} \in B \setminus P$. Conversely, if

$0 \neq x \in F$ with $x^{-1} \notin P$ then $x \notin B$ implies $x^{-1} \in P$, a contradiction, and we have $B = \{x \in F^* \mid x^{-1} \notin P\} \cup \{0\}$. The conditions (i) and (ii) obviously hold for P in this case and to prove (iii) one observes that in both cases $x \notin B$ and $x \in B \setminus P$ one has $x^{-1} \in B$ and $y = x^{-1}xy = y \in P$ follows.

To prove the converse, we assume that the subset P of F satisfies the conditions (i), (ii) and (iii) and we will show that $B = \{x \in F^* \mid x^{-1} \notin P\} \cup \{0\}$ is a valuation subring of F with maximal ideal P.

We observe first that $0, 1$ and -1 are elements of B since -1 can not be in P using (i) and (ii) and hence $-1 = (-1)^{-1} \notin P$. Next, let a, b be elements in B and $a^{-1} \notin P$, $b^{-1} \notin P$ and hence $(ab)^{-1} = b^{-1}a^{-1} \notin P$ follows; *i.e.* $ab \in B$. If $x \in F \setminus B$ and also $x^{-1} \notin B$ then $x^{-1} \in P$, $x \in P$ and $x^{-1} - x \in P$. Since $1 \notin P$ and P is an abelian group, we have $1 + x^{-1} \notin P$, $1 - x \notin P$ and $x^{-1} - x = (1 + x^{-1})(1 - x) \notin P$—a contradiction which shows that $x \in F \setminus B$ implies $x^{-1} \in B$.

To show that B is closed under addition we observe that $a + b = a(1 + a^{-1}b) = b(b^{-1}a + 1)$ and either $a^{-1}b$ or $b^{-1}a$ is in B for $a \neq 0 \neq b$ in B. Since B is multiplicatively closed by the argument above, it remains to show that a in B implies $1 + a$ in B for $a \neq 0$. Since a is in B we have $a^{-1} \notin P$. Assume $1 + a \notin B$. Then $(1 + a)^{-1} \in P$ and $a(1 + a)^{-1} = 1 - (1 + a)^{-1} \notin P$ and therefore $(1 + a)^{-1} = a^{-1}\big(a(1 + a)^{-1}\big) \notin P$, a contradiction, and $1 + a \in B$ follows.

Next we show that P is contained in B: Assume $a \in P$, $a \notin B$. Then $a^{-1} \in B$ by one of the above arguments, and $a = (a^{-1})^{-1} \notin P$—a contradiction. Finally, it will be shown that the set $U(B)$ of units of B equals $B \setminus P$. We have $x \in B \setminus P$ if and only if $x \notin P$ and $x^{-1} \notin P$ which is the case exactly if x^{-1} as well as x is in B. ∎

Associated with a valuation ring B of F with maximal ideal $\mathcal{J}(B) = P$ is the surjective mapping φ from F to $F' \cup \{\infty\}$ with $F' = B/P = \bar{B}$ and $\varphi(a) = \bar{a} \in \bar{B}$ for $a \in B$ and $\varphi(a) = \infty$ otherwise. Such a mapping is called a *place* which in turn can be defined by certain properties, see [A]. Places are obtained through epimorphisms of Desarguesian projective planes, see [M2] and [R].

For a prime P of F we denote with $B(P) = \{x \in F^* \mid x^{-1} \notin P\} \cup \{0\}$ the corresponding valuation ring of F. It follows that $\mathcal{J}\big(B(P)\big) = P$ and $B\big(\mathcal{J}(B)\big) = B$ where P is a prime and B a valuation ring of F. Further, $P_1 \supseteq P_2$ if and only if $B(P_1) \subseteq B(P_2)$ for primes P_1 and P_2 of F. In this case P_2 is a completely prime ideal in $B(P_1)$ and $B(P_2)$ is equal to the localization $B(P_1)_{P_2}$ of $B(P_1)$ at P_2. It follows that the set of primes of F contained in a prime P of F is the set of completely prime ideals of $B(P)$ and hence is totally ordered by inclusion.

For any set of primes $\{P_i \mid i \in I\}$ of F there exists the minimal valuation overring $\tilde{B} = \bigcap B$, with the intersection over all valuation rings B of F with $B \supseteq B(P_i)$ for all $i \in I$. Hence there exists the maximal lower bound $\mathcal{J}(\tilde{B}) = \tilde{P}$ for the set $\{P_i \mid i \in I\}$ with respect to inclusion. We therefore say, (as in the commutative case, see [RI]) that the partial order defined on the set of primes of F by inclusion is a complete tree order and we denote with $T(F)$ the set of primes of F with this partial order. The coarsest prime of F is $P = \{0\}$, with $B(P) = F$, which is also called the *trivial* prime.

We summarize the above discussion:

LEMMA 2.4. *Let F be a skew field and $T(F) = \{P \subset F \mid P \text{ a prime}\}$ the set of primes of F with the partial order defined by inclusion. Then the following properties hold:*

(a) The set $\{P' \mid P' \subseteq P\}$ of primes contained in a fixed prime P is totally ordered.

(b) The union of any ascending chain of primes or the intersection of any descending chain of primes is again a prime of F.

(c) For any set $\{P_i \mid i \in I\}$ of primes of F there exists a greatest lower bound $P_0 \in T(F)$. An upper bound exists for primes P_1 and P_2 only if $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.

(d) For any pair of primes $P \supset P'$ there exists a prime segment $P_1 \supset P_2$ of F with $P \supseteq P_1 \supset P_2 \supseteq P'$.

Since (b) follows directly from the defining properties (i)–(iii) of Theorem 2.3 it remains to prove (d). For $a \in P \setminus P'$ let $P_1 = \bigcap P_i$, P_i prime with $P \supseteq P_i \supset P'$ and $a \in P_i$ and $P_2 = \bigcup P_j$, P_j primes with $a \notin P_j$ but $P \supset P_j \supseteq P'$. ■

REMARK. The set $T(K)$ for a commutative field K consists either of the trivial prime $\{0\}$ only or it is infinite. We have $|T(K)| = 1$ only for the algebraic extensions K of a prime field $GF(p)$, p a prime. All other fields either contain $\mathbb{Q}$ and extensions of the infinitely many primes of $\mathbb{Q}$ or they contain a transcendental extension $GF(p)(x)$ of $GF(p)$ which again leads to infinitely many primes of K. The situation is rather different for skew fields F; in particular, there may not exist a prime P of F with $P \cap F_0 = P_0$ for a given prime P_0 of a sub skew field F_0 of F (see [BG], [CM]). As the quaternions over $\mathbb{Q}$ show, there do exist skew fields with $|T(F)| \neq 1$ and finite. However we do not know which finite tree graphs can be represented as a $T(F)$ for certain skew fields F.

We return to segments $P_1 \supset P_2$ in $T(F)$ and show that the type of a segment can be defined independently of any valuation subring B that contains P_1 and P_2 as ideals.

DEFINITION 2.5. Let $P_1 \supset P_2$ be primes of F with no further primes of F between them. Then $P_1 \supset P_2$ is called a *prime segment* of F, and we denote with A the set $A = A(P_1) = \{a \in P_1 \mid P_1 a P_1 \subset P_1\}$.

We have the following result:

THEOREM 2.6. *Let $P_1 \supset P_2$ be a prime segment of the skew field F and let $B \supset \mathcal{J}(B) \supseteq P_1 \supset P_2$ be a valuation subring of F that contains P_1 and P_2 as ideals. Then $P_1 \supset P_2$ is a prime segment of B and*

*(i) it is of type **i** if and only if $A = P_1$;*

*(ii) it is of type **s** if and only if $A = P_2$;*

*(iii) it is of type **x** if and only if $P_1 \supset A \supset P_2$. In this case $A = Q$, the exceptional prime ideal of B properly between P_1 and P_2.*

PROOF. It follows from the above remarks that $P_1 \supset P_2$ is a prime segment of B. If the segment is of type ***i*** we have $aP_1 = P_1 a$ for all $a \in P_1 \setminus P_2$, hence $P_1 a P_1 \subseteq aP_1 \subset P_1$ since $aj = a$ implies $a = 0$ for $j \in P_1 \subseteq \mathcal{J}(B)$. It follows that $A = P_1$ in this case.

If the segment is of type **s**, then $P_1aP_1 \subseteq P_1$ is an ideal of B that is not contained in P_2 for $a \in P_1 \setminus P_2$. It follows that $P_1aP_1 = P_1$ and $A = P_2$.

We are left with the possibility that the segment is of type **x** and a prime ideal Q of B exists with $P_1 \supset Q \supset P_2$ and no further ideal exists between P_1 and Q. We have $A \supseteq Q$ and for $a \in P_1 \setminus Q$ we obtain $P_1 \supseteq P_1aP_1 = P_1(BaB)P_1 \not\subseteq Q$ since Q is prime and neither P_1 nor BaB is contained in Q. Hence, $A = Q$ follows in this case.

The reverse implications follow from what has been proved so far. ∎

It follows from this result that the type of a prime segment $P_1 \supset P_2$ does not depend on the particular valuation ring containing P_1 and P_2 as ideals. Hence, we can consider the valuation ring $B_1 = B(P_1) \supset \mathcal{J}(B_1) = P_1 \supset P_2$ and the valuation subring $B_1' = B_1/P_2$ of the quotient skew field F_2' of B_1' which is equal to $B(P_2)/P_2$. The type of the prime segment $P_1 \supset P_2$ is equal to the type of the prime segment $P_1/P_2 \supset (0)$ of F_2'. If this prime segment is of type ***i*** then B_1' is invariant and the value group associated with B_1' is a subgroup $H(P_1, P_2)$ of $(\mathbb{R}, +)$, the real numbers with addition as operation. Occasionally, we will label a prime segment of F not only with its type but also with its associated group H in case it is of type ***i***. Next we consider the (right) cofinality type of a prime segment $P_1 \supset P_2$ which we can define as the cofinality type of the set of all right ideals I of a valuation subring $B \supset \mathcal{J}(B) \supseteq P_1 \supset P_2$ with $P_1 \supseteq I \supset P_2$ with respect to the total order $I_1 \geq I_2$ if and only if $I_1 \subseteq I_2$. Since the cofinality type of this set is equal to the cofinality type of the set $\{aP_1 \mid a \in P_1 \setminus P_2\}$ this definition is independent of B and is also equal to the cofinality type of B_1' as defined above; we denote this ordinal by r-cofin(P_1, P_2), and ℓ-cofin(P_1, P_2) is defined similarly.

If the prime segment $P_1 \supset P_2$ is invariant then its right as well as its left cofinality type is ω_0. Since $\bigcap Q^n = P_2$, the same is true for an exceptional prime segment $P_1 \supset Q \supset P_2$. We are interested in the cofinality types of a prime segment of type **s**. We will say that a prime segment is of type ${}_{\omega_\beta}\mathbf{s}_{\omega_\alpha}$ if it is of type **s** with right cofinality type r-cofin$(P_1, P_2) = \omega_\alpha$ and left ℓ-cofin$(P_1, P_2) = \omega_\beta$.

REMARK. We do not know whether in general r-cofin$(P_1, P_2) = \ell$-cofin(P_1, P_2) for prime segments $P_1 \supset P_2$ of a skew field F.

We conclude this section by recalling the definition of the rank of a valuation ring B of F. For each prime segment $P_1 \supset P_2$ of B we choose the prime P_1, and the rank of B is defined as the order type of the set of prime ideals so obtained which is totally ordered under $P < P'$ if and only if $P' \subset P$. We note that this notion of rank agrees with the usual notion if the rank is finite.

3. **Group rings and localizations.** W. Krull in [KR] observed that given any commutative ordered group (A, P) with positive cone P one can construct a valued field (L, V) with associated value group A where the valuation ring V is the localization of the semigroup ring $K[P]$ with respect to the multiplicatively closed set $S = \{\Sigma g r_g \in K[P] \mid r_e \neq 0\}$. The field L is the field of quotients of $K[P]$ and $V = K[P]S^{-1}$ where K is an arbitrary commutative field.

In the non-commutative situation we consider groups G such that the group ring $R[G]$ is again a right Ore domain provided R is a right Ore domain ([PA], [SW], p. 26–27). We recall that a subset S of an integral domain R is a right Ore set if S is multiplicatively closed, does not contain the zero element and $sR \cap rS \neq \emptyset$ for any $r \in R^* = R \setminus \{0\}$ and $s \in S$. The ring R is called a *right Ore domain* if R^* is a right Ore set in R.

DEFINITION 3.1. A subset P of a group G is called a *generalized positive cone* of G if the following conditions hold: $P \cap P^{-1} = \{e\}, P \cup P^{-1} = G, PP \subseteq P$.

We will write (G, P) for the group G with generalized positive cone P. Through $a \leq_r b$ if and only if $ba^{-1} \in P$, a right order '$\leq_r$' is defined on G, *i.e.* $\leq_r$ is a (total) order on G such that $a \leq_r b$ implies $ac \leq_r bc$ for all a, b, c in G. Similarly, P defines a left order '$\leq_\ell$' on G with $a \leq_\ell b$ if and only if $a^{-1}b \in P$. Conversely, given a right order '$\leq_r$' on G one obtains a generalized positive cone $P = \{a \in G \mid e \leq_r a\}$ which in turn also defines the corresponding left order (see [KO]). If in addition $gPg^{-1} = P$ for all g in G, then $\leq_r$ and $\leq_\ell$ agree and (G, P) is an ordered group.

We say that a subset Π of P is a *right ideal* of P if $\Pi P \subseteq \Pi$ and left ideals and (two-sided) ideals of P are defined similarly. A completely prime ideal Ω of P is a proper ideal, $\Omega \subset P$, of P such that $pp' \in \Omega, p, p' \in P$ implies p or p' in Ω. If this last condition holds for ideals of P rather than elements (*i.e.* $\Omega_1\Omega_2 \subseteq \Omega$ for ideals Ω_1, Ω_2 of P implies $\Omega_1 \subseteq \Omega$ or $\Omega_2 \subseteq \Omega$), we say that Ω is *prime*. A completely prime ideal Ω of P is always prime, but the converse does not hold in general. If Ω is prime but not completely prime, it is called an *exceptional prime*. Observe that in any (G, P) the set $P^+ := P \setminus \{e\}$ is a completely prime ideal of P.

We consider the following situation: R is an integral domain, G a group, $\sigma: G \longrightarrow \mathrm{Aut}(R)$ a group homomorphism from G into the automorphism group of R. With $R[G, \sigma] = \{\Sigma g r_g \mid g \in G, r_g \in R\}$ we denote the skew group ring where multiplication is defined by $rg = g\sigma_g(r)$, see also [AT].

THEOREM 3.2. *Let R be a domain and (G, P) a group with generalized cone P such that the skew group ring $R[G, \sigma]$ is a right Ore domain for the homomorphism $\sigma: G \longrightarrow \mathrm{Aut}(R)$. Then the following holds:*

(i) $S = \{\Sigma g r_g \in R[G, \sigma] \mid g \in P,\ r_e \neq 0\}$ is a right Ore set in $R_1 = R[P, \sigma] = \{\Sigma g r_g \in R[G, \sigma] \mid g \in P\}$.

(ii) $B = R_1 S^{-1}$ is a valuation subring of F, the skew field of quotients of $R[G, \sigma]$.

(iii) The non-zero principal right ideals of B have the form gB for some $g \in P$ and $g_1B \supset g_2B$ if and only if $g_1 <_\ell g_2$ with respect to the left order induced by P on G.

PROOF. (i) It follows from the properties of P that R_1 is a subring of $R[G, \sigma]$ and that S is multiplicatively closed.

For $a \in R_1^*$ and $s \in S$ there exist by assumption elements $b = \Sigma g_i r_i$, $c = \Sigma h_j t_j \in R[G, \sigma]$ with $ab = sc \neq 0$. Let g_0 be the element minimal among all the g_i and h_j with $r_i \neq 0$ and $t_j \neq 0$ with respect to the right order in G induced by P. Hence, $b' = \Sigma g_i g_0^{-1} \sigma_{g_0^{-1}}(r_i)$ and $c' = \Sigma h_j g_0^{-1} \sigma_{g_0^{-1}}(t_j)$ are elements in R_1, one of b' or c' is in S and $ab' = sc' \neq 0$. If

$b' \in S$, we are done, otherwise $c' \in S$ and again $b' \in S$ follows, since $ab' \in S$ in this case. Next, we consider the principal right ideals of the ring $B = R_1S^{-1}$. The elements of B have the form as^{-1} for $a \in R_1$, $s \in S$ and we can assume $a = \Sigma g_i r_i \neq 0$. Let g_0 be minimal among all the g_i with $r_i \neq 0$ with respect to the left order induced by P and $s_1 = g_0^{-1}a \in S$ follows. Therefore $as^{-1} = g_0 s_1 s^{-1}$ and $as^{-1}B = g_0 B$, since s_1 and s are units in B. The statement (iii) follows and also that B is a right chain domain.

To prove that B is a left chain domain we observe that any two elements in B can be written with a common denominator, *i.e.* in the form as^{-1}, bs^{-1} for s in S, $a, b \in R_1$. Further we can write $a = a'g_1$, $b = b'g_2$ for $a', b' \in S$, $g_1, g_2 \in P$ if $a \neq 0 \neq b$ where we use the right order induced by P. Therefore, $Bas^{-1} = Bg_1s^{-1}$, $Bbs^{-1} = Bg_2s^{-1}$ and $Bg_1s^{-1} \supset Bg_2s^{-1}$ if and only if $g_2 >_r g_1$ in G. ∎

COROLLARY 3.3. *If $R[G, \sigma]$ is a right and left Ore domain, then S is a right and left Ore system of R_1 and all non-zero left principal ideals of B have the form Bg for g in P with $Bg_1 \supset Bg_2$ if and only if $g_2 >_r g_1$.*

In the above theorem the principal right ideals $\neq (0)$ of B were described uniquely through the elements of P. In general, we are led to the following:

DEFINITION 3.4. We say that the group (G, P) *parameterizes* the valued skew field (F, B) if and only if the generalized cone P is a submonoid of B^*, $P \cap U(B) = e = 1$ and every non-zero element r of B can be written in the form $r = g_1u_1 = u_2g_2$ for $g_1, g_2 \in P$ and $u_1, u_2 \in U(B)$. Here, $B^* = B \setminus \{0\}$, and $U := U(B)$ is the group of units of B.

It follows from this definition that the element r determines the elements g_i and u_i uniquely.

Let (F, B) be a valued skew field parameterized by (G, P). We say that an ideal Ω of P is *U-invariant* if $p \in \Omega$, $u \in U$, $up = p'u'$ with $p' \in P$, $u' \in U$ implies $p' \in \Omega$. This definition is left-right symmetric, since $pu = u''p''$ for $u'' \in U$, $p'' \in P$ implies $u''^{-1}p = p''u^{-1}$. It follows that for an U-invariant ideal Ω of P we have $p \in \Omega$ if and only if $p' \in \Omega$ in the above equation $up = p'u'$. An ideal Ω of P is called *U-prime* if Ω is U-invariant and $\Omega_1\Omega_2 \subseteq \Omega$ for U-invariant ideals $\Omega_1, \Omega_2 \subseteq P$ implies $\Omega_1 \subseteq \Omega$ or $\Omega_2 \subseteq \Omega$.

REMARK. A U-invariant ideal Ω of P which is prime is also U-prime. We were not able to decide whether conversely every U-invariant ideal of P which is U-prime is necessarily prime.

LEMMA 3.5. *Let (F, B) be a valued skew field parameterized by (G, P). Then:*

(a) $\phi(I) = I \cap P$ defines an inclusion preserving correspondence between the set of right (left) ideals $\neq (0)$ of B and the set of right (left) ideals of Ω.

(b) The mapping ϕ also defines a correspondence between the set of non-zero ideals of B and the set of all U-invariant ideals of P such that completely prime ideals in B correspond to completely prime U-invariant ideals in P and prime ideals in B correspond to U-prime U-invariant prime ideals in P.

PROOF. It follows from the assumption that (G,P) parameterizes (F,B) that $(I\cap P)B = I$ for a non-zero right ideal I of B and that $(\Pi B\cap P) = \Pi$ for a right ideal Π of B, which proves (a).

If I is a non-zero ideal of B then $I\cap P = \Omega$ is an ideal of P and $p\in\Omega$, $u\in U$, $up = p'u'$, $p'\in P$, $u'\in U$ implies $p'\in\Omega$, *i.e.* Ω is U-invariant. By (a) we have $I=\Omega B = B\Omega = B\Omega B$.

If conversely, Ω is an U-invariant ideal of P then $r = p_1u_1\in B$ and $pb\in\Omega B$, $p\in\Omega$, $b\in B$, $p_1\in P$, $u_1\in U$ implies $rpb = p_1u_1pb = p_1p'u_1'b\in\Omega B$ for $p'\in\Omega$, $u_1'\in U$. It follows that $\Omega B = B\Omega B$ is an ideal I of B and $I\cap P=\Omega$. The ideal $I\neq(0)$ of B is completely prime if and only if $I\cap P=\Omega$ is completely prime in P. Finally, if $I\neq(0)$ is prime and $\Omega_1\Omega_2\subseteq\Omega = I\cap P$ for U-invariant ideals Ω_1,Ω_2 of P then $(B\Omega_1)(\Omega_2B)\subseteq B\Omega B = I$ and $B\Omega_1$ or Ω_2B is contained in I, hence $\Omega_1\subseteq\Omega$ or $\Omega_2\subseteq\Omega$. Hence, Ω is U-prime.

Conversely, if Ω is a U-prime U-invariant ideal and $I_1I_2\subseteq I=\Omega B$ for non-zero ideals I_1 or I_2 then $I_1 = B\Omega_1$, $I_2=\Omega_2B$ for suitable U-invariant ideals Ω_1,Ω_2 of P. Then $B\Omega_1\Omega_2B\subseteq\Omega B$ and $\Omega_1\Omega_2\subseteq\Omega=\Omega B\cap P$. It follows that Ω_1 or Ω_2 is in Ω and hence I_1 or I_2 is in I, *i.e.* I is a prime ideal of B. ∎

If we apply the Lemma 3.5 to the construction in Theorem 3.2 we obtain the following result, where we assume that (G,P), $R[G,\sigma]$, R_1, S and B are given as in Theorem 3.2.

THEOREM 3.6. *Assume that $R[G,\sigma]$ is a right and left Ore domain. Then the mapping that sends I to $I\cap P$ defines a one to one correspondence between right (left) ideals $\neq(0)$ of B and right (left) ideals of P. Ideals $I\neq(0)$ of B correspond to ideals Ω of P and I is prime or completely prime if and only if the corresponding property holds for Ω.*

PROOF. It follows from Theorem 3.2 and Corollary 3.3 that (F,B) is parameterized by (G,P) and the statement about one-sided ideals follows from Lemma 3.5(a).

It remains to show that all ideals Ω of P are U-invariant where $U=U(B)$ is the unit group of B. We have $U=\{s_1s_2^{-1}\mid s_1,s_2\in S\} = \{s_1^{-1}s_2\mid s_1,s_2\in S\}$. For $s_1^{-1}s_2p = p's_3s_4^{-1}$ for $s_1,s_2,s_3,s_4\in S$ we have $s_2ps_4 = s_1p's_3$ and we must show that $p'\in\Omega$ if and only if p is in Ω. This will follow if we can show that $sh = h's'$ for $s,s'\in S$ implies $h\in\Omega$ if and only if $h'\in\Omega$.

Let $s = r_0+g_1r_1+\cdots+g_nr_n$ be an element in S with $r_i\in R$, $r_0\neq0$, $e\neq g_i\in P$ and assume $h\in P$. Then $r_ih = hr_i'$ for $r_i'=\sigma_h(r_i)\in R$; hence $sh = r_0h+g_1r_1h+\cdots+g_nr_nh = hr_0'+g_1hr_1'+\cdots+g_nhr_n'$. Let i_0 be the index such that $g_{i_0}h = h'$ is minimal among the elements g_ih with $r_i\neq0$ with respect to the left ordering induced by P. Then $sh=h's'$ for an element $s'\in S$ and $h'p_0'=h$ for a suitable element $p_0'\in P$. It follows that $h'\in\Omega$ if h is in Ω since Ω is a left ideal and that $h'\notin\Omega$ if $h\notin\Omega$ since Ω is a right ideal. Our claim follows, ideals Ω of P are $U(B)$-invariant and the statement in the theorem follows from Lemma 3.5. ∎

We consider next a construction of valuation rings which are extensions of given valuation rings, see also [BT2], [M2].

Let (F_0, B_0) be a valued skew field and (G, P) be a group with generalized positive cone P. We assume that there exists an automorphism $\sigma\colon G \longrightarrow \mathrm{Aut}(B_0)$ from G into the automorphism group of B_0 such that the skew group ring $R := B_0[G,\sigma]$ is a right Ore domain with F as skew field of quotients. Further, consider the subring $R_1 := B_0[P,\sigma] = \{\sum g_i r_i \in R \mid g_i \in P \text{ for all } i\}$ of R and the subset S_1 of R_1 consisting of all those elements $\sum g_i r_i \in R_1$ for which at least one of the elements r_i is a unit in B_0.

THEOREM 3.7. *Let (F_0, B_0), (G, P), σ and $R = B_0[G,\sigma]$ and R_1 be given as above. Then*

(a) $S_1 = \{\sum g_i r_i \in R_1 \mid \sum r_i B_0 = B_0\}$ is a right Ore system of R_1 and $B = R_1 S_1^{-1}$ is a valuation subring of F.

(b) The principal right ideals of B have the form $r_0 B$ for r_0 in B_0 and $r_0 B = r_0' B$ if and only if $r_0 B_0 = r_0' B_0$ for r_0, r_0' in B_0.

PROOF. To prove that S_1 is multiplicatively closed we consider $s_1 = \Sigma g_i r_i$, $s_2 = \Sigma h_j t_j \in S_1$ with $g_i <_r g_{i'}$ for $i < i'$ and $h_j <_\ell h_{j'}$ for $j < j'$ with the right and left order determined by P. There exists i_0 minimal with $r_{i_0} \in U(B_0)$ and similarly j_0 minimal with $t_{j_0} \in U(B_0)$. We consider the coefficient of $g_{i_0} h_{j_0}$ in the product $s_1 s_2$ and $g_{i_0} r_{i_0} h_{j_0} t_{j_0}$ contributes the summand $\sigma_{h_{j_0}}(r_{i_0}) t_{j_0} \in U(B_0)$. Any other product $g_i h_j = g_{i_0} h_{j_0}$ must satisfy either $g_i <_r g_{i_0}$ and $r_i, \sigma_{h_j}(r_i) \in \mathcal{J}(B_0)$ or $h_j <_\ell h_{j_0}$ and $t_j \in \mathcal{J}(B_0)$. It follows that the coefficient of $g_{i_0} h_{j_0}$ is a unit in B_0 and $s_1 s_2 \in S_1$, *i.e.* S_1 is multiplicatively closed.

To prove that S_1 satisfies the Ore condition, we choose $0 \neq a \in R_1$, $s \in S_1$ and by assumption there exist $c, b \in B_0[G,\sigma]$ with $ac = sb \neq 0$, $c = \Sigma g_i c_i$, $b = \Sigma h_j b_j$ for g_i, $h_j \in G$ and c_i, $b_j \in B_0$. If g is the minimal element among all the g_i, h_j with $c_i \neq 0$ or $b_j \neq 0$ with respect to the right order defined by P, then $cg^{-1}, bg^{-1} \in B_0[P,\sigma] = R_1$. We have $cg^{-1} = \Sigma g_i' c_i'$, $bg^{-1} = \Sigma h_j' b_j'$ and let $\Sigma B_0 c_i' + \Sigma B_0 b_j' = B_0 w_0$, $c_i' = c_i'' w_0$, $b_j' = b_j'' w_0$ for c_i'', b_j'', $w_0 \in B_0$. Finally, $ac'' = sb''$ for $c'' = \Sigma g_i' c_i''$, $b'' = \Sigma h_j' b_j''$ and at least one of c'', b'' is in S_1. If $c'' \in S_1$ we are done, otherwise $b'' \in S_1$, $sb'' \in S_1$ and $c'' \in S_1$ follows; S_1 is a right Ore system of R_1. The ring $B = B_0[P,\sigma]S_1^{-1}$ exists and every element in B can be written in the form as^{-1} with $a \in R_1 = B_0[P,\sigma]$, $a = r_0 s_1$ for $s_1, s \in S_1$, $r_0 \in B_0$.

It follows that $as^{-1}B = r_0 B$, which proves the first part of (b) and also that B is a right chain domain.

Any two elements in B can be written in the form $s_1 r_1 s^{-1}$ and $s_2 r_2 s^{-1}$ for $s_1, s_2, s \in S_1$ and $r_1, r_2 \in R_0$ and it follows that B is a left chain ring.

Finally, the units of B are of the form $s_1 s_2^{-1}$ for $s_i \in S_1$ and $U(B) \cap B_0 = U(B_0)$ follows. Therefore $r_0 B = r_0' B$ for r_0, r_0' in B_0 only if $r_0 B_0 = r_0' B_0$. ∎

We will now assume that (F_0, B_0) is parameterized by (G_0, P_0) and denote with U_0 the unit group $U(B_0)$ of B_0. We then say that an ideal Ω of P_0 is σ-invariant if $\sigma_g(p) = b_0 = p'u'$ for $p \in \Omega$, $p' \in P_0$, $u' \in U_0$ implies $p' \in \Omega$ for all $g \in G$. If Ω is also U_0-invariant, it follows that $b_0 = u''p''$, $u'' \in U_0$, $p'' \in P_0$ implies $p'' \in \Omega$ as well.

COROLLARY 3.8. *Let (F_0, B_0) be a valued skew field parameterized by (G_0, P_0). Assume that in the notation of Theorem 3.7 the ring $R = B_0[G,\sigma]$ is a right and left Ore domain. Then:*

(a) *(F, B) is a valued skew field parameterized by (G_0, P_0);*
(b) *An ideal Ω of P_0 is $U = U(B)$-invariant if and only if Ω is U_0-invariant and σ-invariant.*

To prove (a) we observe that now all conditions are left-right symmetric and S_1 is a left as well as a right Ore system of $R_1 = B_0[P, \sigma]$. Part (b) of Theorem 3.7 applies to the left principal and the right principal ideals of $B = R_1 S_1^{-1}$ which shows that (G_0, P_0) parameterizes (F, B).

Proof of (b): If Ω is U-invariant then Ω is U_0-invariant and σ-invariant since both U_0 and G are subgroups of U and $g^{-1}p = \sigma_g(p)g^{-1} = p'u'g^{-1}$ for $p \in \Omega$, $g \in G$, $\sigma_g(p) = p'u'$, $p' \in P$, $u' \in U_0$ and $p' \in \Omega$ follows.

If conversely Ω is U_0-invariant and σ-invariant and $s_1^{-1}s_2$ is an arbitrary element in U, then $s_1^{-1}s_2p = as_3^{-1} = p's_4s_3^{-1}$ for $p \in \Omega$, $a \in R_1$, $p' \in P$, $s_4, s_3 \in S_1$. We have $s_2ps_3 = s_1p's_4$ and want to conclude that $p' \in \Omega$. This will follow if we can show that $sh = h's'$ for $s, s' \in S_1$ and $h, h' \in P_0$ implies $h \in \Omega$ if and only if $h' \in \Omega$.

This statement is similar to the statement in the proof of Theorem 3.6, but here we deal with the different Ore system of the construction in Theorem 3.7; some additional calculations are necessary. If $s = \Sigma g_ir_i$, $g_i \in P$, $r_i \in B_0$, $r_i = p_iu_i$ with $p_i \in P_0$ and $u_i \in U_0$ and $h \in P_0$ then $u_ih = h_i'u_i'$ for some $u_i' \in U_0$, $h_i' \in P_0$ and $h_i' \in \Omega$ if and only if $h \in \Omega$ since Ω is U_0-invariant. Since $s \in S_1$ there exists an index i_0 with $p_{i_0} = e \in P_0$. We have $g_ip_ih_i' = q_iv_ig_i$ for some $v_i \in U_0$, $q_i \in P_0$ and $q_iv_i = \sigma_{g_i^{-1}}(p_ih_i')$. If q_{j_0} is minimal among the q_i with respect to the left order induced by P_0 then $sh = q_{j_0}s'$ for some $s' \in S_1$ and $q_{j_0}b_{i_0} = q_{i_0}$ for some $b_{i_0} \in P_0$. If $h \in \Omega$ then $h_{j_0}' \in \Omega$, $p_{j_0}h_{j_0}' \in \Omega$ and $q_{j_0} = h' \in \Omega$. If conversely $q_{j_0} = h' \in \Omega$, then $q_{j_0}b_{i_0} = q_{i_0} \in \Omega$, and $p_{i_0}h_{i_0}' = h_{i_0}' \in \Omega$ since $p_{i_0} = e$ and Ω is σ-invariant; and finally $p = h \in \Omega$ follows since Ω is U_0-invariant. ■

EXAMPLE 3.9. For any given cofinality type ω_α there exists a commutative ordered field $(K, \leq)$ of cofinality type ω_α. (Such a field can be constructed by using a suitable ordered group or see Section 5.) The additive group $G_0 := (K, +)$ with its ordering in K then has cofinality ω_α. The field $F_0 = \mathbb{Q}\{\{G_0\}\}$ of generalized power series of G_0 over $\mathbb{Q}$ contains the valuation ring B_0 of elements with support in P_0, the positive cone of G_0. For the ordered group $G = (K^{>0}, \cdot)$ of positive elements of K under multiplication a homomorphism $\sigma: G \longrightarrow \mathrm{Aut}(B_0)$ exists with $\sigma_g(p_0) = gp_0$ for $g \in G$, $p_0 \in P_0$. Since G is a torsion free abelian group, $R = B_0[G, \sigma]$ is a right and left Ore domain (see also Section 4) and (F, B)—in the notation of Theorem 3.7 and Corollary 3.8—is a valued skew field parameterized by G_0 and hence of cofinality type ω_α. The only ideals Ω of P_0 which are σ-invariant are $P_0^+ = P_0 \setminus \{0\}$ and P_0 itself and hence B is an almost simple valuation domain of cofinality type ω_α. ■

Let (F, B) be a valued skew field and let I be the ordered index set such that $\{P_i' \supset P_i \mid i \in I\}$ is the set of prime segments of B. The order type of I is called the *rank* of B. We define a mapping $\tau: I \longrightarrow \{\boldsymbol{i}, \mathbf{s}, \mathbf{x}\}$ with $\tau(i)$ describing the type of the segment $P_i' \supset P_i$, occasionally we will specify the cofinality types in case $\tau(i) = \mathbf{s}$ by writing ${}_{\omega_{\alpha_i}}\mathbf{s}_{\omega_{\beta_i}}$ instead of $\mathbf{s}$. We then say that (I, τ) is the *type* of (F, B).

In the next example we construct conversely for an arbitrarily given ordered index set I and an arbitrary type function $\tau: I \longrightarrow \{\boldsymbol{i}, \mathbf{s}\}$ a valued skew field (F, B) of type (I, τ) where in addition the cofinality type ω_{α_i} can also be prescribed in case $\tau(i) = \mathbf{s}$, *i.e.* $\tau(i) = {}_{\omega_{\alpha_i}}\mathbf{s}_{\omega_{\alpha_i}}$.

EXAMPLE 3.10. Given a type (I, τ) as defined above. Then there exists a valued skew field (F, B) of type (I, τ).

For $\tau(i) = {}_{\omega_{\alpha_i}}\mathbf{s}_{\omega_{\alpha_i}}$ we construct an ordered field K_i of cofinality type ω_{α_i} which can be obtained as mentioned in 3.9 by using generalized power series rings over suitable ordered groups with the field of rational numbers as coefficients.

For $\tau(i) = \boldsymbol{i}$ we choose $K_i = \mathbb{R}$, the field of real numbers.

Next we form the lexicographically ordered group $G = \bigoplus_{i \in I} G_i$ where $G_i = (K_i, +)$ and the lexicographically ordered group $A = \oplus A_i$ with $A_i = \{id\} = \{1\} \in \mathbb{R}$ if $\tau(i) = \boldsymbol{i}$ and $A_i = (K_i^{>0}, \cdot)$, the multiplicative group of positive elements of K_i, if $\tau(i) = {}_{\omega_{\alpha_i}}\mathbf{s}_{\omega_{\alpha_i}}$.

If P is the generalized positive cone of G with respect to the lexicographical ordering we set $B_0 = \mathbb{Q}\{\{P\}\}$, the generalized (Hahn-Neumann) power series ring, which is parameterized by P. Finally, we observe that $\sigma: A \longrightarrow \mathrm{Aut}(B_0)$ with $\sigma_a(g) = ag = (a_i g_i)$ if $a = (a_i) \in A$, $g = g_i \in P$ defines a homomorphism from A into the group of automorphisms of B_0.

Since A is an abelian torsion free group it follows that $B_0[A, \sigma]$ is a right and left Ore domain and Theorem 3.7 and Corollary 3.8 can be applied in order to obtain a valued skew field (F, B) which is parameterized by (G, P). To determine the type of (F, B) we must determine the segments of σ-invariant completely prime ideals of P (Lemma 3.5 and Corollary 3.8b)).

We claim: Any prime segment $\Pi' \supset \Pi$ of σ-invariant completely prime ideals Π', Π of P is given by $\Pi' = \Pi'(i) = \{\alpha \in P \mid o(\alpha) \leq i\}$ and $\Pi = \Pi(i) = \{\beta \in P \mid o(\beta) < i\}$ for some i, where $o(\alpha) = \min\{i \mid g_i \neq e_i,\ \alpha = (g_i) \in P\}$. We note that $o(\beta\gamma) = \min\{o(\beta), o(\gamma)\}$ for $\beta, \gamma \in P$. From this it follows immediately that $\Pi'(i)$ and $\Pi(i)$ are in fact completely prime ideals of P. Since $\sigma_a\big((p_i)\big) = (a_i p_i)$ for $(p_i) \in G$, $a = (a_i) \in A$ we have $o(g) = o(\sigma_a g)$ for all $a \in A$, $g \in P$ and $\Pi'(i)$ and $\Pi(i)$ are also σ-invariant.

To show that there is no further σ-invariant completely prime ideal Ω between $\Pi'(i)$ and $\Pi(i)$ assume that $\Pi'(i) \supseteq \Omega \supset \Pi(i)$ and Ω contains an element $\alpha = (g_j)$ with $g_j = e_j$ for $j < i$ and $g_i = p_i \in G_i$, $p_i > e_i = 0$. If $\tau(i) = \boldsymbol{i}$ it follows from the archimedean property for $\mathbb{R}$ that for any $\beta \in \Pi'(i)$ there exists an n with $n\beta > \alpha$, $n\beta \in \Omega$ and $\beta \in \Omega$, $\Omega = \Pi'(i)$ follows since Ω is completely prime.

We want to show that the prime segment $\Pi'(i) \supset \Pi(i)$ with $\tau(i) = \boldsymbol{i}$ is in fact invariant. First we show that no σ-invariant prime ideal Q can exist with $\Pi'(i) \supset Q \supset \Pi(i)$ since then there is $\gamma \in \Pi'(i) \setminus Q$, $A(P\gamma) = P(A\gamma)$ is a σ-invariant ideal of P with $A(P\gamma) \not\subseteq Q$ but $[A(P\gamma)]^m \subseteq Q$ for a suitable m. Hence, this segment is not exceptional. Finally, we saw that for $\gamma \in \Pi'(i) \setminus \Pi(i)$, $A(P\gamma)$ is a σ-invariant ideal properly between $\Pi'(i)$ and $\Pi(i)$. It follows that the prime segment of B corresponding to $\Pi'(i) \supset \Pi(i)$ is invariant.

If $\tau(i) = \mathbf{s}$ then, as in the above argument, $A(P\gamma)$ is the smallest σ-invariant ideal containing $\gamma \in \Pi'(i) \setminus \Pi(i)$ and $A(P\gamma) = \Pi'(i)$ follows, *i.e.* $\Pi'(i) \supset \Pi(i)$ corresponds to a simple prime segment of B.

It remains to show that an arbitrary segment of σ-invariant completely prime ideals $\Pi' \supset \Pi$ is of the form $\Pi'(i) \supset \Pi(i)$ for some i. Let $\alpha \in \Pi' \setminus \Pi$, and $o(\alpha) = i$ for some i follows. If there exists an element $\beta \in \Pi$ with $o(\beta) = i$ then $\alpha \in A(P\beta) \subseteq \Pi$ in case $\tau(i) = \mathbf{s}$ and $\alpha^n \in P\beta \subseteq \Pi$ for suitable n in case $\tau(i) = \boldsymbol{i}$—a contradiction and $\Pi \subseteq \Pi(i) \subset \Pi'(i) \subseteq \Pi'$ follows, where the last containment is proved by again considering the cases $\tau(i) = \boldsymbol{i}$ and $\tau(i) = \mathbf{s}$ separately.

We conclude that (F, B) is a valued field of type (I, τ).

4. **Right orders and primes of groups of affine transformations.** In this section we will construct a family of groups (G, P) with generalized cone P to which the construction in Theorem 3.2 can be applied in order to obtain valued skew fields (F, B) parameterized by (G, P) and the results in Theorem 3.6 hold. The type of (F, B) can be computed and conditions can be given so that B is finally simple, Theorem 4.4, or that all segments are simple, Theorem 4.8. In Proposition 4.9 the cofinality types of the simple segments of B are determined. The groups G which we will construct are groups of affine transformations on an ordered K-vector space V over an ordered field K. A first example of this type for $K = \mathbb{R}$ and $V = \mathbb{R}$ was given by Smirnov ([S]). The generalized cones P of G are defined through Dedekind cuts of V and the types of the resulting valued skew field (F, B) are very much influenced by the properties of these Dedekind cuts.

We recall the following definitions:

The pair (K, P_K), where K is a field and P_K is a subset of K, is called an *ordered field* if the following conditions hold:

(i) $P_K \cap -P_K = \{0\}$;

(ii) $P_K \cup -P_K = K$;

(iii) $P_K + P_K \subseteq P_K$;

(iv) $P_K \cdot P_K \subseteq P_K$.

We then write $a \geq b$ if and only if $a - b \in P_K$ for $a, b \in K$ and occasionally we will write $(K, \leq)$ instead of (K, P_K).

Similarly, we say that (V, P_V) is an *ordered K-vector space* if V is a K-vector space with subset P_V such that

(a) $P_V \cap -P_V = \{0\}$;

(b) $P_V \cup -P_V = V$;

(c) $P_V + P_V = P_V$;

(d) $P_K P_V = P_V$.

Here, (K, P_K) is an ordered field and we write $v \leq v'$ if and only if $v' - v \in P_V$ where $v', v \in V$; we also use the notation $(V, \leq)$ instead of (V, P_V).

The following result will be needed to define generalized positive cones for the groups under consideration.

Let (K,P_K) be an ordered field, (V,P_V) an ordered K-vector space and $(V,P_V) \subset (V',P_{V'})$ a proper extension of (V,P_V), *i.e.* $(V',P_{V'})$ is an ordered K-vector space with $V \subset V'$ and $P_{V'} \cap V = P_V$. We recall further that a pair (U,O) of subsets of the linearly ordered set $(V,\leq)$ is called a *Dedekind cut* of $(V,\leq)$ if and only if $U \cup O = V$ and $v < v'$ for any $v \in U$ and any $v' \in O$. The possibilities that $U = \emptyset$ or $O = \emptyset$, the empty set, are not excluded. Two Dedekind cuts (U,O) and (U',O') of V are equal if and only if $U = U'$ and $O = O'$. Every element $\eta \in V' \setminus V$ defines a Dedekind cut (U_η, O_η) of $(V,\leq)$ with $U_\eta = \{v \in V \mid v < \eta\}$ and $O_\eta = \{v \in V \mid v > \eta\}$. It follows from the next result that conversely any Dedekind cut (U,O) of $(V,\leq)$ defines a proper extension of (V,P_V).

THEOREM 4.1. *Let (K,P_K) be an ordered field, (V,P_V) an ordered K-vector space and (U,O) a Dedekind cut of $(V,\leq)$. Then there exists in the K-vector space extension $V_1 = K\beta \oplus V$ with the one additional basis element β of V_1 a unique positive cone P_{V_1} such that (V_1,P_{V_1}) is an extension of (V,P_V) and $v_1 < \beta < v_2$ holds for all $v_1 \in U$ and all $v_2 \in O$.*

PROOF. We define a subset

$$P_1 = \left\{ c\beta + v \in V_1, c \in K, v \in V \;\middle|\; \begin{array}{ll} -\frac{1}{c}v \in U & \text{if } c > 0, \\ -\frac{1}{c}v \in O & \text{if } c < 0, \\ v \geq 0 & \text{if } c = 0 \end{array} \right\}$$

of V_1 and must show that $P_1 = P_{V_1}$ satisfies the conditions (a)-(d) that define an ordered K-vector space. It follows that $\beta - v_1, v_2 - \beta \in P_1$ for any $v_1 \in U$, $v_2 \in O$ and that $P_1 \cap V = P_V$. To show the uniqueness, we assume that (V_1,P_1') is also an ordered K-vector space with $P_1' \cap V = P_V$ and $\beta - v_1, v_2 - \beta \in P_1'$ for all $v_1 \in U$ and $v_2 \in O$. Then $c\beta + v \in P_1'$ if and only $v \in P_V$ for $c = 0$, $-\frac{1}{c}v \in U$ for $c > 0$ and $-\frac{1}{c}v \in O$ for $c < 0$ where $c \in K$, $v \in V$. It follows that $P_1' = P_1$.

We show now that the conditions (a)-(d) hold for P_1.

(a) The element $c\beta + v$, $c \in K$, $v \in V$, is contained in $P_1 \cap -P_1$ if and only if $c\beta + v$, $-c\beta - v \in P_1$. For $c = 0$ this implies $v, -v \in P_V$ and $v = 0$. For $c > 0$ this implies $-\frac{1}{c}v \in U$ and $-\frac{1}{(-c)}(-v) = -\frac{1}{c}v \in O$—a contradiction. The case $c < 0$ is treated similarly.

(b) To show that $P_1 \cup -P_1 = V_1$ we consider an arbitrary element $c\beta + v$ in V_1. For $c = 0$ it follows that either v or $-v$ is contained in P_V and hence in P_1. If $c > 0$ then either $-\frac{1}{c}v$ is in U and $c\beta + v$ is in V_1 or $-\frac{1}{c}v$ is in O. We have $-c < 0$ and $-\frac{1}{c}v = -\frac{1}{(-c)}(-v) \in O$ in this last case and $-c\beta - v \in P_1$ follows. The case $c < 0$ is treaded similarly.

(c) To show that $P_1 + P_1 \subseteq P_1$ we consider the sum of two arbitrary elements in P_1: $(c_1\beta+v_1)+(c_2\beta+v_2) = (c_1+c_2)\beta+(v_1+v_2)$. If $c_1 > 0, c_2 > 0$ then $-\frac{1}{c_1}v_1, -\frac{1}{c_2}v_2 \in U$ and we can assume that $-\frac{1}{c_1}v_1 \leq -\frac{1}{c_2}v_2$ and $-c_2v_1 \leq -c_1v_2$ follows. Since $c_1 + c_2 > 0$ it is enough to show that $-\frac{1}{(c_1+c_2)}(v_1+v_2) \leq -\frac{1}{c_2}v_2$ which holds if and only if $-c_2(v_1+v_2) \leq -(c_1+c_2)v_2$ and this is correct since $-c_2v_1 \leq -c_1v_2$. In the case $c_1 < 0, c_2 < 0$ one has $-\frac{1}{c_1}v_1, -\frac{1}{c_2}v_2 \in O$ and we can assume that $-\frac{1}{c_1}v_1 \leq -\frac{1}{c_2}v_2$. With arguments similar to the ones used in the previous case one proves that $-\frac{1}{c_1}v_1 \leq -\frac{1}{(c_1+c_2)}(v_1+v_2)$, hence $-\frac{1}{(c_1+c_2)}(v_1+v_2) \in O$. If one of the coefficients equals zero, say $c_1 = 0$, then $v_1+v_2 \in P_1$ if $c_2 = 0$ and $-\frac{1}{c_2}(v_1+V_2) \in U$ for $c_2 > 0$ since $-\frac{1}{c_2}v_1 \leq 0$ and $-\frac{1}{c_2}v_2 \in U$. The case $c_2 < 0$ can be treated similarly.

We are left with the case where the coefficients c_i are non-zero, but have opposite sign. We can assume that $c_1 < 0 < c_2$. Then $-\frac{1}{c_2}v_2 \in U$ and $-\frac{1}{c_1}v_1 \in O$, hence $-\frac{1}{c_2}v_2 < -\frac{1}{c_1}v_1$ and $-c_1v_2 > -c_2v_1$. If in addition $c_1+c_2 = 0$, then $c_1 = -c_2$ and $v_2 > -v_1$, $v_2+v_1 > 0$ follows. In case $c_1+c_2 > 0$ the containment $-\frac{1}{(c_1+c_2)}(v_1+v_2) \in U$ follows since $-c_2v_1 < -c_1v_2$ implies $-\frac{1}{(c_1+c_2)}(v_1+v_2) \leq -\frac{1}{c_2}v_2$. A similar argument shows that in the case $c_1+c_2 < 0$ the inequality $-\frac{1}{(c_1+c_2)}(v_1+v_2) \geq -\frac{1}{c_1}v_1$ holds.

(d) Let $0 \neq d \in P_K$ and $c\beta + v \in P_1$. If $c = 0$ then $dv \in P_V \subseteq P_1$ and for $c > 0$ we have $-\frac{1}{c}v \in U$ which implies $-\frac{1}{cd}(dv) \in U$, $dc > 0$ and $dc\beta + dv \in P_1$. A similar argument can be applied if $c < 0$. ∎

As above, let (K, P_K) be an ordered field and (V, P_V) be an ordered K-vector space. We consider the group

$$G = \{(a, v) \mid 0 < a \in K, v \in V\} \text{ with } (a, v)(a', v') = (aa', av' + v)$$

as operation. Then $e = (1, 0)$ is the identity of G and $(a, v)^{-1} = (a^{-1}, -a^{-1}v)$ for any element $(a, v) \in G$. This group can be considered as a group of K-linear affine transformations on V and it is the semidirect product of its normal subgroup

$$H = \{(1, v) \mid v \in V\} \simeq (V, +) \quad \text{and the subgroup}$$
$$L = \{(a, 0) \mid 0 < a \in K\} \simeq (P_K \setminus \{0\}, \cdot).$$

In particular, we have $(a, 0)(1, v) = (1, av)(a, 0)$; *i.e.* $\sigma_a: H \longrightarrow H$ with $\sigma_a(1, v) = (1, av)$ defines an automorphism of H for every $a \in P_K \setminus \{0\}$.

It follows that $R[G]$ is a right and left Ore domain provided R is a right and left Ore domain. This follows, as in the examples in the previous section, from the fact that $R[G] = R[H][L, \sigma]$, H and L are torsion free abelian groups and hence their finitely generated subgroups are direct sums of infinite cyclic groups. Hence, one can apply the result that the Ore extension $R[x, \sigma]$, σ an automorphism of R, is a right and left Ore domain if R has these properties (See [PA], [AT], [WS]).

The group G is an ordered group under the lexicographical ordering, however, we are more interested in the generalized positive cones that can be defined using Dedekind cuts of V.

Let (U, O) be a Dedekind cut of $(V, \leq)$. By Theorem 4.1 there exists an extension $V_1 = K\eta \oplus V$ with $(V_1, P_{V_1}) \supset (V, P_V)$ and $U = \{v \in V \mid v < \eta\}$, $O = \{v \in V \mid \eta < v\}$. We define:

$$P_\eta = \{(a, v) \in G \mid a\eta + v \geq \eta\}.$$

From $a\eta + v \geq \eta$, $a > 0$ in K, follows $\eta \geq a^{-1}\eta - a^{-1}v$ and $P_\eta \cap P_\eta^{-1} = \{e\}$, $P_\eta \cup P_\eta^{-1} = G$ and $P_\eta P_\eta \subseteq P_\eta$, *i.e.* (G, P_η) is a group with generalized cone P_η.

We note that the element η defines the cut (U, O) of $(V, \leq)$, but that elements $\eta' \in V_1 \setminus V$, $\eta \neq \eta'$ can define the same cut of $(V, \leq)$. We consider some special cases:

EXAMPLE 4.2. i) $C = (V, \phi)$, *i.e.* $v < \eta$ in V_1 for all $v \in V$. We write $P_\eta = P_{+\infty}$ in this case and obtain

$$P_{+\infty} = \{(a, v) \in G \mid a > 1 \text{ or } a = 1 \text{ and } v \geq 0\}$$

which is the positive cone of G which belongs to the lexicographical order of G.

ii) $C = (\phi, V)$, in which case we write $P_\eta = P_{-\infty}$ and we obtain

$$P_{-\infty} = \{(a, v) \in G \mid 0 < a < 1 \text{ or } a = 1 \text{ and } v \geq 0\}.$$

This is a generalized cone of G which is invariant under conjugation and therefore defines an order on G. In fact, it is the lexicographical order of G where however the order of the group $(K^{>0}, \cdot) \simeq \{(a, 0) \mid a \in K^{>0}\}$ has been reversed.

iii) Let $s \in V$ be an arbitrary element and we define $U = \{v \in V \mid v \leq s\}$. To $C = (U, V \setminus U)$ belongs then the following

$$P_\eta = P_{s+} = \{(a, v) \in G \mid (a-1)s + v > 0 \text{ or } (a-1)s + v = 0 \text{ and } a \geq 1\}.$$

iv) Let $s \in V$ be an arbitrary element and we define $U = \{v \in V \mid v < s\}$. Then $s \in O = V \setminus U$ and to $C = (U, O)$ belongs

$$P_\eta = P_{s-} = \{(a, v) \in G \mid (a-1)s + v > 0 \text{ or } (a-1)s + v = 0 \text{ and } 0 < a \leq 1\}.$$

The next result shows that only in the cases i) and ii) in Example 4.2 does P_η define an order for G.

LEMMA 4.3. *(G, P_η) is an ordered group if and only if $\eta = \pm\infty$.*

PROOF. We saw in the examples i) and ii) that $P_{+\infty}$ and $P_{-\infty}$ define orders for G.

Conversely, assume that there exist r and s in V with $r < \eta < s$ in V_1. Then for $(a, v) = (2, -r)$ we have $2\eta - r > \eta$, hence $(2, -r) \in P_\eta$. However $(1, s-r)(2, -r)(1, s-r)^{-1} = (1, s-r)(2, -r)(1, r-s) = (2, -s) \notin P_\eta$ since $2\eta - s < \eta$ and P_η does not define an order on G. ∎

The construction described in Theorem 3.2 can be applied to $R = \mathbb{Q}$, the field of rationals, (G, P_η) and $\sigma_a =$ identity for all a in G. One then obtains a valued skew field (F, B_η) parameterized by (G, P_η) and the various types of ideals of B_η are described by the corresponding types of ideals of P_η as is summarized in Theorem 3.6.

The next result gives the conditions under which B_η is finally simple, *i.e.* has a nonzero minimal ideal.

THEOREM 4.4. *The following conditions are equivalent:*
a) B_η is finally simple;
b) There exists a minimal ideal $\Omega_{\min}$ in P_η;
c) There exist elements $r, s \in V$ with $r < \eta < s$ and an element v' in V such that for any v in V there is an a in K with $v \leq av'$.

PROOF. As we noted before, the equivalence of a) and b) follows from Theorem 3.6. To prove that b) implies c) we notice that it follows from Examples 4.2i) and ii) that there exist $r, s \in V$ with $r < \eta < s$ if P_η has a minimal ideal $\Omega_{\min}$. Next, let (a, w) be an arbitrary fixed element in $\Omega_{\min}$. Since by the above observation $r < \eta < s$ for some r, $s \in V$, there exists an element v' in V with $v' > (a-1)\eta + w \geq 0$. Let v be an arbitrary element in V with $v > 0$. Then $P_\eta(1, v)P_\eta \supseteq \Omega_{\min} \ni (a, w)$ and therefore

$$(a, w) = (a_1, v_1)(1, v)(a_2, v_2) = (a_1a_2, a_1v_2 + a_1v + v_1) \quad \text{for some } (a_1, v_1), (a_2, v_2) \in P_\eta.$$

From this we obtain

$$\begin{aligned} v' > (a-1)\eta + w &= (a_1a_2 - 1)\eta + a_1v_2 + a_1v + v_1 \\ &= a_1[(a_2 - 1)\eta + v_2] + [(a_1 - 1)\eta + v_1] + a_1v \end{aligned}$$

and $v' > a_1v$ follows since $a_1 \in K^{>0}$, $(a_i, v_i) \in P_\eta$. Therefore $a_1^{-1}v' > v$ and this proves that b) implies c).

Conversely, assume that condition c) holds. Then there exists an element d in V with $d \geq s - r > 0$ and $d \geq v'$. We claim that $P_\eta(1, d)P_\eta$ is a minimal ideal of P_η. To prove this we show that for any element $(a, v) \in P_\eta$ there exist elements $(a_1, v_1), (a_2, v_2) \in P_\eta$ with $(1, d) = (a_1, v_1)(a, v)(a_2, v_2)$. In V we define an element r_1 as follows:

$$r_1 := r \quad \text{if } r + \frac{d}{2} > \eta \text{ and } r_1 := r + \frac{d}{2} \quad \text{if } r + \frac{d}{2} < \eta.$$

Then $r_1 < \eta < r_1 + \frac{d}{2}$. By assumption for the element $\frac{2}{a}(ar_1 - r_1 + v) \in V$ there is an

$$x \in K^{>0} \quad \text{with} \quad xv' \geq \frac{2}{a}(ar_1 - r_1 + v)$$

and therefore (by $d \geq v'$) $xd \geq \frac{2}{a}(ar_1 - r_1 + v)$, hence $a(x+1)\frac{d}{2} > \frac{ax}{2}d \geq ar_1 - r_1 + v$ since $a > 0$ and $\frac{a}{2}d > 0$. We have $(x+1, -xr_1) \in P_\eta$ since $(x+1-1)\eta - xr_1 = x(\eta - r_1) > 0$, and $(u, w) := \left(\frac{1}{a(x+1)}, \frac{x}{x+1}r_1 - \frac{1}{a(x+1)}v + d\right) \in P_\eta$ since

$$\begin{aligned} &\left(\frac{1}{a(x+1)} - 1\right)\eta + \frac{x}{x+1}r_1 - \frac{1}{a(x+1)}v + d \\ &\quad = \left(\frac{d}{2} + r_1 - \eta\right) + \left(\frac{1}{a(x+1)}(\eta - r_1)\right) + \frac{1}{a(x+1)}\left(a(x+1)\frac{d}{2} - ar_1 + r_1 - v\right) \end{aligned}$$

which is ≥ 0 since each of the three terms in this sum is ≥ 0. Finally, we have $(1, d) = (u, w)(a, v)(x+1, -xr_1)$ which proves the last claim and the theorem. ∎

The condition c) in Theorem 4.4 consists of two parts: η is bounded or equivalently, the cut defined by η is neither (ϕ, V) nor (V, ϕ), and V has a maximal K-archimedean class. Here we say that two elements $0 < v_1, v_2 \in V$ are in the same K-archimedean class if there exist $a_i \in K^{>0}$ with $v_2 \leq a_1 v_1$ and $v_1 \leq a_2 v_2$. With $[v]$ we denote the class of v for $0 < v \in V$, and we set $[-v] = [v]$.

The next result provides information about ideals of P_η generated by one element.

THEOREM 4.5. *i) Let* $(a_1, v_1) \neq (a_2, v_2)$ *be two distinct elements in* $P_\eta^+ = P_\eta \setminus \{(1,0)\}$. *Then* $P_\eta(a_1, v_1)P_\eta = P_\eta(a_2, v_2)P_\eta$ *if and only if there exists an element* s *in* V *with* $\eta < s < a_i\eta + v_i$ *for* $i = 1, 2$ *and the two elements* $(a_1 - 1)\eta + v_1$ *and* $(a_2 - 1)\eta + v_2$ *are contained in the same K-archimedean class of* $V' = K\eta + V$.

ii) If for $\alpha = (a, v) \in P_\eta$ *no* $s \in V$ *exists with* $\eta < s < a\eta + v$ *then* $P_\eta\alpha = \alpha P_\eta = P_\eta\alpha P_\eta$.

PROOF. We can assume that $\eta < a_1\eta + v_1 < a_2\eta + v_2$.

To prove i) we assume first that $P_\eta(a_1, v_1)P_\eta = P_\eta(a_2, v_2)P_\eta$ and hence $(a_1, v_1) \in P_\eta(a_2, v_2)P_\eta$. Therefore $(a_1, v_1) = (u, w)(a_2, v_2)(x, y) = (ua_2x, ua_2y + uv_2 + w)$ for $(u, w), (x, y) \in P_\eta$. We have:

1) $(u - 1)\eta + w \geq 0$
2) $(x - 1)\eta + y \geq 0$;
3) $ua_2x = a_1$;
4) $ua_2y + uv_2 + w = v_1$ and
5) $a_2\eta + v_2 > a_1\eta + v_1$ by assumption.

We multiply 2) by ua_2 and obtain

6) $(ua_2x - ua_2)\eta + ua_2y \geq 0$ and with 1) we have $(ua_2x - 1)\eta + ua_2y + uv_2 + w \geq (ua_2 - u)\eta + uv_2$.

Using 3) and 4) we get $(a_1 - 1)\eta + v_1 \geq u\big((a_2 - 1)\eta + v_2\big)$ *i.e.*

$$c_2[(a_1 - 1)\eta + v_1] \geq [(a_2 - 1)\eta + v_2] \quad \text{with} \quad c_2 = u^{-1} \in K^{>0}.$$

The other inequality needed to show that $(a_1 - 1)\eta + v_1$ and $(a_2 - 1)\eta + v_2$ are in the same K-archimedean class is given by 5) with $c_1 = 1$.

To show that there exists s in V with $\eta < s < a_1\eta + v_1$ we multiply 5) by u and add it to 6):

$$ua_2\eta + uv_2 + (ua_2x - ua_2)\eta + ua_2y > u(a_1\eta + v_1).$$

Using 3) and 4) we obtain $a_1\eta + v_1 - w > u(a_1\eta + v_1)$. Hence, $(1 - u)(a_1\eta + v_1) > w$. From 1) we obtain $(1 - u)\eta \leq w$ and therefore we have

7) $(1 - u)\eta \leq w < (1 - u)(a_1\eta + v_1)$.

Since $\eta < a_1\eta + v_1$ we conclude from 7) that $1 - u > 0$. By dividing 7) by $1 - u$ it follows that $\eta < s < a_1\eta + v_1$ for $s = \frac{1}{1-u}w \in V$ since $\eta \notin V$. To prove the converse we assume that there exists $0 < c_2 \in K$ with

$$(a_2 - 1)\eta + v_2 \leq c_2[(a_1 - 1)\eta + v_1] \quad \text{and} \quad c_2 > 1 \text{ because of 5).}$$

https://doi.org/10.4153/CJM-1995-059-3 Published online by Cambridge University Press

In addition we assume that there exists $s \in V$ with

8) $\eta < s < a_1\eta + v_1$.

We prove next that s can be chosen in such a way that not only 8) holds but also the following inequality:

9) $(a_1 - 1)\eta + v_1 \leq \lambda(a_1\eta + v_1 - s)$ for some $0 < \lambda \in K$.

To see this we consider two cases:

FIRST. $(a_1 - 1)\eta + v_1 \leq \frac{a_1+1}{a_1}(a_1\eta + v_1 - s)$ and 9) is satisfied for $\lambda = \frac{a_1+1}{a_1}$.

SECOND. 10) $(a_1 - 1)\eta + v_1 > \frac{a_1+1}{a_1}(a_1\eta + v_1 - s)$.

Then we replace s by the element $s_1 = \frac{a_1+1}{2a_1}s - \frac{1}{2a_1}v_1 \in V$ and we will show that both 8) and 9) hold for s_1 in place of s and $\lambda = 2$. From 10) we obtain $\frac{a_1+1}{a_1}s - \frac{1}{a_1}v_1 > 2\eta$ and hence $s_1 > \eta$. From $s < a_1\eta + v_1$ multiplied with $\frac{a_1+1}{a_1}$ and $\frac{a_1+1}{a_1}v_1 = v_1 + \frac{1}{a_1}v_1$ follows that $\frac{a_1+1}{a_1}s - \frac{1}{a_1}v_1 < (a_1 + 1)\eta + v_1$ and therefore

11) $2s_1 < (a_1 + 1)\eta + v_1$ which can be rearranged to $(a_1 - 1)\eta + v_1 < 2(a_1\eta + v_1 - s_1)$ *i.e.* to 9) with $\lambda = 2$ and s_1 instead of s.

Further, we obtain from 11) with $0 < (a_1 - 1)\eta + v_1$ that

$$2s_1 < (a_1 + 1)\eta + v_1 + (a_1 - 1)\eta + v_1 = 2a_1\eta + 2v_1, \quad i.e.\ s_1 < a_1\eta + v_1.$$

Therefore $\eta < s_1 < a_1\eta + v_1$ and from now on we will assume that s satisfies 8) and 9) for some $0 < \lambda \in K$ and in fact $1 < \lambda$ since by 8) we have $0 < a_1\eta + v_1 - s < (a_1 - 1)\eta + v_1$. With

$$(u, w) = \left(\frac{1}{\lambda c_2}, \frac{\lambda c_2 - 1}{\lambda c_2}s\right) \in G \quad \text{and}$$
$$(x, y) = \left(a_2^{-1}\lambda c_2 a_1, \quad a_2^{-1}(\lambda c_2 v_1 - \lambda c_2 s + s - v_2)\right) \in G$$

we obtain

12) $(a_1, v_1) = (u, w)(a_2, v_2)(x, y)$. We will show that

13) $(u, w) \in P_\eta$ and $(x, y) \in P_\eta$.

We have $(u - 1)\eta + w = \left(\frac{1}{\lambda c_2} - 1\right)\eta + \frac{\lambda c_2 - 1}{\lambda c_2}s = \frac{\lambda c_2 - 1}{\lambda c_2}(s - \eta) > 0$ since $c_2 > 1, \lambda > 1$ and $s - \eta > 0$.

Next we show that $(x - 1)\eta + y > 0$. From $\eta < s$ we obtain $a_2\eta - s + v_2 < (a_2 - 1)\eta + v_2$ and from $(a_2 - 1)\eta + v_2 \leq c_2[(a_1 - 1)\eta + v_1] \leq c_2\lambda(a_1\eta + v_1 - s)$ using 9) it follows that $a_2\eta - s + v_2 < c_2\lambda(a_1\eta + v_1 - s)$. Therefore,

$$\begin{aligned} a_2[(x - 1)\eta + y] &= a_2[(a_2^{-1}\lambda c_2 a_1 - 1)\eta + a_2^{-1}(\lambda c_2 v_1 - \lambda c_2 s + s - v_2)] \\ &= \lambda c_2(a_1\eta + v_1 - s) - (a_2\eta - s + v_2) > 0 \end{aligned}$$

by the previous inequality. Since $a_2 > 0$, it follows that $(x - 1)\eta + y > 0$ and that $(x, y) \in P_\eta$. From 12) and 13) we conclude that $(a_1, v_1) \in P_\eta(a_2, v_2)P_\eta$. Since by assumption $a_1\eta + v_1 < a_2\eta + v_2$, we have $\eta < a_1^{-1}a_2\eta + a_1^{-1}(v_2 - v_1)$, and therefore $\left(a_1^{-1}a_2, a_1^{-1}(v_2 - v_1)\right) \in P_\eta$ with $(a_1, v_1)\left(a_1^{-1}a_2, a_1^{-1}(v_2 - v_1)\right) = (a_2, v_2)$. It follows that $P_\eta(a_1, v_1)P_\eta = P_\eta(a_2, v_2)P_\eta$, which completes the proof of part i) of Theorem 4.5.

To prove part ii) we assume that for $\alpha = (a, v) \in P_\eta$ no s exists in V with $a\eta + v > s > \eta$ but $a\eta + v > \eta$. We want to prove that $\alpha P_\eta \subseteq P_\eta \alpha$ and assume to the contrary that there exists an element $\beta \in \alpha P_\eta$, $\beta \notin P_\eta \alpha$. Then $P_\eta \beta P_\eta \subseteq P_\eta \alpha P_\eta$ and $\beta\alpha^{-1} \notin P_\eta$ which implies $\alpha\beta^{-1} \in P_\eta$, $\alpha \in P_\eta \beta$. Therefore, $P_\eta \alpha P_\eta \subseteq P_\eta \beta P_\eta$ and $P_\eta \alpha P_\eta = P_\eta \beta P_\eta$, $\alpha \neq \beta$. We apply part i) to conclude that there exists an element $s \in V$ with $\eta < s < a\eta + v$, a contradiction which proves $\alpha P_\eta \subseteq P_\eta \alpha$. The same type of argument proves $P_\eta \alpha \subseteq \alpha P_\eta$ and $\alpha P_\eta = P_\eta \alpha$ follows. ∎

We consider the set $\Delta = \{(a, v) \in P_\eta \mid \exists s \in V : \eta < s < a\eta + v\}$ and show that Δ is a completely prime ideal of P_η, where we assume that $\eta \neq \pm\infty$ (see Example 4.2i), ii); $\Delta = \phi$ in these cases). If $(a, v) \in \Delta$ and $(x, y) \in P_\eta$ then $(a, v)(x, y) = (ax, ay + v)$ and $a(x\eta + y) + v \geq a\eta + v > s > \eta$ for some $s \in V$. From $(x, y)(a, v) = (xa, xv + y)$ follows $xa\eta + xv + y = x(a\eta + v) + y > xs + y > x\eta + y > \eta$ and $(x, y)(a, v) \in \Delta$ since $xs + y \in V$; hence Δ is an ideal of P_η.

That Δ is completely prime follows from the next result.

LEMMA 4.6. *Every ideal Ω of P_η with $\Omega \subseteq \Delta$ is completely prime.*

PROOF. As for ideals in valuation rings (see [BBT]) it is enough to show that $\alpha^2 \in \Omega$, $\alpha \in P_\eta$, implies $\alpha \in \Omega$. To see this assume $\alpha\beta \in \Omega$, $\alpha, \beta \in P_\eta$, and assume $\delta^2 \in \Omega$ implies $\delta \in \Omega$. Either $\beta\gamma = \alpha$ and $\alpha^2 = \alpha\beta\gamma \in \Omega$, $\alpha \in \Omega$ or $\alpha\gamma = \beta$ for some $\gamma \in \Omega$. Then $\beta\alpha\beta\alpha = (\beta\alpha)^2 \in \Omega$, hence $\beta\alpha \in \Omega$, $\beta^2 = \beta\alpha\gamma \in \Omega$ and $\beta \in \Omega$. Next let $\alpha = (a, v) \in P_\eta$ with $(a, v)^2 = \big(a^2, (a+1)v\big) \in \Omega$. By assumption there exists $s \in V$ with $\eta < s < a^2\eta + (a+1)v$. To finish the proof that $(a, v) \in P_\eta(a, v)^2 P_\eta \subseteq \Omega$, we must show by Theorem 4.5 (i) that there exists an element $s_1 \in V$ with $\eta < s_1 < a\eta + v$ and that there exists $c \in K^{>0}$ with $(a^2 - 1)\eta + (a+1)v \leq c[(a-1)\eta + v]$. If $s < a\eta + v$ we are done with $s_1 = s$. Otherwise we have $a\eta + v < s < a^2\eta + (a+1)v$ and $\eta < \frac{s-v}{a} < a\eta + v$ follows and we can set $s_1 = \frac{s-v}{a}$. Finally, for $c = a + 1$ we have $(a^2 - 1)\eta + (a+1)v \leq c[(a-1)\eta + v]$. Hence, $(a, v) \in P_\eta(a, v)^2 P_\eta \subseteq \Omega$ is shown, which proves our claim. ∎

As above, we assume that $\eta \neq \pm\infty$ (see Lemma 4.3 for the case $\eta \pm \infty$). Then Δ is a completely prime ideal of P_η and every one-sided ideal $I \supseteq \Delta$ of P_η is two-sided since it is either equal to Δ or contains elements α with $\alpha \notin \Delta$ and hence $P_\eta \alpha = \alpha P_\eta \supseteq \Delta$ by Theorem 4.5ii). If (F, B_η) is the valued skew field constructed as in Theorem 3.2 parameterized by (G, P_η) and described before Theorem 4.4, then there exists a completely prime ideal D of B_η corresponding to the completely prime ideal Δ of P_η, and the following result holds:

COROLLARY 4.7. *Let $\eta \neq \pm\infty$. Then there exists a completely prime ideal $D \neq (0)$ in B_η such that any prime segment $P_1 \supset P_2$ of B_η is invariant if and only if $P_2 \supseteq D$ and it is simple if and only if $D \supseteq P_1$.*

The result follows from Theorem 3.6 and the above observation that all one-sided ideals of P_η containing Δ and hence all ideals of B_η containing D are two-sided and from Lemma 4.6 which implies that all ideals of B_η contained in D are completely prime. ∎

In the next result we will describe the special situation in which all prime segments of the constructed valuation ring are simple. Let $K, V, \mathcal{C} = (U, O)$, $V_1 = K\eta + V$, (G, P_η), Δ, (F, B_η), D, be as above.

THEOREM 4.8. *The following conditions are equivalent:*

a) All prime segments of B_η are simple;

b) $\Delta = P_\eta \setminus \{1,0\}$;

c) $\mathcal{J}(B_\eta) = D$;

d) V is dense in the ordered vector-space $V_1 = K\eta + V$;

e) The cut $\mathcal{C} = (U, O)$ of $(V, \leq)$ determined by η has the following properties:

(e.1) For every $0 < r \in V$ there exists $u \in U$ with $u + r \in O$.

(e.2) $U = \{u \in V \mid u < \eta\}$ has no largest and $O = \{v \in V \mid \eta < v\}$ has no smallest element.

PROOF. That a), b) and c) are equivalent follows from Corollary 4.7. To prove that b) implies d) we consider two elements $c_i\eta + v_i \in V_1$, $i = 1, 2$, with $c_1\eta + v_1 < c_2\eta + v_2$ and must show that there exists $s \in V$ with $c_1\eta + v_1 < s < c_2\eta + v_2$. If in a first case both $0 < c_1, c_2$, then $\eta < c_1^{-1}c_2\eta + c_1^{-1}(v_2 - v_1)$, $0 < c_1^{-1}c_2$ and we can apply b) to obtain $s' \in V$ with $\eta < s' < c_1^{-1}c_2\eta + c_1^{-1}(v_2 - v_1)$ which in turn leads to $c_1\eta + v_1 < s = c_1s' + v_1 < c_2\eta + v_2$ and $s \in V$.

For the second case we assume that both $c_1, c_2 < 0$. Then $-c_2\eta - v_2 < -c_1\eta - v_1$ and there exists by the first case an $s' \in V$ with $-c_2\eta - v_2 < s' < -c_1\eta - v_1$. Hence, $c_1\eta + v_1 < s = -s' < c_2\eta + v_2$ and $s \in V$.

Finally, we assume that $c_1 = 0$ or $c_2 = 0$ or $c_1c_2 < 0$. We have $c_1\eta + v_1 < \frac{c_1+c_2}{2}\eta + \frac{v_1+v_2}{2} < c_2\eta + v_2$ and we are done if $c_1 + c_2 = 0$. If $\frac{c_1+c_2}{2} \neq 0$ then the coefficients of η are either both greater 0 or both less than zero for one of the following pairs of elements: $(c_1\eta + v_1, \frac{c_1+c_2}{2}\eta + \frac{v_1+v_2}{2})$ and $(\frac{c_1+c_2}{2}\eta + \frac{v_1+v_2}{2}, c_2\eta + v_2)$. If we apply the first or second case to this pair we obtain an element $s \in V$ which lies between the two elements of this pair and hence $c_1\eta + v_1 < s < c_2\eta + v_2$, which proves that b) implies d).

Next we show that d) implies e): To prove (e.1) we consider an arbitrary element r with $0 < r \in V$ and we will exhibit an element $u \in U$ with $u + r \in O$. We have $\eta - r < \eta$ and by d) an element u exists in V with $\eta - r < u < \eta$, hence $u \in U$, $u + r \in O$, *i.e.* (e.1) follows. To show that U has no largest element, assume to the contrary that u_m is maximal in U, hence $u_m < \eta$ and by assumption d) an element $s \in V$ exists with $u_m < s < \eta$, a contradiction. That the second statement of (e.2) holds is proved similarly.

To complete the proof of Theorem 4.8 we assume e) and will prove that b) holds. Let $(1,0) \neq (a, v) \in P_\eta$, hence $\eta < a\eta + v$ and we must show that there exists $s \in V$ with $\eta < s < a\eta + v$. We have $0 < (a-1)\eta + v$ and will show that there exists $w \in V$ with $0 < w < (a-1)\eta + v$. In case $a - 1 = 0$ we can choose $w = \frac{v}{2}$. If $a - 1 > 0$, we have $-v < (a-1)\eta$, $\frac{-v}{a-1} < \eta$ and by (e.2) there exists $w_1 \in V$ with $-\frac{v}{a-1} < w_1 < \eta$ hence $w = (a-1)w_1 + v$ will satisfy $0 < w < (a-1)\eta + v$. If $(a-1) < 0$, then $\eta < \frac{-v}{(a-1)}$ and by (e.2) there exists $w_2 \in V$ with $\eta < w_2 < \frac{-v}{(a-1)}$. For $w = w_2(a-1) + v$ we have $0 < w < (a-1)\eta + v$.

For the element $0 < w$ just found there exists by (e.1) an element $u \in U$ with $u < \eta < u+w$. Finally, since $w < (a-1)\eta+v$, we have $\eta < u+w < \eta+(a-1)\eta+v = a\eta+v$, *i.e.* $s = u+w \in V$ satisfies the condition that shows that $(a, v) \in \Delta$ and b) is proved. This completes the proof of Theorem 4.8. ∎

We single out the special case of Theorem 4.8 in which the valuation ring B_η has a single prime segment which is simple, *i.e.* B_η is nearly simple with $\mathcal{J}(B_\eta)$ and (0) as its only proper ideals.

COROLLARY 4.9. *The ring B_η is nearly simple if and only if the equivalent conditions of Theorem 4.8 are satisfied and in addition the following condition holds:*

(e.3) All positive elements of V are in the same K-archimedean class.

PROOF. If B_η has only one ideal $\neq$ (0), B_η then by Theorem 3.6, $P_\eta(1, v)P_\eta = P_\eta(1, v')P_\eta$ for any $v > 0$, $v' > 0$ in V and v and v' are in the same K-archimedean class by Theorem 4.5(i).

Conversely, let us assume that the conditions in Theorem 4.8 hold and that there is only one K-archimedean class of positive elements in V. Then for $(a, v) \in P_\eta \setminus \{e\}$ we have $w_2 > (a-1)\eta + v > w_1 > 0$ for some elements w_1 and w_2 in V. The assumption implies $kw_1 > w_2$ for some $0 < k \in K$; therefore $kw_1 > (a-1)\eta+v$, and $(a-1)\eta+v$ and w_1 are in the same K-archimedean class in $K\eta + V$. Theorem 4.5(i) can be applied and we have $P_\eta(a, v)P_\eta = P_\eta(1, w_1)P_\eta$ which in turn is equal to $P_\eta(1, u)P_\eta$ for any $0 < u \in V$. So all $(a, v) \in P_\eta \setminus \{e\}$ are in the same ideal of P_η. Hence by Theorem 3.6, B_η has only one proper ideal $\neq$ (0). ∎

As a final result in this section we will determine the cofinality type of the simple segments of the valuation rings B_η constructed above. We also recall that K is an ordered field and we denote with ω_β its cofinality type.

PROPOSITION 4.10. *Let $P_1 \supset P_2$ be a simple prime segment of B_η. Then its right and left cofinality types are both equal to ω_β, the cofinality type of K.*

PROOF. Let $\Omega_1 \supset \Omega_2$ be the completely prime ideals of P_η corresponding to $P_1 \supset P_2$ (Theorem 3.6) and $\Delta \supseteq \Omega_1$ follows by Corollary 4.7. If $(a_0, v_0) \in \Omega_1 \setminus \Omega_2$ then $\Omega_1 = P_\eta(a_0, v_0)P_\eta$ and there exists an element $s \in V$ with $\eta < s < a_0\eta + v_0$. We show next that the K-archimedean classes of $w = (a_0 - 1)s + v_0$ and $(a_0 - 1)\eta + v_0$ as elements of V_1 agree. We observe that $0 < a_0, s < a_0\eta + v_0$ and $\eta < s$ implies

$$\begin{aligned} w = a_0 s - s + v_0 &= a_0 s + \eta + (-\eta - s + v_0) \\ &< a_0(a_0\eta + v_0) + s + (-\eta - s + v_0) \\ &= (a_0 + 1)[(a_0 - 1)\eta + v_0]. \end{aligned}$$

On the other hand, we conclude from $\eta < s$ and $s < a_0\eta + v_0$ that

$$\begin{aligned} a_0[(a_0 - 1)\eta + v_0] &= a_0^2\eta + s + (-s - a_0\eta + a_0 v_0) \\ &< a_0^2 s + (a_0\eta + v_0) + (-s - a_0\eta + a_0 v_0) \\ &= (1 + a_0)[(a_0 - 1)s + v_0], \end{aligned}$$

https://doi.org/10.4153/CJM-1995-059-3 Published online by Cambridge University Press

i.e., $(a_0-1)\eta+v_0 < a_0^{-1}(1+a_0)w$, which proves that w and $(a_0-1)\eta+v_0$ are elements of the same K-archimedean class.

Let $\{c_i\}_{i<\omega_\beta}$ be an ω_β-sequence of elements $0 < c_i$, cofinal in $(K,\leq)$ and we consider the sequence $\{c_iw\}_{i<\omega_\beta}$ for the element $w \in V$ constructed above. Since c_iw, w and $(a_0-1)\eta+v_0$ are all in the same K-archimedean class we have $P_\eta\alpha_iP_\eta = \Omega_1$, for $\alpha_i = (1,c_iw) \in P_\eta$ and $\alpha_i \in \Omega_1 \setminus \Omega_2$ follows for $i < \omega_\beta$. We claim that $\{\alpha_iP_\eta \mid i < \omega_\beta\}$ $(\{P_\eta\alpha_i \mid i < \omega_\beta\})$ is cofinal in the ordered set of all right (left) ideals L of P_η between Ω_1 and Ω_2 with $L_1 \geq L_2$ if and only if $L_1 \subseteq L_2$ defining the order for such right ideals; the order is defined similarly for left ideals. It then follows that ω_β is the right- as well as the left-cofinality type of the segment $P_1 \supset P_2$, since ω_β is itself a cofinality type.

Let $\alpha = (a,v)$ be an element in $\Omega_1 \setminus \Omega_2$. We will show that there exist $i_1, i_2 < \omega_\beta$ with $\alpha_{i_1}P_\eta \subseteq \alpha P_\eta$ and $P_\eta\alpha_{i_2} \subseteq P_\eta\alpha$. We have $\Omega_1 = P_\eta(a,v)P_\eta$ and by Theorem 4.5 it follows that w and $(a-1)\eta+v$ are in the same K-archimedean class and hence $(a-1)\eta+v < cw$ for some $0 < c \in K$. There exists $i_1 < \omega_\beta$ with $c < c_{i_1}$ and $(a-1)\eta+v < cw < c_{i_1}w$ follows, which implies

$$a\eta < c_{i_1}w+\eta-v, \quad \eta < a^{-1}\eta+a^{-1}c_{i_1}w-a^{-1}v$$

and therefore $(a,v)^{-1}(1,c_{i_1}w) = (a^{-1}, a^{-1}c_{i_1}w-a^{-1}v) \in P_\eta$, which in turn implies

$$\alpha_{i_1}P_\eta = (1,c_{i_1}w)P_\eta \subseteq (a,v)P_\eta = \alpha P_\eta.$$

Similarly, there exists $i_2 < \omega_\beta$ with $a^{-1}c < c_{i_2}$ and $(a-1)\eta+v < cw < ac_{i_2}w$ follows which in turn implies $\eta < a^{-1}\eta - a^{-1}v + c_{i_2}w$, *i.e.* $(1,c_{i_2}w)(a,v)^{-1} = (a^{-1}, -a^{-1}v+c_{i_2}w) \in P_\eta$. Therefore, $P_\eta\alpha_{i_2} = P_\eta(1,c_{i_2}w) \subseteq P_\eta(a,v) = P_\eta\alpha$, which proves the above claim and the proposition. ∎

5. **Examples.** In this section we consider various constructions and examples to illustrate the results in the previous section. As in Section 4, we denote with K an ordered field, with V an ordered K-vector space, with $G = \{(a,v) \mid 0 < a \in K, v \in V\}$, and $(a,v)(a',v') = (aa', av'+v)$ as operation, the group of K-linear affine transformations of V. Further, for any Dedekind cut $C = C_\eta = (U,O)$ of V there exists an extension V_1 (as ordered K-vector spaces) of V with $\eta \in V_1 \setminus V$, $V_1 = K\eta + V$ and $U = \{v \in V \mid v < \eta\}$, $O = \{v \in V \mid v > \eta\}$ (see Theorem 4.1). In addition,

$$P_\eta = \{(a,v) \in G \mid (a-1)\eta + v \geq 0\}$$

is a generalized positive cone of G. Therefore, by Theorem 3.2, there exists a valued skew field (F,B_η) parameterized by (G,P_η) such that the ideals of P_η correspond to the ideals $\neq (0)$ of B_η, see Theorem 3.6. The set

$$\Delta = \{(a,v) \in P_\eta \mid \exists s \in V : a\eta + v > s > \eta\}$$

is a completely prime ideal of P_η and corresponds to a completely prime ideal D of B_η which separates the invariant segments of B_η from the simple ones, see Lemma 4.6 and Corollary 4.7.

EXAMPLE 5.1. For any given cofinality type ω_α there exist an ordered field $(K, \leq)$ of this cofinality type and an open Dedekind cut C of $(K, \leq)$ with gap $= \{0\}$, *i.e.* C satisfies the conditions e.2) and e.1) in Theorem 4.8e).

To construct such a field K, let $\{i \mid i < \omega_\alpha\}$ be the set of ordinals less than ω_α. We consider $R = \mathbb{Q}[\{x_i \mid i < \omega_\alpha\}]$, the polynomial ring in the variables x_i, $i < \omega_\alpha$, and we write x^ν for the monomial $x_{i_1}^{n_{i_1}} \cdots x_{i_k}^{n_{i_k}}$ where $\nu = (n_0, n_1, n_2, \ldots, n_i, \ldots)$, $i < \omega_\alpha$, $n_i \in \mathbb{N}_0$, and $n_i \neq 0$ for only finitely many i. The set of multi exponents ν forms an ordered abelian semigroup N under component-wise addition and ordered lexicographically. $R = \mathbb{Q}[\{x^\nu \mid \nu \in N\}]$ can then be considered as the semigroup ring with $x^{\nu_1}x^{\nu_2} = x^{\nu_1+\nu_2}$ and elements $p = \sum c_\nu x^\nu$, $c_\nu \in \mathbb{Q}$, $\nu \in N$, and $c_\nu \neq 0$ for only finitely many ν. The element $p \neq 0$ in R is called *positive* if the coefficient $c_{\nu_0} \neq 0$, for ν_0 minimal, is positive in $\mathbb{Q}$. This defines an order on R and its field L of quotients.

Let $K = \mathbb{Q}(\{x_0 - x_i \mid i < \omega_\alpha\})$ be the subfield of $(L, \leq)$ generated by the differences $x_0 - x_i$, $i < \omega_\alpha$, and the order induced by the order of L. In K we have the elements $x_i - x_{i+1} = (x_0 - x_{i+1}) - (x_0 - x_i)$ and $x_i - x_{i+1} > 0$. Since $\{x_i \mid i < \omega_\alpha\}$ is a zero-sequence in L, the sequence $\{(x_i - x_{i+1}) \mid i < \omega_\alpha\}$ is a zero-sequence in K. Hence, $\{(x_i - x_{i+1})^{-1} \mid i < \omega_\alpha\}$ is cofinal in K and since ω_α is a cofinality type, it follows, that ω_α is the cofinality type of $(K, \leq)$.

We have $x_0 \in L \setminus K$ and x_0 defines a Dedekind cut $C = (U, O)$ of $(K, \leq)$ with $U = \{a \in K \mid a < x_0\}$, $O = \{a \in K \mid a > x_0\}$. We check conditions e.1) and e.2) in Theorem 4.8 for this cut:

e.1) For every $0 < r \in K$ there exists $u \in U$ with $u + r \in O$. This holds because there exists an index $i_1 < \omega_\alpha$ with $x_{i_1} - x_{i_1+1} < \frac{r}{2}$, and then for $u = x_0 - x_{i_1} \in U$ we have $u + r > (x_0 - x_{i_1}) + (2x_{i_1} - 2x_{i_1+1}) = x_0 + x_{i_1} - 2x_{i_1+1} > x_0$, *i.e.* $u + r \in O$, which proves e.1).

To prove e.2), *i.e.* that U has no largest and O has no smallest element, we take $u \in U$, $v \in O$. Then $0 < x_0 - u$, $0 < v - x_0$. We observe that $\{(x_i - 2x_{i+1}) \mid i < \omega_\alpha\}$ is a zero-sequence in L. Hence, there exists an index $i_1 < \omega_\alpha$ with $x_{i_1} < x_0 - u$, $x_{i_1} - 2x_{i_1+1} < v - x_0$. It follows, that the elements $u' = x_0 - x_{i_1} \in U$ and $v' = 2(x_0 - x_{i_1+1}) - (x_0 - x_{i_1}) \in O$ satisfy the conditions $u < u'$ and $v' < v$ proving e.2) and the claim in Example 5.1. ■

In the next example we use the field constructed above and Theorem 4.8 to construct valuation rings of arbitrary rank (see the definition at the end of Section 2) and with simple segments only.

EXAMPLE 5.2. Let $(I, \leq)$ be an arbitrary (not necessarily finite) ordered index set, ω_α a cofinality type. Then there exists a valued field (F, B) such that B has simple prime segments only which are all of cofinality type ω_α and B has rank equal to I^* (the set I with the inverse order of that of I).

We distinguish two cases: a) I has a largest element i_m; b) I does not have a largest element.

a) We choose for the given ω_α an ordered field K as constructed in Example 5.1. For V we choose the K-vector space $K\{I\}$ of all K-valued functions f from I to K with well-ordered support, *i.e.* $\{i \in I \mid f(i) \neq 0\}$ is well-ordered. For each element $f \neq 0$ in V we define $v(f) = \min\{i \in I \mid f(i) \neq 0\}$ and $\partial f = f\big(v(f)\big)$, the first non-zero coefficient of f. An element $f \neq 0$ is then defined as positive if $\partial f > 0$ in K. Let $C = (U, O)$ be a cut of K that satisfies e.1) and e.2) and we define the cut $C_\eta = (U_1, O_1)$ of V such that the set $\{f \in V \mid f(i) = 0 \text{ for } i < i_m, f(i_m) \in U\}$ is cofinal in U_1. If $r > 0$ in V and $v(r) < i_m$, then obviously there exists an f in the above set defining U_1 with $r + f \in O_1$. If $v(r) = i_m$, then such an element exists because C satisfies e.1). It follows from the definition of U_1 and the property e.2) for C that C_η also satisfies e.2). We can apply Theorem 4.8 to obtain a valuation ring B_η with simple segments only. Every simple segment has cofinality type ω_α by Proposition 4.10 and I^* is the rank of B_η by Theorem 4.5 and the fact that V and $K\eta + V$ have the same K-archimedean classes in this case. The valuation ring B_η is finally simple if I has a smallest element.

b) If $(I, \leq)$ has no largest element we take the ordered field K as before, however as vector space we choose the ordered subspace W of the vector space $V = K\{I\}$ in a) of all bounded elements $f \in V$, *i.e.* there exists $i_f \in I$ with $f(j) = 0$ for all $j \geq i_f$. Every element $\eta \in V \setminus W$ defines then a cut $C_\eta = (U_1, O_1)$ with $U_1 = \{w \in W \mid w < \eta\}$, $O_1 = \{w \in W \mid \eta < w\}$ that satisfies conditions e.1) and e.2) of Theorem 4.8. To see this, let $r > 0$ be in W with $v(r) = j_0$. Pick $i_0 > j_0$ in I and an element $u \in U_1$ with $u(i) = \eta(i)$ for $i < i_0$, $u(i_0) < \eta(i_0)$ and $u(j) = 0$. Then $u + r \in O_1$, so e.1) holds. If $u \in U_1$ and $v(\eta - u) = i_0$, then u' with $u'(i) = \eta(i)$ for $i \leq i_0$ and $u'(j) = 0$ otherwise is an element in U_1 with $u < u'$. Similarly one shows that for every $v \in O_1$ we have another $v' \in O_1$ with $v' < v$, which completes the proof of e.2). As in a) one obtains a valuation ring B_η with the desired properties. ∎

Let $(K, \leq)$ be any ordered field, C_ξ a Dedekind cut of $(K, \leq)$; then there exists (by Theorem 4.1) the ordered K-vector space $K[\xi] = K\xi + K$. For every ordered index set $(I, \leq)$ one can consider the ordered K-vector spaces $V = K\{I\} \subset \tilde{V} = K[\xi]\{I\}$ defined as in Example 5.2a). The elements $\eta \in \tilde{V} \setminus V$ define then Dedekind cuts for V.

It follows from Theorem 4.8e) that only the Dedekind cuts that satisfy conditions e.1) and e.2) lead to valuation rings with simple prime segments only. On the other hand, only the cuts $(\phi, V) = C_{-\infty}$ and $(V, \phi) = C_{+\infty}$ of V lead to valuation rings with invariant segments only, see Lemma 4.3, Theorem 4.5, and Lemma 4.6. In all other instances both types of prime segments occur.

We consider the following

EXAMPLE 5.3. Let $(K, \leq)$ be an ordered field, $(I, \leq)$ an ordered index set, V_1 an ordered K-vector space whose K-archimedean classes $[v]$ are indexed by I_1^* and we assume that V_1 has a Dedekind cut C_{η_1} that satisfies conditions e.1) and e.2). Then V_1 is dense in $K\eta_1 + V_1 = V_1'$ and we consider the lexicographical order for the vector spaces $V = V_1 + L \subset V_1' + L$ where L is another K-vector space. Then $\eta = \eta_1 + 0$ is an element in $V_1' + L$, not contained in V and defines a Dedekind cut C_η of V. The elements of the corresponding P_η have the form $(a,v) = (a,f+g)$ with $a \in K$, $a > 0$, $v \in V$, $f \in V_1$, $g \in L$ and $a\eta + f + g \geq \eta$. Hence,

$$P_\eta = \{(a,f+g) \mid a\eta_1 + f > \eta_1\} \cup \{(a,f+g) \mid a\eta_1 + f = \eta_1 \text{ and } g \geq 0\}.$$

We claim: $\Delta = \{(a,f+g) \mid a\eta_1 + f > \eta_1\}$. That the right hand side is a subset of Δ follows from the fact that V_1 is dense in V_1' by Theorem 4.8. Conversely, if $a\eta_1 + f = \eta_1$ and $s = f_1 + g_1 \in V_1$ satisfies $a\eta_1 + f + g > f_1 + g_1 > \eta_1$, then $f_1 = \eta_1 \in V_1$, a contradiction. The invariant prime segments correspond to the prime segments of the subsemigroup $\{(1,g) \in P_\eta \mid g \geq 0, g \in L\}$ of P_η which is isomorphic to $(L \geq 0, +)$. ∎

In the next example we discuss the cuts that don't satisfy condition e.2) (see Example 4.2(iii)).

EXAMPLE 5.4. a) Let $(K, \leq)$ be an ordered field, V an ordered K-vector space and $s \in V$. For $C_\eta = C_{s+} = (U_{s+}, O_{s+})$ with $U_{s+} = \{v \in V \mid v \leq s\}$ and $O_{s+} = \{v \in V \mid s < v\}$ we have:

(i) $P_\eta = P_{s+} = \{(a,v) \in G \mid (a-1)s + v > 0 \text{ or } a \geq 1 \text{ and } (a-1)s + v = 0\}$ and

(ii) $\Delta = \Delta_{s+} = \{(a,v) \in P_{s+} \mid (a-1)s + v > 0\}$.

(i) was already obtained (by Theorem 4.1) in Example 4.2(iii). Further, by Theorem 4.1 it follows that here (for $\eta = s+$) generally $a\eta + v > a_1\eta + v_1$ in $V_1 = K\eta + V$ is equivalent to $as + v > a_1 s + v_1$ or $as + v = a_1 s + v_1$ in V and $a > a_1$ in K. That (ii) is true follows since, on one side $(a-1)s + v > 0$ implies $a\eta + v > as + v > \eta$ with $as + v \in V$. Conversely, if on the other side we would have $a \geq 1$ and $(a-1)s + v = 0$ but $a\eta + v > s' > \eta$ for some $s' \in V$, then $s' > s$ and $s' \leq as + v = s$ follow, a contradiction which proves (ii).

The simple prime segments of P_η and B_η respectively correspond to the K-archimedean classes $[v]$ of non-zero elements v of V, and the invariant prime segments correspond to the archimedean classes $\bar{a}$ of elements $a > 1$ in $(K^{>1}, \cdot)$.

If as a special, and very easy example, we choose $K = \mathbb{Q}$, $V = \mathbb{Q}$, the rational numbers, then $B_\eta = B_{s+} \supset \mathcal{J}(B) \supset D \supset (0)$ is a finally simple ring with one invariant and one simple segment.

If we choose $K_1 = \mathbb{Q}[[t, t^{-1}]]$, the Laurent series ring over $\mathbb{Q}$, and order K_1 by defining $a > 0$ if $a = \sum_{i \geq n_0} q_i t^i \in K_1$, $q_{n_0} > 0$, and $V = K_1 + K_1 + K_1$ ordered lexicographically, then B_{s+} will have two invariant prime segments corresponding to $\bar{q}$ for $q \in \mathbb{Q}$, $q > 1$ and $\overline{t^{-1}}$ as the archimedean classes in $(K_1^{>1}, 0)$ and three simple segments corresponding to the three K_1-archimedean classes of the form $[v]$, $0 \neq v$ in V.

b) Let K and V be as in a), but this time we consider the cut $C_\eta = C_{s-} = (U_{s-}, O_{s-})$ for $s \in V$ with $U_{s-} = \{v \in V \mid v < s\}$, $O_{s-} = \{v \in V \mid s \leq v\}$. Then

$$P_\eta = P_{s-} = \Delta \cup \{(a,v) \in G \mid 0 < a \leq 1 \text{ and } (a-1)s + v = 0\} \quad \text{where}$$
$$\Delta = \{(a,v) \in G \mid (a-1)s + v > 0\}.$$

https://doi.org/10.4153/CJM-1995-059-3 Published online by Cambridge University Press

In this case we have $(a,v) > (a_1, v_1)$ for elements in P_η if and only if

$$as + v > a_1 s + v_1 \quad \text{or} \quad as + v = a_1 s + v_1 \quad \text{and} \quad a < a_1.$$

The simple prime segments correspond to the non-zero K-archimedean classes $[v]$ of V and the invariant prime segments correspond to the archimedean classes $\bar{a}$ of elements a in $(\{a \in K \mid 1 > a > 0\}, \cdot)$.

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