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EXTREMES FOR THE INRADIUS IN THE POISSON LINE TESSELLATION

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Abstract

A Poisson line tessellation is observed in the window $W_\rho := B(0, \pi^{-1/2}\rho^{1/2})$ for $\rho > 0$. With each cell of the tessellation, we associate the inradius, which is the radius of the largest ball contained in the cell. Using the Poisson approximation, we compute the limit distributions of the largest and smallest order statistics for the inradii of all cells whose nuclei are contained in W_ρ as ρ goes to ∞. We additionally prove that the limit shape of the cells minimising the inradius is a triangle.

Keywords: Line tessellation; Poisson point process; extreme value; order statistic

2010 Mathematics Subject Classification: Primary 60D05; 60G70; 60G55
Secondary 60F05; 62G32

1. Introduction

The Poisson line tessellation. Let $\hat{X}$ be a stationary and isotropic Poisson line process of intensity $\hat{\gamma} = \pi$ in $\mathbb{R}^2$ endowed with its scalar product $\langle \cdot, \cdot \rangle$ and its Euclidean norm $|\cdot|$. By $\mathcal{A}$, we shall denote the set of affine lines which do not pass through the origin $0 \in \mathbb{R}^2$. Each line can be written as

$$H(u, t) := \{x \in \mathbb{R}^2 \colon \langle x, u \rangle = t\} \tag{1.1}$$

for some $t \in \mathbb{R}$, $u \in \boldsymbol{S}$, where $\boldsymbol{S}$ is the unit sphere in $\mathbb{R}^2$. When $t > 0$, this representation is unique. The intensity measure of $\hat{X}$ is then given by

$$\mu(\mathcal{E}) := \int_{\boldsymbol{S}} \int_{\mathbb{R}_+} \mathbf{1}_{\{H(u,t) \in \mathcal{E}\}} \, \mathrm{d}t \, \sigma(\mathrm{d}u) \tag{1.2}$$

for all Borel subsets $\mathcal{E} \subseteq \mathcal{A}$, where $\mathcal{A}$ is endowed with the Fell topology (see, for example, [23, p. 563]) and where $\sigma(\cdot)$ denotes the uniform measure on $\boldsymbol{S}$ with the normalisation $\sigma(\boldsymbol{S}) = 2\pi$. The set of closures of the connected components of $\mathbb{R}^2 \setminus \hat{X}$ defines a stationary and isotropic random tessellation with intensity $\gamma^{(2)} = \pi$ (see, for example, [23, Equation (10.46)]) which is the so-called *Poisson line tessellation*, $\mathfrak{m}_{\mathrm{PHT}}$. By a slight abuse of notation, we also write $\hat{X}$ to denote the union of lines. An example of the Poisson line tessellation in $\mathbb{R}^2$ is depicted in Figure 1. Let $B(z, r)$ denote the (closed) disc of radius $r \in \mathbb{R}_+$, centred at $z \in \mathbb{R}^2$ and let $\mathcal{K}$ be the family of convex bodies (i.e. convex compact sets in $\mathbb{R}^2$ with nonempty interiors), endowed with the Hausdorff topology. With each convex body $K \in \mathcal{K}$, we may now define the *inradius*

$$R(K) := \sup\{r \colon B(z, r) \subset K, z \in \mathbb{R}^2, r \in \mathbb{R}_+\}.$$

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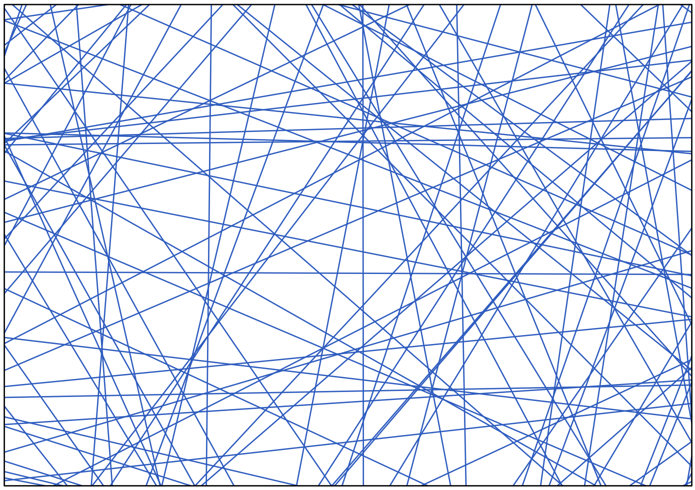

FIGURE 1: A realisation of the Poisson line tessellation truncated to a window.

When there exists a unique $z' \in \mathbb{R}^2$ such that $B(z', R(K)) \subset K$, we define $z(C) := z'$ to be the *incentre* of K. If no such z' exists, we take $z(K) := 0 \in \mathbb{R}^2$. Note that each cell $C \in \mathfrak{m}_{\text{PHT}}$ has a unique z' almost surely. In the rest of the paper we shall use the shorthand $B(K) := B(z(K), R(K))$. To describe the mean behaviour of the tessellation, we recall the definition of the typical cell as follows. Let W be a Borel subset of $\mathbb{R}^2$ such that $\lambda_2(W) \in (0, \infty)$, where λ_2 is the two-dimensional Lebesgue measure. The *typical cell* $\mathcal{C}$ of a Poisson line tessellation $\mathfrak{m}_{\text{PHT}}$ is a random polytope whose distribution is characterised by

$$\mathbb{E}[f(\mathcal{C})] = \frac{1}{\pi \lambda_2(W)} \mathbb{E}\bigg[\sum_{\substack{C \in \mathfrak{m}_{\text{PHT}},\\ z(C) \in W}} f(C - z(C))\bigg] \tag{1.3}$$

for all bounded measurable functions on the set of convex bodies $f\colon \mathcal{K} \to \mathbb{R}$. The typical cell of the Poisson line tessellation has been studied extensively in the literature, including calculations of mean values [16], [17], and distributional results [2] for a number of different geometric characteristics. A long standing conjecture due to Kendall concerning the asymptotic shape of the typical cell conditioned to be large was proved in [12]. The shape of small cells was also considered in [1] for a rectangular Poisson line tessellation. Related results have also been obtained in [11] concerning the approximate properties of random polytopes formed by the Poisson hyperplane process. Global properties of the tessellation have also been established, including, for example, central limit theorems [8], [9].

In this paper we focus on the extremal properties of geometric characteristics for the cells of a Poisson line tessellation whose incentres are contained in a window. The general theory of extreme values deals with stochastic sequences [10] or random fields [15] (more details may be found in the reference works [6] and [21].) To the best of the authors' knowledge, it appears that the first application of extreme value theory in stochastic geometry was given by Penrose; see [19, Chapters 6–8]. More recently, Schulte and Thäle [24] established a theorem to derive the order statistics of a general functional $f_k(x_1, \dots, x_k)$ of k points of a homogeneous Poisson point process, a work which is related to the study of U-statistics. Calka and Chenavier [3] went on to provide a series of results for the extremal properties of cells in the Poisson–Voronoi tessellation, which were then extended by Chenavier [5], who gave a general theorem

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for establishing this type of limit theorem in tessellations satisfying a number of conditions. Unfortunately, none of these methods are directly applicable to the study of extremes for the geometric properties of cells in the Poisson line tessellation, due in part to the fact that even cells which are arbitrarily spatially separated may share lines.

Potential applications. We remark that in addition to the classical references, such as the work by Goudsmit [7] concerning the trajectories of particles in bubble chambers, a number of new and interesting applications of random line processes are emerging in the field of computer science. In particular, recent work by Plan and Vershynin [20] concerns the use of random hyperplane tessellations for dimension reduction with applications to high-dimensional estimation and machine learning, which are important and practical problems facing the computational geometry community at the moment. Notably, [20] points to a lack of results concerning the global properties of tessellations in the traditional stochastic geometry literature, which are of particular interest to this community. Other recent applications for random hyperplanes in computational geometry may also be found in the context of locality sensitive hashing [4]. We believe that our techniques will provide useful tools for the analysis of algorithms in these contexts.

Another potential application field is statistics of point processes in $\mathbb{R}^2$. The key idea would be to identify a point process Φ from the extremes of its underlying line process $\hat{\Phi} := \{H(x), x \in \Phi\}$, where $H(x) := H(u, t)$ for any $x = tu \in \Phi$ with $t \in \mathbb{R}_+$ and $u \in \boldsymbol{S}$. Numerous inference methods have been developed for spatial point processes [18]. A comparison based on the extremes of line tessellations may or may not provide stronger results.

Finally, we note that investigating the extremal properties of cells could also provide a way to describe the regularity of tessellations. For instance, in the finite element method, the quality of the approximation depends on some consistency measurements over the partition; see, for example, [13].

1.1. Contributions

Formally, we shall consider the case in which only a part of the tessellation is observed in the *window* $\boldsymbol{W}_\rho := B(0, \pi^{-1/2}\rho^{1/2})$ for $\rho > 0$. Given a measurable function $f\colon \mathcal{K} \to \mathbb{R}$ satisfying $f(C + x) = f(C)$ for all $C \in \mathcal{K}$ and $x \in \mathbb{R}^2$, we consider the order statistics of $f(C)$ for all cells $C \in \mathfrak{m}_{\mathrm{PHT}}$ such that $z(C) \in \boldsymbol{W}_\rho$ in the limit as $\rho \to \infty$. In this paper we focus on the $f(C) := R(C)$ case in particular because the inradius is one of the rare geometric characteristics for which the distribution of $f(\mathcal{C})$ can be made explicit. More precisely, we investigate the asymptotic behaviour of $m_{\boldsymbol{W}_\rho}[r]$ and $M_{\boldsymbol{W}_\rho}[r]$, which we use respectively to denote the inradii of the rth smallest and the rth largest inballs for fixed $r \geq 1$. Thus for $r = 1$, we have

$$m_{\boldsymbol{W}_\rho}[1] = \min_{\substack{C \in \mathfrak{m}_{\mathrm{PHT}},\\ z(C) \in \boldsymbol{W}_\rho}} R(C) \quad \text{and} \quad M_{\boldsymbol{W}_\rho}[1] = \max_{\substack{C \in \mathfrak{m}_{\mathrm{PHT}},\\ z(C) \in \boldsymbol{W}_\rho}} R(C).$$

The asymptotic behaviours of $m_{\boldsymbol{W}_\rho}[r]$ and $M_{\boldsymbol{W}_\rho}[r]$ are given in the following theorem.

Theorem 1.1. *Let $\mathfrak{m}_{\mathrm{PHT}}$ be a stationary, isotropic Poisson line tessellation in $\mathbb{R}^2$ with intensity π and let $r \geq 1$ be fixed. Then*

(i) *for any $t \geq 0$,*

$$\mathbb{P}\bigg(m_{\boldsymbol{W}_\rho}[r] \geq \frac{t}{2\pi^2\rho}\bigg) \to \mathrm{e}^{-t} \sum_{k=0}^{r-1} \frac{t^k}{k!} \quad \text{as } \rho \to \infty;$$

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(ii) *for any* $t \in \mathbb{R}$,

$$\mathbb{P}\bigg(M_{W_\rho}[r] \leq \frac{1}{2\pi}(\log(\rho)+t)\bigg) \to \mathrm{e}^{-\exp(-t)} \sum_{k=0}^{r-1} \frac{(\mathrm{e}^{-t})^k}{k!} \quad \textit{as } \rho \to \infty.$$

When $r = 1$, the limit distributions are of type II and type III, so $m_{W_\rho}[1]$ and $M_{W_\rho}[1]$ belong to the domains of attraction of Weibull and Gumbel distributions, respectively. The techniques we employ to investigate the asymptotic behaviours of $m_{W_\rho}[r]$ and $M_{W_\rho}[r]$ are quite different. For the cells minimising the inradius, we show that asymptotically $m_{W_\rho}[r]$ has the same behaviour as the rth smallest value associated with a carefully chosen U-statistic. This will allow us to apply the theorem in [25]. The main difficulties we encounter will be in checking the conditions for their theorem, and to deal with boundary effects. The cells maximising the inradius are more delicate since the random variables in question cannot easily be formulated as a U-statistic. Our solution is to use a Poisson approximation, with the method of moments, in order to reduce our investigation to *finite* collections of cells. We then partition the possible configurations of each finite set using a clustering scheme and conditioning on the inter-cell distance.

The shape of cells with a small inradius. It was demonstrated that with high probability the cell which minimises the circumradius for a Poisson–Voronoi tessellation is a triangle [3]. In the following theorem we demonstrate that the analogous result holds for the cells of a Poisson line tessellation with a small inradius. We begin by observing that almost surely there exists a unique cell in $\mathfrak{m}_{\mathrm{PHT}}$ with incentre in W_ρ, say $C_{W_\rho}[r]$, such that $R(C_{W_\rho}[r]) = m_{W_\rho}[r]$. We then consider the random variable $n(C_{W_\rho}[r])$, where, for any (convex) polygon P in $\mathbb{R}^2$, we use $n(P)$ to denote the number of vertices of P.

Theorem 1.2. *Let* $\mathfrak{m}_{\mathrm{PHT}}$ *be a stationary, isotropic Poisson line tessellation in* $\mathbb{R}^2$ *with intensity* π *and let* $r \geq 1$ *be fixed. Then*

$$\mathbb{P}\bigg(\bigcap_{1\leq k\leq r}\{n(C_{W_\rho}[k])=3\}\bigg) \to 1 \quad \textit{as } \rho \to \infty.$$

Remark 1.1. The asymptotic behaviour for the area of all triangular cells with a small area was given in [24, Corollary 2.7]. Applying similar techniques to those which we use to obtain the limit shape of the cells minimising the inradii, and using the fact that

$$\mathbb{P}(\lambda_2(\mathcal{C}) < v) \leq \mathbb{P}\bigg(R(\mathcal{C}) < \Big(\frac{v}{\pi}\Big)^{1/2}\bigg) \quad \text{for all } v > 0,$$

we can also prove that with high probability the cells with a small *area* are triangles. As mentioned in [24, Remark 4] (where a formal proof is not provided), this implies that [24, Corollary 2.7] makes a statement not only about the area of the smallest triangular cell but also about the area of the smallest cell in general.

Remark 1.2. Our theorems are given specifically for the two-dimensional case with a fixed disc-shaped window W_ρ in order to keep our calculations simple. However, Theorem 1.1 holds when the window is any convex body. We believe that our results concerning the largest order statistics may be extended into higher dimensions and more general anisotropic (stationary) Poisson processes using standard arguments. For the case of the smallest order statistics, these generalisations become less evident and may require alternative arguments in places.

1.2. Layout

In Section 2 we shall introduce the general notation and background that will be required throughout the rest of the paper. In Section 3 we provide the asymptotic behaviour of $m_{\mathbf{W}_\rho}[r]$, proving the first part of Theorem 1.1 and Theorem 1.2. In Section 4 we establish some technical lemmas that will be used to derive the asymptotic behaviour of $M_{\mathbf{W}_\rho}[r]$. We conclude in Section 5 by providing the asymptotic behaviour of $M_{\mathbf{W}_\rho}[r]$, finalising the proof of Theorem 1.1.

2. Preliminaries

In this paper we adopt the following notation.

- We shall use $\mathrm{Po}(\tau)$ as a place-holder for a Poisson random variable with mean $\tau > 0$.
- For any pair of functions $f, g : \mathbb{R} \to \mathbb{R}$, we write $f(\rho) \sim g(\rho)$ as $\rho \to \infty$ and $f(\rho) = O(g(\rho))$ to respectively mean that $f(\rho)/g(\rho) \to 1$ as $\rho \to \infty$ and $f(\rho)/g(\rho)$ is bounded for large enough ρ.
- By $\mathcal{B}(\mathbb{R}^2)$ we mean the family of Borel subsets in $\mathbb{R}^2$.
- For any $A \in \mathcal{B}(\mathbb{R}^2)$ and any $x \in \mathbb{R}^2$, we write $x + A := \{x + y : y \in A\}$ and $d(x, A) := \inf_{y\in A} |x - y|$.
- Let E be a measurable set and $K \geq 1$.
  - For any K-tuple of points $x_1, \dots, x_K \in E$, we write $x_{1:K} := (x_1, \dots, x_K)$.
  - By $E^K_{\neq}$, we mean the set of K-tuples of points $x_{1:K}$ such that $x_i \neq x_j$ for all $1 \leq i \neq j \leq K$.
  - For any function $f : E \to F$, where F is a set, and for any $A \subset F$, we write $f(x_{1:K}) \in A$ to imply that $f(x_i) \in A$ for each $1 \leq i \leq K$. In the same spirit, $f(x_{1:K}) > v$ will be used to mean that $f(x_i) > v$ given $v \in \mathbb{R}$.
  - If ν is a measure on E, we write $\nu(\mathrm{d}x_{1:K}) := \nu(\mathrm{d}x_1) \cdots \nu(\mathrm{d}x_K)$.
- Given three lines $H_{1:3} \in \mathcal{A}^3_{\neq}$ in general position (in the sense of [23, p. 128]), we denote by $\triangle(H_{1:3})$ the unique triangle that can be formed by the intersection of the half-spaces induced by the lines H_1, H_2, and H_3. In the same spirit, we denote by $B(H_{1:3})$, $R(H_{1:3})$, and $z(H_{1:3})$ the inball, the inradius, and the incentre of $\triangle(H_{1:3})$, respectively.
- Let $K \in \mathcal{K}$ be a convex body with a unique inball $B(K)$ such that the intersection $B(K) \cap K$ contains exactly three points, x_1, x_2, x_3. In which case we define T_1, T_2, T_3 to be the lines tangent to the border of $B(K)$ intersecting x_1, x_2, x_3, respectively. We now define $\triangle(K) := \triangle(T_{1:3})$, observing that $B(\triangle(K)) = B(K)$.
- For any line $H \in \mathcal{A}$, we write H^+ to denote the half-plane delimited by H and containing $0 \in \mathbb{R}^2$. According to (1.1), we have $H^+(u, t) := \{ x \in \mathbb{R}^2 : \langle x, u\rangle \leq t \}$ for given $t > 0$ and $u \in \mathbf{S}$.
- For any $A \in \mathcal{B}(\mathbb{R}^2)$, we take $\mathcal{A}(A) \subset \mathcal{A}$ to be the set

$$\mathcal{A}(A) := \{H \in \mathcal{A} : H \cap A \neq \varnothing\}.$$

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We also define $\phi\colon \mathcal{B}(\mathbb{R}^2) \to \mathbb{R}_+$ as

$$\phi(A) := \mu(\mathcal{A}(A)) = \int_{\mathcal{A}} \mathbf{1}_{\{H\cap A\neq\varnothing\}}\,\mu(\mathrm{d}H). \tag{2.1}$$

Remark 2.1. Because $\hat{X}$ is a Poisson process, we have, for any $A \in \mathcal{B}(\mathbb{R}^2)$,

$$\mathbb{P}(\hat{X}\cap A = \varnothing) = \mathbb{P}(\#\hat{X}\cap\mathcal{A}(A) = 0) = \mathrm{e}^{-\phi(A)}. \tag{2.2}$$

Remark 2.2. When $A \in \mathcal{B}(\mathbb{R}^2)$ is a convex body, from the Crofton formula [23, Theorem 5.1.1], we have

$$\phi(A) = \ell(A), \tag{2.3}$$

where $\ell(A)$ denotes the perimeter of A. In particular, when $A = B(z, r)$ for some $z \in \mathbb{R}^2$ and $r \geq 0$, we have $\phi(B(z, r)) = \mu(\mathcal{A}(B(z, r))) = 2\pi r$.

A well-known representation of the typical cell. The typical cell of a Poisson line tessellation, as defined in (1.3), can be made explicit in the following sense. For any measurable function $f\colon \mathcal{K} \to \mathbb{R}$, from [23, Theorem 10.4.6], we have

$$\mathbb{E}[f(\mathcal{C})] = \frac{1}{24\pi}\int_0^\infty\int_{S^3}\mathbb{E}[f(C(\hat{X}, u_{1:3}, r))]\mathrm{e}^{-2\pi r}a(u_{1:3})\,\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r, \tag{2.4}$$

where

$$C(\hat{X}, u_{1:3}, r) := \bigcap_{H\in\hat{X}\cap(\mathcal{A}(B(0,r)))^{\mathrm{c}}}\Big\{H^+\cap\bigcap_{j=1}^{3}H^+(u_j, r)\Big\} \tag{2.5}$$

and where $a(u_{1:3})$ is taken to be the area of the convex hull of $\{u_1, u_2, u_3\} \subset \boldsymbol{S}$ when $0 \in \mathbb{R}^2$ is contained in the convex hull of $\{u_1, u_2, u_3\}$ and 0 otherwise. With standard computations, it may be demonstrated that $\int_{S^3} a(u_{1:3})\,\sigma(\mathrm{d}u_{1:3}) = 48\pi^2$, so when $f(C) = R(C)$, we have the following well-known result (see, for example, [23, Theorem 10.4.8]):

$$\mathbb{P}(R(\mathcal{C}) \leq v) = 1 - \mathrm{e}^{-2\pi v} \quad \text{for all } v \geq 0. \tag{2.6}$$

We note that in the following, we occasionally omit the lower bounds in the ranges of sums and unions, and the arguments of functions when they are clear from the context. Throughout this paper we also use c to signify a universal positive constant not depending on ρ but which may depend on other quantities. When required, we assume that ρ is sufficiently large.

3. Asymptotics for cells with a small inradii

3.1. Intermediary results

Let $r \geq 1$ be fixed. In order to avoid boundary effects, we introduce a function $q(\rho)$ such that

$$\frac{\log\rho\cdot q(\rho)}{\rho^2} \to 0 \quad \text{as } \rho\to\infty \quad \text{and} \quad \frac{(q(\rho)^{1/2} - \rho^{1/2})}{\pi^{1/2}} - \varepsilon\log\rho \to +\infty \quad \text{as } \rho\to\infty \tag{3.1}$$

for some $\varepsilon > 0$. We also introduce two intermediary random variables, the first of which relates collections of 3-tuples of lines in $\hat{X}$. Let $\overset{\triangle}{m}_{W_\rho}[r]$ represent the rth smallest value of $R(H_{1:3})$ over all 3-tuples of lines $H_{1:3} \in \hat{X}^3_{\neq}$ such that $z(H_{1:3}) \in \boldsymbol{W}_\rho$ and $\triangle(H_{1:3}) \subset \boldsymbol{W}_{q(\rho)}$. Its asymptotic behaviour is given in the following proposition.

Proposition 3.1. *For any $r \geq 1$ and any $t \geq 0$,*

$$\mathbb{P}\left(\overset{\triangle}{m}_{\boldsymbol{W}_\rho}[r] \geq \frac{t}{2\pi^2\rho}\right) \to \mathrm{e}^t \sum_{k=0}^{r-1} \frac{t^k}{k!} \quad \text{as } \rho \to \infty.$$

The second random variable concerns the cells in $\mathfrak{m}_{\mathrm{PHT}}$. More precisely, we define $\overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r]$ to be the rth smallest value of the inradius over all cells $C \in \mathfrak{m}_{\mathrm{PHT}}$ such that $z(C) \in \boldsymbol{W}_\rho$ and $\triangle(C) \subset \boldsymbol{W}_{q(\rho)}$. We observe that $\overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r] \geq \overset{\triangle}{m}_{\boldsymbol{W}_\rho}[r]$ and $\overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r] \geq m_{\boldsymbol{W}_\rho}[r]$. Actually, in the following result we show that the deviation between these quantities is negligible as ρ goes to ∞.

Lemma 3.1. *For any fixed $r \geq 1$,*

(i) $\mathbb{P}(\overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r] \neq \overset{\triangle}{m}_{\boldsymbol{W}_\rho}[r]) \to 0$ *as* $\rho \to \infty$,

(ii) $\mathbb{P}(m_{\boldsymbol{W}_\rho}[r] \neq \overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r]) \to 0$ *as* $\rho \to \infty$.

Finally, to prove Theorem 1.2, we also investigate the tail of the distribution of the perimeter of a random triangle. To do this, for any 3-tuple of unit vectors $u_{1:3} \in (\boldsymbol{S}^3)_{\neq}$, we write $\triangle(u_{1:3}) := \triangle(H_{1:3})$, where $H_i = H(u_i, 1)$ for any $1 \leq i \leq 3$.

Lemma 3.2. *With the above notation, as v goes to ∞, we have*

$$\int_{\boldsymbol{S}^3} a(u_{1:3})\, \mathbf{1}_{\{l(\triangle(u_{1:3}))>v\}}\, \sigma(\mathrm{d}u_{1:3}) = O(v^{-1}).$$

3.2. Main tool

As stated above, Schulte and Thäle established a general theorem to deal with U-statistics [24, Theorem 1.1]. In this work we make use of a new version of their theorem (see [25]), which we modify slightly to suit our requirements. Let $g\colon \mathcal{A}^3 \to \mathbb{R}$ be a measurable symmetric function and take $\overset{\triangle}{m}_{g,\boldsymbol{W}_\rho}[r]$ to be the rth smallest value of $g(H_{1:3})$ over all 3-tuples of lines $H_{1:3} \in \hat{\boldsymbol{X}}^3_{\neq}$ such that $z(H_{1:3}) \in \boldsymbol{W}_\rho$ and $\triangle(H_{1:3}) \subset \boldsymbol{W}_{q(\rho)}$ (for $q(\rho)$ as in (3.1).) We now define the following quantities for a given $a, t \geq 0$:

$$\alpha_\rho^{(g)}(t) := \frac{1}{6}\int_{\mathcal{A}^3} \mathbf{1}_{\{z(H_{1:3})\in \boldsymbol{W}_\rho\}}\, \mathbf{1}_{\{\triangle(H_{1:3})\subset \boldsymbol{W}_{q(\rho)}\}}\, \mathbf{1}_{\{g(H_{1:3})<\rho^{-a}t\}}\, \mu(\mathrm{d}H_{1:3}), \tag{3.2a}$$

$$r_{\rho,1}^{(g)}(t) := \int_{\mathcal{A}}\left(\int_{\mathcal{A}^2} \mathbf{1}_{\{z(H_{1:3})\in \boldsymbol{W}_\rho\}}\, \mathbf{1}_{\{\triangle(H_{1:3})\subset \boldsymbol{W}_{q(\rho)}\}}\, \mathbf{1}_{\{g(H_{1:3})<\rho^{-a}t\}}\, \mu(\mathrm{d}H_{2:3})\right)^2 \mu(\mathrm{d}H_1), \tag{3.2b}$$

$$r_{\rho,2}^{(g)}(t) := \int_{\mathcal{A}^2}\left(\int_{\mathcal{A}} \mathbf{1}_{\{z(H_{1:3})\in \boldsymbol{W}_\rho\}}\, \mathbf{1}_{\{\triangle(H_{1:3})\subset \boldsymbol{W}_{q(\rho)}\}}\, \mathbf{1}_{\{g(H_{1:3})<\rho^{-a}t\}}\, \mu(\mathrm{d}H_3)\right)^2 \mu(\mathrm{d}H_{1:2}). \tag{3.2c}$$

Theorem 3.1. (Schulte and Thäle [24].) *Let $t \geq 0$ be fixed. Assume that $\alpha_\rho(t)$ converges to $\alpha t^\beta > 0$ for some $\alpha, \beta > 0$, and $r_{\rho,1}(t), r_{\rho,2}(t) \to 0$ as $\rho \to \infty$. Then*

$$\mathbb{P}(\overset{\triangle}{m}{}^{(g)}_{\boldsymbol{W}_\rho}[r] \geq \rho^{-a}t) \to \mathrm{e}^{-\alpha t^\beta} \sum_{k=0}^{r-1} \frac{(\alpha t^\beta)^k}{k!} \quad \text{as } \rho \to \infty.$$

Remark 3.1. Actually, Theorem 3.1 is stated in [25] for a Poisson point process in more general measurable spaces with intensity going to ∞. By scaling invariance, we have written

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their result for a fixed intensity (equal to π) and for the window $\boldsymbol{W}_{q(\rho)} = B(0, \pi^{-1/2}q(\rho)^{1/2})$ with $\rho \to \infty$. We also adapt their result by adding the indicator function $\mathbf{1}_{\{z(H_{1:3})\in \boldsymbol{W}_\rho\}}$ to (3.2a), (3.2b), and (3.2c).

3.3. Proofs

Proof of Proposition 3.1. Let $t \geq 0$ be fixed. We apply Theorem 3.1 with $g = R$ and $a = 1$. First, we compute the quantity $\alpha_\rho(t) := \alpha_\rho^{(R)}(t)$ as defined in (3.2a). Applying a Blaschke–Petkantschin-type change of variables (see, for example, [23, Theorem 7.3.2]), we obtain

$$\begin{aligned}\alpha_\rho(t) &= \frac{1}{24}\int_{\mathbb{R}^2}\int_0^\infty\int_{\boldsymbol{S}^3} a(u_{1:3})\,\mathbf{1}_{\{z\in \boldsymbol{W}_\rho\}}\,\mathbf{1}_{\{z+r\triangle(u_{1:3})\subset \boldsymbol{W}_{q(\rho)}\}}\,\mathbf{1}_{\{r<\rho^{-1}t\}}\,\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r\,\mathrm{d}z\\ &= \frac{1}{24}\int_{\mathbb{R}^2}\int_0^\infty\int_{\boldsymbol{S}^3} a(u_{1:3})\,\mathbf{1}_{\{z\in \boldsymbol{W}_1\}}\,\mathbf{1}_{\{z+r\rho^{-3/2}\triangle(u_{1:3})\subset \boldsymbol{W}_{q(\rho)/\rho}\}}\,\mathbf{1}_{\{r<t\}}\,\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r\,\mathrm{d}z.\end{aligned}$$

We note that the normalisation of μ_1, as defined in [23], is such that $\mu_1 = (1/\pi)\mu$, where μ is given in (1.2). It follows from the monotone convergence theorem that

$$\alpha_\rho(t) \to \frac{1}{24}\int_{\mathbb{R}^2}\int_0^\infty\int_{\boldsymbol{S}^3} a(u_{1:3})\,\mathbf{1}_{\{z\in \boldsymbol{W}_1\}}\,\mathbf{1}_{\{r<t\}}\,\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r\,\mathrm{d}z = 2\pi^2 t \quad \text{as } \rho\to\infty \tag{3.3}$$

since $\lambda_2(\boldsymbol{W}_1) = 1$ and $\int_{\boldsymbol{S}^3} a(u_{1:3})\,\sigma(\mathrm{d}u_{1:3}) = 48\pi^2$. We must now check that

$$r_{\rho,1}(t) \to 0 \quad \text{as } \rho\to\infty, \tag{3.4}$$

$$r_{\rho,2}(t) \to 0 \quad \text{as } \rho\to\infty, \tag{3.5}$$

where $r_{\rho,1}(t) := r_{\rho,1}^{(R)}(t)$ and $r_{\rho,2}(t) := r_{\rho,2}^{(R)}(t)$ are defined in (3.2b) and (3.2c).

Proof of convergence (3.4). Let H_1 be fixed and define

$$G_\rho(H_1) := \int_{\mathcal{A}^2} \mathbf{1}_{\{z(H_{1:3})\in \boldsymbol{W}_\rho\}}\,\mathbf{1}_{\{\triangle(H_{1:3})\subset \boldsymbol{W}_{q(\rho)}\}}\,\mathbf{1}_{\{R(H_{1:3})<\rho^{-1}t\}}\,\mu(\mathrm{d}H_{2:3}).$$

Bounding $\mathbf{1}_{\{\triangle(H_{1:3})\subset \boldsymbol{W}_{q(\rho)}\}}$ by 1, and applying Lemma A.1(i) (given in Appendix A) to $R := \rho^{-1}t$, $R' := \pi^{-1/2}\rho^{1/2}$, and $z' = 0$, we obtain, for large enough ρ,

$$G_\rho(H_1) \leq \frac{c\,\mathbf{1}_{\{d(0,H_1)<\rho^{1/2}\}}}{\rho^{1/2}}.$$

Noting that $r_{\rho,1}(t) = \int_{\mathcal{A}} G_\rho(H_1)^2\mu(\mathrm{d}H_1)$, from (1.2), it follows that

$$r_{\rho,1}(t) \leq \frac{c\int_{\mathcal{A}} \mathbf{1}_{\{d(0,H_1)<\rho^{1/2}\}}\,\mu(\mathrm{d}H_1)}{\rho} = O\biggl(\frac{1}{\rho^{1/2}}\biggr). \tag{3.6}$$

This completes the proof of (3.4). □

Proof of convergence (3.5). Let H_1 and H_2 be such that H_1 intersects H_2 at a unique point, $v(H_{1:2})$. The set $H_1\cup H_2$ divides $\mathbb{R}^2$ into two double-cones with supplementary angles $C_i(H_{1:2})$, $1\leq i\leq 2$; see Figure 2. We then denote by $\theta_i(H_{1:2}) \in [0, \pi/2)$ the half-angle of $C_i(H_{1:2})$ so that $2(\theta_1(H_{1:2}) + \theta_2(H_{1:2})) = \pi$. Moreover, we write

$$E_i(H_{1:2}) := \biggl\{H_3 \in \mathcal{A}\colon z(H_{1:3}) \in \boldsymbol{W}_\rho \cap C_i(H_{1:2}),\ \triangle(H_{1:3}) \subset \boldsymbol{W}_{q(\rho)},\ R(H_{1:3}) < \frac{t}{\rho}\biggr\}.$$

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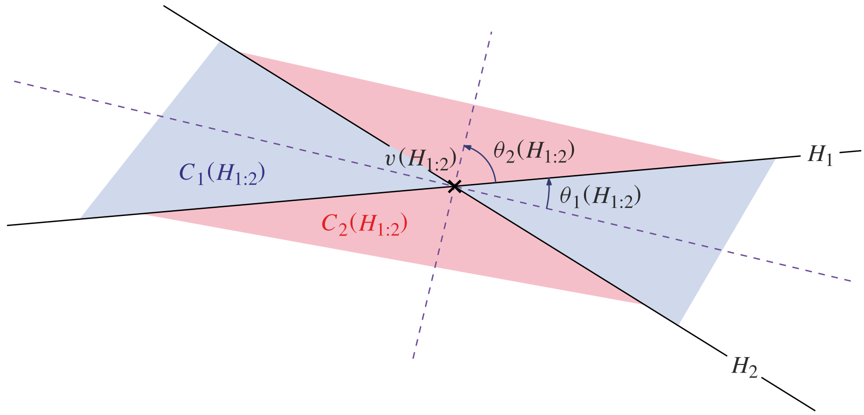

FIGURE 2: Construction of the double-cone for a change of variables.

We provide below a suitable upper bound for $G_\rho(H_1, H_2)$ defined as

$$G_\rho(H_1, H_2) := \int_{\mathcal{A}} \mathbf{1}_{\{z(H_{1:3}) \in \mathbf{W}_\rho\}} \mathbf{1}_{\{\triangle(H_{1:3}) \subset \mathbf{W}_{q(\rho)}\}} \mathbf{1}_{\{R(H_{1:3}) < \rho^{-1} t\}} \, \mu(\mathrm{d}H_3)$$
$$= \sum_{i=1}^{2} \int_{\mathcal{A}} \mathbf{1}_{\{H_3 \in E_i(H_{1:2})\}} \, \mu(\mathrm{d}H_3). \tag{3.7}$$

To do this, we first establish the following lemma.

Lemma 3.3. *Let $H_1, H_2 \in \mathcal{A}$ be fixed and let $H_3 \in E_i(H_{1:2})$ for some $1 \le i \le 2$. Then*

(i) *$H_3 \cap W_{c\rho} \neq \varnothing$ for some c,*

(ii) *$H_3 \cap B(v(H_{1:2}), c\rho^{-1}/\sin\theta_i(H_{1:2})) \neq \varnothing$,*

(iii) *$|v(H_{1:2})| \le c q(\rho)^{1/2}$ for some c.*

Proof. The first statement is a consequence of the fact that

$$d(0, H_3) \le |z(H_{1:3})| + d(z(H_{1:3}), H_3) \le \left(\frac{\rho}{\pi}\right)^{1/2} + \frac{t}{\rho} \le c \cdot \rho^{1/2}.$$

For the second statement, we have

$$d(v(H_{1:2}), H_3) \le |v(H_{1:2}) - z(H_{1:3})| + d(z(H_{1:3}), H_3) \le \frac{R(H_{1:3})}{\sin\theta_i(H_{1:2})} + \frac{t}{\rho}.$$

Since $R(H_{1:3}) = |v(H_{1:2}) - z(H_{1:3})| \sin\theta_i(H_{1:2})$, it follows that

$$d(v(H_{1:2}), H_3) \le \frac{c\rho^{-1}}{\sin\theta_i(H_{1:2})}.$$

Finally, the third statement comes from the fact that $v(H_{1:2}) \in \mathbf{W}_{q(\rho)}$ since $\triangle(H_{1:3}) \subset \mathbf{W}_{q(\rho)}$. This completes the proof of Lemma 3.3. □

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We apply below the first statement of Lemma 3.3 when $\theta_i(H_{1:2})$ is small enough, and the second one otherwise. More precisely, from (3.7) and Lemma 3.3, it follows that

$$G_\rho(H_1, H_2) \le \sum_{i=1}^{2} \phi(\boldsymbol{W}_{c\rho})\, \mathbf{1}_{\{|v(H_{1:2})| \le cq(\rho)^{1/2}\}}\, \mathbf{1}_{\{\sin\theta_i(H_{1:2}) \le \rho^{-3/2}\}} + \phi\biggl(B\biggl(v(H_{1:2}), \frac{c\rho^{-1}}{\sin\theta_i(H_{1:2})}\biggr)\biggr)\, \mathbf{1}_{\{|v(H_{1:2})| \le cq(\rho)^{1/2}\}}\, \mathbf{1}_{\{\sin\theta_i(H_{1:2}) > \rho^{-3/2}\}},$$

where $\phi(\cdot)$ has been defined in (2.1). Applying (2.3) to

$$B := \boldsymbol{W}_{c\rho} = B(0, c^{1/2}\rho^{1/2}) \quad \text{and} \quad B' := B\biggl(v(H_{1:2}), \frac{c\rho^{-1}}{\sin\theta_i(H_{1:2})}\biggr),$$

it follows that

$$G_\rho(H_1, H_2) \le c \sum_{i=1}^{2} \biggl(\rho^{1/2}\, \mathbf{1}_{\{\sin\theta_i(H_{1:2}) \le \rho^{-3/2}\}} + \frac{\rho^{-1}}{\sin\theta_i(H_{1:2})}\, \mathbf{1}_{\{\sin\theta_i(H_{1:2}) > \rho^{-3/2}\}}\biggr) \times \mathbf{1}_{\{|v(H_{1:2})| \le cq(\rho)^{1/2}\}}. \tag{3.8}$$

Applying the fact that

$$r_{\rho,2}(t) = \int_{\mathcal{A}} G_\rho(H_1, H_2)^2 \mu(\mathrm{d}H_{1:2}) \quad \text{and} \quad \biggl(\sum_{i=1}^{2}(a_i + b_i)\biggr)^2 \le 4 \sum_{i=1}^{2}(a_i^2 + b_i^2)$$

for any $a_1, a_2, b_1, b_2 \in \mathbb{R}$, it follows from (3.8) that

$$r_{\rho,2}(t) \le c \sum_{i=1}^{2} \int_{\mathcal{A}^2} \biggl(\rho\, \mathbf{1}_{\{\sin\theta_i(H_{1:2}) \le \rho^{-3/2}\}} + \frac{\rho^{-2}}{\sin^2\theta_i(H_{1:2})}\, \mathbf{1}_{\{\sin\theta_i(H_{1:2}) > \rho^{-3/2}\}}\biggr) \times \mathbf{1}_{\{|v(H_{1:2})| \le cq(\rho)^{1/2}\}}\, \mu(\mathrm{d}H_{1:2}).$$

For any couple of lines $(H_1, H_2) \in \mathcal{A}^2$ such that $H_1 = H(u_1, t_1)$ and $H_2 = H(u_2, t_2)$ for some $u_1, u_2 \in \boldsymbol{S}$ and $t_1, t_2 > 0$, let $\theta(H_1, H_2) \in [-\pi/2, \pi/2)$ be the oriented half-angle between the vectors u_1 and u_2. In particular, the quantity $|\theta(H_{1:2})|$ is equal to $\theta_1(H_{1:2})$ or $\theta_2(H_{1:2})$. This implies that

$$r_{\rho,2}(t) \le 4c \int_{\mathcal{A}^2} \biggl(\rho\, \mathbf{1}_{\{\sin\theta(H_{1:2}) \le \rho^{-3/2}\}} + \frac{\rho^{-2}}{\sin^2\theta(H_{1:2})}\, \mathbf{1}_{\{\sin\theta(H_{1:2}) > \rho^{-3/2}\}}\biggr) \mathbf{1}_{\{\theta(H_{1:2}) \in [0,\pi/2)\}} \times \mathbf{1}_{\{|v(H_{1:2})| \le cq(\rho)^{1/2}\}}\, \mu(\mathrm{d}H_{1:2}). \tag{3.9}$$

With each $v = (v_1, v_2) \in \mathbb{R}^2$, $\beta \in [0, 2\pi)$, and $\theta \in [0, \pi/2)$, we associate two lines H_1 and H_2 as follows. We first define $L(v_1, v_2, \beta)$ as the line containing $v = (v_1, v_2)$ with normal vector $\boldsymbol{\beta}$, where for any $\alpha \in [0, 2\pi)$, we write $\boldsymbol{\alpha} = (\cos\alpha, \sin\alpha)$. Then we define H_1 and H_2 as the lines containing $v = (v_1, v_2)$ with angles θ and $-\theta$ with respect to $L(v_1, v_2, \beta)$, respectively. These lines can be written as $H_1 = H(u_1, t_1)$ and $H_2 = H(u_2, t_2)$ with

$$\begin{aligned}
u_1 &:= u_1(\beta, \theta) := \overrightarrow{\beta - \theta}, \\
t_1 &:= t_1(v_1, v_2, \beta, \theta) := |-\sin(\beta - \theta)v_1 + \cos(\beta - \theta)v_2|, \\
u_2 &:= u_2(\beta, \theta) := \overrightarrow{\beta + \theta}, \\
t_2 &:= t_2(v_1, v_2, \beta, \theta) := |\sin(\beta + \theta)v_1 + \cos(\beta + \theta)v_2|.
\end{aligned}$$

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Denoting by $\overline{\alpha}$, the unique real number in $[0, 2\pi)$ such that $\overline{\alpha} \equiv \alpha \bmod 2\pi$, we define

$$\psi \colon \mathbb{R}^2 \times [0, 2\pi) \times \left[0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R}_+ \times [0, 2\pi),$$
$$(v_1, v_2, \beta, \theta) \longmapsto (t_1(v_1, v_2, \beta, \theta), \overline{\beta - \theta}, t_2(v_1, v_2, \beta, \theta), \overline{\beta + \theta}).$$

Modulo null sets, ψ is a $\mathcal{C}^1$ diffeomorphism with Jacobian $J\psi$ given by $|J\psi(v_1, v_2, \beta, \theta)| = 2\sin 2\theta$ for any point $(v_1, v_2, \beta, \theta)$, where ψ is differentiable. Taking the change of variables as defined above, we deduce from (3.9) that

$$\begin{aligned} r_{\rho,2}(t) &\leq c \int_{\mathbb{R}^2} \int_0^{2\pi} \int_0^{\pi/2} \sin(2\theta)\Big(\rho\, \mathbf{1}_{\{\sin\theta \leq \rho^{-3/2}\}} + \frac{\rho^{-2}}{\sin^2\theta} \mathbf{1}_{\{\sin\theta > \rho^{-3/2}\}}\Big) \\ &\qquad \times \mathbf{1}_{\{|v| \leq c q(\rho)^{1/2}\}}\, \mathrm{d}\theta\, \mathrm{d}\beta\, \mathrm{d}v \\ &= O\bigg(\frac{\log\rho \cdot q(\rho)}{\rho^2}\bigg). \end{aligned} \tag{3.10}$$

As a consequence of (3.1), the last term converges to 0 as ρ goes to ∞, completing the proof of (3.5). □

The above combined with (3.3), (3.6), and Theorem 3.1 concludes the proof of Proposition 3.1. □

Proof of Lemma 3.1(i). Almost surely, there exists a unique triangle with incentre contained in $\boldsymbol{W}_{q(\rho)}$, denoted by $\triangle_{\boldsymbol{W}_\rho}[r]$, such that

$$z(\triangle_{\boldsymbol{W}_\rho}[r]) \in \boldsymbol{W}_\rho \quad \text{and} \quad R(\triangle_{\boldsymbol{W}_\rho}[r]) = \overset{\vartriangle}{m}_{\boldsymbol{W}_\rho}[r].$$

Also, $z(\triangle_{\boldsymbol{W}_\rho}[r])$ is the incentre of a cell of $\mathfrak{m}_{\mathrm{PHT}}$ if and only if $\hat{\boldsymbol{X}} \cap B(\triangle_{\boldsymbol{W}_\rho}[r]) = \varnothing$. Since $\overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r] \geq \overset{\vartriangle}{m}_{\boldsymbol{W}_\rho}[r]$, this implies that

$$\overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r] = \overset{\vartriangle}{m}_{\boldsymbol{W}_\rho}[r] \quad \Longleftrightarrow \quad \text{there exists } 1 \leq k \leq r \text{ such that } \hat{\boldsymbol{X}} \cap B(\triangle_{\boldsymbol{W}_\rho}[k]) \neq \varnothing.$$

In particular, for any $\varepsilon > 0$, we obtain

$$\begin{aligned} \mathbb{P}(\overset{\pentagon}{m}_{\boldsymbol{W}_\rho}[r] \neq \overset{\vartriangle}{m}_{\boldsymbol{W}_\rho}[r]) \leq \sum_{k=1}^{r} (&\mathbb{P}(\hat{\boldsymbol{X}} \cap B(\triangle_{\boldsymbol{W}_\rho}[k]) \neq \varnothing, R(\triangle_{\boldsymbol{W}_\rho}[k]) < \rho^{-1+\varepsilon}) \\ &+ \mathbb{P}(R(\triangle_{\boldsymbol{W}_\rho}[k]) > \rho^{-1+\varepsilon})). \end{aligned} \tag{3.11}$$

The second term of the series converges to 0 as ρ goes to ∞ thanks to Proposition 3.1. For the first term, we obtain, for any $1 \leq k \leq r$,

$$\begin{aligned} &\mathbb{P}(\hat{\boldsymbol{X}} \cap B(\triangle_{\boldsymbol{W}_\rho}[k]) \neq \varnothing, R(\triangle_{\boldsymbol{W}_\rho}[k]) < \rho^{-1+\varepsilon}) \\ &\quad\leq \mathbb{P}\bigg(\bigcup_{H_{1:4} \in \hat{\boldsymbol{X}}^4_{\neq}} \{z(H_{1:3}) \in \boldsymbol{W}_\rho, R(H_{1:3}) < \rho^{-1+\varepsilon}, H_4 \cap B(z(H_{1:3}), \rho^{-1+\varepsilon}) \neq \varnothing\}\bigg) \\ &\quad\leq \mathbb{E}\bigg[\sum_{H_{1:4} \in \hat{\boldsymbol{X}}^4_{\neq}} \mathbf{1}_{\{z(H_{1:3}) \in \boldsymbol{W}_\rho\}} \mathbf{1}_{\{R(H_{1:3}) < \rho^{-1+\varepsilon}\}} \mathbf{1}_{\{H_4 \cap B(z(H_{1:3}), \rho^{-1+\varepsilon}) \neq \varnothing\}}\bigg] \\ &\quad= \int_{\mathcal{A}^4} \mathbf{1}_{\{z(H_{1:3}) \in \boldsymbol{W}_\rho\}} \mathbf{1}_{\{R(H_{1:3}) < \rho^{-1+\varepsilon}\}} \mathbf{1}_{\{H_4 \cap B(z(H_{1:3}), \rho^{-1+\varepsilon}) \neq \varnothing\}}\, \mu(\mathrm{d}H_{1:4}), \end{aligned}$$

where the last line comes from the Mecke–Slivnyak formula [23, Corollary 3.2.3]. Applying the Blaschke–Petkantschin change of variables, we obtain

$$\mathbb{P}(\hat{\boldsymbol{X}} \cap B(\triangle_{\boldsymbol{W}_\rho}[k]) \neq \varnothing,\, R(\triangle_{\boldsymbol{W}_\rho}[k]) < \rho^{-1+\varepsilon})$$
$$\leq c \int_{\boldsymbol{W}_\rho} \int_0^{\rho^{-1+\varepsilon}} \int_{\boldsymbol{S}^3} \int_{\mathcal{A}} a(u_{1:3})\, \mathbf{1}_{\{H_4 \cap B(z,\rho^{-1+\varepsilon}) \neq \varnothing\}}\, \mu(\mathrm{d}H_4)\sigma(\mathrm{d}u_{1:3})\, \mathrm{d}r\, \mathrm{d}z.$$

As a consequence of (2.1) and (2.3), we have

$$\int_{\mathcal{A}} \mathbf{1}_{\{H_4 \cap B(z,\rho^{-1+\varepsilon}) \neq \varnothing\}}\, \mu(\mathrm{d}H_4) = c\rho^{-1+\varepsilon} \quad \text{for any } z \in \mathbb{R}^2.$$

Integrating over $z \in \boldsymbol{W}_\rho$, $r < \rho^{-1+\varepsilon}$, and $u_{1:3} \in \boldsymbol{S}^3$, we obtain

$$\mathbb{P}(\hat{\boldsymbol{X}} \cap B(\triangle_{\boldsymbol{W}_\rho}[k]) \neq \varnothing,\, R(\triangle_{\boldsymbol{W}_\rho}[k]) < \rho^{-1+\varepsilon}) \leq c\rho^{-1+2\varepsilon} \tag{3.12}$$

since $\lambda_2(\boldsymbol{W}_\rho) = \rho$. Taking $\varepsilon < \frac{1}{2}$, we deduce Lemma 3.1(i) from (3.11) and (3.12). □

Proof of Lemma 3.2. Let $u_{1:3} = (u_1, u_2, u_3) \in (\boldsymbol{S}^3)_{\neq}$ be in general position and such that 0 is in the interior of the convex hull of $\{u_1, u_2, u_3\}$, and let $v_i(u_{1:3})$, $1 \leq i \leq 3$, be the three vertices of $\triangle(u_{1:3})$. If $\ell(\triangle(u_{1:3})) > v$ for some $v > 0$ then there exists $1 \leq i \leq 3$ such that $|v_i(u_{1:3})| > v/6$. In particular, we obtain

$$\int_{\boldsymbol{S}^3} a(u_{1:3})\, \mathbf{1}_{\{l(\triangle(u_{1:3}))>v\}}\, \sigma(\mathrm{d}u_{1:3}) \leq \sum_{i=1}^{3} \int_{\boldsymbol{S}^3} a(u_{1:3})\, \mathbf{1}_{\{|v_i(u_{1:3})|>v/6\}}\, \sigma(\mathrm{d}u_{1:3})$$
$$\leq c \int_{\boldsymbol{S}^2} \mathbf{1}_{\{|v(u_{1:2})|>v/6\}}\, \sigma(\mathrm{d}u_{1:2}), \tag{3.13}$$

where $v(u_{1:2})$ is the intersection point between $H(u_1, 1)$ and $H(u_2, 2)$. Besides, if $u_1 = \boldsymbol{\alpha}_1$ and $u_2 = \boldsymbol{\alpha}_2$ for some $\alpha_1, \alpha_2 \in [0, 2\pi)$ such that $\alpha_1 \not\equiv \alpha_2 \bmod \pi$, we obtain with standard computations,

$$|v(u_{1:2})| = \frac{(2(1 + \cos(\alpha_2 - \alpha_1)))^{1/2}}{|\sin(\alpha_2 - \alpha_1)|} \leq \frac{2}{|\sin(\alpha_2 - \alpha_1)|}.$$

This together with (3.13) shows that

$$\int_{\boldsymbol{S}^3} a(u_{1:3})\, \mathbf{1}_{\{l(\triangle(u_{1:3}))>v\}}\, \sigma(\mathrm{d}u_{1:3}) \leq c \int_{[0,2\pi)^2} \mathbf{1}_{\{|\sin(\alpha_2-\alpha_1)|<12\cdot v^{-1}\}}\, \mathrm{d}\alpha_{1:2} = O(v^{-1}).$$

This concludes the proof of Lemma 3.2. □

Proof of Theorem 1.2. Let $\varepsilon \in (0, \frac{1}{3})$ be fixed. For any $1 \leq k \leq r$, we write

$$\mathbb{P}(n(C_{\boldsymbol{W}_\rho}[k]) \neq 3)$$
$$= \mathbb{P}(n(C_{\boldsymbol{W}_\rho}[k]) \geq 4,\, m_{\boldsymbol{W}_\rho}[k] \geq \rho^{-1+\varepsilon}) + \mathbb{P}(n(C_{\boldsymbol{W}_\rho}[k]) \geq 4,\, m_{\boldsymbol{W}_\rho}[k] < \rho^{-1+\varepsilon}).$$

According to Proposition 3.1, Lemma 3.1(i), and the fact that $\overset{\pentagon}{m}_{W_\rho}[k] \geq m_{W_\rho}[k]$, the first term of the right-hand side converges to 0 as ρ goes to ∞. For the second term, from (1.3), we obtain

$$\begin{aligned}\mathbb{P}(n(C_{W_\rho}[k]) \geq 4, m_{W_\rho}[k] < \rho^{-1+\varepsilon}) &\leq \mathbb{P}\Bigg(\min_{\substack{C\in \mathfrak{m}_{\mathrm{PHT}},\\ z(C)\in W_\rho,\, n(C)\geq 4}} R(C) < \rho^{-1+\varepsilon}\Bigg)\\ &\leq \mathbb{E}\Bigg[\sum_{\substack{C\in\mathfrak{m}_{\mathrm{PHT}},\\ z(C)\in W_\rho}} \mathbf{1}_{\{R(C)<\rho^{-1+\varepsilon}\}}\mathbf{1}_{\{n(C)\geq 4\}}\Bigg]\\ &= \pi\rho\,\mathbb{P}(R(\mathcal{C}) < \rho^{-1+\varepsilon}, n(\mathcal{C}) \geq 4). \end{aligned} \tag{3.14}$$

We provide below a suitable upper bound for $\mathbb{P}(R(\mathcal{C}) < \rho^{-1+\varepsilon},\ n(\mathcal{C}) \geq 4)$. Let $r > 0$ and $u_1, u_2, u_3 \in \boldsymbol{S}$ be fixed. We note that the random polygon $C(\hat{\boldsymbol{X}}, u_{1:3}, r)$, as defined in (2.5), satisfies $n(C(\hat{\boldsymbol{X}}, u_{1:3}, r)) \geq 4$ if and only if $\hat{\boldsymbol{X}} \in \mathcal{A}(\triangle(u_{1:3}, r) \setminus B(0, r))$. According to (2.2) and (2.4), this implies that

$$\begin{aligned}&\pi\rho\,\mathbb{P}(R(\mathcal{C}) < \rho^{-1+\varepsilon}, n(\mathcal{C}) \geq 4)\\ &\quad= \frac{\rho}{24}\int_0^{\rho^{-1+\varepsilon}}\int_{\boldsymbol{S}^3}(1-\mathrm{e}^{-\phi(r\triangle(u_{1:3})\setminus B(0,r))})\mathrm{e}^{-2\pi r}a(u_{1:3})\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r\\ &\quad\leq c\rho\int_0^{\rho^{-1+\varepsilon}}\int_{\boldsymbol{S}^3}(1-\mathrm{e}^{-\rho^{-1+\varepsilon}\ell(\triangle(u_{1:3}))})\mathrm{e}^{-2\pi r}a(u_{1:3})\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r \end{aligned}\tag{3.15}$$

since $\phi(r\triangle(u_{1:3}) \setminus B(0,r)) \leq \phi(r\triangle(u_{1:3})) \leq \rho^{-1+\varepsilon}\ell(\triangle(u_{1:3}))$ for all $r \leq \rho^{-1+\varepsilon}$. First, bounding $1-\mathrm{e}^{-\rho^{-1+\varepsilon}\ell(\triangle(u_{1:3}))}$ by 1 and applying Lemma 3.2, we obtain, for any $\alpha > 0$,

$$\begin{aligned}&\rho\int_0^{\rho^{-1+\varepsilon}}\int_{\boldsymbol{S}^3}(1-\mathrm{e}^{-\rho^{-1+\varepsilon}\ell(\triangle(u_{1:3}))})\mathrm{e}^{-2\pi r}a(u_{1:3})\,\mathbf{1}_{\{\ell(\triangle(u_{1:3}))>\rho^{\alpha}\}}\,\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r\\ &\quad\leq c\rho^{1-\alpha}\int_0^{\rho^{-1+\varepsilon}}\mathrm{e}^{-2\pi r}\,\mathrm{d}r\\ &\quad= O(\rho^{\varepsilon-\alpha}). \end{aligned}\tag{3.16}$$

Moreover, bounding this time $1-\mathrm{e}^{-\rho^{-1+\varepsilon}\ell(\triangle(u_{1:3}))}$ by $\rho^{-1+\varepsilon}\ell(\triangle(u_{1:3}))$, we obtain

$$\begin{aligned}&\rho\int_0^{\rho^{-1+\varepsilon}}\int_{\boldsymbol{S}^3}(1-\mathrm{e}^{-\rho^{-1+\varepsilon}\ell(\triangle(u_{1:3}))})\mathrm{e}^{-2\pi r}a(u_{1:3})\,\mathbf{1}_{\{\ell(\triangle(u_{1:3}))\leq\rho^{\alpha}\}}\,\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r\\ &\quad\leq \rho^{\varepsilon+\alpha}\int_0^{\rho^{-1+\varepsilon}}\int_{\boldsymbol{S}^3}\mathrm{e}^{-2\pi r}a(u_{1:3})\,\mathbf{1}_{\{\ell(\triangle(u_{1:3}))\leq\rho^{\alpha}\}}\,\sigma(\mathrm{d}u_{1:3})\,\mathrm{d}r\\ &\quad= O(\rho^{-1+2\varepsilon+\alpha}). \end{aligned}\tag{3.17}$$

Taking $\alpha = (1-\varepsilon)/2$, from (3.15), (3.16), and (3.17), it follows that

$$\pi\rho\,\mathbb{P}(R(\mathcal{C}) < \rho^{-1+\varepsilon}, n(\mathcal{C}) \geq 4) = O(\rho^{-(1-3\varepsilon)/2}).$$

Using this equation this together with (3.14), we obtain

$$\mathbb{P}(n(C_{W_\rho}[k]) \geq 4, m_{W_\rho}[k] < \rho^{-1+\varepsilon}) \to 0 \quad \text{as } \rho \to \infty. \qquad \square$$

Proof of Lemma 3.1(ii). Since $m_{\mathbf{W}_\rho}[r] \neq \overset{\pentagon}{m}_{\mathbf{W}_\rho}[r]$ if and only if $\triangle(C_{\mathbf{W}_\rho}[k]) \cap \mathbf{W}_{q(\rho)}^{\mathrm{c}}$ is nonempty for some $1 \leq k \leq r$, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}(m_{\mathbf{W}_\rho}[r] \neq \overset{\pentagon}{m}_{\mathbf{W}_\rho}[r]) \\ &\leq \sum_{k=1}^{r}(\mathbb{P}(R(C_{\mathbf{W}_\rho}[k]) \geq \rho^{-1+\varepsilon}) + \mathbb{P}(n(C_{\mathbf{W}_\rho}[k]) \neq 3) \\ &\qquad + \mathbb{P}(\triangle(C_{\mathbf{W}_\rho}[k]) \cap \mathbf{W}_{q(\rho)}^{\mathrm{c}} \neq \varnothing, n(C_{\mathbf{W}_\rho}[k]) = 3, R(C_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon})). \quad (3.18)\end{aligned}$$

As in the proof of Theorem 1.2, the first term of the series converges to 0. The same fact is also true for the second term as a consequence of Theorem 1.2. Moreover, for any $1 \leq k \leq r$, we have

$$\begin{aligned}\mathbb{P}(\triangle(C_{\mathbf{W}_\rho}[k]) \cap \mathbf{W}_{q(\rho)}^{\mathrm{c}} \neq \varnothing, n(C_{\mathbf{W}_\rho}[k]) = 3, R(C_{\mathbf{W}_\rho}[k]) < \rho^{-1+\varepsilon}) \\ &\leq \int_{\mathcal{A}^3} \mathbb{P}(\hat{\boldsymbol{X}} \cap \triangle(H_{1:3}) = \varnothing)\, \mathbf{1}_{\{z(H_{1:3}) \in \mathbf{W}_\rho\}}\, \mathbf{1}_{\{\triangle(H_{1:3}) \cap \mathbf{W}_{q(\rho)}^{\mathrm{c}} \neq \varnothing\}}\, \mathbf{1}_{\{R(H_{1:3}) < \rho^{-1+\varepsilon}\}}\, \mu(\mathrm{d}H_{1:3}) \\ &\leq \int_{\mathcal{A}^3} \mathrm{e}^{-\ell(\triangle(H_{1:3}))}\, \mathbf{1}_{\{z(H_{1:3}) \in \mathbf{W}_\rho\}}\, \mathbf{1}_{\{\ell(\triangle(H_{1:3})) > \pi^{-1/2}(q(\rho)^{1/2} - \rho^{1/2})\}}\, \mathbf{1}_{\{R(H_{1:3}) < \rho^{-1+\varepsilon}\}}\, \mu(\mathrm{d}H_{1:3})\end{aligned}$$

according to the Mecke–Slivnyak formula and (2.2), respectively. Using the fact that

$$\mathrm{e}^{-\ell(\triangle(3H_{1:3}))} \leq \mathrm{e}^{-\pi^{-1/2}(q(\rho)^{1/2} - \rho^{1/2})},$$

and applying the Blaschke–Petkantschin formula, we obtain

$$\mathbb{P}(\triangle(C_{\mathbf{W}_\rho}[k]) \cap \mathbf{W}_{q(\rho)}^{\mathrm{c}} \neq \varnothing, n(C_{\mathbf{W}_\rho}[k]) = 3) \leq c\rho^{\varepsilon} \mathrm{e}^{-\pi^{-1/2}(q(\rho)^{1/2} - \rho^{1/2})}.$$

According to (3.1), the last term converges to 0. This together with (3.18) completes the proof of Lemma 3.1(ii). □

Proof of Theorem 1.1(i). The proof is immediate and follows from Proposition 3.1 and Lemma 3.1. □

Remark 3.2. As mentioned in Section 3.1, we introduce an auxiliary function $q(\rho)$ to avoid boundary effects. This addition was necessary to prove the convergence of $r_{\rho,2}(t)$ in (3.10).

4. Technical results

In this section we establish two results which will be needed in order to derive the asymptotic behaviour of $M_{\mathbf{W}_\rho}[r]$.

4.1. Poisson approximation

Consider a measurable function $f\colon \mathcal{K} \to \mathbb{R}$ and a *threshold* v_ρ such that $v_\rho \to \infty$ as $\rho \to \infty$. The cells $C \in \mathfrak{m}_{\mathrm{PHT}}$ such that $f(C) > v_\rho$ and $z(C) \in \mathbf{W}_\rho$ are called the *exceedances*. A classical tool in extreme value theory is to estimate the limiting distribution of the number of exceedances by a Poisson random variable. In our case, we achieve this with the following lemma.

Lemma 4.1. *Let* $\mathfrak{m}_{\mathrm{PHT}}$ *be a stationary, isotropic Poisson line tessellation embedded in* $\mathbb{R}^2$ *and suppose that for any* $K \geq 1$,

$$\mathbb{E}\Bigg[\sum_{\substack{C_{1:K} \in (\mathfrak{m}_{\mathrm{PHT}})^K_{\neq},\\ z(C_{1:K}) \in \boldsymbol{W}_\rho}} \mathbf{1}_{\{f(C_{1:K}) > v_\rho\}}\Bigg] \tau^K \quad \text{as } \rho \to \infty. \tag{4.1}$$

Then

$$\mathbb{P}(M_{f,\boldsymbol{W}_\rho}[r] \leq v_\rho) \to \sum_{k=0}^{r-1} \frac{\tau^k}{k!} \mathrm{e}^{-\tau} \quad \text{as } \rho \to \infty.$$

Proof of Lemma 4.1. Let the number of exceedance cells be denoted by

$$U(v_\rho) := \sum_{\substack{C \in \mathfrak{m}_{\mathrm{PHT}},\\ z(C) \in \boldsymbol{W}_\rho}} \mathbf{1}_{\{f(C) > v_\rho\}}.$$

Let $1 \leq K \leq n$ and let $\left\{ {n \atop K} \right\}$ denote the Stirling number of the second kind. According to (4.1), we have

$$\begin{aligned}
\mathbb{E}[U(v_\rho)^n] &= \mathbb{E}\Bigg[\sum_{K=1}^{n} \left\{ {n \atop K} \right\} U(v_\rho)(U(v_\rho)-1)(U(v_\rho)-2)\cdots(U(v_\rho)-K+1)\Bigg] \\
&= \sum_{K=1}^{n} \left\{ {n \atop K} \right\} \mathbb{E}\Bigg[\sum_{\substack{C_{1:K} \in \mathfrak{m}_{\mathrm{PHT}}{}^K_{\neq},\\ z(C_{1:K}) \in \boldsymbol{W}_\rho}} \mathbf{1}_{\{f(C_{1:K}) > v_\rho\}}\Bigg] \\
&\to \sum_{K=1}^{n} \left\{ {n \atop K} \right\} \tau^K \quad \text{as } \rho \to \infty \\
&= \mathbb{E}[\mathrm{Po}(\tau)^n].
\end{aligned}$$

Thus by the method of moments, $U(v_\rho)$ converges in distribution to a Poisson random variable with mean τ. We note that $M_{f,\boldsymbol{W}_\rho}[r] \leq v_\rho$ if and only if $U(v_\rho) \leq r-1$. This concludes the proof. □

Lemma 4.1 can be generalised for any window $\boldsymbol{W}_\rho$ and for any tessellation in any dimension. A similar method was used to provide the asymptotic behaviour for couples of random variables in the particular setting of a Poisson–Voronoi tessellation; see [3, Proposition 2]. The main difficulty is applying Lemma 4.1, and we deal partially with this in the following section.

4.2. A uniform upper bound for ϕ for the union of discs

Let $\phi\colon \mathcal{B}(\mathbb{R}^2) \to \mathbb{R}_+$ as in (2.1). We evaluate $\phi(B)$ in the particular case where $B = \bigcup_{1\leq i\leq K}^{K} B(z_i, r_i)$ is a finite union of balls centred in z_i and with radius r_i, $1 \leq i \leq K$. Closed-form representations for $\phi(B)$ could be provided but these formulas are not of practical interest to us. We provide below (see Proposition 4.1) some approximations for $\phi(\bigcup_{1\leq i\leq K}^{K} B(z_i, r_i))$ with simple and quasi-optimal lower bounds.

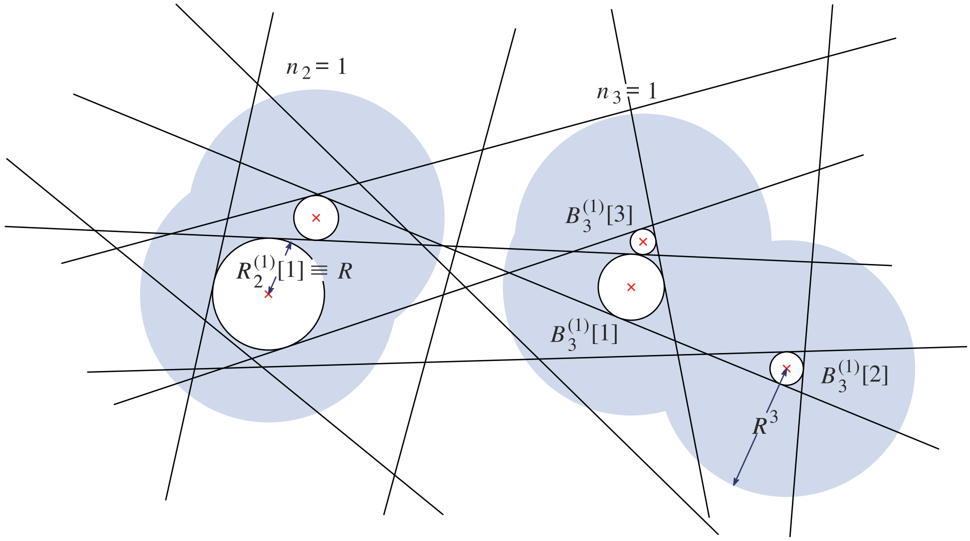

FIGURE 3: Example of connected components for $K = 5$ and $(n_1, \dots, n_K) = (0, 1, 1, 0, 0)$.

4.2.1. *Connected components of cells.* Our bound will follow by splitting collections of discs into a set of connected components. Suppose we are given a threshold v_ρ such that $v_\rho \to \infty$ as $\rho \to \infty$ and $K \geq 2$ discs $B(z_i, r_i)$, satisfying $z_i \in \mathbb{R}^2$, $r_i \in \mathbb{R}_+$, and $r_i > v_\rho$ for all $i = 1, \dots, K$. We take $R := \max_{1 \leq i \leq K} r_i$. The *connected components* are constructed from the graph with vertices $B(z_i, r_i)$, $i = 1, \dots, K$ and edges

$$B(z_i, r_i) \longleftrightarrow B(z_j, r_j) \quad \iff \quad B(z_i, R^3) \cap B(z_j, R^3) \neq \varnothing.$$

On the right-hand side, we have chosen radii of the form R^3 to provide a simpler lower bound in Proposition 4.1. The *size* of a component is the number of discs in that component.

Notation. To refer to the above components, we use the following notation which we highlight for ease of reference.

- For all $k \leq K$, write $n_k := n_k(z_{1:K}, R)$ to denote the number of connected components of size k. Observe that in particular, $\sum_{k=1}^{K} k n_k = K$.
- Suppose that with each component of size k is assigned a unique label $1 \leq j \leq n_k$. We then write $B_k^{(j)} := B_k^{(j)}(z_{1:K}, R)$ to refer to the union of balls in the jth component of size k.
- Within a component, we write $B_k^{(j)}[\ell] := B_k^{(j)}(z_{1:K}, R)[\ell]$, $1 \leq \ell \leq k$, to refer to the ball having the ℓth largest radius in the jth cluster of size k. In particular, we have $B_k^{(j)} = \bigcup_{\ell=1}^{k} B_k^{(j)}[\ell]$. We also write $z_k^{(j)}[\ell]$ and $r_k^{(j)}[\ell]$ as shorthand to refer to the centre and radius of the ball $B_k^{(j)}[\ell]$.

An example is given in Figure 3.

4.2.2. *The uniform upper bound.* In extreme value theory, a classical method to investigate the behaviour of the maximum of a sequence of random variables relies on checking two conditions of the sequence. One such set of conditions is given by Leadbetter [14], who defines the conditions $D(u_n)$ and $D'(u_n)$, which represent an asymptotic property and a local property

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of the sequence, respectively. We shall make use of analogous conditions for the Poisson line tessellation, and it is for this reason that we motivate the different cases concerning spatially separated and spatially close balls in Proposition 4.1.

Proposition 4.1. *Consider a collection of* K *disjoint balls* $B(z_i, r_i)$ *for* $i = 1, \dots, K$ *such that* $r_{1:K} > v_\rho$ *and* $R := \max_{1\le i\le K} r_i$.

(i) *When* $n_{1:K} = (K, 0, \dots, 0)$, *i.e.* $\min_{1\le i,j\le K} |z_i - z_j| > R^3$, *we obtain for large enough* ρ,

$$\phi\Big(\bigcup_{1\le i\le K} B(z_i, r_i)\Big) \ge 2\pi \sum_{i=1}^{K} r_i - \frac{c}{v_\rho}. \tag{4.2}$$

(iia) *For large enough* ρ,

$$\phi\Big(\bigcup_{1\le i\le K} B(z_i, r_i)\Big) \ge 2\pi R + \Big(\sum_{k=1}^{K} n_k - 1\Big) 2\pi v_\rho - \frac{c}{v_\rho};$$

(iib) *when* $R \le (1+\varepsilon)v_\rho$ *for some* $\varepsilon > 0$, *we have for large enough* ρ,

$$\phi\Big(\bigcup_{1\le i\le K} B(z_i, r_i)\Big) \ge 2\pi R + \Big(\sum_{k=1}^{K} n_k - 1\Big) 2\pi v_\rho + \sum_{k=2}^{K} n_k (4 - \varepsilon\pi) v_\rho - \frac{c}{v_\rho}.$$

Remark 4.1. Suppose that $n_{1:K} = (K, 0, \dots, 0)$.

(i) We observe that (4.2) is quasi-optimal since we also have

$$\phi\Big(\bigcup_{1\le i\le K} B(z_i, r_i)\Big) \le \sum_{i=1}^{K} \phi(B(z_i, r_i)) = 2\pi \sum_{i=1}^{K} r_i. \tag{4.3}$$

(ii) Thanks to (2.2), (4.2), and (4.3), we remark that

$$\bigg|\mathbb{P}\Big(\bigcap_{1\le i\le K} \{\hat{X} \cap B(z_i, r_i) = \varnothing\}\Big) - \prod_{1\le i\le K} \mathbb{P}(\hat{X} \cap B(z_i, r_i) = \varnothing)\bigg| \le \frac{c}{v_\rho},$$

which converges to 0 as ρ goes to ∞.

The fact that the events considered in the probabilities above tend to be independent is well known and is related to the fact that the tessellation $\mathfrak{m}_{\mathrm{PHT}}$ satisfies a mixing property; see, for example, the proof of [23, Theorem 10.5.3]. Our contribution is to provide a *uniform rate of convergence* (in the sense that it does not depend on the centres and the radii) when the balls are distant enough (Proposition 4.1(i)), and a suitable *uniform* upper bound for the opposite case (Proposition 4.1(ii)). Proposition 4.1 will be used to check (4.1). Before attacking Proposition 4.1, we first state two lemmas. The first of which deals with the case of just two balls.

Lemma 4.2. *Let* $z_1, z_2 \in \mathbb{R}^2$ *and* $R \ge r_1 \ge r_2 > v_\rho$ *such that* $|z_2 - z_1| > r_1 + r_2$.

(i) *If* $|z_2 - z_1| > R^3$, *we have for large enough* ρ,

$$\mu(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2))) \le \frac{c}{v_\rho}.$$

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(ii) *If* $R \leq (1+\varepsilon)v_\rho$ *for some* $\varepsilon > 0$, *we have for large enough* ρ,

$$\mu(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2))) \leq 2\pi r_2 - (4 - \varepsilon\pi)v_\rho.$$

Actually, closed-form equations for the measure of all lines intersecting two convex bodies can be found in [22, p. 33]. However, Lemma 4.2 is more practical since it provides an upper bound which is independent of the centres and the radii. The following lemma is a generalisation of the previous result.

Lemma 4.3. *Let* $z_{1:K} \in \mathbb{R}^{2K}$ *and* R *such that for all* $1 \leq i \neq j \leq K$, *we have* $R \geq r_i > v_\rho$ *and* $|z_i - z_j| > r_i + r_j$.

(i) *We have*

$$\mu\bigg(\bigcup_{1\leq i\leq K} \mathcal{A}(B(z_i, r_i))\bigg) \geq \sum_{k=1}^{K}\sum_{j=1}^{n_k} 2\pi r_k^{(j)}[1] - \frac{c}{v_\rho}.$$

(ii) *If* $R \leq (1+\varepsilon)v_\rho$ *for some* $\varepsilon > 0$, *we have the following more precise inequality:*

$$\mu\bigg(\bigcup_{1\leq i\leq K} \mathcal{A}(B(z_i, r_i))\bigg) \geq \sum_{k=1}^{K}\sum_{j=1}^{n_k} 2\pi r_k^{(j)}[1] + \sum_{k=2}^{K} n_k(4 - \varepsilon\pi)v_\rho - \frac{c}{v_\rho}.$$

4.3. Proofs

Proof of Proposition 4.1. The proof of (i) follows immediately from (2.1) and Lemma 4.3(i). Using the fact that $r_k^{(j)}[1] > v_\rho$ for all $1 \leq k \leq K$ and $1 \leq j \leq n_k$ such that $r_k^{(j)}[1] \neq R$, we obtain (iia) and (iib) from Lemmas 4.3(i) and 4.3(ii), respectively. □

Proof of Lemma 4.2. As previously mentioned, Santaló [22] provides a general formula for the measure of all lines intersecting two convex bodies. However, to obtain a more explicit representation of $\mu(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2)))$, we write Santaló's result in the particular setting of two balls. According to (1.2) and the fact that μ is invariant under translations, we obtain, with standard computations,

$$\begin{aligned}
&\mu(\mathcal{A}(B(z_1, r_1)) \cap \mathcal{A}(B(z_2, r_2))) \\
&\quad = \int_S \int_{\mathbb{R}_+} \mathbf{1}_{\{H(u,t)\cap B(0,r_1)\neq\varnothing\}} \mathbf{1}_{\{H(u,t)\cap B(z_2-z_1,r_2)\neq\varnothing\}}\, \mathrm{d}t\, \sigma(\mathrm{d}u) \\
&\quad = \int_S \int_{\mathbb{R}_+} \mathbf{1}_{\{t<r_1\}} \mathbf{1}_{\{d(z_2-z_1, H(u,t))<r_2\}}\, dt\, \sigma(\mathrm{d}u) \\
&\quad = \int_{[0,2\pi)} \int_{\mathbb{R}_+} \mathbf{1}_{\{t<r_1\}} \mathbf{1}_{\{|\cos\alpha|z_2-z_1|-t|<r_2\}}\, \mathrm{d}t\, \mathrm{d}\alpha \\
&\quad = 2f(r_1, r_2, |z_2 - z_1|),
\end{aligned}$$

where

$$\begin{aligned}
f(r_1, r_2, h) := (r_1 + r_2)\arcsin\bigg(\frac{r_1 + r_2}{h}\bigg) - (r_1 - r_2)\arcsin\bigg(\frac{r_1 - r_2}{h}\bigg) \\
- h\Bigg(\sqrt{1 - \bigg(\frac{r_1 - r_2}{h}\bigg)^2} - \sqrt{1 - \bigg(\frac{r_1 + r_2}{h}\bigg)^2}\Bigg) \quad \text{for all } h > r_1 + r_2.
\end{aligned}$$

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For any fixed $r_1 \geq r_2$, it may be demonstrated that the function $f_{r_1,r_2}\colon (r_1+r_2,\infty) \to \mathbb{R}_+$, $h \mapsto f(r_1,r_2,h)$ is positive, strictly decreasing and converges to 0 as h tends to ∞. We now consider each of the two cases given above.

(i) Suppose that $|z_2 - z_1| > R^3$. Using the inequalities

$$r_1 + r_2 \leq 2R, \qquad \arcsin\biggl(\frac{r_1+r_2}{|z_2-z_1|}\biggr) \leq \arcsin\biggl(\frac{2}{R^2}\biggr), \qquad r_1 \geq r_2,$$

we obtain, for large enough ρ,

$$f(r_1,r_2,|z_2-z_1|) < f(r_1,r_2,R^3) \leq 4R\arcsin\biggl(\frac{2}{R^2}\biggr) \leq \frac{c}{R} \leq \frac{c}{v_\rho}.$$

(ii) Suppose that $R \leq (1+\varepsilon)v_\rho$. Since $|z_2 - z_1| > r_1 + r_2$, we obtain

$$f(r_1,r_2,|z_2-z_1|) < f(r_1,r_2,r_1+r_2) = 2\pi r_2 + 2(r_1-r_2)\arccos\biggl(\frac{r_1-r_2}{r_1+r_2}\biggr) - 4\sqrt{r_1 r_2}.$$

Using the inequalities

$$r_1 \geq r_2 > v_\rho, \qquad \arccos\biggl(\frac{r_1-r_2}{r_1+r_2}\biggr) \leq \frac{\pi}{2}, \qquad r_1 \leq R \leq (1+\varepsilon)v_\rho,$$

we have

$$f(r_1,r_2,|z_2-z_1|) < 2\pi r_2 + (r_1 - v_\rho)\pi - 4v_\rho \leq 2\pi r_2 - (4-\varepsilon\pi)v_\rho. \qquad \square$$

Proof of Lemma 4.3. (i) Using the notation defined in Section 4.2.1, we note that

$$\bigcup_{1\leq i\leq K} \mathcal{A}(B(z_i,r_i)) = \bigcup_{k\leq K}\bigcup_{j\leq n_k} \mathcal{A}(B_k^{(j)}).$$

From Bonferroni inequalities, we obtain

$$\mu\biggl(\bigcup_{1\leq i\leq K} \mathcal{A}(B(z_i,r_i))\biggr) \geq \sum_{k=1}^{K}\sum_{j=1}^{n_k} \mu(\mathcal{A}(B_k^{(j)})) - \sum_{(k_1,j_1)\neq(k_2,j_2)} \mu(\mathcal{A}(B_{k_1}^{(j_1)}) \cap \mathcal{A}(B_{k_2}^{(j_2)})). \tag{4.4}$$

We begin by observing that for all $1 \leq k_1 \neq k_2 \leq K$ and $1 \leq j_1 \leq n_{k_1}$, $1 \leq j_2 \leq n_{k_2}$, we have

$$\mu(\mathcal{A}(B_{k_1}^{(j_1)}) \cap \mathcal{A}(B_{k_2}^{(j_2)})) \leq \sum_{1\leq \ell_1\leq k_1,\, 1\leq \ell_2\leq k_2} \mu(\mathcal{A}(B_{k_1}^{(j_1)}[\ell_1]) \cap \mathcal{A}(B_{k_2}^{(j_2)}[\ell_2])) \leq \frac{c}{v_\rho} \tag{4.5}$$

when ρ is sufficiently large, with the final inequality following directly from Lemma 4.2(i), taking $r_1 := r_{k_1}^{(j_1)}[\ell_1]$ and $r_2 := r_{k_2}^{(j_2)}[\ell_2]$. In addition

$$\mu(\mathcal{A}(B_k^{(j)})) \geq \mu(\mathcal{A}(B_k^{(j)}[1])) = 2\pi r_k^{(j)}[1]. \tag{4.6}$$

We then deduce Lemma 4.3(i) from (4.4)–(4.6).

(ii) We proceed along the same lines as in the proof of Lemma 4.3(i). The only difference concerns the lower bound for $\mu(\mathcal{A}(B_k^{(j)}))$. We shall consider two cases. For each of the n_1 clusters of size 1, we have $\mu(\mathcal{A}(B_1^{(j)})) = 2\pi r_1^{(j)}[1]$. Otherwise, we obtain

$$\begin{aligned}\mu(\mathcal{A}(B_k^{(j)})) &= \mu\bigg(\bigcup_{\ell=1}^{k} \mathcal{A}(B_k^{(j)}[\ell])\bigg)\\ &\geq \mu(\mathcal{A}(B_k^{(j)}[1]) \cup \mathcal{A}(B_k^{(j)}[2]))\\ &= 2\pi r_k^{(j)}[1] + 2\pi r_k^{(j)}[2] - \mu(\mathcal{A}(B_k^{(j)}[1]) \cap \mathcal{A}(B_k^{(j)}[2]))\\ &\geq 2\pi r_k^{(j)}[1] + (4 - \varepsilon\pi)v_\rho,\end{aligned}$$

which follows from Lemma 4.2(ii). We then deduce Lemma 4.3(ii) from the previous inequality, (4.4), and (4.5). □

5. Asymptotics for cells with large inradii

Notation. We begin this section by introducing the following notation. Let $t \geq 0$ be fixed.

- We shall denote the *threshold* and the mean number of cells having an inradius larger than the threshold respectively as

$$v_\rho := v_\rho(t) := \frac{1}{2\pi}(\log(\pi\rho) + t) \quad \text{and} \quad \tau := \tau(t) := \mathrm{e}^t. \tag{5.1}$$

- For any $K \geq 1$ and for any K-tuple of convex bodies $C_1, \dots, C_K$ such that each C_i has a unique inball, define the events

$$E_{C_{1:K}} := \Big\{\min_{1\leq i\leq K} R(C_i) \geq v_\rho,\, R(C_1) = \max_{1\leq i\leq K} C_i\Big\}, \tag{5.2}$$

$$E^\circ_{C_{1:K}} := \{\text{for all } 1 \leq i \neq j \leq K,\, B(C_i) \cap B(C_j) = \varnothing\}. \tag{5.3}$$

- For any $K \geq 1$, we take

$$I^{(K)}(\rho) := K\,\mathbb{E}\bigg[\sum_{\substack{C_{1:K}\in(\mathfrak{m}_{\mathrm{PHT}})^K_{\neq},\\ z(C_{1:K})\in W_\rho^K}} \mathbf{1}_{E_{C_{1:K}}}\bigg].$$

The proof for Theorem 1.1(ii) will then follow by applying Lemma 4.1 and showing that $I^{(K)}(\rho) \to \tau^k$ as $\rho \to \infty$ for every fixed $K \geq 1$. To begin, we observe that $I^{(1)}(\rho) \to \tau$ as $\rho \to \infty$ as a consequence of (2.6) and (5.1). The rest of this section is devoted to considering the case when $K \geq 2$. Given a K-tuple of cells $C_{1:K}$ in $\mathfrak{m}_{\mathrm{PHT}}$, we use $L(C_{1:K})$ to denote the number lines of $\hat{X}$ (without repetition) which intersect the inballs of the cells. It follows that $3 \leq L(C_{1:K}) \leq 3K$ since the inball of every cell in $\mathfrak{m}_{\mathrm{PHT}}$ intersects exactly three lines (almost surely.) We shall take

$$\{H_1, \dots, H_{L(C_{1:K})}\} := \{H_1(C_{1:K}), \dots, H_{L(C_{1:K})}(C_{1:K})\}$$

to represent the set of lines in $\hat{X}$ intersecting the inballs of the cells $C_{1:K}$. We remark that conditional on the event $L(C_{1:K}) = 3K$, none of the inballs of the cells share any lines in

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common. To apply the bounds we obtained in Section 4.2, we will split the cells up into clusters based on the proximity of their inballs using the procedure outlined in Section 4.2.1. In particular, we define

$$n_{1:K}(C_{1:K}) := n_{1:K}(z(C_{1:K}), R(C_1)).$$

We may now write $I^{(K)}(\rho)$ by summing over events conditioned on the number of clusters of each size and depending on whether or not the inballs of the cells *share* any lines of the process

$$I^{(K)}(\rho) = K \sum_{n_{1:K} \in \mathcal{N}_K} (I_{S^c}^{(n_{1:K})}(\rho) + I_S^{(n_{1:K})}(\rho)), \tag{5.4}$$

where the size of each cluster of size k is represented by a tuple contained in

$$\mathcal{N}_K := \left\{ n_{1:K} \in \mathbf{N}^K : \sum_{k=1}^{K} k n_k = K \right\},$$

and where for any $n_{1:K} \in \mathcal{N}_K$, we write

$$I_{S^c}^{(n_{1:K})}(\rho) := \mathbb{E}\Bigg[\sum_{\substack{C_{1:K} \in (\mathfrak{m}_{\mathrm{PHT}})^K_{\neq}, \\ z(C_{1:K}) \in \mathbf{W}_\rho^K}} \mathbf{1}_{E_{C_{1:K}}} \mathbf{1}_{\{n_{1:K}(C_{1:K}) = n_{1:K}\}} \mathbf{1}_{\{L(C_{1:K}) = 3K\}} \Bigg], \tag{5.5}$$

$$I_{S}^{(n_{1:K})}(\rho) := \mathbb{E}\Bigg[\sum_{\substack{C_{1:K} \in (\mathfrak{m}_{\mathrm{PHT}})^K_{\neq}, \\ z(C_{1:K}) \in \mathbf{W}_\rho^K}} \mathbf{1}_{E_{C_{1:K}}} \mathbf{1}_{\{n_{1:K}(C_{1:K}) = n_{1:K}\}} \mathbf{1}_{\{L(C_{1:K}) < 3K\}} \Bigg]. \tag{5.6}$$

The following proposition deals with the asymptotic behaviours of these functions.

Proposition 5.1. *Using the notation given in (5.5) and (5.6),*

(i) *we have* $I_{S^c}^{(K,0,\dots,0)}(\rho) \to \tau^K$ *as* $\rho \to \infty$*;*

(ii) *for all* $n_{1:K} \in \mathcal{N}_K \setminus \{(K, 0, \dots, 0)\}$*, we have* $I_{S^c}^{(n_{1:K})}(\rho) \to 0$ *as* $\rho \to \infty$*;*

(iii) *for all* $n_{1:K} \in \mathcal{N}_K$*, we have* $I_S^{(n_{1:K})}(\rho) \to 0$ *as* $\rho \to \infty$*.*

The convergences in Proposition 5.1 can be understood intuitively as follows. For Proposition 5.1(i), the inradii of the cells behave as though they are independent, since they are far apart and no line in the process touches more than one of the inballs in the K-tuple (even though two *cells* in the K-tuple may share a line.) For Proposition 5.1(ii), we are able to show that with high probability the inradii of neighbouring cells cannot simultaneously exceed the level v_ρ, due to Proposition 4.1(ii). Finally, to obtain the bound in Proposition 5.1(iii), we use the fact that the proportion of K-tuples of cells which share at least one line is negligible relative to those that do not.

5.1. The graph of configurations

For Proposition 5.1(iii), we will need to represent the dependence structure between the *cells* whose inballs share *lines*. To do this, we construct the following *configuration graph*.

For $K \geq 2$ and $L \in \{3, \dots, 3K\}$, let $V_{\mathrm{C}} := \{1, \dots, K\}$ and $V_{\mathrm{L}} := \{1, \dots, L\}$. We consider the bipartite graph $\boldsymbol{G}(V_{\mathrm{C}}, V_{\mathrm{L}}, E)$ with vertices $V := V_{\mathrm{C}} \sqcup V_{\mathrm{L}}$ and edges $E \subset V_{\mathrm{C}} \times V_{\mathrm{L}}$. Let

$$\Lambda_K := \bigcup_{L \leq 3K} \Lambda_{K,L},$$

where $\Lambda_{K,L}$ represents the collection of all graphs which are isomorphic up to relabelling of the vertices and satisfying

- degree$(v) = 3$ for all $v \in V_{\mathrm{C}}$,
- degree$(w) \geq 1$ for all $w \in V_{\mathrm{L}}$,
- neighbours$(v) \neq$ neighbours(v') for all $(v, v') \in (V_{\mathrm{C}})^2_{\neq}$.

Taking V_{C} to represent the cells and V_{L} to represent the lines in a line process, with edges representing the number of such bipartite graphs is finite since $|\Lambda_{K,L}| \leq 2^{KL}$ so that $|\Lambda_K| \leq 3K2^{(3K^2)}$.

5.2. Proofs

Proof of Proposition 5.1. (i) For any $1 \leq i \leq K$ and the 3-tuple of lines

$$H_i^{(1:3)} := (H_i^{(1)}, H_i^{(2)}, H_i^{(3)}),$$

we recall that $\triangle_i := \triangle_i(H_i^{(1)}, H_i^{(2)}, H_i^{(3)})$ denotes the unique triangle that can be formed by the intersection of the half-spaces induced by the lines $H_i^{(1:3)}$. For brevity, we write $B_i := B(\triangle_i)$ and $H_{1:K}^{(1:3)} := (H_1^{(1:3)}, \dots, H_K^{(1:3)})$. We shall often omit the arguments when they are obvious from the context. Since $\mathbf{1}_{E_{C_{1:K}}} = \mathbf{1}_{E_{B_{1:K}}}$ and since the lines of $\hat{\boldsymbol{X}}$ do not intersect the inballs in their interior, we have

$$\begin{aligned}
I_{S^{\mathrm{c}}}^{(K,0,\dots,0)}(\rho) &= \frac{K}{6^K}\,\mathbb{E}\bigg[\sum_{H_{1:K}^{(1:3)} \in \boldsymbol{X}_{\neq}^{3K}} \mathbf{1}_{\{\{\hat{\boldsymbol{X}} \setminus \cup_{i \leq K, j \leq 3} H_i^{(j)}\} \cap \{\cup_{i \leq K} B_i\} = \varnothing\}} \mathbf{1}_{\{z(B_{1:K}) \in \boldsymbol{W}_\rho^K\}} \\
&\qquad \times \mathbf{1}_{E_{B_{1:K}}} \mathbf{1}_{\{n_{1:K}(B_{1:K}) = (K,0,\dots,0)\}}\bigg] \\
&= \frac{K}{6^K} \int_{\mathcal{A}^{3K}} \mathrm{e}^{-\phi(\bigcup_{i \leq K} B_i)} \mathbf{1}_{\{z(B_{1:K}) \in \boldsymbol{W}_\rho^K\}} \\
&\qquad \times \mathbf{1}_{E_{B_{1:K}}} \mathbf{1}_{\{n_{1:K}(B_{1:K}) = (K,0,\dots,0)\}}\, \mu(\mathrm{d}H_{1:K}^{(1:3)}),
\end{aligned}$$

where the last equality comes from (2.2) and the Mecke–Slivnyak formula. Applying the Blaschke–Petkantschin formula, we obtain

$$\begin{aligned}
I_{S^{\mathrm{c}}}^{(K,0,\dots,0)}(\rho) = \frac{K}{24^K} \int_{(\boldsymbol{W}_\rho \times \mathbb{R}_+ \times \boldsymbol{S}^3)^K} & \mathrm{e}^{-\phi(\bigcup_{i \leq K} B(z_i, r_i))} \prod_{i \leq K} a(u_i^{(1:3)})\, \mathbf{1}_{E_{B_{1:K}}} \\
& \times \mathbf{1}_{\{n_{1:K}(B_{1:K}) = (K,0,\dots,0)\}}\, \mathrm{d}z_{1:K}\, \mathrm{d}r_{1:K}\, \sigma(\mathrm{d}u_{1:K}^{(1:3)}),
\end{aligned}$$

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where we recall that $a(u_i^{(1:3)})$ is the area of the triangle spanned by $u_i^{(1:3)} \in \boldsymbol{S}^3$. From (4.2) and (4.3), we have for any $1 \le i \le K$,

$$\exp\biggl(-2\pi \sum_{i=1}^K r_i\biggr) \mathbf{1}_{E_{B_{1:K}}} \le \exp\biggl(-\phi\biggl(\bigcup_{i\le K} B(z_i, r_i)\biggr)\biggr) \mathbf{1}_{E_{B_{1:K}}} \le \exp\biggl(-2\pi \sum_{i=1}^K r_i\biggr) \exp(c v_\rho^{-1}) \mathbf{1}_{E_{B_{1:K}}}.$$

According to (5.2), this implies that

$$\begin{aligned} I_{S^c}^{(K,0,\dots,0)}(\rho) &\sim \frac{K}{24^K} \int_{(\boldsymbol{W}_\rho \times \mathbb{R}_+ \times \boldsymbol{S}^3)^K} \prod_{i\le K} \mathrm{e}^{-2\pi r_i} a(u_i^{(1:3)}) \mathbf{1}_{\{r_i > v_\rho\}} \mathbf{1}_{\{r_1 = \max_{j\le K} r_j\}} \\ &\qquad \times \mathbf{1}_{\{|z_i - z_j| > r_1^3 \text{ for } j\ne i\}} \,\mathrm{d}z_{1:K}\, \mathrm{d}r_{1:K}\, \sigma(\mathrm{d}u_{1:K}^{(1:3)}) \quad \text{as } \rho\to\infty \\ &= \frac{K\tau^K}{(24\pi)^K} \int_{(\boldsymbol{W}_1 \times \mathbb{R}_+ \times \boldsymbol{S}^3)^K} \prod_{i\le K} \mathrm{e}^{-2\pi r_i'} a(u_i^{(1:3)}) \mathbf{1}_{\{r_1' = \max_{j\le K} r_j'\}} \\ &\qquad \times \mathbf{1}_{\{|z_i' - z_j'| > \rho^{-1/2} r_1'^3 \text{ for } j\ne i\}} \,\mathrm{d}z_{1:K}'\, \mathrm{d}r_{1:K}'\, \sigma(\mathrm{d}u_{1:K}^{(1:3)}), \end{aligned}$$

where the last equality comes from (5.1) and the change of variables $z_i' = \rho^{-1/2} z_i$ and $r_i' = r_i - v_\rho$. From the monotone convergence theorem, it follows that

$$\begin{aligned} I_{S^c}^{(K,0,\dots,0)}(\rho) &\sim \frac{K\tau^K}{(24\pi)^K} \int_{(\boldsymbol{W}_1 \times \mathbb{R}_+ \times \boldsymbol{S}^3)^K} \prod_{i\le K} \mathrm{e}^{-2\pi r_i} a(u_i^{(1:3)}) \\ &\qquad \times \mathbf{1}_{\{r_1 = \max_{j\le K} r_j\}} \,\mathrm{d}z_{1:K}\, \mathrm{d}r_{1:K}\, \sigma(\mathrm{d}u_{1:K}^{(1:3)}) \quad \text{as } \rho\to\infty \\ &= \frac{\tau^K}{(24\pi)^K} \biggl(\int_{(\boldsymbol{W}_1 \times \mathbb{R}_+ \times \boldsymbol{S}^3)^K} a(u_{1:3}) \mathrm{e}^{-2\pi r} \,\mathrm{d}z\, \mathrm{d}r\, \sigma(\mathrm{d}u_{1:3})\biggr)^K \\ &\to \tau^K \quad \text{as } \rho\to\infty, \end{aligned}$$

where the last line follows by integrating over z, r, and $u_{1:3}$, and by using the fact that $\lambda_2(\boldsymbol{W}_1) = 1$ and $\int_{\boldsymbol{S}^3} a(u_{1:3})\, \sigma(\mathrm{d}u_{1:3}) = 48\pi^2$.

(ii) Beginning in the same way as in the proof of Proposition 5.1(i), we have

$$I_{S^c}^{(n_{1:K})}(\rho) = \frac{K}{24^K} \int_{(\boldsymbol{W}_\rho \times \mathbb{R}_+ \times \boldsymbol{S}^3)^K} \exp\biggl(-\phi\biggl(\bigcup_{i\le K} B(z_i, r_i)\biggr)\biggr) \prod_{i\le K} a(u_i^{(1:3)}) \mathbf{1}_{E_{B_{1:K}}} \mathbf{1}_{E_{B_{1:K}}^\circ} \times \mathrm{d}z_{1:K}\, \mathrm{d}r_{1:K}\, \sigma(\mathrm{d}u_{1:K}^{(1:3)}),$$

where the event $E_{B_{1:K}}^\circ$ is defined in (5.3). Integrating over $u_{1:K}^{(1:3)}$, we obtain

$$\begin{aligned} I_{S^c}^{(n_{1:K})}(\rho) &= c \int_{(\boldsymbol{W}_\rho \times \mathbb{R}_+)^K} \exp\biggl(-\phi\biggl(\bigcup_{i\le K} B(z_i, r_i)\biggr)\biggr) \prod_{i\le K} \mathbf{1}_{E_{B_{1:K}}} \mathbf{1}_{E_{B_{1:K}}^\circ} \\ &\qquad \times \mathbf{1}_{\{n_{1:K}(z_{1:K}, r_1) = n_{1:K}\}} \,\mathrm{d}z_{1:K}\, \mathrm{d}r_{1:K} \\ &= I_{S^c, a_\varepsilon}^{(n_{1:K})}(\rho) + I_{S^c, b_\varepsilon}^{(n_{1:K})}(\rho), \end{aligned}$$

where, for any $\varepsilon > 0$, the terms $I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho)$ and $I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho)$ are defined as the term of the first line when we add the indicator that r_1 is larger than $(1+\varepsilon)v_\rho$ in the integral and the indicator for the complement, respectively. We provide below a suitable upper bound for these two terms. For $I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho)$, from Proposition 4.1(iia), we obtain

$$I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho) \le c \int_{(\mathbf{W}_\rho\times\mathbb{R}_+)^K} \exp\biggl(-\biggl(2\pi r_1 + \biggl(\sum_{k=1}^K n_k - 1\biggr)2\pi v_\rho - c v_\rho^{-1}\biggr)\biggr)$$
$$\times \mathbf{1}_{\{r_1>(1+\varepsilon)v_\rho\}}\,\mathbf{1}_{\{r_1=\max_{j\le K} r_j\}}$$
$$\times \mathbf{1}_{\{n_{1:K}(z_{1:K},r_1)=n_{1:K}\}}\,\mathrm{d}z_{1:K}\,\mathrm{d}r_{1:K}.$$

Integrating over $r_{2:K}$ and $z_{1:K}$, we obtain

$$I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho) \le c \int_{(1+\varepsilon)v_\rho}^{\infty} r_1^{K-1} \exp\biggl(-\biggl(2\pi r_1 + \biggl(\sum_{k=1}^K n_k - 1\biggr)2\pi v_\rho\biggr)\biggr)$$
$$\times \lambda_{dK}(\{z_{1:K} \in \mathbf{W}_\rho^K : n_{1:K}(z_{1:K}, r_1) = n_{1:K}\})\,\mathrm{d}r_1. \tag{5.7}$$

Furthermore, for each $n_{1:K} \in \mathcal{N}_K \setminus \{(K,0,\dots,0)\}$, we have

$$\lambda_{dK}(\{z_{1:K} \in \mathbf{W}_\rho^K : n_{1:K}(z_{1:K}, r_1) = n_{1:K}\}) \le c\rho^{\sum_{k=1}^K n_k} r_1^{6(K-\sum_{k=1}^K n_k)}, \tag{5.8}$$

since the number of connected components of $\bigcup_{i=1}^K B(z_i, r_1^3)$ equals $\sum_{k=1}^K n_k$. It follows from (5.7) and (5.8) that there exists a constant $c(K)$ such that

$$I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho) \le c(\rho \mathrm{e}^{-2\pi v_\rho})^{(\sum_{k=1}^K n_k)} \mathrm{e}^{2\pi v_\rho} \int_{(1+\varepsilon)v_\rho}^{\infty} r_1^{c(K)} \mathrm{e}^{-2\pi r_1}\,\mathrm{d}r_1 = O((\log\rho)^{c(K)}\rho^{-\varepsilon}),$$

according to (5.1). For $I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho)$, we proceed exactly as for $I_{S^c,a_\varepsilon}^{(n_{1:K})}(\rho)$, but this time we apply the bound given in Proposition 4.1(iib). We obtain

$$I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho) \le c(\rho\exp(-2\pi v_\rho))^{(\sum_{k=1}^K n_k)} \exp\biggl((2\pi v_\rho - \sum_{k=2}^K n_k(4-\varepsilon\pi)v_\rho\biggr)$$
$$\times \int_{v_\rho}^{(1+\varepsilon)v_\rho} r_1^{c(K)} \exp(-2\pi r_1)\,\mathrm{d}r_1$$
$$= O((\log\rho)^c \rho^{-(4-\varepsilon\pi)/2\pi})$$

since for all $n_{1:K} \in \mathcal{N}_K \setminus \{(K,0,\dots,0)\}$, there exists a $2 \le k \le K$ such that n_k is nonzero. Choosing $\varepsilon < 4/\pi$ ensures that $I_{S^c,b_\varepsilon}^{(n_{1:K})}(\rho) \to 0$ as $\rho \to \infty$.

(iii) Let $\boldsymbol{G} = \boldsymbol{G}(V_{\mathrm{C}}, V_{\mathrm{L}}, E) \in \Lambda_K$, with $|V_{\mathrm{L}}| = L$ and $|V_{\mathrm{C}}| = K$, be a bipartite graph as in Section 5.1. With $\boldsymbol{G}$, we can associate a (unique up to reordering of the lines) way to construct K triangles from L lines by taking V_{C} to denote the set of indices of the triangles, V_{L} to denote the set of indices of the lines, and the edges to represent intersections between them. Besides, let $H_1, \dots, H_L$ be an L-tuple of lines. For each $1 \le i \le K$, let $e_i = \{e_i(0), e_i(1), e_i(2)\}$ be the tuple of neighbours of the ith vertex in V_{C}. In particular,

$$B_i(\boldsymbol{G}) := B(\triangle_i(\boldsymbol{G})) \quad\text{and}\quad \triangle_i(\boldsymbol{G}) := \triangle(H_{e_i(0)}, H_{e_i(1)}, H_{e_i(2)})$$

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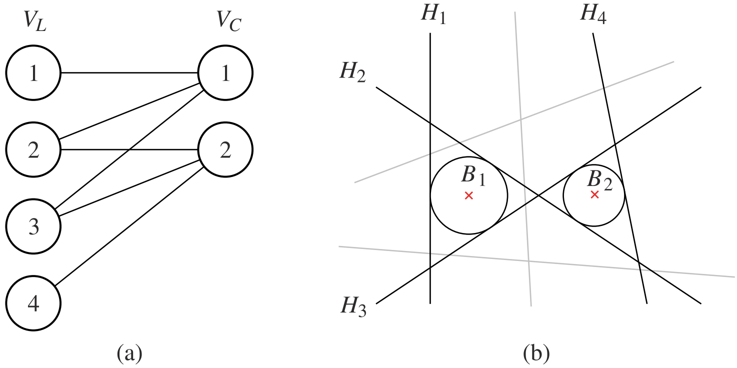

FIGURE 4: Example of configuration of inballs and lines (a), with associated configuration graph (b).

denote the inball and the triangle generated by the 3-tuple of lines with indices in e_i. An example of this configuration graph is given in Figure 4. According to (5.6), we have

$$I_S^{(n_{1:K})}(\rho) = \sum_{\boldsymbol{G}\in\Lambda_K} I_{S_{\boldsymbol{G}}}^{(n_{1:K})}(\rho),$$

where for all $n_{1:K} \in \mathcal{N}_K$ and $\boldsymbol{G} \in \Lambda_K$, we write

$$\begin{aligned} I_{S_{\boldsymbol{G}}}^{(n_{1:K})}(\rho) &= \mathbb{E}\Bigg[\sum_{H_{1:L}\in \boldsymbol{X}_{\neq}^{L}} \mathbf{1}_{\{\{\hat{\boldsymbol{X}}\setminus\cup_{i\le L}H_i\}\cap\{\cup_{i\le K}B_i(\boldsymbol{G})\}=\varnothing\}}\,\mathbf{1}_{\{z(B_{1:K}(\boldsymbol{G}))\in \boldsymbol{W}_\rho^K\}}\,\mathbf{1}_{E_{B_{1:K}(\boldsymbol{G})}} \\ &\qquad\qquad \times \mathbf{1}_{E^\circ_{B_{1:K}(\boldsymbol{G})}}\,\mathbf{1}_{\{n_{1:K}(B_{1:K}(\boldsymbol{G}))=n_{1:K}\}}\Bigg] \\ &= \int_{\mathcal{A}^{|V_L|}} \exp\bigg(-\phi\bigg(\bigcup_{i\le K} B_i(\boldsymbol{G})\bigg)\bigg)\,\mathbf{1}_{\{z(B_{1:K}(\boldsymbol{G}))\in \boldsymbol{W}_\rho^K\}}\,\mathbf{1}_{E_{B_{1:K}(\boldsymbol{G})}} \\ &\qquad \times \mathbf{1}_{E^\circ_{B_{1:K}(\boldsymbol{G})}}\,\mathbf{1}_{\{n_{1:K}(B_{1:K}(\boldsymbol{G}))=n_{1:K}\}}\,\mu(\mathrm{d}H_{1:L}). \end{aligned} \tag{5.9}$$

We now prove that $I_{S_{\boldsymbol{G}}}^{(n_{1:K})}(\rho) \to 0$ as $\rho \to \infty$. Suppose first that $n_{1:K} = (K, 0, \dots, 0)$. In this case, from (5.9), Proposition 4.1(iia), (5.2), and (5.3), we obtain

$$\begin{aligned} I_{S_{\boldsymbol{G}}}^{(K,0,\dots,0)}(\rho) &\le c \int_{\mathcal{A}^L} \mathrm{e}^{-2\pi(R(B_1(\boldsymbol{G}))+(K-1)v_\rho)}\,\mathbf{1}_{\{z(B_{1:K}(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\,\mathbf{1}_{\{R(B_1(\boldsymbol{G}))>v_\rho\}} \\ &\qquad \times \mathbf{1}_{\{R(B_1(\boldsymbol{G}))=\max_{i\le K} R(B_i(\boldsymbol{G}))\}}\,\mathbf{1}_{\{n_{1:K}(B_{1:K}(\boldsymbol{G}))=(K,0,\dots,0)\}}\,\mu(\mathrm{d}H_{1:L}) \\ &\le c\rho^{1/2}\int_{v_\rho}^{\infty} r^{c(K)}\mathrm{e}^{-2\pi r}\,\mathrm{d}r \\ &= O((\log\rho)^{c(K)}\rho^{-1/2}), \end{aligned}$$

where the second inequality is a consequence of (5.1) and Lemma A.2 applied to $f(r) := \mathrm{e}^{-2\pi r}$. Suppose now that $n_{1:K} \in \mathcal{N}_K \setminus \{(K, 0, \dots, 0)\}$. In the same spirit as in the proof of Proposition 5.1(ii), we shall write

$$I_{S_{\boldsymbol{G}}}^{(n_{1:K})}(\rho) = I_{S_{\boldsymbol{G}},a_\varepsilon}^{(n_{1:K})}(\rho) + I_{S_{\boldsymbol{G}},b_\varepsilon}^{(n_{1:K})}(\rho) \tag{5.10}$$

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by adding the indicator that $R(B_1(\boldsymbol{G}))$ is larger than $(1+\varepsilon)v_\rho$ and the opposite in (5.9). For $I_{S_{\boldsymbol{G}},a_\varepsilon}^{(n_{1:K})}(\rho)$, we similarly apply Proposition 4.1(iia) to obtain

$$\begin{aligned} I_{S_{\boldsymbol{G}},a_\varepsilon}^{(n_{1:K})}(\rho) &\le c\int_{\mathcal{A}^L}\exp\biggl(-2\pi\biggl(R(B_1(\boldsymbol{G}))+\biggl(\sum_{k=1}^K n_k-1\biggr)v_\rho\biggr)\biggr) \\ &\qquad\times \mathbf{1}_{\{z(B_{1:K}(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\mathbf{1}_{\{R(B_1(\boldsymbol{G}))>(1+\varepsilon)v_\rho\}} \\ &\qquad\times \mathbf{1}_{\{R(B_1(\boldsymbol{G}))=\max_{i\le K}R(B_i(\boldsymbol{G}))\}}\mathbf{1}_{\{n_{1:K}(B_{1:K}(\boldsymbol{G}))=n_{1:K}\}}\,\mu(\mathrm{d}H_{1:L}) \\ &\le c(\rho\mathrm{e}^{-2\pi v_\rho})^{\sum_{k=1}^K n_k}\rho\int_{(1+\varepsilon)v_\rho}^\infty r^{c(K)}\mathrm{e}^{-2\pi r}\,\mathrm{d}r \\ &= O((\log\rho)^{c(K)}\rho^{-\varepsilon}), \end{aligned} \tag{5.11}$$

where the second inequality follows by applying Lemma A.2. To prove that $I_{S_{\boldsymbol{G}},b_\varepsilon}^{(n_{1:K})}(\rho)$ converges to 0, we proceed exactly as before but this time applying Proposition 4.1(iib). As for $I_{S^{\mathrm{c}},b_\varepsilon}^{(n_{1:K})}(\rho)$, we show that

$$I_{S^{\mathrm{c}},b_\varepsilon}^{(n_{1:K})}(\rho) = O((\log\rho)^{c(K)}\rho^{-(4-\varepsilon\pi)/2\pi})$$

by taking $\varepsilon < 4/\pi$. Using this equation together with (5.10) and (5.11), it follows that $I_{S_{\boldsymbol{G}}}^{(n_{1:K})}(\rho)$ converges to 0 for any $n_{1:K}\in\mathcal{N}_K\setminus\{(K,0,\dots,0)\}$. □

Proof of Theorem 1.1(ii). According to Lemma 4.1, it is now enough to show that for all $K\ge 1$, we have $I^{(K)}(\rho)\to\tau^K$ as $\rho\to\infty$. This fact is a consequence of (5.4) and Proposition 5.1, thus completing the proof. □

Appendix A. Technical lemmas

The following technical lemmas are required for the proofs of Proposition 3.1 and Proposition 5.1(iii).

Lemma A.1. *Let $R, R' > 0$ and let $z'\in\mathbb{R}^d$.*

(i) *For all $H_1\in\mathcal{A}$, we have*

$$G(H_1) := \int_{\mathcal{A}^2}\mathbf{1}_{\{z(H_{1:3})\in B(z',R')\}}\mathbf{1}_{\{R(H_{1:3})<R\}}\,\mu(\mathrm{d}H_{2:3}) \le cRR'\,\mathbf{1}_{\{d(0,H_1)<R+R'\}}.$$

(ii) *For all $H_1, H_2\in\mathcal{A}$, we have*

$$G(H_1,H_2) := \int_{\mathcal{A}}\mathbf{1}_{\{z(H_{1:3})\in B(z',R')\}}\mathbf{1}_{\{R(H_{1:3})<R\}}\,\mu(\mathrm{d}H_3) \le c(R+R').$$

Lemma A.2. *Let $3\le L<3K$ be fixed. For any $\boldsymbol{G}=\boldsymbol{G}(V_{\mathrm{C}},V_{\mathrm{L}},E)\in\Lambda_K$, $n_{1:K}\in\mathcal{N}_{1:K}$, and for any measurable function $f:\mathbb{R}_+\to\mathbb{R}_+$, let*

$$\begin{aligned} F^{(n_{1:K})} := \int_{\mathcal{A}^L} & f(R(B_1(\boldsymbol{G})))\,\mathbf{1}_{\{z(B_{1:K}(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\mathbf{1}_{\{R(B_1(\boldsymbol{G}))>v'_\rho\}}\mathbf{1}_{\{R(B_1(\boldsymbol{G}))=\max_{i\le K}R(B_i(\boldsymbol{G}))\}} \\ &\times \mathbf{1}_{\{n_{1:K}(B_{1:K}(\boldsymbol{G}))=n_{1:K}\}}\,\mu(\mathrm{d}H_{1:L}), \end{aligned}$$

where $v'_\rho\to\infty$. Then for some constant $c(K)$, we have

$$F^{(n_{1:K})} \le \rho^{\min\{\sum_{k=1}^K n_k, K-1/2\}}\int_{v'_\rho}^\infty r^{c(K)}f(r)\,\mathrm{d}r.$$

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Proof of Lemma A.1. (i) The following proof reduces to giving the analogous version of the Blaschke–Petkanschin-type change of variables [23, Theorem 7.3.2]) in which one of the lines is held fixed. We proceed in the same spirit as in the proof of [23, Theorem 7.3.2]. Without loss of generality, we can assume that $z' = 0$ since μ is stationary. Let $H_1 \in \mathcal{A} = H(u_1, t_1)$ be fixed for some $u_1 \in \boldsymbol{S}$ and $t_1 \in \mathbb{R}$. We denote by $\mathcal{A}^2_{H_1} \subset \mathcal{A}^2$ the set of pairs of lines (H_2, H_3) such that H_1, H_2, and H_3 are in general position, and by $P_{H_1} \subset \boldsymbol{S}^2$ the set of pairs of unit vectors (u_2, u_3) such that $0 \in \mathbb{R}^2$ belongs to the interior of the convex hull of $\{u_1, u_2, u_3\}$. Then the mapping

$$\phi_{H_1} : \mathbb{R}^2 \times P_{H_1} \longrightarrow \mathcal{A}^2_{H_1}, \qquad (z, u_2, u_3) \longmapsto (H(u_2, t_2), H(u_3, t_3))$$

with $t_i := \langle z, u_i \rangle + r$ and $r := d(z, H_1)$ is bijective. We can easily prove that its Jacobian $J_{\Phi_{H_1}}(z, u_2, u_3)$ is bounded. Using the fact that $d(0, H_1) \le |z(H_{1:3})| + R(H_{1:3}) < R + R'$ provided that $z(H_{1:3}) \in B(0, R')$ and $R(H_{1:3}) < R$, it follows that

$$\begin{aligned} G(H_1) &\le \int_{\mathbb{R}^2 \times P_{H_1}} |J\phi_{H_1}(z, u_2, u_3)|\, \mathbf{1}_{\{z \in B(0,R')\}}\, \mathbf{1}_{\{d(z,H_1)<R\}}\, \mathbf{1}_{\{d(0,H_1)<R+R'\}}\, \sigma(\mathrm{d}u_{2:3})\, \mathrm{d}z \\ &\le c\lambda_2(B(0, R') \cap (H_1 \oplus B(0, R)))\, \mathbf{1}_{\{d(0,H_1)<R+R'\}} \\ &\le cRR'\, \mathbf{1}_{\{d(0,H_1)<R+R'\}}, \end{aligned}$$

where $A \oplus B$ denotes the Minkowski sum between two Borel sets $A, B \in \mathcal{B}(\mathbb{R}^2)$.

(ii) Let H_1 and H_2 be fixed. Let H_3 be such that $z(H_{1:3}) \in B(z', R')$ and $R(H_{1:3}) < R$. This implies that

$$d(z', H_3) \le |z' - z(H_{1:3})| + d(z(H_{1:3}), H_3) \le R + R'.$$

Integrating over H_3, we obtain

$$G(H_1, H_2) \le \int_{\mathcal{A}} \mathbf{1}_{\{d(z',H_3) \le R+R'\}}\, \mu(\mathrm{d}H_3) \le c(R + R'). \qquad \square$$

Proof of Lemma A.2. Our proof will follow by writing the set of lines $\{1, \dots, |V_{\mathrm{L}}|\}$ as a disjoint union. We take

$$\{1, \dots, |V_{\mathrm{L}}|\} = \bigsqcup_{i=1}^{K} e_i^\star,$$

where $e_i^\star := \{e_i(0), e_i(1), e_i(2)\} \setminus \bigcup_{j<i} \{e_j(0), e_j(1), e_j(2)\}$. In this way, $\{e_i^\star\}_{i \le K}$ may be understood as associating lines of the process with the inballs of the K cells under consideration so that no line is associated with more than one inball. In particular, each inball has between zero and three lines associated with it, $0 \le |e_i^\star| \le 3$ and $|e_1^\star| = 3$ by definition. We now consider two cases depending on the configuration of the clusters, $n_{1:K} \in \mathcal{N}_K$.

Independent clusters. To begin with, we suppose that $n_{1:K} = (K, 0, \dots, 0)$. For convenience, we shall write

$$\mu(\mathrm{d}H_{e_i^\star}) := \prod_{j \in e_i^\star} \mu(\mathrm{d}H_j)$$

for some arbitrary ordering of the elements, and defining the empty product to be 1. It follows from Fubini's theorem that

$$\begin{aligned}F^{(K,0,\dots,0)} = &\int_{\mathcal{A}^3} f(R(B_1(\boldsymbol{G})))\,\mathbf{1}_{\{z(B_1(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\,\mathbf{1}_{\{R(\triangle_1(\boldsymbol{G}))>v'_\rho\}}\\ &\times \int_{\mathcal{A}^{|e_2^\star|}} \mathbf{1}_{\{z(B_2(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\,\mathbf{1}_{\{R(\triangle_2(\boldsymbol{G}))\le R(B_1(\boldsymbol{G}))\}}\\ &\cdots\\ &\qquad\times\bigg[\int_{\mathcal{A}^{|e_K^\star|}} \mathbf{1}_{\{z(B_K(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\,\mathbf{1}_{\{R(\triangle_K(\boldsymbol{G}))\le R(B_1(\boldsymbol{G}))\}}\,\mu(\mathrm{d}H_{e_K^\star})\bigg]\\ &\qquad\times \mu(\mathrm{d}H_{e_{K-1}^\star})\cdots\mu(\mathrm{d}H_{e_1^\star}).\end{aligned}\tag{A.1}$$

We now consider three possible cases for the innermost integral in (A.1).

(i) If $|e_K^\star| = 3$, the integral equals $cR(B_1(\boldsymbol{G}))\rho$ after a Blaschke–Petkanschin change of variables.

(ii) If $|e_K^\star| = 1, 2$, the integral is bounded by $c\rho^{1/2}R(B_1(\boldsymbol{G}))$ thanks to Lemma A.1 applied with $R := R(B_1(\boldsymbol{G}))$, $R' := \pi^{-1/2}\rho^{1/2}$.

(iii) If $|e_K^\star| = 0$, the integral decays and we may bound the indicators by 1. To simplify our notation we just assume the integral is bounded by $c\rho^{1/2}R(B_1(\boldsymbol{G}))$.

To distinguish these cases, we define $x_i := \mathbf{1}_{\{|e_i^\star|<3\}}$ for each $2 \le i \le K$, giving

$$\begin{aligned}F^{(K,0,\dots,0)} \le c\rho^{1-x_K/2} &\int_{\mathcal{A}^3} R(B_1(\boldsymbol{G}))^{c(K)} f(R(B_1(\boldsymbol{G})))\,\mathbf{1}_{\{R(\triangle_1(\boldsymbol{G}))>v'_\rho\}}\,\mathbf{1}_{\{z(B_1(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\\ &\times \int_{\mathcal{A}^{|e_2^\star|}} \mathbf{1}_{\{R(\triangle_2(\boldsymbol{G}))\le R(B_1(\boldsymbol{G}))\}}\,\mathbf{1}_{\{z(B_2(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\\ &\cdots\\ &\times\bigg[\int_{\mathcal{A}^{|e_{K-1}^\star|}} \mathbf{1}_{\{R(\triangle_K(\boldsymbol{G}))\le R(B_1(\boldsymbol{G}))\}}\,\mathbf{1}_{\{z(B_{K-1}(\boldsymbol{G}))\in \boldsymbol{W}_\rho\}}\,\mu(\mathrm{d}H_{e_{K-1}^\star})\bigg]\\ &\times \mu(\mathrm{d}H_{e_{K-2}^\star})\cdots\mu(\mathrm{d}H_{e_1^\star}).\end{aligned}$$

Recursively applying the same bound, we obtain

$$F^{(K,0,\dots,0)} \le c\rho^{\sum_{i=2}^K(1-x_i/2)} \int_{\mathcal{A}^3} R(H_{1:3})^{c(K)} f(R(H_{1:3}))\,\mathbf{1}_{\{R(H_{1:3})>v'_\rho\}}\,\mathbf{1}_{\{z(H_{1:3})\in \boldsymbol{W}_\rho\}}\,\mu(\mathrm{d}H_{1:3}).$$

From the Blaschke–Petkanschin formula, it follows that

$$F^{(K,0,\dots,0)} \le c\rho^{(K-(1/2)\sum_{i=2}^K x_i)} \int_{v'_\rho}^\infty r^{c(K)} f(r)\,\mathrm{d}r.$$

Since, by assumption $|V_{\mathrm{L}}| < 3K$, it follows that $x_i = 1$ for some $i > 1$. This implies that

$$F^{(K,0,\dots,0)} \le c\rho^{K-1/2} \int_{v'_\rho}^\infty r^{c(K)} f(r)\,\mathrm{d}r$$

as required.

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Dependent clusters. We now focus on the case in which $n_{1:K} \in \mathcal{N}_K \setminus \{(K, 0, \dots, 0)\}$. We proceed in the same spirit as before. For any $1 \le i \ne j \le K$, we write $B_i(\boldsymbol{G}) \nleftrightarrow B_j(\boldsymbol{G})$ to specify that the balls $B(z(B_i(\boldsymbol{G})), R(B_1(\boldsymbol{G}))^3)$ and $B(z(B_j(\boldsymbol{G})), R(B_1(\boldsymbol{G}))^3)$ are not in the same connected component of $\bigcup_{l=1}^{K} B(z(B_l(\boldsymbol{G})), R(B_1(\boldsymbol{G}))^3)$. Then we choose a unique 'delegate' convex for each cluster using the following indicator:

$$\alpha_i(B_{1:K}(\boldsymbol{G})) := \mathbf{1}_{\{\text{for all } i<j, B_j(\boldsymbol{G}) \nleftrightarrow B_i(\boldsymbol{G})\}}.$$

It follows that $\sum_{k=1}^{K} n_k = \sum_{i=1}^{K} \alpha_i(B_{1:K}(\boldsymbol{G}))$ and $\alpha_1(B_{1:K}(\boldsymbol{G})) = 1$. The set of all possible ways to select the delegates is given by

$$A_{n_{1:K}} := \bigg\{\alpha_{1:K} \in \{0,1\}^K : \sum_{i=1}^{K} \alpha_i = \sum_{k=1}^{K} n_k\bigg\}.$$

Then we have

$$\begin{aligned} F^{(n_{1:K})} = \sum_{\alpha_{1:K} \in A_{n_{1:K}}} & \int_{\mathcal{A}^3} f(R(B_1(\boldsymbol{G})))\, \mathbf{1}_{\{z(B_1(\boldsymbol{G})) \in \boldsymbol{W}_\rho\}} \mathbf{1}_{\{R(\triangle_1(\boldsymbol{G})) > v'_\rho\}} \\ & \times \int_{\mathcal{A}^{|e_2^\star|}} \mathbf{1}_{\{z(B_2(\boldsymbol{G})) \in \boldsymbol{W}_\rho\}} \mathbf{1}_{\{R(\triangle_2(\boldsymbol{G})) \le R(B_1(\boldsymbol{G}))\}} \mathbf{1}_{\{\alpha_2(B_{1:K}(\boldsymbol{G})) = \alpha_2\}} \\ & \qquad \cdots \\ & \qquad \times \bigg[\int_{\mathcal{A}^{|e_K^\star|}} \mathbf{1}_{\{z(B_K(\boldsymbol{G})) \in \boldsymbol{W}_\rho\}} \mathbf{1}_{\{R(\triangle_K(\boldsymbol{G})) \le R(B_1(\boldsymbol{G}))\}} \mathbf{1}_{\{\alpha_K(B_{1:K}(\boldsymbol{G})) = \alpha_K\}} \mu(\mathrm{d}H_{e_K^\star})\bigg] \\ & \qquad \times \mu(\mathrm{d}H_{e_{K-1}^\star}) \cdots \mu(\mathrm{d}H_{e_1^\star}). \end{aligned}$$

For this part, we similarly split the inner-most integral into multiple cases and recursively bound.

(i) When $\alpha_K = 1$, the integral equals $cR(B_1(\boldsymbol{G}))\rho$ if $e_K^\star = 3$ thanks to the Blaschke–Petkanschin formula, and is bounded by $c\rho^{1/2} R(B_1(\boldsymbol{G}))$ otherwise thanks to Lemma A.1. In particular, we bound the integral by $cR(B_1(\boldsymbol{G}))^{c(K)} \rho^{\alpha_K}$.

(ii) When $\alpha_K = 0$, the integral equals $cR(B_1(\boldsymbol{G}))^7$ if $e_K^\star = 3$ and is otherwise bounded by $cR(B_1(\boldsymbol{G}))^{5/2}$ for similar arguments. In this case, we can also bound the integral by

$$cR(B_1(\boldsymbol{G}))^{c(K)} \rho^{\alpha_K}.$$

Proceeding in the same way and recursively for all $2 \le i \le K$, we obtain

$$\begin{aligned} F^{(n_{1:K})} \le c \sum_{\alpha_{1:K} \in A_{n_{1:K}}} \rho^{\sum_{i=2}^{K} \alpha_i} \int_{\mathcal{A}^3} & R(B_1(\boldsymbol{G}))^{c(K)} f(R(H_{1:3})) \\ & \times \mathbf{1}_{\{z(B_1(\boldsymbol{G})) \in \boldsymbol{W}_\rho\}} \mathbf{1}_{\{R(B_1(\boldsymbol{G})) > v'_\rho\}} \mu(\mathrm{d}H_{1:3}). \end{aligned}$$

From the Blaschke–Petkanschin formula, we obtain

$$F^{(n_{1:K})} \le c \sum_{\alpha_{1:K} \in A_{n_{1:K}}} \rho^{\sum_{i=2}^{K} \alpha_i} \rho \int_{v'_\rho}^{\infty} r^{c(K)} f(r)\, \mathrm{d}r \le c\rho^{\sum_{k=1}^{K} n_k} \int_{v'_\rho}^{\infty} r^{c(K)} f(r)\, \mathrm{d}r,$$

since

$$\sum_{i=2}^{K} \alpha_i + 1 = \sum_{i=1}^{K} \alpha_i = \sum_{k=1}^{K} n_k. \qquad \square$$

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