Nagoya Math. J., **242** (2021), 1–34
DOI 10.1017/nmj.2019.18

HERZ–MORREY SPACES ON THE UNIT BALL WITH VARIABLE EXPONENT APPROACHING 1 AND DOUBLE PHASE FUNCTIONALS

YOSHIHIRO MIZUTA, TAKAO OHNO
AND TETSU SHIMOMURA

Abstract. Our aim in this paper is to deal with integrability of maximal functions for Herz–Morrey spaces on the unit ball with variable exponent $p_1(\cdot)$ approaching 1 and for double phase functionals $\Phi_d(x,t) = t^{p_1(x)} + a(x)t^{p_2}$, where $a(x)^{1/p_2}$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$ and $1/p_2 = 1 - \theta/N > 0$. We also establish Sobolev type inequality for Riesz potentials on the unit ball.

§1. Introduction

Let $\mathbf{R}^N$ denote the N-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r. For a locally integrable function f on an open set $G \subset \mathbf{R}^N$, we consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{G\cap B(x,r)} |f(y)|\, dy.$$

Following Kováčik and Rákosník [19], we consider a positive continuous function $p_1(\cdot)$ on $\mathbf{B} = B(0,1)$ and a measurable function f satisfying

$$\int_{\mathbf{B}} |f(y)|^{p_1(y)} dy < \infty.$$

In this paper we are concerned with $p_1(\cdot)$ such that

$$p_1(x) = 1 + \frac{a}{\log(e + 1/|x|)} + \frac{b \log(\log(e + 1/|x|))}{\log(e + 1/|x|)} \tag{1.1}$$

for $x \in \mathbf{B}$, where $a \geqslant 0$ and $b \geqslant 0$.

Cruz-Uribe *et al.* [8] proved the maximal operator M is not bounded on $L^{p(\cdot)}(G)$ if $\inf_G p(x) = 1$, where G is a bounded set. Hästö [17] proved that

Received September 25, 2018. Revised May 20, 2019. Accepted May 20, 2019.
2010 Mathematics subject classification. 46E30, 42B25.

the maximal operator M is bounded from $L^{p(\cdot)}(G)$ with variable exponent approaching 1 to $L^1(G)$ when G satisfies a certain regular condition. As an extension of Hästö [17], Futamura and Mizuta [13] proved the following:

THEOREM A. *Let $p_1(\cdot)$ be as in (1.1). If $b \geqslant 1/N$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbf{B}} Mf(x)\,dx < \infty$$

for all $f \in L^{p_1(\cdot)}(\mathbf{B})$.

We also refer to [23, 24] for integrability of maximal functions for generalized Lebesgue spaces with variable exponent approaching 1.

For $0 < \alpha < N$, we define the Riesz potential of order α of a locally integrable function f on an open set $G \subset \mathbf{R}^N$ by

$$I_\alpha f(x) = \int_G |x-y|^{\alpha-N} f(y)\,dy.$$

Our first aim in this paper is to deal with integrability of $I_\alpha f$ for generalized Lebesgue spaces $L^{p_1(\cdot)}(\mathbf{B})$ on the unit ball with variable exponent approaching 1 (Theorem 3.2). To do so, we use the boundedness of the maximal operator (Lemma 2.4). The sharpness of Theorem 3.2 will be discussed in Remarks 3.3 and 3.4.

Regarding regularity theory of differential equations, Baroni *et al.* [3, 9] studied a double phase functional

$$\Phi(x,t) = t^p + a(x)t^q, \quad x \in \mathbf{R}^N, t \geqslant 0,$$

where $1 < p < q$, $a(\cdot)$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0, 1]$ (see also [11, 12]). In [9], Colombo and Mingione showed the boundedness of the maximal operator on $L^\Phi(\Omega)$ when $\Phi(x,t) = t^p + a(x)t^q$, $q > p > 1$, $\Omega \subset \mathbf{R}^N$ is bounded, $a \in C^\theta(\bar{\Omega})$ is nonnegative and $q < (1 + \theta/N)p$. Further, in [18], Hästö showed the boundedness of the maximal operator on $L^\Phi(\Omega)$ in case $q \leqslant (1 + \theta/N)p$.

As an application of integrability of $I_\alpha f$ (Theorem 3.2), we shall study integrability of maximal functions for the double phase functional given by

$$\Phi_d(x,t) = t^{p_1(x)} + a(x)t^{p_2}$$

for $x \in \mathbf{B}$ and $t \geqslant 0$, where $a(x)^{1/p_2}$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0, 1]$ and $1/p_2 = 1 - \theta/N > 0$ (Theorem 3.7). In fact, we shall give $\|Mf\|_{L^{\Phi_{1,d}}(\mathbf{B})} \leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})}$. Our result extends Theorem A and [18, Theorem 4.7]. See also [7, 16]. For the sharpness of Theorem 3.7, see Remark 3.8. We also discuss integrability of $I_\alpha f$ for the double phase functional Φ_d (Theorem 3.11). In fact, we shall show $\|I_\alpha f\|_{L^{\Psi_{\alpha,d}}(\mathbf{B})} \leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})}$. See Section 3 for the definitions of $\Phi_{1,d}$ and $\Psi_{\alpha,d}$.

Let $A(r) = \mathbf{B} \cap [B(0,2r) \setminus B(0,r)]$ and let $\omega(r) : (0,\infty) \to (0,\infty)$ be almost monotone on $(0,\infty)$ satisfying the doubling condition. For $0 < q \leqslant \infty$, we define Herz–Morrey spaces $\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})$ of all measurable functions f on the unit ball $\mathbf{B}$ such that

$$\|f\|_{\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})} = \left(\int_0^1 \left(\omega(r)\|f\|_{L^{p_1(\cdot)}(A(r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{\mathcal{H}^{p_1(\cdot),\infty,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(r)\|f\|_{L^{p_1(\cdot)}(A(r))} < \infty$$

when $q = \infty$. In [21, 22], the boundedness of the maximal and Riesz potential operators were studied for Herz–Morrey spaces with variable exponents in a way different from Almeida and Drihem [2]. See also Samko [26]. There are several Morrey type spaces related to our nonhomogeneous central Morrey type spaces; for example, Morrey spaces by Adams and Xiao [1], local Morrey type spaces by Burenkov *et al.* [4–6, 14, 15].

Next we deal with integrability of maximal functions for Herz–Morrey spaces $\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})$ on the unit ball with variable exponent approaching 1 (Theorem 4.4), as an extension of Theorem A and [18, Theorem 4.7]. The sharpness of Theorem 4.4 will be discussed in Remark 4.8. We also establish norm inequalities for the Riesz potential operator $f \to I_\alpha f$ from $\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})$ to the Herz–Morrey–Orlicz space $\mathcal{H}^{\Psi_\alpha,q,\omega}(\mathbf{B})$ (Theorem 4.14). See Section 3 for the definition of Ψ_α.

Our final goal is to study norm inequalities for the maximal operator in the frame of double phase functionals, as well as the Riesz potential operators (see Theorems 4.12 and 4.15).

Throughout this paper, let C denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \leqslant g \leqslant Ch$ for some constant $C > 0$.

§2. Integrability of maximal functions

For a positive continuous nonincreasing function k on $(0,\infty)$, assume that there exists a constant $\varepsilon_0 > 0$ such that:

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

(k1) $(\log(e+1/r))^{-\varepsilon_0}k(r)$ is nondecreasing on $(0,1)$;
(k2) $k(1) \geqslant e^{\varepsilon_0}$.

By (k1), we see that

$$k(r) \leqslant k(r^2) \leqslant Ck(r) \quad \text{whenever } 0<r<1, \tag{2.1}$$

which implies the doubling condition on k, that is, there exists a constant $C \geqslant 1$ such that $k(r) \leqslant Ck(2r)$ for all $0<r<1$. Further, (k1) and (k2) imply that $\log k(r)/(\log(e+1/r))$ is nondecreasing on $(0,1)$ [23, Lemma 2.1].

Our typical example of k is

$$k(r) = a(\log(e+1/r))^b$$

for $0<r<1$, where $a \geqslant e^b(\log(e+1))^{-b}$ and $b>0$.

LEMMA 2.1. (Cf. [25, Lemmas 2.1 and 2.2])

(1) *For $a>0$, there exists a constant $C \geqslant 1$ such that*

$$C^{-1}k(r) \leqslant k(r^a) \leqslant Ck(r)$$

for all $0<r<1$.

(2) *For $b>0$, there exists a constant $C \geqslant 1$ such that*

$$r_1^b k(r_1) \leqslant C r_2^b k(r_2)$$

for all $0<r_1<r_2<1$.

We consider a positive convex function Φ on $(0,\infty)$ satisfying:

(Φ0) $\Phi(0) = \lim_{r\to 0}\Phi(r) = 0$;
(Φ1) Φ is doubling on $(0,\infty)$; namely there exists a constant $C \geqslant 1$ such that

$$\Phi(2r) \leqslant C\Phi(r) \quad \text{for all } r>0.$$

For an open set $G \subset \mathbf{R}^N$ and $f \in L^1_{\mathrm{loc}}(G)$, we define the norm

$$\|f\|_{L^\Phi(G)} = \inf\left\{\lambda>0 : \int_G \Phi\left(\frac{|f(y)|}{\lambda}\right)\,dy \leqslant 1\right\}.$$

For a locally integrable function f on $\mathbf{B}$, we consider the maximal function

$$Mf(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{\mathbf{B}\cap B(x,r)}|f(y)|\,dy.$$

The following lemma is an extension of Stein [27].

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

LEMMA 2.2. (Cf. [23, Theorem 3.1]) *Let $\Phi \in C^1((0,\infty))$ be a nondecreasing positive function on $(0,\infty)$ satisfying* ($\Phi 0$), ($\Phi 1$) *and*

($\Phi; k$) *there exists a constant $C > 0$ such that*

$$\int_1^t \frac{\Phi'(s)}{s}\, ds \leqslant Ck(t^{-1})$$

for all $t > 1$.

If f is a locally integrable function on $\mathbf{B}$ such that

$$\int_{\mathbf{B}} |f(x)| k(|f(x)|^{-1})\, dx \leqslant 1,$$

then there exists a constant $C > 0$ such that

$$\int_{\mathbf{B}} \Phi(Mf(x))\, dx \leqslant C.$$

Consider a function $p_1(\cdot)$ such that $p_1(0) = 1$ and

$$p_1(x) = 1 + \frac{\log k(|x|)}{\log(e + 1/|x|)}$$

for $x \in \mathbf{B} \setminus \{0\}$. Here note that $\lim_{|x|\to 0} p_1(x) = p_1(0)$. For an open set $G \subset \mathbf{R}^N$, we define the $L^{p_1(\cdot)}$-norm of a function $f \in L^1_{\mathrm{loc}}(G)$ by

$$\|f\|_{L^{p_1(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \left(\frac{|f(y)|}{\lambda}\right)^{p_1(y)} dy \leqslant 1\right\};$$

see [10].

In view of [23, Lemma 2.4], we know the following result.

LEMMA 2.3. *Let f be a measurable function on $\mathbf{B}$ with $\|f\|_{L^{p_1(\cdot)}(\mathbf{B})} \leqslant 1$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbf{B}} |f(x)| k(|f(x)|^{-1})^N\, dx \leqslant C.$$

By Lemmas 2.2 and 2.3, we have the following result.

LEMMA 2.4. *Let $\Phi \in C^1((0,\infty))$ be a positive convex function on $(0,\infty)$ satisfying* ($\Phi 0$), ($\Phi 1$) *and* ($\Phi; k^N$). *Then there exists a constant $C > 0$ such that*

$$\|Mf\|_{L^\Phi(\mathbf{B})} \leqslant C\|f\|_{L^{p_1(\cdot)}(\mathbf{B})}$$

for all $f \in L^{p_1(\cdot)}(\mathbf{B})$.

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

§3. Integrability of Riesz potentials

For $0 < \alpha < N$, we define the Riesz potential of order α of a locally integrable function f on $\mathbf{B}$ by

$$I_\alpha f(x) = \int_{\mathbf{B}} |x-y|^{\alpha-N} f(y)\, dy.$$

LEMMA 3.1. *There is a constant $C > 0$ such that*

$$\int_{B(x,r)} |f(y)|\, dy \leqslant C k(r)^{-N} \|f\|_{L^{p_1(\cdot)}(\mathbf{B})}$$

for all $x \in \mathbf{B}$, $0 < r < 1$ and $f \in L^{p_1(\cdot)}(\mathbf{B})$.

Proof. Let $x \in \mathbf{B}$ and $0 < r < 1$. Let f be a measurable function on $\mathbf{B}$ satisfying $\|f\|_{L^{p_1(\cdot)}(\mathbf{B})} \leqslant 1$. We find by Lemmas 2.1 and 2.3

$$\begin{aligned}\int_{B(x,r)} |f(y)|\, dy &\leqslant C r^N (rk(r))^{-N} + \int_{B(x,r)} |f(y)| \left(\frac{k(|f(y)|^{-1})}{k\left((rk(r))^N\right)} \right)^N dy \\ &\leqslant C \left\{ k(r)^{-N} + k(r)^{-N} \int_{\mathbf{B}} |f(y)| k(|f(y)|^{-1})^N\, dy \right\} \\ &\leqslant C k(r)^{-N},\end{aligned}$$

which proves the result. □

In what follows, let us assume that Φ is a positive convex function in $C^1((0,\infty))$ satisfying $(\Phi0)$, $(\Phi1)$ and $(\Phi; k^N)$ when it is not specially mentioned. Set

$$\Psi_\alpha(t) = \Phi\left(\left(tk(t^{-1})^\alpha\right)^{N/(N-\alpha)} \right).$$

THEOREM 3.2. *There exists a constant $C > 0$ such that*

$$\int_{\mathbf{B}} \Psi_\alpha(|I_\alpha f(x)|)\, dx \leqslant C$$

for all measurable functions f on $\mathbf{B}$ such that $\|f\|_{L^{p_1(\cdot)}(\mathbf{B})} \leqslant 1$.

Proof. Let f be a measurable function on $\mathbf{B}$ such that $\|f\|_{L^{p_1(\cdot)}(\mathbf{B})} \leqslant 1$. For $x \in \mathbf{B}$ and $0 < r \leqslant 1$, write

$$\begin{aligned}I_\alpha f(x) &= \int_{B(x,r)} |x-y|^{\alpha-N} f(y)\, dy + \int_{\mathbf{B}\setminus B(x,r)} |x-y|^{\alpha-N} f(y)\, dy \\ &= I_1(x) + I_2(x).\end{aligned}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

Then note that

$$|I_1(x)| \leqslant C r^{\alpha} M f(x).$$

We have by Lemmas 3.1 and 2.1

$$\begin{aligned}|I_2(x)| &\leqslant C \int_r^2 \left(\int_{B(x,t)} |f(y)|\, dy \right) t^{\alpha-N} \frac{dt}{t} \\ &\leqslant C \int_r^2 t^{\alpha-N} k(t)^{-N} \frac{dt}{t} \\ &\leqslant C r^{\alpha-N} k(r)^{-N}.\end{aligned}$$

Now we establish

$$|I_\alpha f(x)| \leqslant C \left\{ r^{\alpha} M f(x) + r^{\alpha-N} k(r)^{-N} \right\}.$$

If $Mf(x) \leqslant 1$, then, taking $r = 1$, we have

$$|I_\alpha f(x)| \leqslant C.$$

Next consider the case $Mf(x) > 1$. Since $\sup_{t>1} t^{-1/N} k(t^{-1})^{-1} \leqslant c_1$ for some constant $c_1 > 0$, taking $r = c_1^{-1} Mf(x)^{-1/N} k(Mf(x)^{-1})^{-1} \leqslant 1$, we find

$$|I_\alpha f(x)| \leqslant C Mf(x)^{1-\alpha/N} k(Mf(x)^{-1})^{-\alpha}.$$

Therefore we obtain

$$\left(|I_\alpha f(x)| k(|I_\alpha f(x)|^{-1})^{\alpha} \right)^{N/(N-\alpha)} \leqslant C \left\{ Mf(x) + 1 \right\}.$$

In view of $(\Phi 1)$ and Lemma 2.4, we have

$$\int_{\mathbf{B}} \Psi_\alpha(|I_\alpha f(x)|)\, dx \leqslant C \left\{ \int_{\mathbf{B}} \Phi(Mf(x))\, dx + 1 \right\} \leqslant C,$$

which proves the result. □

REMARK 3.3. For $\beta < -1$, consider the function

$$f(y) = |y|^{-N} (\log(e + 1/|y|))^{\beta} \chi_{\mathbf{B}}(y),$$

where χ_E the characteristic function of a measurable set $E \subset \mathbf{R}^N$. Then we have the following:

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

(1) there exists a constant $C > 0$ such that

$$\int_{\mathbf{B}} f(y)\, dy \leqslant C;$$

(2) there exists a constant $C > 0$ such that

$$Mf(x) \geqslant C|x|^{-N}(\log(e + 1/|x|))^{\beta+1}$$

for all $x \in \mathbf{B}$;

(3) there exists a constant $C > 0$ such that

$$I_\alpha f(x) \geqslant C|x|^{\alpha-N}(\log(e + 1/|x|))^{\beta+1}$$

for all $x \in \mathbf{B}$.

Let $\gamma > -1$. If $-\gamma - 2 < \beta < -1$, then (2) implies

$$Mf(x)(\log(e + Mf(x)))^{\gamma} \geqslant C|x|^{-N}(\log(e + 1/|x|))^{\beta+\gamma+1} \quad \text{for } x \in \mathbf{B},$$

so that

$$\int_{\mathbf{B}} Mf(x)(\log(e + Mf(x)))^{\gamma}\, dx = \infty.$$

Thus the maximal operator $M : f \mapsto Mf$ is not bounded from $L^1(\mathbf{B})$ to $L^{\widetilde{\Phi}}(\mathbf{B})$, where $\widetilde{\Phi}(t) = t(\log(e + t))^{\gamma}$ with $\gamma > -1$.

If $-(\gamma + 1)(N - \alpha)/N - 1 < \beta < -1$, then (3) implies

$$I_\alpha f(x)^{N/(N-\alpha)}(\log(e + I_\alpha f(x)))^{\gamma} \geqslant C|x|^{-N}(\log(e + 1/|x|))^{(\beta+1)N/(N-\alpha)+\gamma}$$

for $x \in \mathbf{B}$, so that

$$\int_{\mathbf{B}} I_\alpha f(x)^{N/(N-\alpha)}(\log(e + I_\alpha f(x)))^{\gamma}\, dx = \infty.$$

Thus the Riesz potential operator $I_\alpha : f \mapsto I_\alpha f$ is not bounded from $L^1(\mathbf{B})$ to $L^{\widetilde{\Psi}_\alpha}(\mathbf{B})$, where $\widetilde{\Psi}_\alpha(t) = t^{N/(N-\alpha)}(\log(e + t))^{\gamma}$ with $\gamma > -1$.

REMARK 3.4. Let $k(r) = a(\log(e + 1/r))^{b}$ with $a \geqslant e^{b}(\log(e + 1))^{-b}$ and $b \geqslant 1/N$. Then

$$\Phi(r) = r(\log(e + r))^{bN-1}$$

satisfies $(\Phi; k^N)$ and

$$\Psi_\alpha(r) \sim r^{N/(N-\alpha)}(\log(e + r))^{b\alpha N/(N-\alpha)+bN-1}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

for $0 < r < 1$. Then Lemma 2.2 and Theorem 3.2 hold for the above Φ and Ψ_α.

Consider the function

$$f(y) = |y|^{-N}(\log(e + 1/|y|))^{\beta}\chi_{\mathbf{B}}(y).$$

Then we have the following:

(1) if $\beta + bN + 1 < 0$, then there exists a constant $C > 0$ such that

$$\int_{\mathbf{B}} f(y)^{p_1(y)}\, dy \leqslant C;$$

(2) if $\beta + 1 < 0$, then there exists a constant $C > 0$ such that

$$Mf(x) \geqslant C|x|^{-N}(\log(e + 1/|x|))^{\beta+1}$$

for all $x \in \mathbf{B}$;

(3) if $\beta + 1 < 0$, then there exists a constant $C > 0$ such that

$$I_\alpha f(x) \geqslant C|x|^{\alpha-N}(\log(e + 1/|x|))^{\beta+1}$$

for all $x \in \mathbf{B}$.

Hence, for $\beta = -b'N - 1$ and $b' > b$, (2) implies

$$Mf(x)(\log(e + Mf(x)))^{b'N-1} \geqslant C|x|^{-N}(\log(e + 1/|x|))^{\beta+b'N} \quad \text{for } x \in \mathbf{B},$$

so that

$$\int_{\mathbf{B}} Mf(x)(\log(e + Mf(x)))^{b'N-1}\, dx = \infty.$$

For $\beta = -bN - \alpha(b' - b) - 1$ and $b' > b$, (3) implies

$$\begin{aligned}&\left(I_\alpha f(x)(\log(e + I_\alpha f(x)))^{\alpha b'}\right)^{N/(N-\alpha)} \\ &\qquad \geqslant C|x|^{-N}(\log(e + 1/|x|))^{(\beta+1+\alpha b')N/(N-\alpha)}\end{aligned}$$

for $x \in \mathbf{B}$, so that

$$\begin{aligned}&\int_{\mathbf{B}} \Phi\left(\left(I_\alpha f(x)(\log(e + I_\alpha f(x)))^{\alpha b'}\right)^{N/(N-\alpha)}\right) dx \\ &\qquad \geqslant C\int_{\mathbf{B}} |x|^{-N}(\log(e + 1/|x|))^{-1}\, dx = \infty.\end{aligned}$$

Thus the exponents of the log terms of Φ and Ψ_α are sharp.

Let us consider a double phase functional given by

$$\Phi_d(x,t) = t^{p_1(x)} + a(x)t^{p_2} = t^{p_1(x)} + (b(x)t)^{p_2}$$

for $x \in \mathbf{B}$ and $t \geqslant 0$, where:

- $b(x)$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$;
- $1/p_2 = 1 - \theta/N > 0$;
- $b(x) = a(x)^{1/p_2}$;

(cf. [9, 18]). For an open set $G \subset \mathbf{R}^N$ and $f \in L^1_{\mathrm{loc}}(G)$, we define the norm

$$\|f\|_{L^{\Phi_d}(G)} = \inf\left\{\lambda > 0 : \int_G \Phi_d\left(y, \frac{|f(y)|}{\lambda}\right)\, dy \leqslant 1\right\}.$$

For a locally integrable function f on $\mathbf{B}$ and $0 \leqslant \sigma < N$, we consider the fractional maximal function

$$M_\sigma f(x) = \sup_{r>0} \frac{r^\sigma}{|B(x,r)|} \int_{\mathbf{B}\cap B(x,r)} |f(y)|\, dy.$$

In view of Theorem 3.2 we find the following:

LEMMA 3.5. *There exists a constant $C > 0$ such that*

$$\|M_\theta f\|_{L^{p_2}(\mathbf{B})} \leqslant C\|f\|_{L^{p_1(\cdot)}(\mathbf{B})}$$

for all $f \in L^{p_1(\cdot)}(\mathbf{B})$.

In fact, it suffices to note

$$M_\theta f(x) \leqslant C(I_\theta|f|)(x)$$

and

$$t^{p_2} \leqslant C\left(tk(t^{-1})^\theta\right)^{p_2} \leqslant C\Phi\left(\left(tk(t^{-1})^\theta\right)^{N/(N-\theta)}\right) = C\Psi_\theta(t)$$

for all $t \geqslant 1$. Now we apply Theorem 3.2.

LEMMA 3.6. *There exists a constant $C > 0$ such that*

$$\|bMf\|_{L^{p_2}(\mathbf{B})} \leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})}$$

for all $f \in L^{\Phi_d}(\mathbf{B})$.

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

Proof. Let $f \in L^{\Phi_d}(\mathbf{B})$. Note that

$$\begin{aligned}
&b(x)\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy \\
&\quad = \frac{1}{|B(x,r)|}\int_{B(x,r)}[b(x)-b(y)]|f(y)|\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)}b(y)|f(y)|\,dy \\
&\quad \leqslant Cr^{\theta}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)}b(y)|f(y)|\,dy,
\end{aligned}$$

so that

$$b(x)Mf(x) \leqslant CM_{\theta}f(x) + M[bf](x). \tag{3.1}$$

Hence, by Lemma 3.5 and the boundedness of maximal operator on $L^{p_2}(\mathbf{B})$, we give

$$\begin{aligned}
\|bMf\|_{L^{p_2}(\mathbf{B})} &\leqslant C\{\|M_{\theta}f\|_{L^{p_2}(\mathbf{B})} + \|M[bf]\|_{L^{p_2}(\mathbf{B})}\} \\
&\leqslant C\{\|f\|_{L^{p_1(\cdot)}(\mathbf{B})} + \|bf\|_{L^{p_2}(\mathbf{B})}\} \\
&\leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})},
\end{aligned}$$

as required. ∎

Set

$$\Phi_{1,d}(x,t) = \Phi(t) + (b(x)t)^{p_2}.$$

By Lemmas 2.4 and 3.6, we establish the following result.

THEOREM 3.7. *There exists a constant $C > 0$ such that*

$$\|Mf\|_{L^{\Phi_{1,d}}(\mathbf{B})} \leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})}$$

for all $f \in L^{\Phi_d}(\mathbf{B})$.

REMARK 3.8. Consider $b_{\theta}(x) = \min\{\max\{x_N^{\theta}, 0\}, 1\}$ for $x = (x', x_N) \in \mathbf{B}$. In Remark 3.3, if $0 < \theta_1 < \theta$, then

$$\int_{\mathbf{B}} |b_{\theta_1}(x)Mf_1(x)|^{p_2}\,dx = \infty,$$

where $f_1(y) = f(y)$ when $y_N < 0$ and $f_1(y) = 0$ when $y_N \geqslant 0$ with $y = (y', y_N) \in \mathbf{B}$.

The optimality of θ and p_2 is also considered in [11, 12].

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

Set

$$\frac{1}{p_2^*} = \frac{1}{p_2} - \frac{\alpha}{N} = 1 - \frac{\alpha+\theta}{N}.$$

LEMMA 3.9. *If $\alpha + \theta < N$, then there exists a constant $C > 0$ such that*

$$\|I_{\alpha+\theta} f\|_{L^{p_2^*}(\mathbf{B})} \leqslant C\|f\|_{L^{p_1(\cdot)}(\mathbf{B})}$$

for all $f \in L^{p_1(\cdot)}(\mathbf{B})$.

In fact, it suffices to note

$$t^{p_2^*} \leqslant C\left(tk(t^{-1})^{\alpha+\theta}\right)^{p_2^*} \leqslant C\Phi\left(\left(tk(t^{-1})^{\alpha+\theta}\right)^{p_2^*}\right) = C\Psi_{\alpha+\theta}(t)$$

for all $t \geqslant 1$. Now we apply Theorem 3.2.

LEMMA 3.10. *If $\alpha + \theta < N$, then there exists a constant $C > 0$ such that*

$$\|bI_\alpha f\|_{L^{p_2^*}(\mathbf{B})} \leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})}$$

for all $f \in L^{\Phi_d}(\mathbf{B})$.

Proof. Let $f \in L^{\Phi_d}(\mathbf{B})$ be a nonnegative function. Note that

$$\begin{aligned}
&b(x)\int_{\mathbf{B}} |x-y|^{\alpha-N} f(y)\,dy \\
&\quad= \int_{\mathbf{B}} [b(x)-b(y)]|x-y|^{\alpha-N} f(y)\,dy + \int_{\mathbf{B}} |x-y|^{\alpha-N} b(y)f(y)\,dy \\
&\quad\leqslant C\int_{\mathbf{B}} |x-y|^{\alpha+\theta-N} f(y)\,dy + \int_{\mathbf{B}} |x-y|^{\alpha-N} b(y)f(y)\,dy,
\end{aligned}$$

so that

$$b(x)I_\alpha f(x) \leqslant CI_{\alpha+\theta} f(x) + I_\alpha[bf](x).$$

Hence, by Lemma 3.9 and the Sobolev inequality on $L^{p_2}(\mathbf{B})$, we give

$$\begin{aligned}
\|bI_\alpha f\|_{L^{p_2^*}(\mathbf{B})} &\leqslant C\{\|I_{\alpha+\theta} f\|_{L^{p_2^*}(\mathbf{B})} + \|I_\alpha[bf]\|_{L^{p_2^*}(\mathbf{B})}\} \\
&\leqslant C\{\|f\|_{L^{p_1(\cdot)}(\mathbf{B})} + \|bf\|_{L^{p_2}(\mathbf{B})}\} \\
&\leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})},
\end{aligned}$$

as required. □

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

Set

$$\Psi_{\alpha,d}(x,t) = \Psi_\alpha(t) + (b(x)t)^{p_2^*}.$$

Theorem 3.2 and Lemma 3.10, we establish the following result.

THEOREM 3.11. *If $\alpha + \theta < N$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{\Psi_{\alpha,d}}(\mathbf{B})} \leqslant C\|f\|_{L^{\Phi_d}(\mathbf{B})}$$

for all $f \in L^{\Phi_d}(\mathbf{B})$.

§4. Herz–Morrey spaces

We consider a measurable function $\omega(r) : (0,\infty) \to (0,\infty)$ satisfying the following conditions ($\omega1$) and ($\omega2$):

($\omega1$) $\omega(\cdot)$ is almost monotone on $(0,\infty)$; that is, $\omega(\cdot)$ is almost increasing on $(0,\infty)$ or $\omega(\cdot)$ is almost decreasing on $(0,\infty)$; namely there exists a constant $c_1 > 0$ such that

$$\omega(r) \leqslant c_1\omega(s) \quad \text{for all } 0 < r < s$$

or

$$\omega(s) \leqslant c_1\omega(r) \quad \text{for all } 0 < r < s,$$

respectively;

($\omega2$) $\omega(\cdot)$ is doubling on $(0,\infty)$; that is, there exists a constant $C > 1$ such that

$$C^{-1}\omega(r) \leqslant \omega(2r) \leqslant C\omega(r) \quad \text{for all } r > 0.$$

For $0 < q \leqslant \infty$, we define Herz–Morrey spaces $\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})$ of all measurable functions f on $\mathbf{B}$ such that

$$\|f\|_{\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})} = \left(\int_0^1 \left(\omega(r)\|f\|_{L^{p_1(\cdot)}(A(r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{\mathcal{H}^{p_1(\cdot),\infty,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(r)\|f\|_{L^{p_1(\cdot)}(A(r))} < \infty$$

when $q = \infty$, where $A(r) = \mathbf{B} \cap [B(0,2r) \setminus B(0,r)]$. We refer the reader to [1] for Morrey spaces. When $\omega(r) = r^\nu$, we simply write $\mathcal{H}^{p_1(\cdot),q,\nu}(\mathbf{B})$ for $\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})$.

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

Further, for $0 < q \leqslant \infty$, we define Herz–Morrey–Orlicz spaces $\mathcal{H}^{\Phi,q,\omega}(\mathbf{B})$ of all measurable functions f on $\mathbf{B}$ such that

$$\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{B})} = \left(\int_0^1 \left(\omega(r) \|f\|_{L^{\Phi}(A(r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{\mathcal{H}^{\Phi,\infty,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(r) \|f\|_{L^{\Phi}(A(r))} < \infty$$

when $q = \infty$.

For fundamental properties of Herz–Morrey spaces, we have the following.

LEMMA 4.1. *For* $0 < q_1 < q_2 < \infty$,

$$\mathcal{H}^{p_1(\cdot),q_1,\omega}(\mathbf{R}^N) \subset \mathcal{H}^{p_1(\cdot),q_2,\omega}(\mathbf{R}^N) \subset \mathcal{H}^{p_1(\cdot),\infty,\omega}(\mathbf{R}^N).$$

For later use we prepare the following result.

LEMMA 4.2. *Let* $\Phi(r)$ *be a positive convex function on* $(0, \infty)$ *satisfying* $(\Phi 0)$ *and* $(\Phi 1)$. *Then there is a constant* $C > 0$ *such that*

$$\|\chi_{A(r)}\|_{L^{\Phi}(\mathbf{B})} \leqslant C\{\Phi^{-1}(r^{-N})\}^{-1}$$

for all $0 < r < 1$.

We consider the following two types of conditions for $\omega(r)$:

$(\omega 1; \xi)$ $r \mapsto r^{\varepsilon_1+\xi}\omega(r)$ is almost decreasing on $(0, 1]$ for some $\varepsilon_1 > 0$;
$(\omega 2; \mu)$ $r \mapsto r^{-\varepsilon_2+\mu}\omega(r)$ is almost increasing on $(0, 1]$ for some $\varepsilon_2 > 0$.

REMARK 4.3. $(\omega 1; \xi)$ implies that there exists a constant $0 < \varepsilon_1' < \varepsilon_1$ such that $r \mapsto r^{\varepsilon_1'+\xi}k(r)^N\omega(r)$ is almost decreasing on $(0, 1]$. Similarly, $(\omega 2; \mu)$ implies that there exists a constant $0 < \varepsilon_2' < \varepsilon_2$ such that $r \mapsto r^{-\varepsilon_2'+\mu}k(r)^N\omega(r)$ is almost increasing on $(0, 1]$.

In view of Almeida and Drihem [2], we know that the maximal operator $M : f \to Mf$ is bounded in $\mathcal{H}^{p,q,\nu}(\mathbf{R}^N)$, when $1 < p < \infty$. The case $p = 1$ is treated in the following.

THEOREM 4.4. *Assume that* $\omega(r)$ *satisfies* $(\omega 1; 0)$ *and* $(\omega 2; N)$. *Suppose further:*

($\Phi k1$) *there exist constants* $0<\tilde{\varepsilon}_1<\varepsilon_1$ *and* $C>0$ *such that*

$$\left(\int_t^1\left(\left\{\Phi^{-1}(r^{-N})\right\}^{-1}r^{-\tilde{\varepsilon}_1-N}k(r)^{-N}\right)^q\frac{dr}{r}\right)^{1/q}\leqslant Ct^{-\tilde{\varepsilon}_1}$$

for all $0<t<1$ *when* $0<q<\infty$ *and*

$$\sup_{0<r<1}\left(\left\{\Phi^{-1}(r^{-N})\right\}^{-1}r^{-N}k(r)^{-N}\right)\leqslant C$$

when $q=\infty$;

($\Phi k2$) *there exist constants* $0<\tilde{\varepsilon}_2<\varepsilon_2$ *and* $C>0$ *such that*

$$\left(\int_0^t\left(\left\{\Phi^{-1}(r^{-N})\right\}^{-1}r^{\tilde{\varepsilon}_2-N}k(r)^{-N}\right)^q\frac{dr}{r}\right)^{1/q}\leqslant Ct^{\tilde{\varepsilon}_2}$$

for $0<t<1$ *when* $0<q<\infty$ *and*

$$\sup_{0<r<1}\left(\left\{\Phi^{-1}(r^{-N})\right\}^{-1}r^{-N}k(r)^{-N}\right)\leqslant C$$

when $q=\infty$.

Then there exists a constant $C>0$ *such that*

$$\|Mf\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{B})}\leqslant C\|f\|_{\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})}$$

for all $f\in\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})$.

REMARK 4.5. Let $k(r)=a(\log(e+1/r))^b$ with $a\geqslant e^b(\log(e+1))^{-b}$ and $b\geqslant 1/N$. Then we can take

$$\Phi(r)=r(\log(e+r))^{bN-1}.$$

In this case,

$$\left\{\Phi^{-1}(r^{-N})\right\}^{-1}\sim r^N(\log(e+1/r))^{bN-1}\quad\text{when } 0<r<1.$$

To show Theorem 4.4, we prepare the estimates of Hardy type operators:

LEMMA 4.6. (Cf. [20, Lemma 3.4]) *Let* $\beta\in\mathbf{R}$. *If* $\omega(r)$ *satisfies* ($\omega1; N-\beta$) *and* $0<\varepsilon<\varepsilon_1$, *then*

$$H_\beta^- f(r)\equiv r^{-\beta}\int_{B(0,r)}|y|^{\beta-N}|f(y)|\,dy$$
$$\leqslant Cr^{-\varepsilon-N}k(r)^{-N}\omega(r)^{-1}\left(\int_0^r\left(t^\varepsilon\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q\frac{dt}{t}\right)^{1/q}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

for all $0 < r < 1$ *and* $f \in L^1_{\rm loc}(\mathbf{B})$ *when* $0 < q < \infty$ *and*

$$H^-_\beta f(r) \leqslant Cr^{-N}k(r)^{-N}\omega(r)^{-1} \sup_{0<t<r} \left(\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)$$

for all $0 < r < 1$ *and* $f \in L^1_{\rm loc}(\mathbf{B})$ *when* $q = \infty$.

Proof. Let $0 < r < 1$, $0 < \varepsilon < \varepsilon_1$ and $f \in L^1_{\rm loc}(\mathbf{B})$. Then we have by Lemma 3.1,

$$\begin{aligned}\int_{B(0,r)} |y|^{\beta-N}|f(y)|\,dy &\leqslant C\int_{B(0,r)} \left(\int_{|y|/2}^{|y|} t^{\beta-N}\,\frac{dt}{t}\right)|f(y)|\,dy\\ &\leqslant C\int_0^r t^{\beta-N}\left(\int_{A(t)}|f(y)|\,dy\right)\frac{dt}{t}\\ &\leqslant C\int_0^r t^{\beta-N}k(t)^{-N}\|f\|_{L^{p(\cdot)}(A(t))}\,\frac{dt}{t}.\end{aligned}$$

In case $1 < q < \infty$, by Hölder's inequality and $(\omega1; N-\beta)$, we have

$$\begin{aligned}&\int_{B(0,r)} |y|^{\beta-N}|f(y)|\,dy\\ &\leqslant C\left(\int_0^r \left(t^{-\varepsilon+\beta-N}k(t)^{-N}\omega(t)^{-1}\right)^{q'}\,\frac{dt}{t}\right)^{1/q'}\\ &\quad\times\left(\int_0^r \left(t^{\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)^{q}\,\frac{dt}{t}\right)^{1/q}\\ &\leqslant Cr^{-\varepsilon_1+\beta-N}k(r)^{-N}\omega(r)^{-1}\left(\int_0^r (t^{-\varepsilon+\varepsilon_1})^{q'}\,\frac{dt}{t}\right)^{1/q'}\\ &\quad\times\left(\int_0^r \left(t^{\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)^{q}\,\frac{dt}{t}\right)^{1/q}\\ &\leqslant Cr^{-\varepsilon+\beta-N}k(r)^{-N}\omega(r)^{-1}\left(\int_0^r \left(t^{\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)^{q}\,\frac{dt}{t}\right)^{1/q}.\end{aligned}$$

In case $0 < q \leqslant 1$, by $(\omega1; N-\beta)$ and Minkowski's inequality, we have

$$\begin{aligned}&\int_{B(0,r)} |y|^{\beta-N}|f(y)|\,dy\\ &\leqslant Cr^{-\varepsilon+\beta-N}k(r)^{-N}\omega(r)^{-1}\int_0^r t^{\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t)\cap B(0,r))}\frac{dt}{t}\end{aligned}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

$$\begin{aligned}
&\leqslant Cr^{-\varepsilon+\beta-N}k(r)^{-N}\omega(r)^{-1}\\
&\quad\times \int_0^r \left(\int_{t/\sqrt{2}}^{t\sqrt{2}} \left(s^{\varepsilon}\omega(s)\|f\|_{L^{p(\cdot)}(A(s)\cap B(0,r))}\right)^q \frac{ds}{s}\right)^{1/q} \frac{dt}{t}\\
&\leqslant Cr^{-\varepsilon+\beta-N}k(r)^{-N}\omega(r)^{-1}\\
&\quad\times \left(\int_0^{r\sqrt{2}} \left(s^{\varepsilon}\omega(s)\|f\|_{L^{p(\cdot)}(A(s)\cap B(0,r))}\right)^q \left(\int_{s/\sqrt{2}}^{s\sqrt{2}} \frac{dt}{t}\right)^q \frac{ds}{s}\right)^{1/q}\\
&\leqslant Cr^{-\varepsilon+\beta-N}k(r)^{-N}\omega(r)^{-1} \left(\int_0^r \left(s^{\varepsilon}\omega(s)\|f\|_{L^{p(\cdot)}(A(s))}\right)^q \frac{ds}{s}\right)^{1/q}.
\end{aligned}$$

In case $q=\infty$, by $(\omega 1; N-\beta)$, we have

$$\begin{aligned}
&\int_{B(0,r)} |y|^{\beta-N}|f(y)|\,dy\\
&\leqslant C\left(\int_0^r t^{-\varepsilon+\beta-N}k(t)^{-N}\omega(t)^{-1}\frac{dt}{t}\right)\sup_{0<t<r}\left(t^{\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)\\
&\leqslant Cr^{-\varepsilon+\beta-N}k(r)^{-N}\omega(r)^{-1}\sup_{0<t<r}\left(t^{\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)\\
&\leqslant Cr^{\beta-N}k(r)^{-N}\omega(r)^{-1}\sup_{0<t<r}\left(\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right).
\end{aligned}$$

Therefore, we obtain the required result. ∎

LEMMA 4.7. (Cf. [20, Lemma 3.5]) *Let $\beta\in\mathbf{R}$. If $\omega(r)$ satisfies $(\omega 2; N-\beta)$ and $0<\varepsilon<\varepsilon_2$, then*

$$\begin{aligned}
H_\beta^+ f(r) &\equiv r^{-\beta}\int_{\mathbf{B}\setminus B(0,r)} |y|^{\beta-N}|f(y)|\,dy\\
&\leqslant Cr^{\varepsilon-N}k(r)^{-N}\omega(r)^{-1}\left(\int_{r/2}^1 \left(t^{-\varepsilon}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right)^{1/q}
\end{aligned}$$

for all $0<r<1$ and $f\in L^1_{\mathrm{loc}}(\mathbf{B})$ when $0<q<\infty$ and

$$H_\beta^+ f(r)\leqslant Cr^{-N}k(r)^{-N}\omega(r)^{-1}\sup_{r/2<t<1}\left(\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)$$

for all $0<r<1$ and $f\in L^1_{\mathrm{loc}}(\mathbf{B})$ when $q=\infty$.

Proof. We show only the case when $1 < q < \infty$. Let $0 < r < 1$, $0 < \varepsilon < \varepsilon_2$ and $f \in L^1_{\mathrm{loc}}(\mathbf{B})$. By Lemma 3.1,

$$\begin{aligned}
\int_{\mathbf{B}\setminus B(0,r)} |y|^{\beta-N}|f(y)|\,dy &\leqslant C\int_{\mathbf{B}\setminus B(0,r)} \left(\int_{|y|/2}^{|y|} t^{\beta-N}\,\frac{dt}{t}\right)|f(y)|\,dy \\
&\leqslant C\int_{r/2}^{1} t^{\beta-N}\left(\int_{A(t)}|f(y)|\,dy\right)\frac{dt}{t} \\
&\leqslant C\int_{r/2}^{1} t^{\beta-N}k(t)^{-N}\|f\|_{L^{p(\cdot)}(A(t))}\,\frac{dt}{t}.
\end{aligned}$$

By Hölder's inequality and $(\omega 2; N-\beta)$, we have

$$\begin{aligned}
&\int_{r/2}^{1} t^{\beta-N}k(t)^{-N}\|f\|_{L^{p(\cdot)}(A(t))}\,\frac{dt}{t} \\
&\quad\leqslant C\left(\int_{r/2}^{1}\left(t^{\varepsilon+\beta-N}k(t)^{-N}\omega(t)^{-1}\right)^{q'}\,\frac{dt}{t}\right)^{1/q'} \\
&\qquad\times\left(\int_{r/2}^{1}\left(t^{-\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)^{q}\,\frac{dt}{t}\right)^{1/q} \\
&\quad\leqslant Cr^{\varepsilon_2+\beta-N}k(r)^{-N}\omega(r)^{-1}\left(\int_{r/2}^{1}(t^{\varepsilon-\varepsilon_2})^{q'}\,\frac{dt}{t}\right)^{1/q'} \\
&\qquad\times\left(\int_{r/2}^{1}\left(t^{-\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)^{q}\,\frac{dt}{t}\right)^{1/q} \\
&\quad\leqslant Cr^{\varepsilon+\beta-N}k(r)^{-N}\omega(r)^{-1}\left(\int_{r/2}^{1}\left(t^{-\varepsilon}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)^{q}\,\frac{dt}{t}\right)^{1/q},
\end{aligned}$$

which gives the required result. □

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. We show only the case when $0 < q < \infty$, because the remaining case is easily obtained. Let f be a measurable function on $\mathbf{B}$ such that $\|f\|_{\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})} \leqslant 1$. For $0 < r < 1$, write

$$f = f\chi_{B(0,r/2)} + f\chi_{\tilde{A}(r)} + f\chi_{\mathbf{B}\setminus B(0,4r)} = f_1 + f_2 + f_3,$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

where $\tilde{A}(r) = A(r/2) \cup A(r) \cup A(2r)$. Note here that

$$\begin{aligned} Mf_1(x) &\leqslant \sup_{t \geqslant r/2} \frac{1}{|B(x,t)|} \int_{B(x,t)\cap B(0,r/2)} |f(y)|\, dy \\ &\leqslant C r^{-N} \int_{B(0,r/2)} |f(y)|\, dy \\ &\leqslant C H_N^- f(r) \end{aligned}$$

for $x \in A(r)$. Hence we obtain by Lemmas 4.2 and 4.6, $(\Phi k1)$ and Fubini's theorem

$$\begin{aligned} &\int_0^1 \left(\omega(r) \|Mf_1\|_{L^\Phi(A(r))}\right)^q \frac{dr}{r} \\ &\leqslant C \int_0^1 \left(\left\{\Phi^{-1}(r^{-N})\right\}^{-1} r^{-\tilde{\varepsilon}_1 - N} k(r)^{-N}\right)^q \\ &\quad \times \left(\int_0^r \left(t^{\tilde{\varepsilon}_1} \omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\ &\leqslant C \int_0^1 \left(t^{\tilde{\varepsilon}_1} \omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \\ &\quad \times \left(\int_t^1 \left(\left\{\Phi^{-1}(r^{-N})\right\}^{-1} r^{-\tilde{\varepsilon}_1 - N} k(r)^{-N}\right)^q \frac{dr}{r}\right) \frac{dt}{t} \\ &\leqslant C \int_0^1 \left(\omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t} \\ &\leqslant C. \end{aligned} \tag{4.1}$$

Since $f_3 = 0$ in case $1/4 \leqslant r < 1$ and

$$Mf_3(x) \leqslant C \int_{\mathbf{B}\setminus B(0,4r)} |y|^{-N} |f(y)|\, dy \leqslant C H_0^+ f(4r)$$

for $x \in A(r)$ in case $0 < r < 1/4$, we obtain by Lemmas 4.2 and 4.7, $(\Phi k2)$ and Fubini's theorem

$$\begin{aligned} &\int_0^1 \left(\omega(r) \|Mf_3\|_{L^\Phi(A(r))}\right)^q \frac{dr}{r} \\ &\leqslant C \int_0^1 \left(\left\{\Phi^{-1}(r^{-N})\right\}^{-1} r^{\tilde{\varepsilon}_2 - N} k(r)^{-N}\right)^q \end{aligned}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

$$
\begin{aligned}
&\quad \times \left(\int_r^1 \left(t^{-\tilde{\varepsilon}_2} \omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))} \right)^q \frac{dt}{t} \right) \frac{dr}{r} \\
&\leqslant C \int_0^1 \left(t^{-\tilde{\varepsilon}_2} \omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))} \right)^q \\
&\quad \times \left(\int_0^t \left(\{\Phi^{-1}(r^{-N})\}^{-1} r^{\tilde{\varepsilon}_2 - N} k(r)^{-N} \right)^q \frac{dr}{r} \right) \frac{dt}{t} \\
&\leqslant C \int_0^1 \left(\omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))} \right)^q \frac{dt}{t} \\
&\leqslant C. \qquad (4.2)
\end{aligned}
$$

For f_2, by Lemma 2.4

$$
\|Mf_2\|_{L^\Phi(A(r))} \leqslant C\|f_2\|_{L^{p_1(\cdot)}(\mathbf{B})} = C\|f\|_{L^\Phi(\tilde{A}(r))},
$$

so that

$$
\int_0^1 \left(\omega(r) \|Mf_2\|_{L^\Phi(A(r))} \right)^q \frac{dr}{r} \leqslant C. \tag{4.3}
$$

By (4.1), (4.2) and (4.3)

$$
\int_0^1 \left(\omega(r) \|Mf\|_{L^\Phi(A(r))} \right)^q \frac{dr}{r} \leqslant C. \qquad \square
$$

REMARK 4.8. Let $k(r) = a(\log(e + 1/r))^b$ with $a \geqslant e^b(\log(e+1))^{-b}$ and $b \geqslant 1/N$ and $\omega(r) = r^\nu$ with $-N < \nu < 0$. Then note that $\Phi(r) = r(\log(e + r))^{bN-1}$ by Remark 3.4. Let $b_1 > b$ and $0 < r < 1/2$. We choose $0 < \lambda_2 < \lambda_1 < 1$ such that $b_1 > (1 + \lambda_1)b$. For $|x_0| = r$, consider

$$
f(y) = |x_0 - y|^{-N} \chi_{C(x_0, r, \lambda_1)},
$$

where $C(x_0, r, \lambda_1) = \{y \in A(r) : r^{1+\lambda_1} < |x_0 - y| < r\}$. Note that

$$
\begin{aligned}
\frac{\log k(|y|)}{\log(e + 1/|y|)} \log f(y) &\leqslant \frac{\log k(|y|)}{\log(e + 1/|y|)} \log \left((|y|/2)^{-(1+\lambda_1)N} \right) \\
&\leqslant \frac{\log k(|y|)}{\log(e + 1/|y|)} ((1 + \lambda_1)N \log(1/|y|) + C) \\
&\leqslant \log \left(k(|y|)^{(1+\lambda_1)N} \right) + C \\
&\leqslant \log \left(k(|x_0 - y|)^{(1+\lambda_1)N} \right) + C
\end{aligned}
$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

for $y \in C(x_0, r, \lambda_1)$ since

$$|x_0 - y| < r < |y| < 2r < 2|x_0 - y|^{1/(1+\lambda_1)}.$$

Then we have

$$\begin{aligned}\int_{\mathbf{B}} f(y)^{p_1(y)}\, dy &= \int_{C(x_0,r,\lambda_1)} |x_0 - y|^{-N} \exp\left(\frac{\log k(|y|)}{\log(e + 1/|y|)} \log f(y)\right) dy \\ &\leqslant C \int_{C(x_0,r,\lambda_1)} |x_0 - y|^{-N} (\log(e + 1/|x_0 - y|))^{(1+\lambda_1)bN}\, dy \\ &\leqslant C \int_{r^{1+\lambda_1}}^{r} (\log(e + 1/t))^{(1+\lambda_1)bN} \frac{dt}{t} \\ &\leqslant C(\log(e + 1/r))^{(1+\lambda_1)bN+1}.\end{aligned}$$

Set

$$f_r(y) = r^{-\nu}(\log(e + 1/r))^{-(1+\lambda_1)bN-1} f(y).$$

We see that

$$\|f_r\|_{L^{p_1(\cdot)}(\mathbf{B})} \leqslant Cr^{-\nu},$$

so that

$$\begin{aligned}\int_0^1 \left(t^{\nu}\|f_r\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t} &\leqslant \int_{r/2}^{2r} \left(t^{\nu}\|f_r\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t} \\ &\leqslant C\left(r^{\nu}\|f_r\|_{L^{p_1(\cdot)}(\mathbf{B})}\right)^q \leqslant C\end{aligned}$$

when $0 < q < \infty$ and

$$\begin{aligned}\sup_{0<t<1} \left(t^{\nu}\|f_r\|_{L^{p_1(\cdot)}(A(t))}\right) &\leqslant \sup_{r/2<t<2r} \left(t^{\nu}\|f_r\|_{L^{p_1(\cdot)}(A(t))}\right) \\ &\leqslant Cr^{\nu}\|f_r\|_{L^{p_1(\cdot)}(\mathbf{B})} \leqslant C\end{aligned}$$

when $q = \infty$. Therefore $\|f_r\|_{\mathcal{H}^{p_1(\cdot),q,\nu}(\mathbf{B})} \leqslant C$ for all $0 < r < 1/2$.

On the other hand, for $x \in C(x_0, r, \lambda_2)$ we find

$$\begin{aligned}Mf(x) &\geqslant \frac{C}{|B(x_0, |x_0 - x|)|} \int_{B(x_0,|x_0-x|)} f(y)\, dy \\ &\geqslant C|x_0 - x|^{-N} \int_{r^{1+\lambda_1}}^{|x_0-x|} \frac{dt}{t}\end{aligned}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

$$\begin{aligned}&\geqslant C|x_0-x|^{-N}\int_{|x_0-x|^{(1+\lambda_1)/(1+\lambda_2)}}^{|x_0-x|}\frac{dt}{t}\\&\geqslant C|x_0-x|^{-N}\log(e+1/|x_0-x|).\end{aligned}$$

Then we obtain

$$\begin{aligned}&\int_{A(r)} Mf_r(x)/\left(r^{-\nu}(\log(e+1/r))^{(b_1-(1+\lambda_1)b)N}\right)\\&\qquad\times\left(\log\left(e+Mf_r(x)/\left(r^{-\nu}(\log(e+1/r))^{(b_1-(1+\lambda_1)b)N}\right)\right)\right)^{b_1N-1}\,dx\\&=\int_{A(r)}(\log(e+1/r))^{-(b_1N+1)}Mf(x)\\&\qquad\times\left(\log\left(e+(\log(e+1/r))^{-(b_1N+1)}Mf(x)\right)\right)^{b_1N-1}\,dx\\&\geqslant C(\log(e+1/r))^{-(b_1N+1)}\int_{C(x_0,r,\lambda_2)}|x_0-x|^{-N}\log(e+1/|x_0-x|)\\&\qquad\times\left(\log\left(e+|x_0-x|^{-N}\left(\log(e+1/|x_0-x|)\right)^{-b_1N}\right)\right)^{b_1N-1}\,dx\\&\geqslant C(\log(e+1/r))^{-(b_1N+1)}\\&\qquad\times\int_{C(x_0,r,\lambda_2)}|x_0-x|^{-N}(\log(e+1/|x_0-x|))^{b_1N}\,dx\\&\geqslant C.\end{aligned}$$

It follows that

$$\|Mf_r\|_{L^{\Phi_1}(A(r))}\geqslant Cr^{-\nu}(\log(e+1/r))^{(b_1-(1+\lambda_1)b)N},$$

where $\Phi_1(r)=r(\log(e+r))^{b_1N-1}$. Hence

$$\begin{aligned}\int_0^1\left(t^{\nu}\|Mf_r\|_{L^{\Phi_1}(A(t))}\right)^q\frac{dt}{t}&\geqslant\int_{3r/4}^{3r/2}\left(t^{\nu}\|Mf_r\|_{L^{\Phi_1}(A(t))}\right)^q\frac{dt}{t}\\&\geqslant C\left(r^{\nu}\|Mf_r\|_{L^{\Phi_1}(A(r))}\right)^q\\&\geqslant C(\log(e+1/r))^{(b_1-(1+\lambda_1)b)Nq}\end{aligned}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

when $0 < q < \infty$ and

$$\sup_{0<t<1} \left(t^{\nu} \|Mf_r\|_{L^{\Phi_1}(A(t))} \right) \geqslant C r^{\nu} \|Mf_r\|_{L^{\Phi_1}(A(r))}$$
$$\geqslant C(\log(e+1/r))^{(b_1-(1+\lambda_1)b)N}$$

when $q = \infty$, so that $\|Mf_r\|_{\mathcal{H}^{\Phi_1,q,\nu}(\mathbf{B})} \to \infty$ as $r \to 0$.

Thus Theorem 4.4 is the best possible.

REMARK 4.9. Let $k(r) = a(\log(e+1/r))^b$ with $a \geqslant e^b(\log(e+1))^{-b}$ and $b \geqslant 1/N$ and $\Phi(r) = r(\log(e+r))^{bN-1}$ as in Remark 3.4 . If $\|f\|_{\mathcal{H}^{p_1(\cdot),1,\nu}(\mathbf{B})} \leqslant 1$ and $-N < \nu < 0$, then we can find a constant $C > 0$ such that

$$\int_{\mathbf{B}} |x|^{\nu} Mf(x)(\log(e+Mf(x)))^{bN-1}\, dx \leqslant C.$$

We shall show this. We may assume that $\|Mf\|_{L^{\Phi}(A(r))} \leqslant 1$ for all $0 < r < 1$ by Lemma 4.1 and Theorem 4.4. Then note that

$$1 = \int_{A(r)} \Phi\left(Mf(x)/\|Mf\|_{L^{\Phi}(A(r))} \right) dx$$
$$\geqslant (\|Mf\|_{L^{\Phi}(A(r))})^{-1} \int_{A(r)} \Phi(Mf(x))\, dx.$$

Therefore, by Fubini's theorem and Theorem 4.4

$$\begin{aligned}\int_{\mathbf{B}} |x|^{\nu}\Phi(Mf(x))\, dx &\leqslant C \int_{\mathbf{B}} \Phi(Mf(x)) \left(\int_{|x|/2}^{|x|} t^{\nu}\, \frac{dt}{t} \right) dx \\ &= C \int_0^1 t^{\nu} \left(\int_{A(t)} \Phi(Mf(x))\, dx \right) \frac{dt}{t} \\ &\leqslant C \int_0^1 t^{\nu} \|Mf\|_{L^{\Phi}(A(t))}\, \frac{dt}{t} \\ &= C\|Mf\|_{\mathcal{H}^{\Phi,1,\nu}(\mathbf{B})} \\ &\leqslant C,\end{aligned}$$

which proves the result.

As in the proof of Lemmas 4.6 and 4.7, we have the following results by using

$$t^{-N} \int_{A(t)} |f(y)|\, dy \leqslant C t^{-N/p} \|f\|_{L^p(A(t))}$$

for $t > 0$ and $p \geqslant 1$.

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

LEMMA 4.10. (Cf. [20, Lemma 3.4]) *Let $\beta \in \mathbf{R}$ and $p \geqslant 1$. If $\omega(r)$ satisfies $(\omega1; N/p - \beta)$ and $0 < \varepsilon < \varepsilon_1$, then*

$$H_\beta^- f(r) \leqslant C r^{-\varepsilon - N/p} \omega(r)^{-1} \left(\int_0^r \left(t^\varepsilon \omega(t) \|f\|_{L^p(A(t))} \right)^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L^1_{\mathrm{loc}}(\mathbf{B})$ when $0 < q < \infty$ and

$$H_\beta^- f(r) \leqslant C r^{-N/p} \omega(r)^{-1} \sup_{0<t<r} \left(\omega(t) \|f\|_{L^p(A(t))} \right)$$

for all $0 < r < 1$ and $f \in L^1_{\mathrm{loc}}(\mathbf{B})$ when $q = \infty$.

LEMMA 4.11. (Cf. [20, Lemma 3.5]) *Let $\beta \in \mathbf{R}$ and let $p \geqslant 1$. If $\omega(r)$ satisfies $(\omega2; N/p - \beta)$ and $0 < \varepsilon < \varepsilon_2$, then*

$$H_\beta^+ f(r) \leqslant C r^{\varepsilon - N/p} \omega(r)^{-1} \left(\int_{r/2}^1 \left(t^{-\varepsilon} \omega(t) \|f\|_{L^p(A(t))} \right)^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L^1_{\mathrm{loc}}(\mathbf{B})$ when $0 < q < \infty$ and

$$H_\beta^+ f(r) \leqslant C r^{-N/p} \omega(r)^{-1} \sup_{r/2<t<1} \left(\omega(t) \|f\|_{L^p(A(t))} \right)$$

for all $0 < r < 1$ and $f \in L^1_{\mathrm{loc}}(\mathbf{B})$ when $q = \infty$.

For $0 < q \leqslant \infty$, we define Herz–Morrey–Musielak–Orlicz spaces $\mathcal{H}^{\Phi_d, q, \omega}(\mathbf{B})$ of all measurable functions f on $\mathbf{B}$ such that

$$\|f\|_{\mathcal{H}^{\Phi_d,q,\omega}(\mathbf{B})} = \left(\int_0^1 \left(\omega(r) \|f\|_{L^{\Phi_d}(A(r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{\mathcal{H}^{\Phi_d,\infty,\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(r) \|f\|_{L^{\Phi_d}(A(r))} < \infty$$

when $q = \infty$.

THEOREM 4.12. *Assume that $\omega(r)$ satisfies $(\omega1; 0)$ and $(\omega2; N - \theta)$. Suppose further $(\Phi k1)$ and $(\Phi k2)$ hold. Then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\mathcal{H}^{\Phi_{1,d},q,\omega}(\mathbf{B})} \leqslant C \|f\|_{\mathcal{H}^{\Phi_d,q,\omega}(\mathbf{B})}$$

for all $f \in \mathcal{H}^{\Phi_d,q,\omega}(\mathbf{B})$.

Proof. We show only the case when $0 < q < \infty$, because the remaining case is easily obtained. Let f be a measurable function on $\mathbf{B}$ such that $\|f\|_{\mathcal{H}^{\Phi_d, q, \omega}(\mathbf{B})} \leqslant 1$. Note that $(\omega1; 0)$ and $(\omega2; N - \theta)$ imply $(\omega1; N/p_2 - N)$ and $(\omega2; N)$, respectively.

By Theorem 4.4, we have

$$\int_0^1 \left(\omega(r)\|Mf\|_{L^{\Phi}(A(r))}\right)^q \frac{dr}{r} \leqslant C.$$

For $0 < r < 1$, write

$$f = f\chi_{B(0,r/2)} + f\chi_{\tilde{A}(r)} + f\chi_{\mathbf{B}\setminus B(0,4r)} = f_1 + f_2 + f_3,$$

where $\tilde{A}(r) = A(r/2) \cup A(r) \cup A(2r)$. Here note from (3.1) that

$$\begin{aligned}
b(x)Mf_1(x) &\leqslant CM_\theta f_1(x) + M[bf_1](x) \\
&\leqslant C\left\{\sup_{t \geqslant r/2} \frac{t^\theta}{|B(x,t)|}\int_{B(x,t)\cap B(0,r/2)} |f(y)|\,dy + H_N^-[bf](r)\right\} \\
&\leqslant C\left\{r^{-N+\theta}\int_{B(0,r/2)} |f(y)|\,dy + H_N^-[bf](r)\right\} \\
&\leqslant C\left\{r^\theta H_N^- f(r) + H_N^-[bf](r)\right\}
\end{aligned}$$

for $x \in A(r)$. Let $0 < \tilde{\varepsilon}_1 < \varepsilon_1$ and $0 < \tilde{\varepsilon}_2 < \varepsilon_2$. Hence we obtain by Lemmas 4.2, 4.6 and 4.10 and Fubini's theorem

$$\begin{aligned}
&\int_0^1 \left(\omega(r)\|bMf_1\|_{L^{p_2}(A(r))}\right)^q \frac{dr}{r} \\
&\leqslant C\Bigg\{\int_0^1 \left(r^{-\tilde{\varepsilon}_1 - N + N/p_2 + \theta} k(r)^{-N}\right)^q \\
&\quad \times \left(\int_0^r \left(t^{\tilde{\varepsilon}_1}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\
&\quad + \int_0^1 \left(r^{-\tilde{\varepsilon}_1 - N/p_2 + N/p_2}\right)^q \left(\int_0^r \left(t^{\tilde{\varepsilon}_1}\omega(t)\|bf\|_{L^{p_2}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r}\Bigg\} \\
&\leqslant C\Bigg\{\int_0^1 \left(t^{\tilde{\varepsilon}_1}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \left(\int_t^1 r^{-\tilde{\varepsilon}_1 q} \frac{dr}{r}\right) \frac{dt}{t} \\
&\quad + \int_0^1 \left(t^{\tilde{\varepsilon}_1}\omega(t)\|bf\|_{L^{p_2}(A(t))}\right)^q \left(\int_t^1 r^{-\tilde{\varepsilon}_1 q} \frac{dr}{r}\right) \frac{dt}{t}\Bigg\}
\end{aligned}$$

$$\leqslant C \int_0^1 \left(\omega(t)\|f\|_{L^{\Phi_d}(A(t))}\right)^q \frac{dt}{t}$$

$$\leqslant C. \tag{4.4}$$

Since $f_3 = 0$ in case $1/4 \leqslant r < 1$ and

$$\begin{aligned} & b(x)Mf_3(x) \\ & \leqslant C\left\{\sup_{t>0}\frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(0,4r)}|x-y|^{\theta}|f(y)|\,dy + H_0^+[bf](4r)\right\} \\ & \leqslant C\left\{\int_{\mathbf{B}\cap B(0,4r)}|y|^{\theta-N}|f(y)|\,dy + H_0^+[bf](4r)\right\} \\ & \leqslant C\left\{r^{\theta}H_\theta^+ f(4r) + H_0^+[bf](4r)\right\} \end{aligned}$$

for $x \in A(r)$ in case $0 < r < 1/4$, we obtain by Lemmas 4.2, 4.7 and 4.11 and Fubini's theorem

$$\begin{aligned} & \int_0^1 (\omega(r)\|bMf_3\|_{L^{p_2}(A(r))})^q \frac{dr}{r} \\ & \leqslant C\bigg\{\int_0^1 (r^{\tilde{\varepsilon}_2}k(r)^{-N})^q \left(\int_r^1 \left(t^{-\tilde{\varepsilon}_2}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right)\frac{dr}{r} \\ & \quad + \int_0^1 r^{\tilde{\varepsilon}_2 q}\left(\int_r^1 (t^{-\tilde{\varepsilon}_2}\omega(t)\|bf\|_{L^{p_2}(A(t))})^q \frac{dt}{t}\right)\frac{dr}{r}\bigg\} \\ & \leqslant C\int_0^1 \left(\omega(t)\|f\|_{L^{\Phi_d}(A(t))}\right)^q \frac{dt}{t} \\ & \leqslant C. \end{aligned} \tag{4.5}$$

For f_2, by Lemma 3.6

$$\|bMf_2\|_{L^{p_2}(A(r))} \leqslant C\|f_2\|_{L^{\Phi_d}(\mathbf{B})} = C\|f\|_{L^{\Phi_d}(\tilde{A}(r))},$$

so that

$$\int_0^1 (\omega(r)\|bMf_2\|_{L^{p_2}(A(r))})^q \frac{dr}{r} \leqslant C. \tag{4.6}$$

By (4.4), (4.5) and (4.6)

$$\int_0^1 (\omega(r)\|bMf\|_{L^{p_2}(A(r))})^q \frac{dr}{r} \leqslant C. \qquad \square$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

REMARK 4.13. Let $k(r) = a(\log(e + 1/r))^b$ with $a \geqslant e^b(\log(e+1))^{-b}$ and $b \geqslant 1/N$ and $\Phi(r) = r(\log(e+r))^{bN-1}$ as in Remark 3.4. If $\|f\|_{\mathcal{H}^{\Phi_d,1,\nu}(\mathbf{B})} \leqslant 1$ and $-N+\theta < \nu < 0$, then we can find a constant $C > 0$ such that

$$\int_{\mathbf{B}} |x|^{\nu} \left\{ Mf(x)(\log(e + Mf(x)))^{bN-1} + (b(x)Mf(x))^{p_2} \right\} dx \leqslant C.$$

THEOREM 4.14. *Assume that $\omega(r)$ satisfies $(\omega 1; 0)$ and $(\omega 2; N-\alpha)$. Suppose:*

$(\Psi_\alpha k1)$ *there exist constants $0 < \tilde{\varepsilon}_1 < \varepsilon_1$ and $C > 0$ such that*

$$\left(\int_t^1 \left(\left\{ \Psi_\alpha^{-1}(r^{-N}) \right\}^{-1} r^{-\tilde{\varepsilon}_1 + \alpha - N} k(r)^{-N} \right)^q \frac{dr}{r} \right)^{1/q} \leqslant C t^{-\tilde{\varepsilon}_1}$$

for all $0 < t < 1$ when $0 < q < \infty$ and

$$\sup_{0<r<1} \left(\left\{ \Psi_\alpha^{-1}(r^{-N}) \right\}^{-1} r^{\alpha - N} k(r)^{-N} \right) \leqslant C$$

when $q = \infty$;

$(\Psi_\alpha k2)$ *there exist constants $0 < \tilde{\varepsilon}_2 < \varepsilon_2$ and $C > 0$ such that*

$$\left(\int_0^t \left(\left\{ \Psi_\alpha^{-1}(r^{-N}) \right\}^{-1} r^{\tilde{\varepsilon}_2 + \alpha - N} k(r)^{-N} \right)^q \frac{dr}{r} \right)^{1/q} \leqslant C t^{\tilde{\varepsilon}_2}$$

for $0 < t < 1$ when $0 < q < \infty$ and

$$\sup_{0<r<1} \left(\left\{ \Psi_\alpha^{-1}(r^{-N}) \right\}^{-1} r^{\alpha - N} k(r)^{-N} \right) \leqslant C$$

when $q = \infty$.

Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\mathcal{H}^{\Psi_\alpha,q,\omega}(\mathbf{B})} \leqslant C \|f\|_{\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})}$$

for all $f \in \mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ such that $\|f\|_{\mathcal{H}^{p_1(\cdot),q,\omega}(\mathbf{B})} \leqslant 1$. For $x \in \mathbf{B}$, set

$$\begin{aligned}
I_\alpha f(x) &= \int_{B(0,|x|/2)} |x-y|^{\alpha-N} f(y)\,dy \\
&\quad + \int_{(B(0,2|x|)\cap \mathbf{B})\setminus B(0,|x|/2)} |x-y|^{\alpha-N} f(y)\,dy \\
&\quad + \int_{\mathbf{B}\setminus B(0,2|x|)} |x-y|^{\alpha-N} f(y)\,dy \\
&= u_1(x)+u_2(x)+u_3(x).
\end{aligned}$$

Let $0<r<1$. Since

$$\begin{aligned}
u_1(x) &\leqslant C|x|^{\alpha-N} \int_{B(0,|x|/2)} f(y)\,dy \\
&\leqslant Cr^\alpha H_N^- f(r)
\end{aligned}$$

for $x\in A(r)$, using Lemma 4.6 we have

$$u_1(x) \leqslant Cr^{-\tilde{\varepsilon}_1+\alpha-N} k(r)^{-N}\omega(r)^{-1} \left(\int_0^r \left(t^{\tilde{\varepsilon}_1}\omega(t)\|f\|_{L^{p(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right)^{1/q}.$$

Hence we obtain by Lemma 4.2, $(\Psi_\alpha k1)$ and Fubini's theorem

$$\begin{aligned}
&\int_0^1 \left(\omega(r)\|u_1\|_{L^{\Psi_\alpha}(A(r))}\right)^q \frac{dr}{r} \\
&\leqslant C\int_0^1 \left(\left\{\Psi_\alpha^{-1}(r^{-N})\right\}^{-1} r^{-\tilde{\varepsilon}_1+\alpha-N} k(r)^{-N}\right)^q \\
&\quad \times \left(\int_0^r \left(t^{\tilde{\varepsilon}_1}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\
&\leqslant C\int_0^1 \left(t^{\tilde{\varepsilon}_1}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \\
&\quad \times \left(\int_t^1 \left(\left\{\Psi_\alpha^{-1}(r^{-N})\right\}^{-1} r^{-\tilde{\varepsilon}_1+\alpha-N} k(r)^{-N}\right)^q \frac{dr}{r}\right) \frac{dt}{t} \\
&\leqslant C\int_0^1 \left(\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t} \\
&\leqslant C. \qquad (4.7)
\end{aligned}$$

Similarly, since

$$\begin{aligned}
u_3(x) &\leqslant C\int_{\mathbf{B}\setminus B(0,2|x|)} |y|^{\alpha-N} f(y)\,dy \\
&\leqslant Cr^\alpha H_\alpha^+ f(2r)
\end{aligned}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

for $x \in A(r)$, we see by Lemma 4.7,

$$u_3(x) \leqslant C r^{\tilde{\varepsilon}_2+\alpha-N} k(r)^{-N} \omega(r)^{-1} \left(\int_r^1 \left(t^{-\tilde{\varepsilon}_2} \omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))} \right)^q \frac{dt}{t} \right)^{1/q}.$$

Hence we obtain by Lemma 4.2, $(\Psi_\alpha k2)$ and Fubini's theorem

$$\begin{aligned}
&\int_0^1 \left(\omega(r) \|u_3\|_{L^{\Psi_\alpha}(A(r))} \right)^q \frac{dr}{r} \\
&\leqslant C \int_0^1 \left(\left\{ \Psi_\alpha^{-1}(r^{-N}) \right\}^{-1} r^{\tilde{\varepsilon}_2+\alpha-N} k(r)^{-N} \right)^q \\
&\quad \times \left(\int_r^1 \left(t^{-\tilde{\varepsilon}_2} \omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))} \right)^q \frac{dt}{t} \right) \frac{dr}{r} \\
&\leqslant C \int_0^1 \left(t^{-\tilde{\varepsilon}_2} \omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))} \right)^q \\
&\quad \times \left(\int_0^t \left(\left\{ \Psi_\alpha^{-1}(r^{-N}) \right\}^{-1} r^{\tilde{\varepsilon}_2+\alpha-N} k(r)^{-N} \right)^q \frac{dr}{r} \right) \frac{dt}{t} \\
&\leqslant C \int_0^1 \left(\omega(t) \|f\|_{L^{p_1(\cdot)}(A(t))} \right)^q \frac{dt}{t} \\
&\leqslant C. \qquad (4.8)
\end{aligned}$$

Let $\tilde{A}(r) = A(r/2) \cup A(r) \cup A(2r)$. Since $|u_2(x)| \leqslant C I_\alpha (f \chi_{(B(0,4r)\cap \mathbf{B})\setminus B(0,r/2)})(x)$ for $x \in A(r)$, we have by Theorem 3.2

$$\|u_2\|_{L^{\Psi_\alpha}(A(r))} \leqslant C \|f\|_{L^{p_1(\cdot)}(\tilde{A}(r))},$$

so that

$$\int_0^1 \left(\omega(r) \|u_2\|_{L^{\Psi_\alpha}(A(r))} \right)^q \frac{dr}{r} \leqslant C. \qquad (4.9)$$

Thus, by (4.7), (4.8) and (4.9), we obtain the required result. □

THEOREM 4.15. *Suppose $\alpha + \theta < N$. Assume that $\omega(r)$ satisfies $(\omega 1; 0)$ and $(\omega 2; N - \alpha - \theta)$. Suppose further $(\Psi_\alpha k1)$ and $(\Psi_\alpha k2)$ hold. Then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{\Psi_{\alpha,d},q,\omega}(\mathbf{B})} \leqslant C \|f\|_{\mathcal{H}^{\Phi_d,q,\omega}(\mathbf{B})}$$

for all $f \in \mathcal{H}^{\Phi_d,q,\omega}(\mathbf{B})$.

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

Proof. We show only the case when $0 < q < \infty$, because the remaining case is easily obtained. Let f be a nonnegative measurable function on $\mathbf{B}$ such that $\|f\|_{\mathcal{H}^{\Phi_d,q,\omega}(\mathbf{B})} \leqslant 1$. Note that $(\omega1; 0)$ and $(\omega2; N-\alpha-\theta)$ imply $(\omega1; -\theta)$ and $(\omega2; N-\alpha)$, respectively.

By Theorem 4.14, we have

$$\int_0^1 \left(\omega(r)\|I_\alpha f\|_{L^{\Psi_\alpha}(A(r))}\right)^q \frac{dr}{r} \leqslant C.$$

For $x \in \mathbf{B}$, set

$$\begin{aligned} I_\alpha f(x) &= \int_{B(0,|x|/2)} |x-y|^{\alpha-N} f(y)\,dy \\ &\quad + \int_{(B(0,2|x|)\cap \mathbf{B})\setminus B(0,|x|/2)} |x-y|^{\alpha-N} f(y)\,dy \\ &\quad + \int_{\mathbf{B}\setminus B(0,2|x|)} |x-y|^{\alpha-N} f(y)\,dy \\ &= u_1(x) + u_2(x) + u_3(x). \end{aligned}$$

Let $0 < r < 1$. Let $0 < \tilde{\varepsilon}_1 < \varepsilon_1$ and $0 < \tilde{\varepsilon}_2 < \varepsilon_2$. Since

$$\begin{aligned} b(x)u_1(x) &\leqslant Cb(x)|x|^{\alpha-N} \int_{B(0,|x|/2)} f(y)\,dy \\ &\leqslant C\left\{|x|^{\alpha+\theta-N}\int_{B(0,|x|/2)} f(y)\,dy + |x|^{\alpha-N}\int_{B(0,|x|/2)} b(y)f(y)\,dy\right\} \\ &\leqslant C\left\{r^{\alpha+\theta}H_N^- f(r) + r^\alpha H_N^-[bf](r)\right\} \end{aligned}$$

for $x \in A(r)$, we obtain by Lemmas 4.2, 4.6 and 4.10 and Fubini's theorem

$$\begin{aligned} &\int_0^1 \left(\omega(r)\|bu_1\|_{L^{p_2^*}(A(r))}\right)^q \frac{dr}{r} \\ &\leqslant C\Big\{\int_0^1 \left(r^{-\tilde{\varepsilon}_1+N/p_2^*+\alpha+\theta-N} k(r)^{-N}\right)^q \\ &\quad \times \left(\int_0^r \left(t^{\tilde{\varepsilon}_1}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\ &\quad + \int_0^1 \left(r^{-\tilde{\varepsilon}_1+N/p_2^*+\alpha-N/p_2}\right)^q \end{aligned}$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

$$
\begin{aligned}
&\times \left(\int_0^r \left(t^{\tilde{\varepsilon}_1}\omega(t)\|bf\|_{L^{p_2}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \Bigg\} \\
&\leqslant C\Bigg\{ \int_0^1 \left(t^{\tilde{\varepsilon}_1}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \left(\int_t^1 r^{-\tilde{\varepsilon}_1 q} \frac{dr}{r}\right) \frac{dt}{t} \\
&\quad + \int_0^1 \left(t^{\tilde{\varepsilon}_1}\omega(t)\|bf\|_{L^{p_2}(A(t))}\right)^q \left(\int_t^1 r^{-\tilde{\varepsilon}_1 q} \frac{dr}{r}\right) \frac{dt}{t} \Bigg\} \\
&\leqslant C \int_0^1 \left(\omega(t)\|f\|_{L^{\Phi_d}(A(t))}\right)^q \frac{dt}{t} \\
&\leqslant C. \qquad (4.10)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
b(x)u_3(x) &\leqslant Cb(x) \int_{\mathbf{B}\setminus B(0,2|x|)} |y|^{\alpha-N} f(y)\,dy \\
&\leqslant C \left\{ \int_{\mathbf{B}\setminus B(0,2|x|)} |y|^{\alpha-N+\theta} f(y)\,dy + \int_{\mathbf{B}\setminus B(0,2|x|)} |y|^{\alpha-N} b(y) f(y)\,dy \right\} \\
&\leqslant C \left\{ r^{\alpha+\theta} H^{+}_{\alpha+\theta} f(2r) + r^{\alpha} H^{+}_{\alpha}[bf](2r) \right\}
\end{aligned}
$$

for $x \in A(r)$, we obtain by Lemmas 4.2, 4.7 and 4.11 and Fubini's theorem

$$
\begin{aligned}
&\int_0^1 \left(\omega(r)\|bu_3\|_{L^{p_2^*}(A(r))}\right)^q \frac{dr}{r} \\
&\leqslant C\Bigg\{ \int_0^1 \left(r^{\tilde{\varepsilon}_2} k(r)^{-N}\right)^q \left(\int_r^1 \left(t^{-\tilde{\varepsilon}_2}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\
&\quad + \int_0^1 r^{\tilde{\varepsilon}_2 q} \left(\int_r^1 \left(t^{-\tilde{\varepsilon}_2}\omega(t)\|bf\|_{L^{p_2}(A(t))}\right)^q \frac{dt}{t}\right) \frac{dr}{r} \Bigg\} \\
&\leqslant C\Bigg\{ \int_0^1 \left(t^{-\tilde{\varepsilon}_2}\omega(t)\|f\|_{L^{p_1(\cdot)}(A(t))}\right)^q \left(\int_0^t r^{\tilde{\varepsilon}_2 q} \frac{dr}{r}\right) \frac{dt}{t} \\
&\quad + \int_0^1 \left(t^{-\tilde{\varepsilon}_2}\omega(t)\|bf\|_{L^{p_2}(A(t))}\right)^q \left(\int_0^t r^{\tilde{\varepsilon}_2 q} \frac{dr}{r}\right) \frac{dt}{t} \Bigg\} \\
&\leqslant C \int_0^1 \left(\omega(t)\|f\|_{L^{\Phi_d}(A(t))}\right)^q \frac{dt}{t} \\
&\leqslant C. \qquad (4.11)
\end{aligned}
$$

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

Let $\tilde{A}(r) = A(r/2) \cup A(r) \cup A(2r)$. Since $u_2(x) \leqslant C I_\alpha(f \chi_{(B(0,4r)\cap \mathbf{B})\setminus B(0,r/2)})(x)$ for $x \in A(r)$, we have by Lemma 3.10

$$\|bu_2\|_{L^{p_2^*}(A(r))} \leqslant C\|f\|_{L^{\Phi_d}(\tilde{A}(r))},$$

so that

$$\int_0^1 \left(\omega(r)\|bu_2\|_{L^{p_2^*}(A(r))}\right)^q \frac{dr}{r} \leqslant C. \tag{4.12}$$

Thus, by (4.10), (4.11) and (4.12), we obtain the required result. □

REMARK 4.16. Let $k(r) = a(\log(e+1/r))^b$ with $a \geqslant e^b(\log(e+1))^{-b}$ and $b \geqslant 1/N$. Then

$$\Psi_\alpha(r) \sim r^{N/(N-\alpha)}(\log(e+r))^{b\alpha N/(N-\alpha)+bN-1}$$

as in Remark 3.4. Let $\alpha + \theta < N$. If $\|f\|_{\mathcal{H}^{\Phi_d,1,\nu}(\mathbf{B})} \leqslant 1$ and $-N+\alpha+\theta < \nu < 0$, then we can find a constant $C > 0$ such that

$$\begin{aligned}\int_{\mathbf{B}} |x|^\nu \Big\{ I_\alpha f(x)^{N/(N-\alpha)}&(\log(e + I_\alpha f(x)))^{b\alpha N/(N-\alpha)+bN-1} \\ &+ (b(x) I_\alpha f(x))^{p_2^*}\Big\}\, dx \\ \leqslant C.&\end{aligned}$$

REFERENCES

[1] D. R. Adams and J. Xiao, *Morrey spaces in harmonic analysis*, Ark. Mat. **50**(2) (2012), 201–230.
[2] A. Almeida and D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl. **394**(2) (2012), 781–795.
[3] P. Baroni, M. Colombo and G. Mingione, *Non-autonomous functionals, borderline cases and related function classes*, St Petersburg Math. J. **27** (2016), 347–379.
[4] V. I. Burenkov, A. Gogatishvili, V. S. Guliyev and R. Ch. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55**(8–10) (2010), 739–758.
[5] V. I. Burenkov, A. Gogatishvili, V. S. Guliyev and R. Ch. Mustafayev, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35**(1) (2011), 67–87.
[6] V. I. Burenkov and H. V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Stud. Math. **163**(2) (2004), 157–176.
[7] F. Colasuonno and M. Squassina, *Eigenvalues for double phase variational integrals*, Ann. Mat. Pura Appl. (4) **195**(6) (2016), 1917–1959.

https://doi.org/10.1017/nmj.2019.18 Published online by Cambridge University Press

[8] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Ser. Math. **28** (2003), 223–238; **29** (2004), 247–249.

[9] M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), 443–496.

[10] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces With Variable Exponents*, Lecture Notes in Mathematics **2017**, Springer, Heidelberg, 2011.

[11] L. Esposito, F. Leonetti and G. Mingione, *Sharp regularity for functionals with (p, q) growth*, J. Differential Equations **204**(1) (2004), 5–55.

[12] I. Fonseca, J. Malý and G. Mingione, *Scalar minimizers with fractal singular sets*, Arch. Ration. Mech. Anal. **172**(2) (2004), 295–307.

[13] T. Futamura and Y. Mizuta, *Maximal functions for Lebesgue spaces with variable exponent approaching* 1, Hiroshima Math. J. **36**(1) (2006), 23–28.

[14] V. S. Guliyev, J. J. Hasanov and S. G. Samko, *Maximal, potential and singular operators in the local "complementary" variable exponent Morrey type spaces*, J. Math. Anal. Appl. **401**(1) (2013), 72–84.

[15] V. S. Guliyev, S. G. Hasanov and Y. Sawano, *Decompositions of local Morrey-type spaces*, Positivity **21**(1) (2017), 1223–1252.

[16] P. Harjulehto, P. Hästö and A. Karppinen, *Local higher integrability of the gradient of a quasiminimizer under generalized Orlicz growth conditions*, Nonlinear Anal. **177** (2018), 543–552.

[17] P. Hästö, *The maximal operator in Lebesgue spaces with variable exponent near* 1, Math. Nachr. **280**(1–2) (2007), 74–82.

[18] P. Hästö, *The maximal operator on generalized Orlicz spaces*, J. Funct. Anal. **269**(12) (2015), 4038–4048; Corrigendum to *The maximal operator on generalized Orlicz spaces*, J. Funct. Anal. **271**(1) (2016), 240–243.

[19] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618.

[20] F. Y. Maeda, Y. Mizuta and T. Shimomura, *Variable exponent weighted norm inequality for generalized Riesz potentials on the unit ball*, Collect. Math. **69** (2018), 377–394.

[21] Y. Mizuta and T. Ohno, *Sobolev's theorem and duality for Herz–Morrey spaces of variable exponent*, Ann. Acad. Sci. Fenn. Math. **39** (2014), 389–416.

[22] Y. Mizuta and T. Ohno, *Herz–Morrey spaces of variable exponent, Riesz potential operator and duality*, Complex Var. Elliptic Equ. **60**(2) (2015), 211–240.

[23] Y. Mizuta, T. Ohno and T. Shimomura, *Integrability of maximal functions for generalized Lebesgue spaces with variable exponent*, Math. Nachr. **281** (2008), 386–395.

[24] Y. Mizuta, T. Ohno and T. Shimomura, "*Integrability of maximal functions for generalized Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$*", in *Potential Theory and Stochastics in Albac*, Theta Series in Advanced Mathematics **11**, Theta, Bucharest, 2009, 193–202.

[25] Y. Mizuta and T. Shimomura, *Differentiability and Hölder continuity of Riesz potentials of Orlicz functions*, Analysis (Munich) **20**(3) (2000), 201–223.

[26] S. Samko, *Variable exponent Herz spaces*, Mediterr. J. Math. **10**(4) (2013), 2007–2025.

[27] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

Yoshihiro Mizuta
4-13-11 Hachi-Hom-Matsu-Minami
Higashi-Hiroshima 739-0144
Japan
yomizuta@hiroshima-u.ac.jp

Takao Ohno
Faculty of Education
Oita University
Dannoharu Oita-city 870-1192
Japan
t-ohno@oita-u.ac.jp

Tetsu Shimomura
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524
Japan
tshimo@hiroshima-u.ac.jp