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A new class of Banach space with the drop property

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We discuss a new class of Banach spaces which are wider than the strongly convex spaces introduced by Congxin Wu and Yongjin Li. We prove that the new class of Banach spaces lies strictly between either the class of uniformly convex spaces and strongly convex spaces or the class of fully k-convex spaces and strongly convex spaces. The new class of Banach spaces has inclusive relations with neither the class of locally uniformly convex spaces nor the class of nearly uniformly convex spaces. We obtain in addition some characterizations of this new class of Banach spaces.

1. Introduction

In 1936, Clarkson [2] introduced the concept of uniform convex Banach spaces. Consequently, some methods were found to investigate the property of Banach space from the geometric structure of the unit sphere in Banach space. This initiated the study of convexity of Banach space. Since convexity has a striking intuitive geometric meaning, many mathematicians were attracted to this field of study. Smoothness, later introduced as a dual notion of convexity, is closely related to the various properties of differentiability of norm (a kind of special convex function). This prompted further in-depth study of smoothness and the further development of the smooth theory of Banach space. As two important properties of geometry in Banach space, both convexity and smoothness not only have promoted the development of geometric theories of Banach space, but are also widely applied to such fields as control theory, operator theory, optimal approximation theory and fixed-point theory.

In 1936, the concept of uniformly rotund (UR) Banach spaces was first introduced by Clarkson. In 1955, the locally uniformly rotund (LUR) Banach spaces were introduced by Lovaglia [10] as a generalization of UR spaces. Indeed, this is the local version of UR spaces and need not be reflexive in general. Later, two more generalizations of uniform convexity were introduced for Banach spaces. In 1979, Sullivan [16] introduced the k-uniformly rotund (kUR) spaces, and in 1980 Huff [7] introduced the nearly uniformly convex (NUC) Banach spaces. By 'fixing' one variable, Sullivan gave the local version of kUR spaces, i.e. locally k-uniformly

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rotund (LkUR) spaces. In 1994, Kutzarova and Lin [8] gave the local version of NUC spaces, i.e. locally nearly uniformly convex (LNUC) spaces. Obviously, NUC spaces imply LNUC spaces. Fan and Glicksberg [5, 6] extended the fully 2-convex (2R) Banach spaces which were introduced by Šmulian [15], and studied the fully k-convex (kR) Banach spaces. It is well known that kUR and kR spaces imply reflexivity. In 1993, Wu and Li [17] introduced the strongly convex spaces. In 1998, Wulede and Wu [19] introduced the k-strongly convex spaces as a generalization of strongly convex Banach spaces and proved that kUR spaces imply k-strongly convex spaces. About kUR, LkUR and kR spaces, we have the following chain of implications [4, 5, 10, 21]:

$$\mathrm{2R} \implies \cdots \implies k\mathrm{R} \implies (k+1)\mathrm{R};$$
$$\mathrm{UR} \implies \mathrm{LUR} = \mathrm{L1UR} \implies \mathrm{L2UR} \implies \cdots \implies \mathrm{L}k\mathrm{UR} \implies \mathrm{L}(k+1)\mathrm{UR}.$$

In 1972, Daneš [3] proved the so-called 'drop theorem'. For any Banach space $(X, \|\cdot\|)$ and every closed set $C \subset X$ disjoint from closed unit ball $U(X)$ of X, there exists a point $x \in C$ such that $D(x, U(X)) \cap C = \{x\}$, where the set $D(x, U(X))$, the convex hull of x and $U(X)$, is called the drop generated by $x \notin U(X)$. In 1987, modifying the assumption of the Daneš drop theorem, Rolewicz [13] began the study of the drop property for the closed unit ball. He defined the norm $\|\cdot\|$ to have the drop property if, for every closed set $C \subset X$ disjoint from $U(X)$, there exists an $x \in C$ such that $D(x, U(X)) \cap C = \{x\}$. Later, two characterizations of the drop property were obtained. Montesinos [11] proved that $(X, \|\cdot\|)$ having the drop property is equivalent to reflexive space X possessing the Kadec–Klee property. In 1990, Banaś [1] gave the characterization of the drop property as follows: $(X, \|\cdot\|)$ has the drop property if and only if, for any functional f of norm 1, $\lim_{\epsilon \to 0} \alpha(F(f, \epsilon)) = 0$ holds, where $\alpha(F(f, \epsilon))$ denotes the Kuratowski measure of non-compactness of the set

$$F(f, \epsilon) = \{x \colon x \in X,\ \|x\| \leqslant 1,\ f(x) \geqslant 1 - \epsilon\}.$$

Here we consider a new class of convexity, i.e. uniform extreme convexity, and discuss its relation to the drop property, the strong convexity, the full k-convexity and the uniform convexity, as well as the relations between the local uniform convexity and near uniform convexity. We also give some characterizations of uniform extreme convexity. Throughout this paper X denotes an infinite-dimensional real Banach space with the norm $\|\cdot\|$. The symbol X^* denotes the dual of the space X. $U(X)$ and $S(X)$ denote the closed unit ball and the unit sphere of X, respectively. $S(X^*)$, $S(X^{**})$ and $S(X^{***})$ denote the unit spheres of X^*, X^{**} and X^{***}, respectively.

A Banach space X is said to be a UR space [2] if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that, for any norm-1 elements x and y, $\|x+y\| > 2-\delta$ and $\|x-y\| < \epsilon$.

A Banach space X is said to be an LUR space [10] if for any norm-1 element x and $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, x) > 0$ such that, for any norm-1 element y, $\|x + y\| > 2 - \delta$ and $\|x - y\| < \epsilon$.

A Banach space X is said to be a kUR ($k \geqslant 1$) space [16] if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for all norm-1 elements $x_1, x_2, \dots, x_{k+1}$, $\|x_1 + x_2 + \cdots +$

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$x_{k+1}\| > (k+1) - \delta$, and

$$A(x_1, x_2, \dots, x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_1, \dots, f_k \in S(X^*)\right\} < \epsilon.$$

A Banach space X is said to be an NUC space [7] if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for any sequence $\{x_n\} \subset U(X)$, $\operatorname{sep}(x_n) > \epsilon$, we have $\operatorname{co}(\{x_n\}) \cap (1-\delta)U(X) \neq \emptyset$, where $\operatorname{sep}(x_n) = \inf\{\|x_n - x_n\| : n \neq m\}$ and $\operatorname{co}(\{x_n\})$ means the convex hull of $\{x_n\}$.

A Banach space X is said to be an LNUC space [8] if for any norm-1 element x and $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, x) > 0$ such that, for any sequence $\{x_n\} \subset U(X)$, $\operatorname{sep}(x_n) > \epsilon$, we have $\operatorname{co}(\{x\} \cup \{x_n\}) \cap (1-\delta)U(X) \neq \emptyset$, where $\operatorname{co}(\{x\} \cup \{x_n\})$ means the convex hull of $\{x\}$ and $\{x_n\}$.

A Banach space X is said to be a kR space ($k \geqslant 2$) [5] if, for any sequence $\{x_n\}$ in X such that

$$\lim_{n_1,\dots,n_k\to\infty} \frac{1}{k}\|x_{n_1} + x_{n_2} + \cdots + x_{n_k}\| = 1,$$

$\{x_n\}$ is a Cauchy sequence in X.

A Banach space X is said to be a strongly convex space [17] if, for any $x \in S(X)$, $\{x_n\} \subset S(X)$ and for a certain functional $f \in S_x$ such that $f(x_n) \to 1$, $n \to \infty$, then $\|x_n - x\| \to 0$, $n \to \infty$, where $S_x = \{f : f \in S(X^*),\ f(x) = 1\}$.

A Banach space X is said to be a k strongly convex space [20] if, for any norm-1 element x, $\epsilon > 0$ and for any functional $f \in S_x$, there is a $\delta(x, f, \epsilon) > 0$ such that for all norm-1 elements $x_1, \dots, x_k$ and $f(x + x_1 + \cdots + x_k) > (k+1) - \delta$, we have $A(x, x_1, \dots, x_k) < \epsilon$.

Theorem 1.1 (Kadec–Klee property). *If any $x \in S(X)$, $\{x_n\} \subset S(X)$ such that $x_n \xrightarrow{w} x$, $n \to \infty$, and $\|x_n\| \to \|x\|$, $n \to \infty$, then $\|x_n - x\| \to 0$, $n \to \infty$, where $x_n \xrightarrow{w} x$, $n \to \infty$, means that $f(x_n) \to f(x)$, $n \to \infty$, for all $f \in X^*$.*

In §§ 2 and 3, we shall use the following four lemmas.

Lemma 1.2 (Banaś [1]; Montesinos [11]). *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X has the drop property;*

(ii) *for any $f \in S(X^*)$, $\lim_{\epsilon\to 0} \alpha(F(f, \epsilon)) = 0$;*

(iii) *the reflexive space X has the Kadec–Klee property.*

Lemma 1.3 (Wulede [18]; Wu and Li [17]). *Let X be a Banach space. Then*

(i) *if X is a strictly convex k-strongly convex (respectively, LUR or kR) space, X is a strongly convex space,*

(ii) *if X is a strongly convex space, X is a strictly convex space having the Kadec–Klee property.*

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LEMMA 1.4 (Wu and Li [17]). *X is a strongly convex space if and only if for any $\epsilon > 0, x \in S(X)$ and $f \in S_x$ there exists a $\delta(x, f, \epsilon) > 0$ such that, for any $y \in S(X)$, $f(x+y) > 2-\delta$, we have $\|x-y\| < \epsilon$.*

LEMMA 1.5 (Wu and Li [17]). *Let X be a reflexive Banach space. X is then strongly convex if and only if every point of $S(X)$ is a denting point of $U(X)$, i.e. if every $x \in S(X)$, then $x \notin \overline{\text{co}}(M(x,\epsilon))$ for all $\epsilon > 0$, where $M(x,\epsilon) = \{y\colon y \in U(X),\ \|y-x\| \geqslant \epsilon\}$.*

2. The definition and characterizations of uniformly extremely convex spaces

DEFINITION 2.1. A Banach space X is said to be a uniformly extremely convex space if, for any sequences $\{x_n\}$, $\{y_n\}$ consisting of elements of norm-1 for a certain functional f of norm 1, and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = 1$ holds, $\|x_n - y_n\| \to 0$, $n \to \infty$.

We shall need the following simple but useful proposition.

PROPOSITION 2.2. *X is uniformly extremely convex space if and only if, for any $\epsilon > 0$, $f \in S(X^*)$, there exists a $\delta(\epsilon, f) > 0$ such that, for any $x, y \in S(X)$, $f(x+y) > 2-\delta$, we have $\|x-y\| < \epsilon$.*

Proof of necessity. Suppose the contrary. Then there exist $\epsilon_0 > 0$, $f_0 \in S(X^*)$ and $x, y \in S(X)$ such that for any $\delta = 1/n$, $n \in \mathbb{N}$, we have $f_0(x+y) > 2 - 1/n$, but $\|x-y\| \geqslant \epsilon_0$. Take $x_n = x$, $y_n = y$. Then $\{x_n\}, \{y_n\} \subset S(X)$, $\|x_n - y_n\| \geqslant \epsilon_0$ and $2 - 1/n < f_0(x_n + y_n) \leqslant 2$. It follows that

$$\lim_{n\to\infty} f_0(x_n) = \lim_{n\to\infty} f_0(y_n) = 1.$$

On the other hand, by the definition of the uniformly extremely convex space, we have $\|x_n - y_n\| \to 0$, $n \to \infty$; this contradicts the statement that $\|x_n - y_n\| \geqslant \epsilon_0$. □

Proof of sufficiency. If, for any $\{x_n\}, \{y_n\} \subset S(X)$, $f \in S(X^*)$,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = 1$$

holds, then $f(x_n + y_n) \to 2$, $n \to \infty$. Therefore, for any $\delta > 0$, there exists an integer $N_0 \in \mathbb{N}$ such that, for all $n \geqslant N_0$, $f(x_n + y_n) > 2 - \delta$. For any $\epsilon > 0$, by the conditions given in proposition 2.2, we have $\|x_n - y_n\| < \epsilon$; this means that $\|x_n - y_n\| \to 0$, $n \to \infty$. □

THEOREM 2.3. *X is uniformly extremely convex if and only if X is strictly convex and has the drop property.*

Proof of necessity. Suppose that X is a uniformly extremely convex space. Then, as a simple consequence of lemma 1.4 and proposition 2.2 we know that X is a strongly convex space. Hence, X is strictly convex and has the Kadec–Klee property. Now we prove that X is reflexive. For any $f \in S(X^*)$, there exists $\{x_n\} \subset S(X)$ satisfying $f(x_n) \to 1$, $n \to \infty$. Hence, for each $\epsilon > 0$ there exists an integer $N > 0$ such that $f(x_n) > 1 - \frac{1}{2}\delta(\epsilon)$ as $n > N$. It follows that $f(x_n + x_m) > 2 - \delta(\epsilon)$, $n, m > N$, where $\delta(\epsilon)$ is the function required in the definition of the uniformly extremely convex space. From the definition of the uniformly extremely convex space, we have $\|x_n - x_m\| < \epsilon$ for any $n, m > N$. This means that x_n is a Cauchy sequence in Banach space X; hence, there exists an $x \in S(X)$ such that $\|x_n - x\| \to 0$, $n \to \infty$. Furthermore, $f(x) = 1$. From James's theorem we know that X is reflexive. □

Proof of sufficiency. If X is not uniformly extremely convex, then there exist $f_0 \in S(X^*)$, $\{x_n^0\}, \{y_n^0\} \subset S(X)$ such that $\lim_{n\to\infty} f_0(x_n^0) = \lim_{n\to\infty} f_0(y_n^0) = 1$, but there exist some $\epsilon_0 > 0$ and a subsequence $\{n_k\} \subset \{n\}$ satisfying $\|x_{n_k}^0 - y_{n_k}^0\| \geqslant \epsilon_0$. On the other hand, X is reflexive since X has the drop property. Therefore, $\{x_{n_k}^0\}$ has a weak convergent subsequence, without loss of generality, and, letting the weak convergent subsequence itself be $\{x_{n_k}^0\}$, we have

$$x_{n_k}^0 \xrightarrow{w} x^0, \quad k \to \infty,$$

where $x^0 \in X$. Similarly, $y_{n_k}^0$ has a weak convergent subsequence $y_{n_{k_j}}^0$ such that

$$y_{n_{k_j}}^0 \xrightarrow{w} y^0, \quad j \to \infty,$$

where $y^0 \in X$. By the Banach–Mazur theorem, we know that $\|x^0\| \leqslant 1$, $\|y^0\| \leqslant 1$. As

$$\lim_{n\to\infty} f_0(x_n^0) = \lim_{n\to\infty} f_0(y_n^0) = 1,$$

we know that $f_0(x^0) = f_0(y^0) = 1$; thus, $x^0, y^0 \in S(X)$ and $1 \geqslant \|\frac{1}{2}(x^0 + y^0)\| \geqslant f_0(\frac{1}{2}(x^0 + y^0)) = 1$. By the conditions given in theorem 2.3, we know that X is strictly convex and has the Kadec–Klee property. It follows that $x^0 = y^0$ and $\|x_{n_{k_j}}^0 - x^0\| \to 0$, $\|y_{n_{k_j}}^0 - x^0\| \to 0$, $j \to \infty$. Furthermore, $\|x_{n_{k_j}}^0 - y_{n_{k_j}}^0\| \to 0$, $j \to \infty$. This contradicts $\|x_{n_k}^0 - y_{n_k}^0\| \geqslant \epsilon_0$. □

Corollary 2.4. *Let X be a Banach space. Then the following are equivalent:*

(i) *X is uniformly extremely convex;*

(ii) *X is strongly convex and reflexive;*

(iii) *X is strictly convex and, for any $f \in S(X^*)$, $\lim_{\epsilon\to 0} \alpha(F(f, \epsilon)) = 0$;*

(iv) *if, for any sequences $\varphi_n, \psi_n \in S(X^{**})$ and a certain functional f of norm 1, $\lim_{n\to\infty} \varphi_n(f) = \lim_{n\to\infty} \psi_n(f) = 1$ holds, then $\|\varphi_n - \psi_n\| \to 0$, $n \to \infty$.*

Proof.
(ii) $\Longrightarrow$ (i). This follows immediately from lemmas 1.2 and 1.3 and theorem 2.3.

(i) $\Longrightarrow$ (ii). This follows immediately from proposition 2.2, lemma 1.4 and theorem 2.3.

The equivalence between (i) and (iii) follows immediately from theorem 2.3 and lemma 1.2. The equivalence between (i) and (iv) follows immediately from definition 2.1 and the reflexivity of X. □

Theorem 2.5. *X is a uniformly extremely convex space if and only if X is reflexive and every point of $S(X)$ is a denting point of $U(X)$.*

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Proof. This follows immediately from lemma 1.5 and corollary 2.4. □

THEOREM 2.6. *X is a uniformly extremely convex space if and only if, for each $f \in S(X^*)$, the limit*

$$\lim_{t\to 0^+} \frac{\|f+tF\| + \|f-tF\| - 2}{t} = 0 \tag{2.1}$$

exists uniformly for $F \in S(X^{***})$.

Proof. If (2.1) does not hold, then there exist $\epsilon_0, F_k \in S(X^{***})$ and t_k with $0 < t_k < 1/k$ such that

$$\frac{\|f+t_kF_k\| + \|f-t_kF_k\| - 2}{t_k} \geqslant \epsilon_0.$$

Hence, from $\|f-t_kF_k\| + \|f+t_kF_k\| - 2 \geqslant \epsilon_0 t_k$, it follows that

$$2 + \epsilon_0 t_k - 2t_k^2 \leqslant \|f+t_kF_k\| + \|f-t_kF_k\| - 2t_k^2. \tag{2.2}$$

For each t_k let u_k, v_k be chosen so that $u_k, v_k \in S(X^{**})$ and

$$f(u_k) + t_kF_k(u_k) > \|f+t_kF_k\| - t_k^2, \tag{2.3}$$

$$f(v_k) - t_kF_k(v_k) > \|f-t_kF_k\| - t_k^2. \tag{2.4}$$

Clearly, $u_k(f) \to 1$ and $v_k(f) \to 1$ as $k \to \infty$. However, combining (2.3) and (2.4), we have

$$\|f+t_kF_k\| + \|f-t_kF_k\| - 2t_k^2 < f(u_k+v_k) + t_kF_k(u_k-v_k) \leqslant 2 + t_kF_k(u_k-v_k). \tag{2.5}$$

Combining (2.5) and (2.2), we have

$$2 + \epsilon_0 t_k - 2t_k^2 < 2 + t_kF_k(u_k-v_k).$$

It follows that

$$\epsilon_0 - 2t_k < F_k(u_k-v_k) \leqslant \|F_k\|\|u_k-v_k\|.$$

Let $k \to \infty$. Then we have

$$\epsilon_0 \leqslant \|u_k - v_k\|. \tag{2.6}$$

On the other hand, from the definition of uniformly extremely convex space, we also have $\|u_k - v_k\| \to 0$; this contradicts (2.6).

Suppose that for any $f \in S(X^*)$, $\{\varphi_k\}$ and $\{\psi_k\} \subset S(X^{**})$ we have $\varphi_k(f) \to 1$, $\psi_k(f) \to 1$, $k \to \infty$. Then, by (2.1), we know that for $\epsilon > 0$ there exists T, $0 < T < 1$, such that

$$\left|\frac{\|f+tF\| + \|f-tF\| - 2}{t}\right| < \frac{\epsilon}{3}$$

for all $F \subset S(X^{***})$ and $0 < t < T$. Take t satisfying $0 < t < \min\{\epsilon, T\}$. Then there exists an integer $K > 0$ such that for any $k > K$ we have

$$\left|\frac{\varphi_k(f) - 1}{t}\right| < \frac{\epsilon}{3}, \qquad \left|\frac{\psi_k(f) - 1}{t}\right| < \frac{\epsilon}{3}.$$

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On the other hand,

$$\left|\frac{\|f+tF\|+\|f-tF\|-2}{t}\right| \geqslant \frac{|f(\varphi_k)+tF(\varphi_k)|+|f(\psi_k)-tF(\psi_k)|-2}{t}$$
$$\geqslant \frac{f(\varphi_k)+tF(\varphi_k)+f(\psi_k)-tF(\psi_k)-2}{t}$$
$$= \frac{f(\varphi_k)-1}{t}+\frac{f(\psi_k)-1}{t}+F(\varphi_k-\psi_k).$$

Finally, for any $k>K$ we have

$$F(\varphi_k-\psi_k) \leqslant \left|\frac{\|f+tF\|+\|f-tF\|-2}{t}\right| + \left|\frac{f(\varphi_k)-1}{t}\right| + \left|\frac{f(\psi_k)-1}{t}\right| < \epsilon,$$

and, similarly, $F(\psi_k-\varphi_k)<\epsilon$. Hence, we have $\|\varphi_k-\psi_k\|<\epsilon$. From corollary 2.4 we know that X is a uniformly extremely convex space. □

Corollary 2.7. *X is a uniformly extremely convex space if and only if X is reflexive and, for each $f \in S(X^*)$, the limit*

$$\lim_{t\to 0^+}\frac{\|f+tg\|+\|f-tg\|-2}{t}=0$$

exists uniformly for $g\in S(X^*)$.

Proof. The necessity can be immediately obtained from theorems 2.3 and 2.6. The sufficiency is immediate from theorem 2.6 and the reflexivity of X. □

3. The relations between uniform extreme convexity and various other types of convexity

Theorem 3.1. *If X is a UR space, then X is a uniformly extremely convex space.*

Proof. It is clear from definition of UR and proposition 2.2 that UR space implies uniformly extremely convex space. □

Theorem 3.2. *If X is a kR space, then X is a uniformly extremely convex space.*

Proof. Noticing the fact that kR space implies reflexivity, it is easy to see that the kR space implies a uniformly extremely convex space from lemma 1.3 and corollary 2.4. □

The converse implication of the theorem 3.1 is not true.

Example 3.3. There exists a uniformly extremely convex space X which is not an LUR space.

Let $E=(l_2,\|\cdot\|)$. For $x=(a_1,a_2,\dots)\in E$, define

$$\|x\|^2=\{|a_1|+(a_2^2+a_3^2+\cdots)^{1/2}\}^2+\left\{\left(\frac{a_2}{2}\right)^2+\cdots+\left(\frac{a_n}{n}\right)^2+\cdots\right\}.$$

It follows from [5] that $X=(\Sigma\oplus E)_{l_2}$ is a 2R space, but is not an LkUR space [12]. Furthermore, X is not an LUR space; hence, X is not a UR space. On the other

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hand, by theorem 3.2 we know that 2R space implies uniformly extremely convex space.

Example 3.4. There exists a strongly convex space X which is not a uniformly extremely convex space.

In fact, we consider a non-reflexive LUR space X [4,10]. Then, by lemma 1.3, we know that X is a strongly convex space, but is not a uniformly extremely convex space because X is non-reflexive.

By corollary 2.4, theorem 3.1, and examples 3.3 and 3.4, we have the following.

Remark 3.5.

(i) The class of uniformly extremely convex spaces lies strictly between the class of uniformly convex spaces and the class of strongly convex spaces.

(ii) There are no inclusion relations between the class of locally uniformly convex spaces and the class of uniformly extremely convex spaces.

The converse implication of the theorem 3.2 is not true.

Example 3.6. There exists a uniformly extremely convex space X which is not a kR space for every $k \geqslant 2$.

Let $k \geqslant 2$ be an integer, and let $i_1 < i_2 < \cdots < i_k$. For each $x = (a_1, a_2, \dots) \in l_2$, define

$$\|x\|^2_{i_1,\dots,i_k} = \left(\sum_{j=1}^{k} |a_{i_j}|\right)^2 + \sum_{i \neq i_1,\dots,i_k} a_i^2,$$

and let $X_{i_1,\dots,i_k} = (l_2, \|\cdot\|_{i_1,\dots,i_k})$. For $x \in l_2$, let

$$\|x\|_1 = \sup_{i_1<i_2<\cdots<i_k} \|x\|_{i_1,\dots,i_k}, \quad X_1 = (l_2, \|x\|_1).$$

Define

$$\|x\|^2 = \|x\|_1^2 + \sum_{i=1}^{\infty} \frac{a_i^2}{2^i} \quad \text{for all } x = (a_1, a_2, \dots) \in l_2.$$

It follows from [9] that $(X_1, \|\cdot\|)$ is a strictly convex kUR space, but is not a kR space. Hence, X is a k-strongly convex space. By lemma 1.3, we know that X is a strongly convex space. Noticing that kUR space implies reflexivity, it follows from corollary 2.4 that X is a uniformly extremely convex space.

By theorem 3.2, corollary 2.4 and example 3.4 and 3.6, we have the following.

Remark 3.7. The class of uniformly extremely convex spaces lies strictly between the classes of fully k-convex spaces and strongly convex spaces.

Example 3.8. There exists a uniformly extremely convex space X which is not an NUC space.

Let $(X, \|\cdot\|)$ be a 2R space with normalized basis $\{e_n\}$. Define,

$$\text{for all } x = \sum_{n=1}^{\infty} a_n e_n \text{ in } X, \quad |||x||| = \left\{ \left(|a_1| + \left\| \sum_{n=2}^{\infty} a_n e_n \right\| \right)^2 + \sum_{n=2}^{\infty} \left(\frac{a_n}{n} \right)^2 \right\}^{1/2}.$$

Then, by [8], we know that $(X, |||\cdot|||)$ is a 2R Banach space, but is not an LNUC space. Furthermore, X is not an NUC space. On the other hand, by theorem 3.2 we know that X is uniformly extremely convex space.

Example 3.9. There exists an NUC space X which is not a uniformly extremely convex space.

In fact, we consider a non-strictly convex NUC space X [14]. Then by theorem 2.3 we know that X is not a uniformly extremely convex space.

By Examples 3.8 and 3.9, we have the following.

Remark 3.10. There are no inclusion relations between the class of nearly uniformly convex spaces and the class of uniformly extremely convex spaces.

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