Proceedings of the Royal Society of Edinburgh, page 1 of 18
DOI:10.1017/prm.2022.89

A geometric interpretation of Ranicki duality

Frank Connolly
Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA (connolly.1@nd.edu)

(Received 25 April 2022; accepted 23 November 2022)

Consider a commutative ring R and a simplicial map, $X \xrightarrow{\pi} K$, of finite simplicial complexes. The simplicial cochain complex of X with R coefficients, $\Delta^* X$, then has the structure of an (R, K) chain complex, in the sense of Ranicki . Therefore it has a Ranicki-dual (R, K) chain complex, $T\Delta^* X$. This (contravariant) duality functor $T : \mathcal{B}R_K \to \mathcal{B}R_K$ was defined algebraically on the category of (R, K) chain complexes and (R, K) chain maps.

Our main theorem, 8.1, provides a natural (R, K) chain isomorphism:

$$T\Delta^* X \cong C(X_K)$$

where $C(X_K)$ is the cellular chain complex of a CW complex X_K. The complex X_K is a (nonsimplicial) subdivision of the complex X. The (R, K) structure on $C(X_K)$ arises geometrically.

Keywords: Manifolds; Surgery; K-Theory

1. Introduction; description of results

This article is an addition to a theory of blocked surgery, pioneered by Ranicki, augmented by others in [**1**, **4**, **5**, **6**, **8**, **9**, **16**, **17**], and still in a developing state.

Let R be a commutative ring; let K be a finite simplicial complex. In [**16**] Ranicki introduced the category of (R, K) chain complexes and chain maps denoted $\mathcal{B}R_K$ here. He also defined algebraically, a contravariant functor $T : \mathcal{B}R_K \to \mathcal{B}R_K$.[1]

The simplest geometric example of an (R, K) chain complex arises from a K-*space* (X, π). This is a finite simplicial complex X and a simplicial map, $\pi : X \to K$. In that case, the simplicial cochains on X (with R coefficients) form an (R, K) chain complex denoted $\Delta^* X$.

At the same time, (X, π) specifies a regular CW complex X_K, which is a (nonsimplicial) subdivision of X. We show that the cellular chain complex (with R coefficients) of X_K forms a second (R, K) chain complex $C(X_K)$.

Our main theorem, theorem 8.1, exhibits a geometrically defined chain isomorphism between $C(X_K)$ and $T\Delta^* X$. Roughly put:

$$T\Delta^* X = C(X_K).$$

[1]The duality functor T for (R, K^{op}) complexes seems to play a lesser role at present in the geometric contexts of interest here.

It is also our aim to give a transparent definition of this duality functor T, a clear treatment of Ranicki's natural transformation $e : T^2 \to id.$ and a simple proof that $e_C : T^2C \to C$ is an (R, K) chain equivalence for all C.

Our larger goal is to facilitate applications of Ranicki's theory to geometric questions such as the topological rigidity of non-positively curved groups as in [**4**, **5**, **9**, **10**] when those groups have elements of finite order.

The vehicle for such applications would be a full blown *K-blocked surgery theory* of which there are only hints in [**16**]. This would start with a degree-one normal map between closed manifolds, $(f, b) : (M, \nu(M)) \to (X, \xi)$ (as in [**2**]) together with a reference map, $\pi : X \to K$ as above. It would seek an L-theoretic obstruction to finding a normal cobordism of (f, b) to a '*K-blocked homotopy equivalence*,' $M' \to X$. But we will not pursue this here or even define the terms precisely.

In the classical case ($K =$ point; [**2**, **12**, **18**, **19**, **20**]) one has the 'surgery obstruction' $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ to such a normal cobordism. But this functor $L_n()$, was generalized in [**16**] to yield obstruction groups $L_n(\mathcal{A})$ for any 'category-with-chain-involution' $(A, *, \epsilon)$. Here A is an additive category, $\mathcal{B}A \xrightarrow{*} \mathcal{B}A$ is a contravariant functor satisfying certain conditions, on the category $\mathcal{B}A$, of finite chain complexes in A, and $\epsilon : (*)^2 \to id$, is an equivalence in the homotopy category of $\mathcal{B}A$.

Ranicki, in [**16**], then starts with a finite complex K and a category with chain involution, $\mathcal{A} = (A, *, \varepsilon)$ as above. He then constructs the additive category A_K of K-blocked objects from A, and K-blocked A-maps. From $*$, and ε, he defines the *Ranicki Duality Functor* $T : \mathcal{B}(A_K) \to \mathcal{B}(A_K)$, and the natural transformation $e : T^2 \to id_{\mathcal{B}(A_K)}$. This construction allows one to define the surgery obstruction groups, $L_n(\mathcal{A}_K)$ where $\mathcal{A}_K = (A_K, T, e)$.

This seems to apply directly to a K-blocked normal map, $M^n \xrightarrow{(f,b)} X^n \xrightarrow{\pi} K$. Here the relevant category seems to be $A = A(R)$, the category of finitely generated free modules over a fixed commutative ring R. We write AR_K for $(AR)_K$ and $\mathcal{B}R_K$ for $\mathcal{B}(AR_K)$. Its objects are (R, K)-*chain complexes*. So the simplicial cochain complexes of X and M denoted Δ^*X and Δ^*M, and the simplicial chain complexes, $\Delta X'$ and $\Delta M'$, are (R, K)-chain complexes. (See § 3). Thus the L-groups of $(\mathcal{A}R_K, T, e)$ seem likely to be useful.

However, Ranicki's definition of $\mathcal{B}R_K \xrightarrow{T} \mathcal{B}R_K$ was only a starting point. Indeed his assertion in [**16**] of the *crucial* theorem that $(\mathcal{A}R_K, T, e)$ is a category with chain involution was only proved in 2018 (by Adams-Florou and Macko, [**1**]).

This paper interprets Ranicki's notions geometrically. Section 2 fixes chain-complex conventions. Section 3 reviews Ranicki's concepts concerning (R, K) complexes while attempting to simplify notation. In § 5 we introduce the (R, K) chain complex $C \otimes_K D$, defined if D is an (R, K) complex and C is an (R, K^{op}) complex. This complex $C \otimes_K D$ is a certain quotient of $C \otimes_R D$.

Our definition (see 6.1) of the Ranicki dual TC, of an (R, K) complex C, is:

$$TC = C^* \otimes_K \Delta^* K.$$

In § 7 we show, using work of M. Cohen [**3**], that each K-space (X, π) defines a certain regular CW-complex X_K, whose cellular chain complex has a natural

(R, K) structure. Therefore from each K-space (X, π) we obtain three (R, K) chain complexes:

(1) $\Delta^* X$, the simplicial cochain complex of X (definition 4.1).

(2) $C(X_K)$, the cellular chain complex of the CW complex X_K (§ 8).

(3) $\Delta X'$, the simplicial chain complex of X', the barycentric subdivision of X (definition 7.2).

This paper shows that these three are closely related by T. Our main result, theorem 8.1, exhibits an isomorphism of (R, K) chain complexes:

$$\Phi_X : T\Delta^* X \cong C(X_K)$$

Then, using [**13**], we prove there are (R, K) chain homotopy equivalences:

$$T\Delta X' \simeq \Delta^* X; \quad C(X_K) \simeq \Delta X'.$$

When X is a pl-manifold, and $C = C(X_K)$, Poincare duality then becomes an n-cycle in the (R, K^{op}) complex, $Hom_{(R,K)}(TC, C)$.

This CW complex X_K is a subdivision of X, and X' is a simplicial subdivision of X_K. In fact, for each simplex S of X and each face σ of $\pi(S) \in K$, there is a single cell S_σ of X_K. Specifically, if $D(\sigma, \pi(S))$ is the dual cell of σ in $\pi(S)$:

$$S_\sigma = (\pi \mid S)^{-1} |D(\sigma, \pi(S))| \,.$$

The author is indebted to Jim Davis for his helpful comments. He also wants to acknowledge, with thanks, the referee's suggestions for improving the text.

2. Chain complex conventions

Throughout this paper, R denotes a fixed commutative ring; AR is the additive category of finitely generated free R modules.

For any additive category A we will write $\mathcal{B}A$ for the additive category of finite chain complexes, $C = \{C_q, \partial_q\}_{q\in\mathbb{Z}}$ and chain maps $f = \{f_q : C_q \to D_q\}_{q\in\mathbb{Z}}$ from A. (*Finite* means: $C_q = 0$ for all but finitely many q). We abbreviate $\mathcal{B}(AR)$ to $\mathcal{B}R$.

As usual two chain maps $f, g : C \to D$ are chain homotopic if there is a sequence of A maps, $h = \{h_q : C_q \to D_{q+1}\}$, for which $d^D_{q+1} h_q + h_{q-1} d^C_q = g_q - f_q \ \forall q$.

We regard A as the full subcategory of $\mathcal{B}A$ consisting of chain complexes concentrated in degree zero.

Let $C, D \in Ob(\mathcal{B}R)$. The complexes $C \otimes_R D$, and $Hom_R(C, D)$ in $Ob(\mathcal{B}R)$, are:

$$\boxed{(C \otimes_R D)_q = \sum_{r\in\mathbb{Z}} C_r \otimes_R D_{q-r}; \quad Hom_R(C, D)_q = \sum_{r\in\mathbb{Z}} Hom_R(C_r, D_{q+r})} \text{ and:}$$

$$\boxed{d^{C\otimes D}(x \otimes y) = d^C x \otimes y + (-1)^{|x|} x \otimes d^D y; \quad d^{Hom}\phi = d^D \circ \phi - (-1)^{|\phi|} \phi \circ d^C}$$

The *evaluation map*, $eval_{C,D} : Hom_R(C, D) \otimes_R C \longrightarrow D$ is the *R-chain map:*

$$eval_{C,D}(f \otimes x) = f(x).$$

Note that $eval_{R,D} : Hom_R(R, D) \otimes_R R \cong D$.

Write $ev_C : C^* \otimes C \to R$ for $eval_{C,R}$.

The contravariant functor $\mathcal{B}R \xrightarrow{*} \mathcal{B}R$ is : $C^* = Hom(C, R)$; $f^* = Hom(f, 1_R)$.

Therefore we have:

$$(C^*)_{-q} = Hom_R(C_q, R); \quad d^{C^*}_{-q} = (-1)^{q+1}(d^C_{q+1})^* : (C^*)_{-q} \to (C^*)_{-q-1}.$$

The functor $*$ comes with a natural equivalence, $\varepsilon : (*)^2 \to 1_{\mathcal{B}R}$. Specifically, the chain isomorphism $\epsilon_C : C^{**} \to C$ is characterized by the identity:

$$a(\varepsilon_C(\alpha)) = (-1)^q \alpha(a) \quad \forall \alpha \in (C^{**})_q,\ a \in (C^*)_{-q}.$$

3. Basic definitions for (R, K) chain complexes

DEFINITION 3.1. Let K be a *finite poset with partial order* $\leqslant$.

K^{op} denotes the same set with the opposite partial order.

(Later we will specialize to the case when K is a finite simplicial complex).

(1) An *(R,K) module* is an ordered pair $M = (M(K), \{M(\sigma)\}_{\sigma \in K})$ such that:
  (a) $M(K)$ and each $M(\sigma)$ are R-modules in $Ob(\mathcal{A}R)$;
  (b) $M(K) = \oplus_{\sigma \in K} M(\sigma)$.
  More generally, for any $S \subset K$ we write: $M(S) = \oplus_{\sigma \in S} M(\sigma)$.

(2) An (R, K) *map* $M \xrightarrow{f} N$ of (R, K) modules is a map $M(K) \xrightarrow{f} N(K)$ of R modules, whose components, $f(\tau, \sigma) : M(\sigma) \to N(\tau)$, satisfy:

$$f(\tau, \sigma) = 0 \text{ unless } \tau \geqslant \sigma.$$

(3) The additive category of (R, K) maps and modules is written $\mathcal{A}R_K$.
  We abbreviate the category of chain complexes, $\mathcal{B}(\mathcal{A}R_K)$, to $\mathcal{B}R_K$.

(4) An object $C = \{C_q, \partial_q\}_{q \in \mathbb{Z}}$ of $\mathcal{B}R_K$ is an *(R, K) chain complex*. We then write $C(K)$ for $\{C_q(K), \partial_q\}_{q \in \mathbb{Z}}$, an R-chain complex in $ob(\mathcal{B}R)$.
  Note: $C \in ob(\mathcal{B}R_K)$ is specified by specifying the R complex $C(K)$ and the required collection $\{C_q(\sigma)\}_{\sigma \in K, q \in \mathbb{Z}}$ of R submodules.

(5) Let $C, D \in ob(\mathcal{B}R_K)$. $Hom_{(R,K)}(C, D)$ is the (R, K^{op}) complex such that:
  (a) $Hom_{(R,K)}(C, D)(K)$ is the subcomplex of $Hom_R(C(K), D(K))$ given by those $f = \{f_q : C_q \to D_{q+|f|}\}_{q \in \mathbb{Z}}$ for which each f_q is an (R, K) map.
  (b) $Hom_{(R,K)}(C, D)_p(\sigma)$ is the set of $f \in Hom_{(R,K)}(C, D)(K)_p$ satisfying:

$$f_q \mid_{C_q(\tau)} = 0 \text{ if } \tau \neq \sigma,\ \forall\, q.$$

(6) We say a sequence of chain maps $0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ in $\mathcal{B}R_K$ is *exact* if for each $\sigma \neq \tau$, $i(\sigma, \tau) = 0$, $j(\sigma, \tau) = 0$, and, for all q, the corresponding sequence, $0 \to C'_q(\sigma) \to C_q(\sigma) \to C''_q(\sigma) \to 0$. is an exact sequence in $\mathcal{A}R$. We then say i is an (R, K) *monomorphism* and j is an (R, K) *epimorphism*.

(7) Note that $*$ specifies a contravariant functor, $\mathcal{B}R_K \xrightarrow{*} \mathcal{B}R_{K^{op}}$, provided that we define $(C^*)_q(\sigma)$ as $(C_{-q}(\sigma))^*$ and d^{C^*} as $d^{C(K)^*}$ for $C \in ob(\mathcal{B}R_K)$. $\mathcal{B}R_K \xrightarrow{*} \mathcal{B}R_{K^{op}}$ preserves exactness and homotopy. The transformation $\varepsilon_C : C^{**} \to C$ of § 2 is an (R, K) isomorphism, for all $C \in ob(\mathcal{B}R_K)$.

(8) We say $S \subset K$ is *full in* K if, whenever $\rho, \tau \in S$, then:

$$\{\sigma \in K \mid \rho \leqslant \sigma \leqslant \tau\} \subset S.$$

Let C be an (R, K) complex. Let S be a full subset of K. We define $\partial_q^{C(S)} : C_q(S) \to C_{q-1}(S)$ by:

$$\partial_q^{C(S)} x = \sum_{\tau \in S} \partial^C(\tau, \sigma)x, \quad \forall x \in C_q(\sigma), \forall \tau, \sigma \in S.$$

Then $C(S) := \{C_q(S), \partial_q^{C(S)}\}_{q\in\mathbb{Z}}$ is an R chain complex. But in many cases, it is neither a subcomplex nor a quotient complex of $C(K)$.

4. *K* spaces and their chain complexes

For the rest of this paper, K denotes a finite simplicial complex.

A simplicial complex K is a poset so the above definitions apply. In this case $\sigma \leqslant \tau$ means that the simplex σ is a face (not necessarily proper) of the simplex τ.

$\Delta_*(K; R) = \{\Delta_q(K; R), \partial_q\}_{q\in\mathbb{Z}}$ denotes the simplicial chain complex of K.

$\Delta^*(K; R) = Hom_R(\Delta_*(X; R), R)$ denotes the simplicial cochain complex of K.

One can choose a basis, bK for $\Delta_*(K; R)$ consisting of one oriented q-simplex, $\sigma = \langle v_0 \ldots, v_q\rangle \in \Delta_q(K; R)$ for each q-simplex with vertices $v_0, \ldots v_q$, of K. Recall: $\langle v_0, \ldots, v_q\rangle = sgn(\pi)\langle v_{\pi(0)}, \ldots, v_{\pi(q)}\rangle$ for each $\pi \in S_{q+1}$. The oriented q-simplex $\sigma \in \Delta_q(K; R)$ defines a dual cochain $\sigma^* \in \Delta^*(K; R)_{-q}$ such that $\sigma^*(\tau) = 0$ for all $\tau \neq \pm\sigma$, and $\sigma^*(\sigma) = 1$.

One then defines $\sigma^{**} \in \Delta_q(K; R)^{**}$ by: $\varepsilon(\sigma^{**}) = \sigma$.

Each simplex $\sigma \in K$ defines subcomplexes, $\overline{\sigma}$ and $\partial\sigma$, and a subset $st(\sigma)$:

$$\overline{\sigma} = \{\tau \in K \mid \tau \leqslant \sigma\}; \quad \partial\sigma = \{\tau \in K \mid \tau < \sigma\}; \quad st(\sigma) = \{\tau \in K \mid \tau \geqslant \sigma\}$$

The incidence number $[\tau, \sigma] \in \{1, -1, 0\}$ is defined for any oriented simplices σ, τ of K. It satisfies: $\partial_q(\sigma) = \sum_{\tau\in bK}[\sigma, \tau]\tau$ for any basis, bK of oriented simplices of K. $[\sigma, \tau] \neq 0$ iff τ is a codimension-one face of σ.

Definition 4.1. (K-spaces, Δ^*X and ΔX)

Let K be a finite simplicial complex. A K-*space* is a pair (X, π) where X is a finite simplicial complex and $|X| \xrightarrow{\pi} |K|$ is a simplicial map, $X \to K$. A *map of K-spaces*, $(X, \pi_X) \to (Y, \pi_Y)$ is a simplicial map $f : |X| \to |Y|$ satisfying: $\pi_Y f = \pi_X$.

Let (X, π) be a K-space.

ΔX denotes the (R, K^{op}) complex for which $\Delta X(K) = \Delta_*(X; R)$. For each $\sigma \in K$, $(\Delta X)_p(\sigma)$ is the submodule generated by oriented p-simplices in $\Delta_p(X; R)$ whose underlying p-simplex, $S \in X$, satisfies $\sigma = \pi(S) \in K$.

By definition, $\Delta^*X = (\Delta X)^*$. Therefore $\Delta^*X(K) = Hom_R(\Delta_*(X; R), R) = \Delta^*(X; R)$, the simplicial cochain complex of X. For each $\sigma \in K$, $(\Delta^*X)_{-p}(\sigma)$ is

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therefore the submodule spanned by all S^* for which $S \in \Delta_p(X; R)$ is an oriented simplex and $\sigma = \pi(S) \in K$.

A map $f : X \to Y$ of K-spaces induces an (R, K) chain map $f^* : \Delta^* Y \to \Delta^* X$ and an (R, K^{op}) chain map $f_* : \Delta Y \to \Delta X$.

The next lemma will be used in § 6.

LEMMA 4.2. *Suppose $S \in K$ and there is no $\tau \in K$ for which $S < \tau$. The K-space $(\overline{S}, inclusion)$ specifies the (R, K) complex $\Delta^*\overline{S}$. Then $\Delta^*\overline{S}(st(\sigma))$ is a contractible R-complex for all $\sigma \in K$ such that $\sigma \neq S$. Also $\Delta^*\overline{S}(st(S)) = RS^*$.*

Proof. It is obvious that $\Delta^*\overline{S}(st(S)) = RS^*$ (after orienting S) and that $\Delta^*\overline{S}(st(\sigma)) = 0$ if σ is not a face of S. So we assume $\sigma < S$. Let τ be the complementary face of σ in S. Then the joins, $\overline{S} = \sigma * \tau$ and $\partial\sigma * \tau$ are contractible simplicial complexes. Note $st(\sigma) = \overline{S} - \partial\sigma * \tau$. Consequently, $\Delta^*\overline{S}(st(\sigma)) = \Delta^*(\sigma * \tau, \partial\sigma * \tau; R)$ is a contractible chain complex. □

5. $C \otimes_K D$ and the isomorphism $Hom_{(R,K)}(D, C^*) \cong (C \otimes_K D)^*$

Throughout this section, C denotes an (R, K^{op}) complex and D denotes an (R, K) complex.

We will first define two (R, K) complexes: $C \otimes_R D$ and a quotient of this, $C \otimes_K D$.

In K, the star of any simplex , $st(\sigma)$, as well as $K - st(\sigma)$ are *full* in K. Moreover the chain complex $C(K - st(\sigma))$ is a subcomplex of $C(K)$ and $C(st(\sigma))$ is a quotient complex. These fit into a short exact sequence of chain maps in $\mathcal{B}R$:

$$0 \to C(K - st(\sigma)) \xrightarrow{i_\sigma} C(K) \xrightarrow{p_{st(\sigma)}} C(st(\sigma)) \to 0$$

Here $C(K) \xrightarrow{p_{st(\sigma)}} C(st(\sigma))$ is defined by: $p_{st(\sigma)}|_{C_q(st(\sigma))} = 1_{C_q(st(\sigma))}$; and $p_{st(\sigma)}|_{C(K-st(\sigma))} = 0$.

(For the (R, K) complex D, we get $0 \to D(st(\sigma)) \to D(K) \to D(K - st(\sigma)) \to 0$).

DEFINITION 5.1. ($C \otimes_K D$, $C \otimes_R D$, and $C \otimes_R D \xrightarrow{\pi_{C,D}} C \otimes_K D$).

Let C be an (R, K^{op}) complex and D be an (R, K) complex.

(1) Let $C \otimes_R D$ be the (R, K) complex for which:

$$(C \otimes_R D)(K) = C(K) \otimes_R D(K);$$
$$(C \otimes_R D)_q(\rho) = (C(K) \otimes_R D(\rho))_q \quad \forall \rho \in K, q \in \mathbb{Z}$$

(2) Let $C \otimes_K D$ be the (R, K) complex for which:
   (a) $(C \otimes_K D)_q(K) = \sum_{\rho \in K} (C(st(\rho)) \otimes_R D(\rho))_q \ \forall q \in \mathbb{Z}$

   (b) $(C \otimes_K D)(\rho) = C(st(\rho)) \otimes_R D(\rho) \ \forall \ \rho \in K$

   (c) The map $C \otimes_R D \xrightarrow{\pi_{C,D}} C \otimes_K D$ is an (R, K) chain epimorphism, if we define $\pi_{C,D}$ by requiring that $\pi_{C,D}(\sigma, \rho) = 0$ for $\sigma \neq \rho$ and:

$$\pi_{C,D}(\rho, \rho) = p_{st(\rho)} \otimes_R 1_{D(\rho)} : C(K) \otimes_R D(\rho) \longrightarrow C(st(\tau)) \otimes_R D(\rho).$$

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Expicitly, for any $\rho \leqslant \tau$ and $x \otimes_R y \in C_r(\tau) \otimes_R D_{q-r}(\rho) \subset (C \otimes_K D)_q(\rho)$, we have

$$\boxed{d^{C\otimes_K D}(x \otimes y) = \sum_{\{\sigma|\rho\leqslant\sigma\leqslant\tau\}} d^C(\sigma,\tau)x \otimes y + (-1)^r x \otimes d^D(\sigma,\rho)y.} \tag{5.1}$$

We now show that $(C \otimes_K D)^*$ is a convenient expression for $Hom_{(R,K)}(D, C^*)$:

LEMMA 5.2. *There is a natural isomorphism Ψ of functors, denoted,*

$$\Psi_{C,D} : Hom_{(R,K)}(D, C^*) \cong (C \otimes_K D)^* \tag{5.2}$$

for any $(C, D) \in Ob(\mathcal{B}R_{K^{op}} \times \mathcal{B}R_K)$.

Proof. Suppose f is in $Hom_{(R,K)}(D, C^*)_q(\sigma)$ for some $\sigma \in K$ and $q \in \mathbb{Z}$. Define an R-map, $\Psi(f) : C(st(\sigma)) \otimes D(\sigma)_{-q} \to R$, by the formula:

$$\Psi(f)(x \otimes y) = (-1)^{|x||y|} f(y)(x) \quad \text{for } x \otimes y \in (C \otimes_K D)_{-q}(\sigma).$$

The same formula yields 0, if $x \otimes y$ is in $(C \otimes_K D)(\tau)_{-q}$ for $\tau \neq \sigma$. One easily sees that this rule (i.e. $f \mapsto \Psi(f)$ gives an isomorphism,

$$\Psi_{C,D} : Hom_{(R,K)}(D, C^*) \xrightarrow{\cong} (C \otimes_K D)^*$$

of (R, K^{op}) complexes for all $(C, D) \in Ob(\mathcal{B}R_{K^{op}} \times \mathcal{B}R_K)$. Naturality is obvious. □

6. Ranicki Duality and the (R, K) chain equivalence $e : T^2 \to 1_{\mathcal{B}R_K}$

DEFINITION 6.1. Ranicki Duality is the contravariant functor $\mathcal{B}R_K \xrightarrow{T} \mathcal{B}R_K$ defined for a chain complex $C \in Ob(\mathcal{B}R_K)$ and a (R, K) chain map, $f : C \to D$ by:

$$TC = C^* \otimes_K \Delta^* K \quad Tf = f^* \otimes_K 1_{\Delta^* K}$$

$\Delta^* K$ comes from the K-space, $(K, 1_K)$. After examining [16], p. 75 and p. 26, lines -6 to -4 one can see that this is in agreement with the definition indicated there, up to isomorphism and differences in sign conventions. In particular compare our formula for $d^{C\otimes_K D}$ with that on p.26, line -5 of [16].

COROLLARY 6.2. *T is an exact homotopy functor.*

Proof. By lemma 5.2, $TC = C^* \otimes_K \Delta^* K$ is isomorphic to $Hom_{(R,K)}(\Delta^* K, C)^*$ (since $\varepsilon_C : C^{**} \cong C$ for all C). But $C \mapsto C^*$ and $C \mapsto Hom(\Delta^* K, C)$ are both exact homotopy functors. The result follows. □

We now want to show that $T^2 C$ and C are (R, K)-chain equivalent. See 6.5.

DEFINITION 6.3. (of $E_C : Hom_{(R,K)}(\Delta^* K, C) \otimes_K \Delta^* K \to C$).
Let C be an (R, K) complex.

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Consider the evaluation chain map, $eval_{A,B} : Hom_R(A, B) \otimes_R A \to B$, when $A = \Delta^* K(K)$ and $B = C(K)$. Its restriction to $(Hom_{(R,K)}(\Delta^* K, C) \otimes_R \Delta^* K)(K)$, denoted E'_C, is an (R, K) chain map,

$$E'_C : Hom_{(R,K)}(\Delta^* K, C) \otimes_R \Delta^* K \to C$$

(by definition of an (R, K) map). Moreover, for each $\sigma \in K$, E'_C annihilates $Hom_{(R,K)}(\Delta^* K, C)(K - st(\sigma)) \otimes_R \Delta^* K(\sigma)$. Therefore E'_C descends uniquely to an (R, K) chain map,

$$E_C : Hom_{(R,K)}(\Delta^* K, C) \otimes_K \Delta^* K \to C, \quad E_C(f \otimes \sigma^*) = f(\sigma^*).$$

satisfying: $E'_C = E_C \circ \pi_{H,\,\Delta^* K}$. Here $H = Hom_{(R,K)}(\Delta^* K, C)$ (see 5.1).

E is obviously natural in C.

For each (R, K) complex C, define

$$\Psi_{C^*} = \Psi_{C^*,\Delta^* K} : Hom_{(R,K)}(\Delta^* K, C^{**}) \xrightarrow{\cong} (C^* \otimes_K \Delta^* K)^*$$

In view of lemma 5.2. we have an (R, K) chain isomorphism:

$$\Psi_{C^*} \otimes 1_{\Delta^* K} : Hom_{(R,K)}(\Delta^* K, C^{**}) \otimes_K \Delta^* K \xrightarrow{\cong} (C^* \otimes_K \Delta^* K)^* \otimes_K \Delta^* K = T^2 C.$$

Definition 6.4. For each (R, K) complex C define $e_C : T^2 C \to C$ by

$$e_C = \varepsilon_C \circ E_{C^{**}} \circ (\Psi_{C^*} \otimes 1_{\Delta^* K})^{-1} :$$
$$(C^* \otimes_K \Delta^* K)^* \otimes_K \Delta^* K \to Hom_{(R,K)}(\Delta^* K, C^{**}) \otimes_K \Delta^* K \to C^{**} \to C.$$

Note e_C is an (R, K) chain epimorphism and e is a natural transformation.

Theorem 6.5. *$e_C : T^2 C \longrightarrow C$ is an (R, K) chain equivalence, for each (R, K) complex C.*

Proof. By [**16**] (proposition 4.7), we need only prove that $e_C(\sigma, \sigma) : T^2 C(\sigma) \to C(\sigma)$ is an R-chain equivalence, for all $\sigma \in K$. (No proof of this proposition appears in [**16**]. A brief proof appears in Appendix 2).

Case I: Assume there is a simplex $S \in K$ for which: $C(\sigma) = 0 \ \forall \ \sigma \neq S$.

We need only show $e_C(S, S)$ is a chain isomorphism, and $T^2 C(\sigma)$ is contractible for $\sigma \neq S$. We compute, for all $\sigma \in K$, in view of the restriction on C:

$$TC(st(\sigma)) = (C^* \otimes_K \Delta^* K)(st(\sigma)) = (C^* \otimes_R \Delta^* \overline{S})(st(\sigma))$$
$$= C^*(S) \otimes_R \Delta^* \overline{S}(st(\sigma))$$
$$\text{So: } T^2 C(\sigma) \cong C^{**}(S) \otimes_R \Delta^{**} \overline{S}(st(\sigma)) \otimes_R R\sigma^*$$

So for $\sigma \neq S$, $T^2 C(\sigma)$ is contractible because $\Delta^{**} \overline{S}(st(\sigma))$ is contractible by 4.2.

Next we prove that the map

$$e_C(S, S) = \varepsilon_C(S, S) \circ E_{C^{**}}(S, S) \circ (\Psi^*_C \otimes 1_{\Delta^* K})^{-1}(S, S)$$

is an isomorphism, or equivalently that $E_C(S, S)$ is an isomorphism.

Assume S has been oriented. Because $C(\sigma) = 0$ for $\sigma \neq S$,

$$E_C(S,S) : [Hom_{(R,K)}(\Delta^* K, C) \otimes_K \Delta^* K](S) \to C(S)$$

is simply: $eval_{RS^*,C(S)} : Hom_R(RS^*, C(S)) \otimes_R RS^* \to C(S)$.

This is a chain isomorphism as observed in § 2. So $e_C(\sigma,\sigma)$ is a chain isomorphism for $\sigma = S$ and a chain equivalence for $\sigma \neq S$. This completes the proof in Case I.

Case II (the general case): For any $C \neq 0$ in $\mathcal{B}R_K$ one can choose some $S \in K$ for which $C(S) \neq 0$, and an exact sequence $0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ for which $i(S,S) : C'(S) \to C(S)$ is an isomorphism, and $C'(\sigma) = 0$ for $\sigma \neq S$. For example, choose S to be of maximum dimension among $\{\sigma \in K \mid C(\sigma) \neq 0\}$).

The argument is by induction on the number n, of $\sigma \in K$, for which $C(\sigma) \neq 0$.

If $n = 1$, Case I applies. If $n > 1$, by induction, $e_{C''}(\sigma,\sigma)$ and $e_{C'}(\sigma,\sigma)$ are R chain equivalences. Also the commuting diagram below has exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^2C'(\sigma) & \longrightarrow & T^2C(\sigma) & \longrightarrow & T^2C''(\sigma) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle e_{C'}(\sigma,\sigma)} & & \downarrow{\scriptstyle e_C(\sigma,\sigma)} & & \downarrow{\scriptstyle e_{C''}(\sigma,\sigma)} & & \\
0 & \longrightarrow & C'(\sigma) & \longrightarrow & C(\sigma) & \longrightarrow & C''(\sigma) & \longrightarrow & 0
\end{array}$$

Therefore $e_C(\sigma,\sigma)$ is an R-chain equivalence for all σ. This completes the proof. □

Note: The first proof of the above theorem appeared in [**1**].

7. Construction of the ball complex X_K

The purpose of this section is to construct the complex X_K advertised in the introduction and establish its properties.

Definition 7.1. (of X'): Let X be a finite simplicial complex in a euclidean space, with vertex set V_X. Its underlying polyhedron is: $|X| = \cup\{\sigma \mid \sigma \in X\}$. For each $p \geqslant 0$, X_p denotes the set of p-simplices of X.

If $|X|$ is pl-homeomorphic to I^n we say $|X|$ or X is a pl n-ball and write ∂X for the subcomplex for which $|\partial X| = \partial |X|$.

Each p-simplex $\sigma \in X$ is the convex hull, $[v_0, v_1, \ldots, v_p]$, of its vertices in V_X. Its *barycenter* is $\hat{\sigma} := \frac{1}{p+1} \sum_{i=0}^{p} v_i \in \sigma^\circ$.

Choose a point $b\sigma \in \sigma^\circ$, the interior of σ, for each $\sigma \in X$.

The derived complex X' is defined as the unique simplicial subdivision of X for which $V_{X'} = \{b\sigma \mid \sigma \in X\}$. X' has one p-simplex, $[b\sigma_0, b\sigma_1 \ldots b\sigma_p]$, for each decreasing sequence of simplices $\sigma_0 > \cdots > \sigma_p$ of X.

If $\sigma_0 > \cdots > \sigma_p$, the ordered $p+1$ tuple $(b\sigma_0, b\sigma_1, \ldots, b\sigma_p)$ then specifies an oriented p-simplex in $\Delta_p(X'; R)$ *which we denote* $\langle \sigma_0, \sigma_1 \ldots, \sigma_p \rangle$ (suppressing the barycenters for concision).

These form a canonical basis for $\Delta_p(X'; R)$ (in contrast to $\Delta_p(X; R)$).

Because we want to use the McCrory cap product, we follow the orderings of [**13**] regarding simplices of X'.

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Definition 7.2. (of $\Delta X'$): Let (X,π) be a K-*space*. The *derived complexes of* (X,π) are the simplicial subdivisions X' of X, and K' of K whose vertex sets $\{b\sigma \mid \sigma \in K\}$ and $\{bS \mid S \in X\}$ are chosen as follows:

$$\text{If } \sigma \in K, \quad b\sigma := \hat{\sigma} \in \sigma^\circ;$$

$$\text{If } S \in X \text{ and } \sigma = \pi(S), \quad bS := \text{ centroid of } (S \cap \pi^{-1}(\hat{\sigma})) \in S^\circ.$$

By construction, $\pi(V_{X'}) \subset V_{K'}$. So π is also a simplicial map from X' to K', because π is linear on each simplex of X'.

X' provides a second geometric example, $\Delta X'$, of an (R,K) complex:
We define $\Delta X'$ by,

(1) $\Delta X'(K) = \Delta_*(X';R)$.

(2) For each $\sigma \in K, p \in \mathbb{Z}$, $(\Delta X')_p(\sigma)$ is the submodule of $\Delta_p(X';R)$ spanned by all $\langle Q^0, \dots Q^p\rangle$ in X' for which $\sigma = \pi(Q^p)$.

It is straightforward to see that $\Delta X'$ is an (R,K) complex.

The dual cone of a simplex $\sigma \in K$, denoted $D(\sigma,K)$, is a subcomplex of K' first defined in [**15**], § 7. It is a pl ball if K is a pl-manifold). It gives rise to several 'dual' subcomplexes in K' and X' which we define now.

Definition 7.3. Let (X,π) be a K-space. Suppose $\sigma,\tau \in K$, $T \in X$.

(1) $D(\sigma,K) := \{\langle\sigma_0,\sigma_1,\dots,\sigma_p\rangle \in K' \mid \sigma_p \geqslant \sigma\}$

(2) $D(\sigma,\tau) := \{\langle\sigma_0,\sigma_1,\dots,\sigma_p\rangle \in K' \mid \sigma_p \geqslant \sigma,\ \tau \geqslant \sigma_0\}$, the dual cell of σ in τ.

(3) $D_\sigma T := \{\langle S_0,S_1,\dots,S_p\rangle \in X' \mid \sigma \leqslant \pi(S_p),\ S_0 \leqslant T\}$

(4) $T_\sigma := |D_\sigma T|$. (Therefore, $T_\sigma = (\pi \mid T)^{-1}|D(\sigma,\pi(T))|$).

Of course, $D(\sigma,\tau) = \emptyset$ unless $\sigma \leqslant \tau$, and $D_\sigma T = \emptyset$ unless $\sigma \leqslant \pi(T)$.
$D_\sigma T$ is a subcomplex of X'. $D(\sigma,K)$ and $D(\sigma,\tau)$ are subcomplexes of K'.

Lemma 7.4. *Let (X,π) be a K-space. Suppose $\sigma \in K$, $T \in X$, and $\sigma \leqslant \pi(T)$.*

(1) *$T_\sigma = |D_\sigma T|$ is a pl ball. $dim(T_\sigma) = dim(T) - dim(\sigma)$.*

(2) *$\partial D_\sigma T = \partial^i D_\sigma T \cup \partial^o D_\sigma T$, (the inner and outer boundaries) where:*

$$\partial^i D_\sigma T = \cup\{D_\rho T \ \mid \sigma < \rho\}; \quad \partial^o D_\sigma T = \cup\{D_\sigma S \ \mid S < T\}$$

(3) *Suppose $\sigma < \pi(T)$. Then $|\partial^i D_\sigma T|$ and $|\partial^o D_\sigma T|$ are pl balls of dimension $dim(D_\sigma T) - 1$, and*

$$\partial(\partial^i D_\sigma T) = \partial(\partial^o D_\sigma T) = \partial^i D_\sigma T \cap \partial^o D_\sigma T\ .$$

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Proof. of (1): For each vertex v of τ note that,

$$|D(v,\tau)| = \{x \in \tau \mid a_v(x) \geqslant a_w(x), \text{ for all vertices } w \text{ of } \tau\}.$$

where $a_v : |K| \to [0,1]$ denotes the barycentric coordinate function defined by the vertex v. This is a convex subset of τ. So

$$|D(\sigma,\tau)| = \cap_{v \in V(K)} |D(v,\tau)|$$

is also convex. Therefore $T_\sigma = (\pi_{|T})^{-1}(|D(\sigma,\tau)|)$ is also convex since $\pi_{|T} : T \to \tau$ is simplicial. So T_σ is a compact convex polyhedron and therefore a pl ball.

Since $|D(\sigma,\tau)| \cap \tau^\circ \neq \emptyset$, this operator $(\pi_{|T})^{-1}$ preserves codimension:

$$dim(\tau) - dim(D(\sigma,\tau)) = dim(T) - dim(D_\sigma T).$$

Since $dim(D(\sigma,\tau)) = dim(\tau) - dim(\sigma)$, we get: $dim(D_\sigma T) = dim(T) - dim(\sigma)$. □

Proof. of (2): See [3], proposition 5.6(2), applied to $\pi_{|\overline{T}} : \overline{T} \to \pi(\overline{T})$. □

Proof. of (3): The equation in (3), and the fact that $|\partial^i D_\sigma T|$ and $|\partial^o D_\sigma T|$ are both pl manifolds, are proved in [3] [proposition 5.6 (3),(4)]. To show $|\partial^i D_\sigma T|$ is a pl ball, it suffices to note that it collapses to the vertex bT, and so $|\partial^i D_\sigma T|$ is a regular neighbourhood of bT in $|\partial D_\sigma T|$ (by 3.30 of [14]). Then by 3.13 of [14], $\partial^o D_\sigma T$ is also a pl ball. □

Definition 7.5. ([14] p.27) A *ball complex* is a finite collection $Z = \{B_i\}_{i \in I}$ of pl balls in a euclidean space, such that each point of $|Z| := \cup\{B \mid B \in Z\}$ lies in the interior of precisely one ball of Z, and the boundary of each $B \in Z$ is a union of balls of lesser dimension of Z. Therefore $(|Z|, Z)$ is a regular CW-complex.

Let Z and Y be ball complexes A pl map $f : |Z| \to |Y|$ is a *map of ball complexes* if for each ball B of Z, $f(B)$ is a ball of Y.

Definition 7.6. Let (X,π) be a K-space. We define

$$X_K = \{T_\sigma \mid \sigma \in K,\ T \in X,\ \sigma \leqslant \pi(T)\}$$

Theorem 7.7. *Let (X,π) be a K-space. Then X_K is a ball complex. Moreover X' is a simplicial subdivision of X_K. Also, X_K is a subdivision of X.*

Let $f : (X,\pi_X) \to (Y,\pi_Y)$ is a map of K-spaces. The induced map $f' : X' \to Y'$ of derived complexes is then a map of ball complexes, $f_K : X_K \to Y_K$.

Proof. (The induced map f' means the simplicial map $f' : X' \to Y'$ for which $f'(bS) = b(f(S))$ for each $S \in X$.) By lemma 7.4 the boundary of each T_σ is a

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union of balls of X_K with smaller dimension and

$$T^\circ_\sigma = \coprod \{A^\circ \mid A = \langle S_0, \ldots, S_p\rangle \in D_\sigma T,\ A \notin \partial^i D_\sigma T,\ A \notin \partial^o D_\sigma T\}.$$

This can be rewritten as:

$$T^\circ_\sigma = \coprod \{A^\circ \mid A = \langle S_0, \ldots, S_p\rangle \in X',\ \sigma = \pi(S_p),\ T = S_0\}, \tag{7.1}$$

By equation (7.1), for each $A \in X'$ there is a unique $T_\sigma \in X_K$ for which $A^\circ \subset T^\circ_\sigma$. Therefore : $|X'| = \coprod\{T^\circ_\sigma \mid T_\sigma \in X_K\} = |X_K|$.

This proves that X_K is a ball complex and that X' is a subdivision of X_K. Because $T_\sigma \subset T$, we see X_K is a subdivision of X.

Now let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map of K-spaces. For each simplex $S \in X$ we see $f(S) \in Y$ because f is simplicial. For each face σ of $\pi_X(S)$ in K, we see from the definitions that $f'(D_\sigma S) = D_\sigma f(S)$. So f' is a map of ball complexes, $f_K : X_K \to Y_K$. □

8. The isomorphism $\Phi_X : T\Delta^* X \cong C(X_K)$

Our main theorem is:

THEOREM 8.1. *For each K-space (X, π) the cellular chain complex of X_K with R coefficients, denoted $C(X_K)$, comes with a natural (R, K)complex structure. There is defined (below) an isomorphism of (R, K) chain complexes:*

$$\Phi_X : T\Delta^* X \cong C(X_K)\ .$$

For each map $f : (X, \pi_X) \to (Y, \pi_Y)$ of K-spaces, the square below commutes.

$$\begin{array}{ccc} T(\Delta^* X) & \xrightarrow{T(f^*)} & T(\Delta^* Y) \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ C(X_K) & \xrightarrow{f_K} & C(Y_K) \end{array}$$

Proof. Choose a basis bK of oriented cells for $\Delta_*(K; R)$. Choose next, a basis b_*X of oriented cells for $\Delta_*(X; R)$. But choose the orientations in b_*X so that if $T \in b_*X$ and $\sigma \in bK$ are both q-cells, and if $\pi_*(T) = \pm\sigma \in \Delta_q(K; R)$, then:

$$\pi_*(T) = (-1)^{dim(\sigma)}\sigma \in \Delta_q(K; R).$$

We call such a pair, (bK, b_*X) *an orientation for* (X, π).

Our first task is to construct the cellular chain complex $C_*(X_K; R)$ as the underlying R-complex of an (R, K) complex $C(X_K)$. Define

$$C(X_K) = \Delta X \otimes_K \Delta^* K; \quad C_*(X_K; R) = (\Delta X \otimes_K \Delta^* K)(K)$$

For each oriented simplex $\rho \in bK$ and oriented simplex $T \in b_*X$, define

$$[T_\rho] = T \otimes_K \rho^* \in C_{|T|-|\sigma|}(X_K; R) \quad (\text{where } |\sigma| = dim(\sigma)).$$

(The geometric intuition for this definition is the fact that, the map C_X, of corollary 9.3, takes $T \otimes_K \rho^*$ to a fundamental cycle, in $\Delta X'$, for the cell $D_\rho T$, whose underlying space is T_ρ).

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Define $bX_K = \{[T_\rho] \mid T \in b_*X,\ \rho \in bK,\ T_\rho \leqslant \pi(T)\}$. Then bX_K is an R-basis for $C_*(X_K;R)$ in bicorrespondence with the cells of X_K. Write ∂_q for the boundary map in $C_*(X_K;R)$, namely: $\partial_q = (d^{\Delta X \otimes_K \Delta^* K})_q$.

But to justify these definitions, we must check that $C_*(X_K;R)$ does compute the cellular homology of X_K. It suffices to check, for any $[T_\rho] \in bX_K$, that $\partial_q([T_\rho])$ is a sum with ± 1 coefficients of those $[S_\sigma] \in bX_K$ which are $(q-1)$-faces of T_ρ. (See [7], for example.)

All proper faces of T_ρ have the form T_σ, for $\rho < \sigma$, or S_ρ, for $S < T$.

Suppose $[T_\rho] \in bX_K$. So $T \in b_*X$, $\rho \in bK$. Set $\tau = \pi(T) \in K$. By (5.1):

$$\begin{aligned}\partial_q[T_\rho] &= d^{\Delta X \otimes_K \Delta^* K}(T \otimes_K \rho^*)\\ &= \sum_{\{\sigma \mid \rho \leqslant \sigma \leqslant \tau\}} \{(d^{\Delta X}(\sigma,\tau)T) \otimes \rho^* + (-1)^{|T|} T \otimes d^{\Delta^* K}(\sigma,\rho)\rho^*\}\\ &= \sum_{S<T}[T,S][S_\rho] + (-1)^{1+|T_\rho|}\sum_{\rho<\sigma}[\sigma,\rho][T_\sigma]\end{aligned}$$

which is as required.

This completes the construction of the cellular chain complex of X_K, as an (R,K) complex, $C(X_K)$.

The (R,K) isomorphism, $\Phi_X : T\Delta^*X \cong C(X_K)$ is simply:

$$\Phi_X := (\varepsilon_{\Delta X} \otimes_K 1_{\Delta^* K}) : T\Delta^* X = \Delta^{**}X \otimes_K \Delta^* K \longrightarrow \Delta X \otimes_K \Delta^* K = C(X_K).$$

Naturality of Φ is obvious from the naturality of ε. □

9. The McCrory cap product, $\Delta^* X$ and $\Delta X'$

We now use the work of McCrory [13] to construct, for any K-space, (X,π), an (R,K) chain monomorphism $C(X_K) \xrightarrow{C_X} \Delta X'$. serving two purposes.

First, it defines an (R,K) chain homotopy equivalence, $T\Delta^*X \simeq \Delta X'$.

Second, C_X identifies $C(X_K)$ with that (R,K) subcomplex of $\Delta X'$ which admits a basis consisting of one fundamental q-cycle, in $\Delta_q(D_\sigma T, \partial D_\sigma T) \subset \Delta_q(X')$, for each q-cell T_σ of X_K. (This will complete our geometric interpretation of T).

Let K be a finite simplicial complex. McCrory (see [13], and also [11]) defines a map, $c' : \Delta_*(K;R) \otimes_R \Delta^*(K;R) \to \Delta_*(K';R)$ which he shows is chain homotopic to the composite,

$$\Delta_*(K;R) \otimes_R \Delta^*(K;R) \xrightarrow{\cap} \Delta_*(K;R) \xrightarrow{Sd} \Delta_*(K';R)$$

where $\cap$ denotes the Whitney-Cech cap product. We will write c_K for c'. We repeat his definition here with appropriate sign changes because McCrory's sign conventions differ slightly from ours.

For any q-simplex, $Q = \langle Q^0, Q^1, \ldots Q^q\rangle$ of K' *in which each* Q_i *is oriented*, McCrory then defines

$$\varepsilon(Q) = [Q^0, Q^1][Q^1, Q^2]\ldots[Q^{q-1}, Q^q].$$

This is independent of the orientations on $Q_1, Q_2, \ldots Q_{q-1}$. If $q = 0$, set $\varepsilon(Q) = 0$.

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For any n-simplex τ and $(n-q)$-simplex σ of K, each simplex $Q = \langle Q^0, Q^1, \dots Q^q\rangle$ of $D(\sigma,\tau)_q$ satisfies: $Q^0 = \tau$; $Q^q = \sigma$. Therefore, $\varepsilon(Q)$ makes sense if τ and σ are oriented simplices chosen from some basis bK of oriented simplices for $\Delta_*(K;R)$ (but not if $\sigma = -\tau$).

The *McCrory Cap Product*, $\Delta_*(K;R)\otimes_R \Delta^*(K;R) \xrightarrow{c_K} \Delta_*(K';R)$ is the map defined by:

$$\boxed{c_K(\tau\otimes\sigma^*) = \sum_{Q\in D(\sigma,\tau)_q} (-1)^{dim(\sigma)}\varepsilon(Q)Q}$$

for any oriented simplices σ,τ in some basis bK. Here $q = dim(\tau) - dim(\sigma)$. Note this is zero unless $\sigma \leqslant \tau$. Note that c_K does not change if we change the basis.

c_K is a chain map. We reprove this in Appendix I, § A, because of the sign changes and because McCrory's proof, [**13**] p.155 lines 7-8, is only a sketch.

Now suppose (X,π) is a K-space.

Note that if T and σ are oriented simplices of X and K and $q = dim(T) - dim(\sigma) \neq 0$:

$$c_X(T\otimes_R \pi^*\sigma^*) = \sum_{Q\in(D_\sigma,T)_q} (-1)^{dim(\sigma)}\varepsilon(Q)Q \ \in \Delta_q X'(\sigma)$$

(because $D_\sigma T = \cup\{D(S,T) \mid S\in X, dim(S) = dim(\sigma), \pi(S) = \sigma\}$). This formula still makes sense and is true if $q = 0$ and $\pi_*(T) \neq -\sigma$).

In this way, $c_X \circ (1\otimes\pi^*)$ defines an (R,K) chain map,

$$c_X\circ(1\otimes\pi^*) : \Delta X\otimes_R \Delta^*K \longrightarrow \Delta X'$$

Proposition 9.1. *There is a unique* (R,K) *chain map*

$$C_X : C(X_K) = \Delta X\otimes_K \Delta^*K \longrightarrow \Delta X'$$

satisfying:

$$c_X\circ(1\otimes\pi^*) = C_X\circ\pi_{\Delta X,\Delta^*K}$$

C_X *is an* (R,K) *monomorphism. For all* q*-cells* T_σ *of* X_K*, with* $q\neq 0$,

$$C_X(T\otimes_K\sigma^*) = \sum_{Q\in(D_\sigma T)_q} (-1)^{dim(\sigma)}\varepsilon(Q)Q.$$

For a 0*-cell* T_σ*, of* X_K*, with* $T\in\Delta_n(X;R), \sigma\in\Delta_n(K;R)$ *oriented so that* $\pi_*(T) = \sigma$*, then*

$$C_X(T\otimes_K\sigma^*) = (-1)^{dim(T)}\langle T\rangle, \quad (\langle T\rangle \textit{ is the barycenter } bT \textit{ of } T).$$

Proof. Note that $c_X(T\otimes_R\pi^*\sigma^*) = 0$ unless $\pi(T)\geqslant\sigma$. Also $c_X(T\otimes_R\pi^*\sigma^*)\in \Delta X'(\sigma)$ for all $\sigma\in K$ and $T\in X$ because each q-cell $Q\in D_\sigma T$ is in $\Delta_q X'(\sigma)$ if $q = dim(T_\sigma)$.

So $c_X\circ(1\otimes\pi^*) : \Delta X\otimes_R\Delta^*K\to\Delta X'$ is an (R,K) chain map annihilating each $\Delta X(K - st(\sigma))\otimes_R\Delta^*K(\sigma)$. Hence there is a unique (R,K) chain map

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monomorphism, $\Delta X \otimes_K \Delta^* K \xrightarrow{C_X} \Delta X'$ such that $c_X \circ (1 \otimes \pi^*) = C_X \circ \pi_{\Delta X, \Delta^* K}$. The calculation follows if $q \neq 0$. If $q = 0$, then $(\pi_{|T})^* \sigma^* = T^*$, so

$$C_X(T \otimes_K \sigma^*) = c_X(T \otimes T^*) = (-1)^{dim(T)} \sum_{Q \in D(T,T)_0} Q = (-1)^{dim(T)} \langle T \rangle$$

Clearly C_X is natural in (X, π). □

REMARK 9.2. If we choose an orientation $(bK, b_* X)$ for X_K, then for each 0-cell $T_\sigma = T_{\pi(T)}$ of X_K, with $[T_\sigma] \in bX_K$, we have $C_X([T_\sigma]) = \langle T \rangle \in \Delta_0(X'; R)$.

COROLLARY 9.3. *For each q-cell T_σ of X_K, $C_X(T \otimes \sigma^*)$ is a fundamental cycle, in $\Delta_q(D_\sigma T, \partial D_\sigma T; R)$ for the q-manifold $D_\sigma T$.*

Proof. $C_X(T \otimes_K \sigma^*)$ is a fundamental cycle in $\Delta_q(D_\sigma T, \partial D_\sigma T; R)$ since C_X is a chain map and since each $Q \in (D_\sigma T)_q$ appears with coefficient ± 1 in $C_X(T \otimes_K \sigma^*)$. □

THEOREM 9.4. *For each K-space (X, π), the map $C(X_K) \xrightarrow{C_X} \Delta X'$ is an (R, K) chain homotopy equivalence.*

Proof. By 9.3, for all T_σ, C_X restricts to a homotopy equivalence,

$$C_*(T_\sigma, \partial T_\sigma; R) \to \Delta_*(D_\sigma(T), \partial D_\sigma(T); R)$$

and it takes chains on any subcomplex of X_K to chains on its subdivision. By an induction-excision argument on the number of cells in the subcomplex one sees C_X yields a homology equivalence and then a chain homotopy equivalence on each such subcomplex. So $C_X(\sigma, \sigma)$ is an R-chain equivalence for each σ. Therefore C_X is an (R, K) chain equivalence. □

Together, 9.4 and 8.1 clearly prove:

COROLLARY 9.5. *$T\Delta^* X \xrightarrow{C_X \Phi_X} \Delta X'$ is an (R, K) chain homotopy equivalence. Consequently $e_{\Delta^* X} \circ T(C_X \Phi_X)$ is an explicit (R, K) chain homotopy equivalence,*

$$T\Delta X' \simeq \Delta^* X.$$

Appendix A.

We must prove:

PROPOSITION A.1. *$\Delta_*(K; R) \otimes_R \Delta^*(K; R) \xrightarrow{c_K} \Delta_*(K'; R)$ is a chain map. That is to say, for any oriented simplices σ, τ in some basis bK for ΔK, with $p = dim(\tau) - dim(\sigma)$,*

$$d^{K'} c_K(\tau \otimes \sigma^*) = c_K\{d^K \tau \otimes \sigma^* + (-1)^{dim(\tau)} \tau \otimes d^{\Delta^*(K)} \sigma^*\}$$

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where, by the definitions,

$$d^K\tau = \sum_{\rho \in b_K} [\tau, \rho]\rho, \quad d^{\Delta^*(K)}\sigma^* = (-1)^{dim(\sigma)+1} \sum_{\rho \in b_K} [\rho, \sigma]\rho^*$$

and for any p-simplex $Q = \langle Q^0, Q^1, \dots Q^p \rangle$ of K',

$$d^{K'}Q = \sum_{i=0}^{p} (-1)^i d^i(Q); \quad d^i(Q) = \langle Q^0, Q^1, \dots \hat{Q}^i \dots Q^p \rangle$$

Proof. We first prove: $d^o c(\tau \otimes \sigma^*) = c(d^K\tau \otimes \sigma^*)$, where $c = c_K$.

$$\begin{aligned} d^0 c(\tau \otimes \sigma) &= (-1)^{dim(\sigma)} \sum_{Q \in D(\sigma,\tau)_p} \varepsilon(Q)\langle Q^1, \dots Q^p \rangle \\ &= (-1)^{dim(\sigma)} \sum_{\rho \in b_K} [\tau, \rho] \sum_{P \in D(\sigma,\rho)} \varepsilon(P)P \\ &= c(\sum_{\rho \in b_K} [\tau, \rho]\rho \otimes \sigma^*) = c(d^K\tau \otimes \sigma^*). \end{aligned}$$

Next we show: $(-1)^p d^p c(\tau \otimes \sigma^*) = (-1)^{dim(\tau)} c(\tau \otimes d^{\Delta^*(K)}\sigma^*)$:

$$\begin{aligned} (-1)^p d^p c(\tau \otimes \sigma^*) &= (-1)^{p+dim(\sigma)} \sum_{Q \in D(\sigma,\tau)_p} \varepsilon(Q)\langle \tau, Q^1 \dots Q^{p-1} \rangle \\ &= (-1)^{p+1} c(\tau \otimes \sum_{\rho \in b_K} [\rho, \sigma]\rho^*) = (-1)^{dim(\tau)} c(\tau \otimes d^{\Delta^*(K)}\sigma^*) \end{aligned}$$

Finally we prove $d^i c(\tau \otimes \sigma^*) = 0$ for $0 < i < p$.

For such i and for $Q \in D(\sigma, \tau)$ note $d^iQ = \langle \tau, \dots \sigma \rangle \in D(\sigma, \tau) - \partial D(\sigma, \tau)$. So suppose P is a $p - 1$ simplex of the form d^iQ in the p manifold $D(\sigma, \tau)$. Then there is exactly one other $S \in D(\sigma, \tau)_p$ having Q as a face. We can identify S by listing the vertices of τ as $v_0, \dots v_n$ so that $Q^j = [v_j, \dots v_n]$ for all j. Define $S^i = [v_0 \dots v_{i-1}, v_{i+1} \dots v_n]$ and define $S^j = Q^j$ for $j \neq i$. Then $S := \langle S^0, S^1, \dots S^p \rangle$ in $D(\sigma, \tau)_p$ satisfies $d^iS = P$; $\varepsilon(S) = -\varepsilon(Q)$ so P must appear with zero coefficient in $d^i c(\tau \otimes \sigma^*)$ for all $p - 1$ simplices P. So $d^i c(\tau \otimes \sigma^*) = 0$. □

Appendix B.

We must prove the following result of Ranicki and Weiss:

Proposition B.1. *Let $i : A \to B$ be an (R, K) chain map in $\mathcal{B}R_K$ for some finite poset K. Then i is a chain equivalence in $\mathcal{B}R_K$ if and only if $i(\sigma, \sigma)$ is a chain equivalence in $\mathcal{B}R$ for all $\sigma \in K$.*

https://doi.org/10.1017/prm.2022.89 Published online by Cambridge University Press

LEMMA B.2. *[(The Contraction Principle)]: For any additive category A, with the split exact structure, and any exact sequence of chain complexes in $\mathcal{B}A$,*

$$0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$$

C'' is contractible if and only if f has a left inverse $r : C \to C'$ which is a chain homotopy inverse of f.

Proof. For any $h'' \in Hom_A(C'', C'')_1$ there is an $h \in Hom_A(C, C)_1$ such that $gh = h''g$ and $hf = 0$. Then h'' is a contraction of C'' iff h is a chain homotopy from 1_C to a chain map $\rho : C \to C$ for which $\rho = fr$ for some chain map $r : C \to C'$. r satisfies $rf = 1_{C'}$. So r is a left inverse of f and fr is chain homotopic to 1_C. □

Proof. of B.1: First assume i is a chain equivalence. Note, for each $\sigma \in K$, the functor $B \to B(\sigma)$ is an additive functor $AR_K \to A_R$. So it induces a homotopy functor $\mathcal{B}R_K \to \mathcal{B}R$. Therefore $i(\sigma, \sigma)$ is a chain equivalence for each $\sigma \in K$.

Conversely suppose $i(\sigma, \sigma)$ is a chain equivalence in $\mathcal{B}R$ for all $\sigma \in K$. We prove that i is a chain equivalence in $\mathcal{B}R_K$. Replacing B by the mapping cylinder of i if necessary, we can assume i fits into an exact sequence, $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$.

By B.2 then, each $C(\sigma)$ is contractible, and we have only to prove the claim that C is contractible. The proof is by induction on the number, $n(C)$, of $\sigma \in K$ for which $C(\sigma) \neq 0$. If $n = 0$ we are done. We can assume this claim is proved for complexes C' for which $0 \leqslant n(C') < n(C)$.

There is some $\rho \in K$ for which $C(\rho) \neq 0$, and an exact sequence of the form:

$$0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$$

for which $f(\rho, \rho)$ is an isomorphism, and $g(\sigma, \sigma)$ is an isomorphism for all $\sigma \neq \rho$. (For example pick ρ to be maximal in $\{\sigma \in K \mid C(\sigma) \neq 0\}$). C' is contractible because $C(\rho)$ is contractible. But C'' is contractible by induction, so that f is a chain equivalence, by B.2. So C is contractible as claimed. □

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