In $\triangle ABE$ we have

$$\frac{c}{\sin \frac{1}{2}\gamma} = \frac{a + b}{\sin\left(\frac{1}{2}\pi + \frac{1}{2}(\alpha - \beta)\right)} = \frac{a + b}{\cos \frac{1}{2}(\alpha - \beta)}.$$

The formulae of Mollweide and Newton may also be quickly obtained from the figures used in the two proofs of the law of tangents in [1, pp. 66-67]. In addition, alternative proofs of the formulae of Mollweide and Newton can be found in [1, pp. 62-63].

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Reference

1. R. B. Nelsen, *Proofs Without Words III*, Mathematical Association of America, Washington, DC (2015).

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106.41 Infinitely many series arising from $\cos^2 x + \sin^2 x = 1$

In this Note we will exhibit some convergent series and identities hidden in the most famous Pythagorean identity: $\cos^2 x + \sin^2 x = 1$, for all complex numbers x. In order to obtain these results we will consider the expansion of the functions sine and cosine into infinite products.

In [1], W. F. Eberlein proved Euler's formula for the infinite product of the sine function using only simple arguments of analysis without considering the Weierstrass theorem on infinite products or Fourier series. Writing $\cos x = \frac{\sin 2x}{2 \sin x}$, we have the formula for the cosine expansion as infinite product as follows

$$\sin x = x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right), \qquad \cos x = \prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{(2n - 1)^2\pi^2}\right).$$

Inserting these products into $\cos^2 x + \sin^2 x = 1$, we have:

$$x^2 \prod_{n=1}^{\infty}\left(1 - \frac{2x^2}{n^2\pi^2} + \frac{x^4}{n^4\pi^4}\right) + \prod_{n=1}^{\infty}\left(1 - \frac{8x^2}{(2n-1)^2\pi^2} + \frac{16x^4}{(2n-1)^4\pi^4}\right) = 1. \quad (1)$$

The central idea of this Note is to investigate the coefficients of x^n in the (1). Clearly, for $n > 0$ these coefficients are equal to zero, but the most important matter here is how to obtain this zero.

Searching for the coefficient of x^2 in (1), we find:

$$x^2\left(1 - \frac{8}{1^2\pi^2} - \frac{8}{3^2\pi^2} - \frac{8}{5^2\pi^2} - \ldots\right) = 0x^2,$$

thus

$$\sum_{i \text{ odd}} \frac{1}{i^2} = \frac{\pi^2}{8}.$$

In that way, we have obtained the limit of a well-known series involving quotients of odd terms. In fact, we get this result without using, as usual, Parseval's identity (see Rudin [2] and Arfken [3] for more detailed information).

Checking the terms involving x^4, we have

$$-\frac{2}{1^2\pi^2} - \frac{2}{3^2\pi^2} - \frac{2}{5^2\pi^2} - \ldots,$$

from the coefficients of x^4 in the first infinite product of (1), and the next infinite sums are related to the contribution of the second product:

$$\left(\frac{8^2}{1^23^2\pi^2} + \frac{8^2}{1^25^2\pi^2} + \frac{8^2}{1^27^2\pi^2} + \ldots\right) + \left(\frac{16}{1^4\pi^4} + \frac{16}{3^4\pi^4} + \frac{16}{5^4\pi^4} + \ldots\right).$$

So we have the following equation

$$-\frac{2}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i^2} + \frac{64}{\pi^4}\sum_{\substack{i<j \\ i,j \text{ odd}}}\frac{1}{i^2j^2} + \frac{16}{\pi^4}\sum_{i \text{ odd}}\frac{1}{i^4} = 0.$$

As we know that,

$$\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6} \text{ and } \sum_{i \text{ odd}}\frac{1}{i^4} = \frac{\pi^4}{96}, \tag{2}$$

it follows that

$$\sum_{\substack{i<j \\ i,j \text{ odd}}}\frac{1}{i^2j^2} = \frac{\pi^4}{384}.$$

The last equation of (2) can be found using Parseval's identity. Similarly, considering terms in x^6, we have:

$$x^6\left(\frac{2^2}{1^22^2\pi^4} + \frac{2^2}{1^23^2\pi^4} + \frac{2^2}{1^24^2\pi^4} + \ldots\right) + x^6\left(\frac{1}{1^4\pi^4} + \frac{1}{2^4\pi^4} + \frac{1}{3^4\pi^4} + \ldots\right)$$
$$+ x^6\left(-\frac{8^3}{1^23^25^2\pi^6} - \frac{8^3}{1^23^27^2\pi^6} - \frac{8^3}{1^23^29^2\pi^6} - \ldots\right)$$
$$+ x^6\left(-\frac{128}{1^23^4\pi^6} - \frac{128}{1^25^4\pi^6} - \frac{128}{1^27^4\pi^6} - \ldots\right)$$
$$+ x^6\left(-\frac{128}{1^43^2\pi^6} - \frac{128}{1^45^2\pi^6} - \frac{128}{1^47^2\pi^6} - \ldots\right) = 0x^6.$$

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This leads to

$$-\frac{8^3}{\pi^6}\sum_{i<j<k \text{ odd}}\frac{1}{i^2j^2k^2} = -\frac{4}{\pi^4}\sum_{i<j}\frac{1}{i^2j^2}$$

$$-\frac{1}{\pi^4}\sum_{i=1}^{\infty}\frac{1}{i^4} + \frac{128}{\pi^6}\left(\sum_{i<j \text{ odd}}\frac{1}{i^2j^4} + \sum_{i<j \text{ odd}}\frac{1}{i^4j^2}\right),$$

or more concisely

$$\sum_{\substack{i<j<k\\ i,j,k \text{ odd}}}\frac{1}{i^2j^2k^2} = \frac{\pi^2}{128}\sum_{i<j}\frac{1}{i^2j^2} + \frac{\pi^2}{512}\sum_{i=1}^{\infty}\frac{1}{i^4} - \frac{1}{4}\sum_{\substack{i,j \text{ odd}\\ i,j \text{ distinct}}}\frac{1}{i^2j^4}.$$

For the equation generated by the search for the coefficient of x^8 in (1), we get the following identity.

$$\sum_{i<j<k<l \text{ odd}}\frac{1}{i^2j^2k^2l^2} = \frac{\pi^2}{512}\sum_{i<j<k}\frac{1}{i^2j^2k^2} + \frac{\pi^2}{2048}\sum_{i,j \text{ dixtinct}}\frac{1}{i^2j^4}$$

$$-\frac{1}{4}\sum_{\substack{i,j,k \text{ odd}\\ i,j,k \text{ distinct}}}\frac{1}{i^4j^2k^2} - \frac{1}{16}\sum_{i<j \text{ odd}}\frac{1}{i^4j^4}.$$

A natural question that arises is how to find an identity involving series linked to the coefficient of the general term x^n, for n even. We have an identity involving sums as follows.

$$\sum_{i_1<1_2<\ldots<i_m \text{ odd}}\frac{1}{i_1^2 i_2^2\ldots\, i_m^2}.$$

For this purpose we make use of compositions. The A-restricted composition of a positive integer n is an ordered collection of elements in A whose sum is n. For instance, if $A = \{1, 2\}$, the compositions of $n = 5$ are: $2 + 2 + 1$, $2 + 1 + 2$, $1 + 2 + 2$, $2 + 1 + 1 + 1$, $1 + 2 + 1 + 1$, $1 + 1 + 2 + 1$, $1 + 1 + 1 + 2$ and $1 + 1 + 1 + 1 + 1$. Detailed information on A-restricted compositions can be found in Sills [4], and Heubach and Mansour [5].

In our problem, let us consider $A = \{2, 4\}$, and n an even positive integer. With $a_m = -1/(m^2\pi^2)$ and $b_m = -4/((2m-1)^2\pi^2)$, the equation (1) can be rewritten as

$$x^2\prod_{n=1}^{\infty}\left(1 + 2a_nx^2 + (a_n)^2x^4\right) + \prod_{n=1}^{\infty}\left(1 + 2b_nx^2 + (b_n)^2x^4\right) = 1. \qquad (3)$$

Let us consider $C(m, A) = \{c_n \mid m \in \mathbb{N}\}$ the set of all A-restricted compositions of m. We will use A-restricted compositions to obtain the expansion of the products in (3) and search for identities hidden in the coefficients of x^n, for n an even positive integer. Searching for the coefficient of x^n, we find the following equation.

$$\sum_{c(n-2) \in C(n-2,A)} \sum_{\substack{m \geqslant 1 \\ 0 \leqslant j_i < j_2 < \ldots < j_r \\ 0 \leqslant l_1 < l_2 < \ldots < l_s}} 2^r a(m+j_1) \ldots a(m+j_r) a(m+l_1)^2 \ldots a(m+l_s)^2$$

$$+ \sum_{c(n) \in C(n,A)} \sum_{\substack{m \geqslant 1 \\ 0 \leqslant j_i < j_2 < \ldots < j_p \\ 0 \leqslant l_1 < l_2 < \ldots < l_q}} 2^p a(m+j_1) \ldots a(m+j_p) a(m+l_1)^2 \ldots a(m+l_q)^2 = 0$$

where the composition of $n - 2$, $c(n - 2)$ has r parts of 2 and s parts equal to 4, and $c(n)$ has p parts of 2 and q of 4. We consider $c(n - 2)$ in the first sum because the term x^2 in the first infinite product of (3) means that 2 is always part of the composition.

For example, if $n = 8$, for the composition of 6, $2 + 4$, in the first infinite product of (1), we have the series:

$$2a(1)(a(2))^2 + 2a(1)(a(3))^2 + \ldots = -\frac{2}{\pi^6} \sum_{i<j} \frac{1}{i^2 j^4},$$

while for the composition of 8, $2 + 4 + 2$, in the second product, we have:

$$2b(1)(b(2))^2 2b(3) + 2b(1)(b(3))^2 2b(4) + \ldots = \frac{1024}{\pi^8} \sum_{i<j<k \text{ odd}} \frac{1}{i^2 j^4 k^2}.$$

References

1. W. F. Eberlein, On Euler's infinite product for the sine, *Journal of Mathematical Analysis and Applications*, **58**(1) (1977) pp. 147-151.
2. W. Rudin, *Real and Complex Analysis* (3rd edn.), McGraw Hill (1987).
3. G. Arfken, Fourier Series, Ch. 14 in *Mathematical Methods for Physicists* (3rd ed.) Orlando, FL: Academic Press (1985) pp. 760-793.
4. A. V. Sills, Compositions, Partitions, and Fibonacci Numbers, *Fibonacci Quarterly* **40,** (2011) pp. 348-354.
5. S. Heubach, T. Mansour, Compositions of n with parts in a set, *Congressus Numerantium* **168** (2004) pp. 33-51.

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