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ANNIHILATOR-STABILITY AND TWO QUESTIONS OF NICHOLSON

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Abstract. An element a in a ring R is left annihilator-stable (or left AS) if, whenever $Ra + \mathrm{l}(b) = R$ with $b \in R$, $a - u \in \mathrm{l}(b)$ for a unit u in R, and the ring R is a left AS ring if each of its elements is left AS. In this paper, we show that the left AS elements in a ring form a multiplicatively closed set, giving an affirmative answer to a question of Nicholson [*J. Pure Appl. Alg.* **221** (2017), 2557–2572.]. This result is used to obtain a necessary and sufficient condition for a formal triangular matrix ring to be left AS. As an application, we provide examples of left AS rings R over which the triangular matrix rings $\mathbb{T}_n(R)$ are not left AS for all $n \geq 2$. These examples give a negative answer to another question of Nicholson [*J. Pure Appl. Alg.* **221** (2017), 2557–2572.] whether $R/J(R)$ being left AS implies that R is left AS.

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1. Introduction. Throughout, rings are associative with unity. An element a in a ring R is left uniquely generated (or left UG) if, whenever $Ra = Rb$, $a = ub$ for a unit u in R, and the ring R is a left UG ring if each of its elements is left UG. In [**10**], a left UG element is called an element with the left unique generator property. The study of left UG rings was initialed in 1949 by Kaplansky [**9**] in his work on matrices admitting diagonal reduction and has been continued by a number of authors; see, for instance, [**1**, **2**, **3**, **4**, **7**, **10**, **11**, **13**, **15**]. Left UG rings are closely related to directly finite rings, unit-regular rings, internal cancellation property, and stable range one property.

Kaplansky [**9**] gave some first known examples of left UG rings: any ring whose zero divisors are contained in its Jacobson radical (e.g., domains and local rings) [**9**, Lemma 2.1], commutative principal ideal rings and commutative artinian rings, and the matrix rings over a left Hermite domain [**9**, Theorem 3.8]. Commutative UG rings, under the name of associate rings or strongly associate rings, have been extensively discussed in [**1**, **2**, **3**, **6**, **15**]. For example, every commutative ring is embeddable in a commutative UG ring by [**15**, Theorem 14]; every commutative p.p. ring is UG by [**1**, Theorem 11]. Khurana and Lam [**10**, Theorem 6.2] showed that every regular element in a ring is unit-regular iff every regular element in the ring is left UG and, independently, Marks [**11**, Theorem] proved that a regular ring is unit-regular iff it is left UG. An earlier result of Hartwig and Luh [**7**, Theorem 2B], generalizing the two results, states that a regular element a in a ring R is unit-regular iff, whenever $Ra = Rb$ with b unit-regular, $a = ub$ for a unit u in R. In [**13**], Nicholson introduced left annihilator-stable (or left AS) elements and rings as natural generalizations of elements and rings with left stable range 1. By [**13**], every left AS ring is directly finite, and the matrix rings $\mathbb{M}_n(R)$ are left *AS* in case $R/J(R)$ is unit-regular and idempotents lift modulo the Jacobson radical of R. Various characterizations of left *AS* rings were obtained in [**4**, Corollary 4.4] and [**13**, Theorem 5]. Particularly, Canfell [**4**, Corollary 4.4] showed that a ring is left *AS* iff it is left *UG*. Thus, Marks' Theorem [**11**, Theorem] can be restated as follows: a regular ring is unit-regular iff it is left *AS*. The element-wise version of this result is obtained by Nicholson [**13**, Lemma 24]: an element in a ring is unit-regular iff it is regular and left *AS*.

This paper is a continuation of the study of left *AS* elements and left *AS* rings. Section 2 is mainly about properties of left *AS* elements. Though left *AS* and left *UG* are not equivalent element-wise, a *UG* element does represent a sort of annihilator-stability: an element $b \in R$ is left *UG* iff whenever $Ra + \mathrm{l}(b) = R$ with $a \in R$, $a - u \in \mathrm{l}(b)$ for a unit u. In Section 2, we show that the left *AS* elements in a ring form a multiplicatively closed set, giving an affirmative answer to a question of Nicholson [**13**]. Using this result, we establish a necessary and sufficient condition for a formal triangular matrix ring to be left *AS* in Section 3 and further produce examples of left *AS* rings R over which the triangular matrix rings $\mathbb{T}_n(R)$ are not left *AS* for all $n \geq 2$. These examples give a negative answer to another well-motivated question of Nicholson [**13**] whether $R/J(R)$ being left *AS* implies that R is left *AS*. While it remains open whether R being a commutative *UG* ring implies that $R[[t]]$ is *UG*, our concluding result shows that there exists a *UG* ring R such that $R[[t]]$ is not *UG*.

We denote by $J(R)$ and $U(R)$ the Jacobson radical and the unit group of R, respectively. For an element a in a ring R, $\mathrm{l}(a)$ is the left annihilator of a in R. An element a in a ring R is regular if $a = aba$ for some $b \in R$ and is unit-regular if $a = aua$ for some $u \in U(R)$. The ring R is (unit-) regular if every element in R is (unit-) regular.

2. Left *AS* elements. In this section, we compare left *UG* elements with left *AS* elements. As a main result, we show that the left *AS* elements in a ring form a multiplicatively closed set, answering a question of Nicholson [**13**] in the affirmative. The following result was proved by Canfell [**4**, Corollary 4.4] (also see [**13**, Theorem 5]).

THEOREM 2.1. *[4] A ring is left UG iff it is left AS.*

We recall a theorem of Marks [**11**, Theorem].

THEOREM 2.2. *[11] A regular ring is unit-regular iff it is left UG*

Thus, Theorem 2.2 can be restated as

COROLLARY 2.3. *A regular ring is unit-regular iff it is left AS.*

The next result of Nicholson [**13**], an element-wise version of Corollary 2.3, shows the significance of left *AS* elements.

THEOREM 2.4. *[13] An element $a \in R$ is unit-regular iff it is regular and left AS.*

As noticed by Nicholson [**13**], left *UG* and left *AS* are not equivalent for elements. In fact, in the proof of [**13**, Theorem 6] (note that [**13**, Theorem 6] has been corrected in [**14**]), Nicholson gave a left *AS* element that is not left *UG* in a commutative ring. In [**3**, Example 3.5(2)], the authors showed that in $C(\mathbb{R})$, the ring of all continuous real-valued functions on $\mathbb{R}$, there is a *UG* element that is not *AS*.

In order to detect the relation between left *UG* and left *AS*, we give the following definition.

DEFINITION 2.5. An element $b \in R$ is called left modified AS (or left MAS) if $Ra + \mathrm{l}(b) = R$, $a \in R$, implies $a - u \in \mathrm{l}(b)$ for some $u \in U(R)$, and the ring R is left MAS if every element in R is left MAS.

Obviously, a ring is left AS iff it is left MAS. Thus, the next statement may be viewed as an element-wise version of Theorem 2.1.

PROPOSITION 2.6. *An element $b \in R$ is left UG iff b is left MAS.*

Proof. ($\Rightarrow$). Let $Ra + \mathrm{l}(b) = R$. Then $Rab = Rb$, so, by hypothesis, $b = uab$ for some $u \in U(R)$, i.e., $u^{-1}b = ab$. Thus, $a - u^{-1} \in \mathrm{l}(b)$. So b is left MAS.

($\Leftarrow$). Let $Ra = Rb$. Then $a = xb$ and $b = ya$ where $x, y \in R$. So $b = yxb$ or $(1 - yx) \in l(b)$. Thus, $Ryx + \mathrm{l}(b) = R$, and so $Rx + \mathrm{l}(b) = R$. As b is left MAS, $x - u \in \mathrm{l}(b)$ for some $u \in U(R)$. Hence, $a = xb = ub$, i.e., $b = u^{-1}a$. So, b is left UG. □

We write $Z_r(R)$ for the right singular ideal of R and $\mathrm{ureg}(R)$ for the set of all unit-regular elements in R. For convenience, let $\mathrm{as}_l(R)$ be the set of all left AS elements in R. Some notable properties of $\mathrm{as}_l(R)$ are proved in [13, Example 13; Lemma 35]: (1) $J(R) \cup Z_r(R) \cup \mathrm{ureg}(R) \subseteq \mathrm{as}_l(R)$; (2) $\mathrm{as}_l(R) + J(R) = \mathrm{as}_l(R)$.

An element $a \in R$ is called a left $SR1$ element if whenever $Ra + Rb = R$, $b \in R$, $a - u \in Rb$ for a unit u in R (see [13]). Naturally, left $SR1$ elements are left AS. Motivated by the result that the product of two $SR1$ elements is again $SR1$ [5, Lemma 17], the following question is raised by Nicholson [13, Question 1]:

QUESTION 2.7. [13] Is the product of two left AS elements again left AS?

This question is answered in the affirmative. Noting that the product of two UG elements need not be UG (see [3, Example 3.11]), the next result gives a surprising contrast.

THEOREM 2.8. *If $a, b \in R$ are left AS, then ab is left AS.*

Proof. Assume that $Rab + \mathrm{l}(c) = R$ with $c \in R$. Then $1 = rab + x$ where $r \in R$ and $x \in \mathrm{l}(c)$, so $c = rabc$. From $Rab + \mathrm{l}(c) = R$, it follows that $Rb + \mathrm{l}(c) = R$. Since b is left AS, $b - u \in \mathrm{l}(c)$ for some unit $u \in R$. Thus, $bc = uc$, and so $abc = auc$ and $c = rabc = rauc$. Hence, $1 - rau \in \mathrm{l}(c)$, so $Rau + \mathrm{l}(c) = R$. Since a is left AS and u is a unit, au is left AS by [13, Lemma 12]. It follows that $au - v \in \mathrm{l}(c)$ for a unit v in R. Thus, $auc = vc$. As $auc = abc$, we obtain that $abc = vc$, i.e., $ab - v \in \mathrm{l}(c)$. Hence, ab is left AS. □

As seen in the next section, Theorem 2.8, together with the other known properties of left AS elements, is quite useful in constructing new examples of left AS rings. But directly from Theorem 2.8, we see an important fact that $\mathrm{as}_l(R)$ possesses an algebraic structure.

COROLLARY 2.9. *For a ring R, $\mathrm{as}_l(R)$ is a submonoid of the multiplicative monoid $(R, \cdot)$.*

3. Going up and going down. Let B be a subring of A with $1_B = 1_A$. As usual, it is interesting to know if the left AS property is a going-up property or a going-down property, that is, (1) if B is left AS, does it imply that every element of B is left AS in A? and (2) if every element of B is left AS in A, does it imply that B is left AS?

By [15, Theorem 14], every commutative ring is embeddable in a commutative UG ring. So, a subring of a left UG ring need not be left UG. Hence, a subring of a left AS ring need not be left AS, and this shows that the left AS property is not a going-down property.

For the going-up, consider a more restrictive situation.

QUESTION 3.1. Let B be a subring of A with $1_B = 1_A$ such that $A = B + J(A)$. If B is left AS, does it imply that every element of B is left AS in A?

It would be convenient to give examples of left AS rings if the anwser to Question 3.1 was in the affirmative. Meanwhile, Question 3.1 is related to another question of Nicholson [**13**, Question 3]. A ring is clean if every element is the sum of an idempotent and a unit (see [**12**]).

QUESTION 3.2. [**13**] If $R/J(R)$ is left AS does it follow that R is left AS? What if R is exchange? Clean?

Note that R has $SR1$ iff $R/J(R)$ has $SR1$. Moreover, by [**13**, Corollary 19], if $R/J(R)$ is unit-regular and idempotents lift modulo $J(R)$, then the matrix ring $\mathbb{M}_n(R)$ is left AS for all $n \geq 1$; in particular, R is left AS. Furthermore, every unit-regular ring is clean. So, Question 3.2 is well motivated.

To answer Questions 3.1 and 3.2, we first prove a necessary and sufficient condition for a triangular matrix ring to be left AS as an application of Theorem 2.8.

THEOREM 3.3. *Let A, B be rings and M be an (A, B)-bimodule. The following are equivalent:*

(1) The triangular matrix ring $R := \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is left AS.*

(2)(a) Whenever $(1 - a'a)r = 0$ and $(1 - a'a)x \in Ms$, $a, a', r \in A$, $s \in B$ and $x \in M$, there exists a unit $u \in U(A)$ such that $(1 - ua)r = 0$ and $(1 - ua)x \in Ms$.

(b) B is left AS.

Proof. $(1) \Rightarrow (2)$. Suppose that $(1 - a'a)r = 0$ and $(1 - a'a)x \in Ms$, where $a, a', r \in A$, $s \in B$ and $x \in M$. Let $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} r & x \\ 0 & s \end{pmatrix}$ and write $(1 - a'a)x = x's$ with $x' \in M$. Then, $\begin{pmatrix} 1 - a'a & -x' \\ 0 & 0 \end{pmatrix} \in \mathrm{l}(\beta)$, and it follows that $R\alpha + \mathrm{l}(\beta) = R$. As α is left AS, there is a unit $\gamma := \begin{pmatrix} u & y \\ 0 & v \end{pmatrix}$ in R such that $\alpha - \gamma \in \mathrm{l}(\beta)$. It follows that u is a unit in A, $(a - u)r = 0$ and $(a - u)x \in Ms$. Hence, $(1 - u^{-1}a)r = 0$ and $(1 - u^{-1}a)x \in Ms$; so $(2a)$ holds.

It follows from [**13**, Theorem 3] that B is left AS.

$(2) \Rightarrow (1)$. As $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \subseteq J(R)$, to show (1) it suffices to show that every $\alpha = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R$ is left AS in R by [**13**, Lemma 35]. As $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, we only need to show that both $\alpha_1 := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha_2 := \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ are left AS in R by Theorem 2.8.

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Assume that $R\alpha_1 + \mathrm{l}(\beta) = R$ where $\beta = \begin{pmatrix} r & x \\ 0 & s \end{pmatrix}$. Then, there exists $\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \in R$ such that

$$0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \alpha_1 \right) \beta$$
$$= \begin{pmatrix} (1-a'a)r & (1-a'a)x - x's \\ 0 & -b's \end{pmatrix}.$$

That is, $(1-a'a)r = 0$ and $(1-a'a)x \in Ms$. By (2)(a), there exists $u \in U(A)$ such that $(1-ua)r = 0$ and $(1-ua)x \in Ms$. Write $(1-ua)x = ys$ with $y \in M$. Then $\gamma := \begin{pmatrix} u^{-1} & -u^{-1}y \\ 0 & 1 \end{pmatrix}$ is a unit in R and $\alpha_1 - \gamma \in \mathrm{l}(\beta)$. So α_1 is left AS in R.

Assume that $R\alpha_2 + \mathrm{l}(\beta) = R$ where $\beta = \begin{pmatrix} r & x \\ 0 & s \end{pmatrix}$. Then there exists $\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \in R$ such that

$$0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \alpha_2 \right) \beta$$
$$= \begin{pmatrix} (1-a')r & (1-a')x - x'bs \\ 0 & (1-b'b)s \end{pmatrix}.$$

Thus, $(1-a')r = 0$ and $(1-a')x \in Mbs$. By (2)(a), there exists $u \in U(A)$ such that $(1-u)r = 0$ and $(1-u)x \in Mbs$. Write $(1-u)x = ybs$ with $y \in M$. Moreover, from $(1-b'b)s = 0$, we see that $Bb + \mathrm{l}(s) = B$. Hence, by (2)(b), $b - v \in \mathrm{l}(s)$ for some $v \in U(B)$. Now $\gamma := \begin{pmatrix} u & yb \\ 0 & v \end{pmatrix}$ is a unit in R and $\alpha_2 - \gamma \in \mathrm{l}(\beta)$. So α_2 is left AS in R. □

COROLLARY 3.4. *The upper triangular matrix ring* $\mathbb{T}_n(\mathbb{Z})$ $(n \geq 2)$ *is not left AS.*

Proof. By [**13**, Theorem 30], it suffices to show that $\mathbb{T}_2(\mathbb{Z})$ is not left AS. Considering Theorem 3.3(2a) and considering $a' = 2$, $a = 3$, $x = 1$ and $s = 5$ with $A = B = M = \mathbb{Z}$, we have $(1 - 2 \cdot 3) \cdot 1 \in 5\mathbb{Z}$. That is,

$$(1-a'a)x \in Ms.$$

We next see that a' cannot be replaced by a unit u in A. $\mathbb{Z}$ has two units 1 and -1. If $a' = 1$, then $-2 = (1-a'a)x \in 5\mathbb{Z}$, a contradiction. If $a' = -1$, then $4 = (1-a'a)x \in 5\mathbb{Z}$, again a contradiction. So, Theorem 3.3(2a) is not satisfied. Hence, $\mathbb{T}_2(\mathbb{Z})$ is not left AS. □

In the next example, we give a direct proof that $\mathbb{T}_2(\mathbb{Z})$ is not left AS.

EXAMPLE 3.5. Let $R = \mathbb{T}_2(\mathbb{Z})$. If $p \in \mathbb{Z}$ is a prime, then $\alpha := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ is not left AS in R.

Proof. We have $R\alpha = \begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Take $1 < q \in \mathbb{Z}$ such that

$$\gcd\big(q,\ p(p^2-1)\big) = 1.$$

Let $\beta = \begin{pmatrix} 0 & 1 \\ 0 & q \end{pmatrix} \in R$. Then, $\mathrm{l}(\beta) = \left\{ \begin{pmatrix} qn & -n \\ 0 & 0 \end{pmatrix} : n \in \mathbb{Z} \right\}$. As $\gcd(q, p) = 1$, $p\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}$, so $R\alpha + \mathrm{l}(\beta) = R$.

We next show that for any unit $\gamma = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in R, $\alpha - \gamma \notin \mathrm{l}(\beta)$. Assume that $\alpha - \gamma \in \mathrm{l}(\beta)$. Then, $0 = \begin{pmatrix} p-x & -y \\ 0 & 1-z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & (p-x)-qy \\ 0 & q(1-z) \end{pmatrix}$. It follows that $p - x = qy$. As γ is a unit in R, $x = \pm 1$. But, this would yield $p-1 = qy$ or $p+1 = qy$ in $\mathbb{Z}$, contradicting the choice of q. Hence, $\alpha - \gamma \notin \mathrm{l}(\beta)$ for any unit γ in R. So, α is not left *AS* in R. □

EXAMPLE 3.6. Let $R = \begin{pmatrix} \mathbb{Z}_2 & M \\ 0 & \mathbb{Z} \end{pmatrix}$, where M is a $(\mathbb{Z}_2, \mathbb{Z})$-bimodule. Then, R is left *AS*.

Proof. Since $\mathbb{Z}$ is left *AS*, by Theorem 3.3, it suffices to verify that whenever $(\bar{1} - a'a)r = 0$ and $(\bar{1} - a'a)x \in Ms$, $a, a', r \in \mathbb{Z}_2$, $s \in \mathbb{Z}$ and $x \in M$, we have $(\bar{1} - a)r = 0$ and $(\bar{1} - a)x \in Ms$. This is certainly the case if $a' = \bar{1}$. So, we can assume that $a' = \bar{0}$, which implies that $r = \bar{0}$. Thus, $(\bar{1} - a)r = 0$, and $x = (\bar{1} - a'a)x \in Ms$. It follows that $(1 - a)x \in Ms$. □

We now give answers to both Questions 3.1 and 3.2.

THEOREM 3.7. *(1) The answer to Question 3.1 is in the negative.*
(2) There exists a ring R such that $R/J(R)$ is left AS, but R is not left AS.
(3) If R is an exchange ring, then R is left AS iff $R/J(R)$ is left AS.

Proof. (1) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $R = S + J(R)$, where $S = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$. Here $S \cong \mathbb{Z} \times \mathbb{Z}$ is left *AS*. But, the element $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ in S is not left *AS* in R by Example 3.5. Hence, the answer to Question 3.1 is in the negative.

(2) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $J(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$, so $R/J(R) \cong \mathbb{Z} \times \mathbb{Z}$ is left *AS*. But R is not left *AS* by Example 3.5.

(3) Since R is an exchange ring, R being left *AS* iff R having SR1 and, respecively, $R/J(R)$ being left *AS* iff $R/J(R)$ having SR1 by Theorem 2.1 and [**10**, Theorem 6.5]. Moreover, it is known that R has SR1 iff $R/J(R)$ has SR1, so it follows that R is left *AS* iff $R/J(R)$ is left *AS*. □

REMARK 3.8. Corollary 3.4 disproves [13, Theorem 36], [13, Lemma 37] and [13, Theorem 38].

It is unknown whether every left AS ring is right AS (see [4, Remark 4.9], [10, p. 218], [13, Question 4]). However, we have

THEOREM 3.9. *A left AS element in a ring need not be right AS.*

Proof. Let $R = \mathbb{T}_2(\mathbb{Z})$ and $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in R$, where $p \in \mathbb{Z}$ is a prime. By Example 3.5, α is not left AS in R. We next show that α is right AS in R.

Suppose that $\alpha R + \mathrm{r}(\beta) = R$, where $\beta = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in R. As $\alpha R = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, it follows from $\alpha R + \mathrm{r}(\beta) = R$ that $x = 0$, and so $\beta = \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix}$. Then $\alpha - I_2 = \begin{pmatrix} p-1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{r}(\beta)$. So, α is right AS in R. $\square$

It is open whether R being commutative UG implies that $R[[t]]$ is UG (see [1, Question 21]). We end the paper by giving a UG ring R such that $R[[t]]$ is not UG. Note that $\mathbb{M}_n(\mathbb{Z})$ is a UG ring by [9, Theorem 3.8].

EXAMPLE 3.10. Let $R = \mathbb{M}_n(\mathbb{Z})$ $(n \geq 2)$. Then, $R[[t]]$ is not left UG.

Proof. As $\mathbb{M}_n(\mathbb{Z})[[t]] \cong \mathbb{M}_n(\mathbb{Z}[[t]])$, it suffices to show that $\mathbb{M}_2(\mathbb{Z}[[t]])$ $(\cong \mathbb{M}_2(\mathbb{Z})[[t]])$ is not left AS by Theorem 2.1 and [13, Theorem 30]. Hence, we can assume that $n = 2$.

Let $\alpha = a_0 + a_1 t$ and $\beta = b_0 + b_1 t$, where $a_0 = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$, $a_1 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$, $b_0 = \begin{pmatrix} 0 & 0 \\ 17 & 17 \end{pmatrix}$, and $b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and let $a = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$
\begin{aligned}
(1 - \alpha a)\beta &= [(1 - a_0 a) - a_1 a t](b_0 + b_1 t) \\
&= (1 - a_0 a) b_0 + [(1 - a_0 a) b_1 - a_1 a b_0] t - a_1 a b_1 t^2 \\
&= 0.
\end{aligned}
$$

So , $R[[t]]a + \mathrm{l}(\beta) = R[[t]]$. We next show that for any unit $\gamma = r_0 + r_1 t + \cdots$ in $R[[t]]$, $a - \gamma \notin \mathrm{l}(\beta)$. Assume that $a - \gamma \in \mathrm{l}(\beta)$. Then, it follows that $(a - r_0) b_0 = 0$ and $(a - r_0) b_1 = r_1 b_0$ with r_0 a unit in R. Write $r_0 = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$. From $(a - r_0) b_0 = 0$, it follows that $u_2 = 0$ and $u_4 = 1$. So $r_0 = \begin{pmatrix} u_1 & 0 \\ u_3 & 1 \end{pmatrix}$ and $u_1 = \pm 1$. Thus, from $(a - r_0) b_1 = r_1 b_0$, it follows that $5 - u_1 = 17k$ for some $k \in \mathbb{Z}$. But this is impossible as $u_1 = \pm 1$. Therefore, we have proved that a is not left AS in $R[[t]]$. $\square$

By Example 3.10, the ring $\mathbb{M}_n(\mathbb{Z})[[t]]$ $(n \geq 2)$ also satisfies Theorem 3.7(2).

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