

PAPER

# Freely adjoining monoidal duals

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## Abstract

Given a monoidal category  $\mathcal{C}$  with an object  $J$ , we construct a monoidal category  $\mathcal{C}[J^\vee]$  by freely adjoining a right dual  $J^\vee$  to  $J$ . We show that the canonical strong monoidal functor  $\Omega: \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  provides the unit for a biadjunction with the forgetful 2-functor from the 2-category of monoidal categories with a distinguished dual pair to the 2-category of monoidal categories with a distinguished object. We show that  $\Omega: \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  is fully faithful and provide coend formulas for homs of the form  $\mathcal{C}[J^\vee](U, \Omega A)$  and  $\mathcal{C}[J^\vee](\Omega A, U)$  for  $A \in \mathcal{C}$  and  $U \in \mathcal{C}[J^\vee]$ . If  $\mathbb{N}$  denotes the free strict monoidal category on a single generating object 1, then  $\mathbb{N}[1^\vee]$  is the free monoidal category Dpr containing a dual pair  $- \dashv +$  of objects. As we have the monoidal pseudopushout  $\mathcal{C}[J^\vee] \simeq \text{Dpr} \times_{\mathbb{N}} \mathcal{C}$ , it is of interest to have an explicit model of Dpr: we provide both geometric and combinatorial models. We show that the (algebraist's) simplicial category  $\Delta$  is a monoidal full subcategory of Dpr and explain the relationship with the free 2-category Adj containing an adjunction. We describe a generalization of Dpr which includes, for example, a combinatorial model Dseq for the free monoidal category containing a duality sequence  $X_0 \dashv X_1 \dashv X_2 \dashv \dots$  of objects. Actually, Dpr is a monoidal full subcategory of Dseq.

**Keywords:** Autonomization; monoidal dual; string diagram; adjunction; biadjoint

## 1. Introduction

From the higher categorical point of view, dual objects in monoidal categories are a special case of adjoint morphisms in bicategories and so the concept basically goes back to the counit-unit definition of adjunction due to Kan (1958). The special case was made explicit for different uses by Kelly (1972) and Saavedra Rivano (1972). A symmetric monoidal category in which every object has a dual, Kelly called *compact* and Saavedra Rivano called *rigid*. Much later, without the symmetry requirement and so requiring both left and right duals, Joyal–Street (Joyal and Street 1988) used the term *autonomous*.

This project began by our seeking a construction which freely adjoined dual objects to a given monoidal category  $\mathcal{C}$  with some chosen objects to have the duals. The existence was not in doubt. We wanted the construction to be sufficiently explicit for us to prove various properties of it and its relationship to  $\mathcal{C}$ . While considering this, we came across the work of A. Delpeuch (2019) in which he uses the string diagrams as in the unpublished work (Joyal and Street 1988) of A. Joyal and the second author to construct the free autonomous (rigid) monoidal category  $\text{Auton}\mathcal{C}$  on  $\mathcal{C}$ : all objects are given duals. He proves that the canonical functor  $\mathcal{C} \rightarrow \text{Auton}\mathcal{C}$  is fully faithful. This was also one of our observations.

Since our constructions and techniques are quite different from those of Delpuch (2019), we feel they should be of independent interest. Moreover, we deal with situations where fewer duals are adjoined than in  $\text{Auton}\mathcal{C}$  and we instigate the study of general Hom sets in such categories. We shall now describe the contents of this paper in more detail.

After a brief review in Section 2 of duality for objects in a monoidal category and associated string diagrams, we describe in Section 3 a geometric model of the free monoidal category  $\text{Dpr}$  containing a dual pair  $- \dashv +$ . The objects are words in the symbols  $-$  and  $+$  while the morphisms are string diagrams. We should think of these morphisms as in normal form since, when the string diagrams are composed via vertical stacking, the result must be reduced to normal form by tugging (that is, by applying the snake identities).

The goal of Section 4 is to present a non-skeletal combinatorial model of  $\text{Dpr}$ . The objects are pairs  $(M, S)$  where  $M$  is a finite linearly ordered set and  $S$  is a subset; we think of  $M$  as the set of positions for the letters  $-$  and  $+$  in the word and  $S$  tells where there is a  $+$ . Morphisms  $(A, B) : (M, S) \rightarrow (N, T)$  consist of subsets  $A \subseteq M$  and  $B \subseteq N$ , which tell the positions in the domain that are joined to positions in the codomain by a string; there are conditions on  $(A, B)$ .

In Section 5, we show that the (algebraist’s) simplicial category  $\Delta$  is a monoidal full subcategory of  $\text{Dpr}$  and explain the relationship with the free 2-category  $\text{Adj}$  containing an adjunction.

As an interlude, in Section 6, we generalize the construction of  $\text{Dpr}$  to include, for example, a combinatorial model  $\text{Dseq}$  for the free monoidal category containing a duality sequence  $X_0 \dashv X_1 \dashv X_2 \dashv \dots$  of objects.

Section 7 constructs the free monoidal category  $\mathcal{C}[J^\vee]$  adjoining a right dual to a given object  $J$  in a monoidal category  $\mathcal{C}$ . Section 8 proves the universal property. Sections 9 proves that the canonical strong monoidal functor  $\Omega : \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  is full by appealing to the details of the construction. Section 10 proves that  $\Omega : \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  is faithful by using a more transcendental argument. Section 11 provides formulas for some homs in  $\mathcal{C}[J^\vee]$  other than those coming from  $\mathcal{C}$ . In Section 12, we demonstrate that adjoining a dual corresponds to taking a pseudopushout with  $\text{Dpr}$ . In Section 13, we explain how our results extend to  $K$ -linear categories. In Section 14, we discuss the link between our results and the principle of autonomization.

## 2. Dual Pairs

For the definition of monoidal category  $\mathcal{V}$  (also called tensor category) see Eilenberg and Kelly (1966), Lane (1971), Joyal and Street (1991). We use  $\mathbb{I}$  for the tensor unit and  $\otimes$  for the binary tensor product. We will make use of the fact that every monoidal category is monoidally equivalent to a strict monoidal category (that is, one in which the associativity and unitality isomorphisms are identities).

For a linearly ordered set  $M = \{m_1 < m_2 < \dots < m_k\}$  and  $X_m \in \mathcal{V}$  for all  $m \in M$ , we put

$$\bigotimes_{m \in M} X_m = X_{m_1} \otimes X_{m_2} \otimes \dots \otimes X_{m_k}.$$

Duality for objects in  $\mathcal{V}$  was defined in Saavedra Rivano (1972), Kelly (1972), Joyal and Street (1988). We will make use of the string diagrams for monoidal categories as explained in Joyal and Street (1988, 1991) except that we will read from top to bottom rather than left to right or bottom to top.

A *right dual* for an object  $X$  of  $\mathcal{V}$  is an object  $Y$  equipped with morphisms  $\varepsilon : X \otimes Y \rightarrow \mathbb{I}$  and  $\eta : \mathbb{I} \rightarrow Y \otimes X$ , called *counit* and *unit*, such that the composites

$$X \xrightarrow{X \otimes \eta} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes X} X \text{ and } Y \xrightarrow{\eta \otimes Y} Y \otimes X \otimes Y \xrightarrow{Y \otimes \varepsilon} Y \tag{1}$$

are identities. As this is a special case of adjunction in a bicategory, we use the notation  $X \dashv Y$  and call this a *dual pair*. Here are the string diagrams expressing that the composites in (1) are identities.

When there is no ambiguity, we denote counits by cups  $\cup$  and units by caps  $\cap$ . So (1) becomes the more geometrically “obvious” operation of pulling the ends of the strings as in (3). These are sometimes called the *snake equations*.

Dualities tensor. If  $X \dashv Y$  and  $Z \dashv W$ , then  $X \otimes Z \dashv W \otimes Y$ ; the counit and unit are the nested cups and nested caps as shown in (4).

In particular, if  $X \dashv Y$ , then  $X^{\otimes n} \dashv Y^{\otimes n}$  with counit and unit given by  $n$  nested cups and  $n$  nested caps.

If  $X \dashv Y$  and  $A \dashv B$  in  $\mathcal{V}$ , recall that morphisms  $A \rightarrow X$  and  $Y \rightarrow B$  are *mates* if they are exchanged under the bijection  $\mathcal{V}(A, X) \cong \mathcal{V}(Y, B)$ .

We remind the reader that duals invert.

**Proposition 1** (Day and Pastro 2008; Joyal and Street 1988; Saavedra Rivano 1972). *Suppose  $X \dashv Y$  is a dual pair in a monoidal category  $\mathcal{A}$ . If  $\sigma : S \Rightarrow T : \mathcal{A} \rightarrow \mathcal{V}$  is a monoidal natural transformation between strong monoidal natural transformations, then the components  $\sigma_X$  and  $\sigma_Y$  of  $\sigma$  are invertible. Indeed,  $\sigma_X^{-1}$  is the mate of  $\sigma_Y$  and  $\sigma_Y^{-1}$  is the mate of  $\sigma_X$  under the dualities  $SX \dashv SY$  and  $TX \dashv TY$ .*

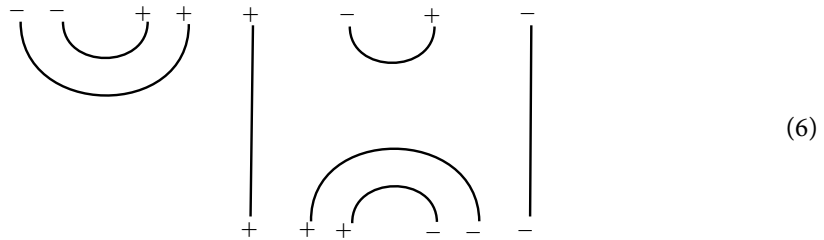
A morphism  $(u, v) : (X, Y, \varepsilon, \eta) \rightarrow (X_1, Y_1, \varepsilon_1, \eta_1)$  of dual pairs in  $\mathcal{V}$  consists of morphisms  $u : X \rightarrow X_1$  and  $v : Y_1 \rightarrow Y$  in  $\mathcal{V}$  such that  $\varepsilon_1 \circ (u \otimes 1) = \varepsilon \circ (1 \otimes v)$ ; see (5). In fact,  $u$  and  $v$  are mates under the dualities so determine each other. This defines a monoidal category  $\text{Dp}\mathcal{V}$  of dual pairs in  $\mathcal{V}$ .

A monoidal category is called *autonomous* when every object has a left and a right dual. The terms *rigid* and *compact* are all used in the literature to mean the existence of duals. The free

autonomous monoidal category on any autonomous tensor scheme was described geometrically in Joyal and Street (1988). The free autonomous monoidal category on any monoidal category was described geometrically in Delpeuch (2019). Here we are interested in combinatorial descriptions of related free monoidal structures.

### 3. Geometric Model of Dpr

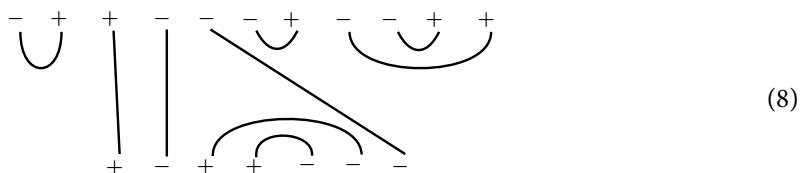
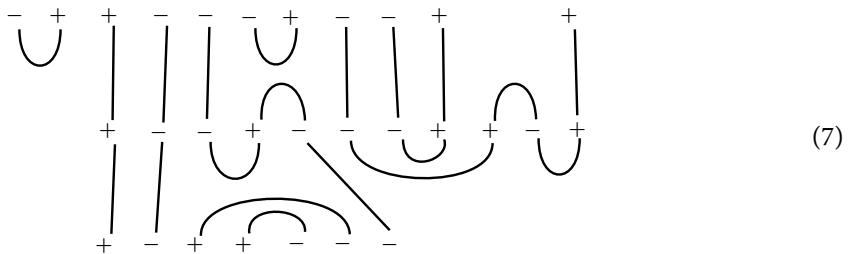
Our interest here is in the free monoidal category  $Dpr$  containing a duality pair of objects. A presentation of this in terms of tensor schemes (in the sense of Joyal and Street 1991) takes the generating tensor scheme to have two objects  $-$  and  $+$  and two morphisms  $\varepsilon : -+ \rightarrow \emptyset$  and  $\eta : \emptyset \rightarrow +-$  and subjects them to the snake equations. The objects of  $Dpr$  are therefore elements of the free monoid  $\{-, +\}^*$  on the two symbols  $-$  and  $+$ . As strings, each morphism consists of a sequence of nested cups on the domain word, a sequence of nested caps on the codomain word, and lines between the remaining symbols in the words without any crossings of lines, cups, or caps. The cups in the domain are directed from  $-$  to  $+$ , the caps in the codomain are directed from  $+$  to  $-$ , and the lines are directed either from a  $-$  in the domain to a  $-$  in the codomain or from a  $+$  in the codomain to a  $+$  in the domain. Diagram (6) is an example of a morphism from  $--++++-+-$  to  $++++--$ ; diagrams (7) and (8) give other examples.



Composition is performed by vertical stacking followed by employing the snake equations to straighten out the snaking into cups, caps, and lines. For example, the composable morphisms

$$(-++-- --+-++) \rightarrow (+---+---++-+) \rightarrow (+-++----)$$

as shown stacked vertically in (7) have composite as shown in (8).



**Remark 2.** The string diagrams we have described are in “normal form.” Composition is done by vertical stacking and then moving to normal form. Readers requiring a geometric model which accommodates the diagrams arising from vertical stacking of the normal diagrams *per se* should see Joyal and Street (1988). Winding numbers are involved.

**Remark 3.** These string diagrams and their composition are much like those of the Temperley–Lieb algebra (Temperley and Lieb 1971) except that their directed nature prohibits the creation of loops. They are also particularly non-tangled tangles as occurring in Yetter (1988) and Shum (1994).

**4. Combinatorial Definition of Dpr**

A subset  $K$  of an ordered set  $M$  is an *interval* when  $a \leq b \leq c$  and  $a, c \in K$  imply  $b \in K$ . For  $a, c \in M$ , we have the interval  $[a, c] = \{b \in M : a \leq b \leq c\}$ .

Let  $S$  be a subset of a linearly ordered set  $M$  and let  $K$  be a finite interval in  $M$ . We say that  $K$  is an *S-cup* [respectively, *S-cap*] when  $K$  has even cardinality and  $S \cap K$  is a final [respectively, initial] segment of  $K$  containing exactly half of the elements of  $K$ . Notice that, if  $K$  and  $L$  are both *S-cups* or both *S-caps* and  $K \cap L \neq \emptyset$ , then either  $K \subseteq L$  or  $L \subseteq K$ .

Suppose  $M$  and  $S$  are as above and  $a \in M$ . We write  $a^\cup$  for the element of  $M$  for which the interval  $[a, a^\cup]$  in  $M$  is an *S-cup*. Such an element  $a^\cup$  may not exist but it will if  $M$  is a union of *S-cups* and  $a \notin S$ . Similarly, we write  $a^\cap$  for the element of  $M$  for which the interval  $[a, a^\cap]$  in  $M$  is an *S-cap*.

**Example.** If  $M = \{m_1 < m_2 < m_3 < m_4\}$ ,  $S = \{m_3, m_4\}$ , then  $m_1^\cup = m_4$  and  $m_2^\cup = m_3$ .

Suppose  $H \subseteq M$  and  $x, x' \in M$ . A sequence

$$x = x_0, x_1, \dots, x_n = x'$$

is said to be *snaking in H* out of  $x$  into  $x'$  when  $x_i \in H$  for  $0 < i < n$  and either  $x_{m+1} = x_m^\cup$  or  $x_{m+1} = x_m^\cap$  for  $0 \leq m < n$ . We write  $x \rightsquigarrow_H x'$  when such a sequence exists.

If  $M$  and  $N$  are linearly ordered sets and  $A \subseteq M, B \subseteq N$  are subsets of the same cardinality  $\#A = \#B$ , then there is a unique order-preserving bijection  $A \rightarrow B$ ; we call  $a \in A$  and  $b \in B$  *cobbers* when they correspond under this bijection.

We now describe a monoidal category *Dpr* which is equivalent to the geometric version of Section 3. An object  $(M, S)$  consists of a finite linearly ordered set  $M$  and a subset  $S \subseteq M$ . Of course, we can also think of such an object as a function  $\chi_S : M \rightarrow \{-, +\}$  for which  $S$  is the inverse image of  $+$ .

A morphism  $(A, B) : (M, S) \rightarrow (N, T)$  in *Dpr* consists of subsets  $A \subseteq M$  and  $B \subseteq N$  such that

- (i)  $\#A = \#B$ ,
- (ii) for all cobbers  $a \in A$  and  $b \in B, a \in S$  if and only if  $b \in T$ ,
- (iii)  $A' = M \setminus A$  is a union of *S-cup* intervals in  $M$ ,
- (iv)  $B' = N \setminus B$  is a union of *T-cap* intervals in  $N$ .

The identity morphism of  $(M, S)$  is  $(M, M) : (M, S) \rightarrow (M, S)$ .

The composite

$$(M, S) \xrightarrow{(E,F)} (P, U) = ((M, S) \xrightarrow{(A,B)} (N, T) \xrightarrow{(C,D)} (P, U))$$

is defined by

$$E = \{x \in A : \text{the cobber } y \in B \text{ of } x \text{ has } y \rightsquigarrow_{B' \cap C} y' \text{ with } y' \in C\} \\ \cup \{x' \in A : \text{the cobber } y' \in B \text{ of } x' \text{ has } y \rightsquigarrow_{B' \cap C} y' \text{ with } y \in C\}$$

and

$$F = \{z \in D : \text{the cobber } y \in C \text{ of } z \text{ has } y \rightsquigarrow_{B' \cap C} y' \text{ with } y' \in B\} \\ \cup \{z' \in D : \text{the cobber } y' \in C \text{ of } z' \text{ has } y \rightsquigarrow_{B' \cap C} y' \text{ with } y \in B\}.$$

The intersection of the two sets whose union gives the definition of  $E$  contains precisely those  $x \in A$  whose cobber  $y \in B$  is also in  $C$ . The intersection of the two sets whose union is  $F$  contains precisely those  $z \in D$  whose cobber  $y \in C$  is also in  $B$ . The sets  $E$  and  $F$  have the same cardinality: the cobber in  $F$  of an  $x \in E$  in the first set of the union is the cobber in  $D$  of the  $y' \in C$ , while the cobber in  $F$  of an  $x' \in E$  in the second set of the union is the cobber in  $D$  of the  $y \in C$ .

The complement  $E'$  of  $E$  is a union of  $S$ -cups; indeed, we have the disjoint union

$$E' = A' \cup \{x, x' \in A : \text{the cobbers } y, y' \in B \text{ of } x, x' \text{ have } y \rightsquigarrow_{B' \cap C} y'\}.$$

Similarly,

$$F' = D' \cup \{z, z' \in D : \text{the cobbers } y, y' \in C \text{ of } z, z' \text{ have } y \rightsquigarrow_{B' \cap C} y'\}.$$

**Proposition 4.** *A morphism  $(A, B) : (M, S) \rightarrow (N, T)$  in Dpr is invertible if and only if  $A = M, B = N$ .*

The tensor product for Dpr is defined by componentwise ordinal sum which we denote by  $+$ . So

$$(M, S) \otimes (M_1, S_1) = (M + M_1, S + S_1) \text{ and } (A, B) \otimes (A_1, B_1) = (A + A_1, B + B_1).$$

We write  $\mathbf{n}$  for the ordinal  $\{0, 1, \dots, n - 1\}$ . The unit for this tensor product is  $\mathbb{I} = (\mathbf{0}, \emptyset)$ .

We shall now distinguish a dual pair in Dpr. We have two objects  $- = (\mathbf{1}, \emptyset)$  and  $+ = (\mathbf{1}, \{0\})$  of Dpr. Then  $- \otimes + = (\mathbf{2}, \{1\})$  and  $+ \otimes - = (\mathbf{2}, \{0\})$ . Put  $\varepsilon = (\emptyset, \emptyset) : - \otimes + \rightarrow \mathbb{I}$  and  $\eta = (\emptyset, \emptyset) : \mathbb{I} \rightarrow + \otimes -$ . It is an easy exercise in the definition of composition in Dpr to see that the composites (1) (with  $X = -$  and  $Y = +$ ) are identities. So  $- \dashv +$ .

Every object of Dpr is uniquely isomorphic to one of the form  $(\mathbf{n}, S)$ . For every finite linearly ordered set  $M$ , we have  $(M, \emptyset) \dashv (M, M)$  in Dpr by tensoring  $- \dashv +$   $\#M$  times.

Let us call *elementary* those morphisms in Dpr of the form

$$(\emptyset, \emptyset) : (M, S) \rightarrow (N, T)$$

where  $M$  and  $N$  are not both empty.

**Proposition 5.** *Every morphism of Dpr is uniquely of the form*

$$f_1 \otimes e_1 \otimes f_2 \otimes e_2 \otimes \dots \otimes e_{k-1} \otimes f_k$$

where  $f_1, \dots, f_k$  are invertible morphisms and  $e_1, \dots, e_{k-1}$  are elementary morphisms.

*Proof.* Take any morphism  $(A, B) : (M, S) \rightarrow (N, T)$  of Dpr. Then we can write  $A = K_1 + K_2 + \dots + K_k$  and  $B = L_1 + L_2 + \dots + L_k$  where the  $K_i$  are intervals in  $M$  which are maximal as subsets of  $A$ , the  $L_j$  are intervals in  $N$  which are maximal as subsets of  $B$ , and if  $a \in K_i$  in  $A$  has cobber  $b \in L_j$  in  $B$  then  $i = j$ . For  $1 \leq i < k$ , let  $E_i \subseteq A'$  consist of all the elements between  $K_i$  and  $K_{i+1}$  and let  $F_i \subseteq B'$  consist of all the elements between  $L_i$  and  $L_{i+1}$ ; not both are empty or else  $K_i$  and  $K_{i+1}$  or  $L_i$  and  $L_{i+1}$  would intersect. Then we have  $f_i : (K_i, S \cap K_i) \rightarrow (L_i, T \cap L_i)$  and  $e_i : (E_i, S \cap E_i) \rightarrow (F_i, T \cap F_i)$  as desired. □

For monoidal categories  $\mathcal{A}$  and  $\mathcal{V}$ , write  $\text{StMon}_g(\mathcal{A}, \mathcal{V})$  for the groupoid of strong monoidal functors from  $\mathcal{A}$  to  $\mathcal{V}$  and monoidal natural isomorphisms. Recall (from around (5)) the category  $\text{Dp}^{\mathcal{V}}$  of dual pairs in  $\mathcal{V}$ . Write  $\text{Dp}_g^{\mathcal{V}}$  for its subcategory consisting of all the objects but only the invertible morphisms.

**Proposition 6.** *For any monoidal category  $\mathcal{V}$ , the functor*

$$\text{StMon}_g(\text{Dpr}, \mathcal{V}) \longrightarrow \text{Dp}_g^{\mathcal{V}}, \quad \Phi \mapsto (\Phi - \dashv \Phi +)$$

*is an equivalence of groupoids.*

*Proof.* Take a dual pair  $X \dashv Y$  in  $\mathcal{V}$ . Define  $\text{Dpr} \xrightarrow{\Phi} \mathcal{V}$  on objects by

$$\Phi(M, S) = \bigotimes_{m \in M} Z_m$$

where  $Z_m = X$  for  $m \notin S$  and  $Z_m = Y$  for  $m \in S$ . Define  $\Phi$  to take invertible morphisms (see Proposition 4) of  $\text{Dpr}$  to identity morphisms in  $\mathcal{V}$ . In accord with the desire for  $\Phi$  to be a strong monoidal functor, a nested cup  $\mathbb{I} \rightarrow (M, S)$  with  $\#M = 2h$  must be taken to the counit of  $X^{\otimes h} \dashv Y^{\otimes h}$ . A nested cap  $(N, T) \rightarrow \mathbb{I}$  with  $\#N = 2k$  must be taken to the unit of  $X^{\otimes k} \dashv Y^{\otimes k}$ . The effect of  $\Phi$  on a general morphism is then forced by Proposition 5. The fact that  $\Phi$  preserves composition follows from the snake equations for the dual pairs  $X^{\otimes k} \dashv Y^{\otimes k}$ . Clearly the functor of the Proposition takes this  $\Phi$  to  $X \dashv Y$ . □

### 5. Relationship to the Free Adjunction

The 2-category freely generated by an adjunction was first considered in Auderset (1974) with an explicit model  $\text{Adj}$  described in Schanuel and Street (1986). The 2-category  $\text{Adj}$  has two objects which we will now denote by  $\flat$  and  $\sharp$ . The hom category  $\text{Adj}(\flat, \flat)$  is the usual algebraist’s  $\Delta$ : the objects are the finite ordinals  $\mathbf{n} = \{0, 1, \dots, n - 1\}$  and the morphisms order-preserving functions. The hom category  $\text{Adj}(\sharp, \sharp)$  is the category  $\Delta_{\top, \perp}$ : the objects are the non-empty ordinals  $\mathbf{n}$  and the morphisms order-preserving functions which preserve first and last elements. The hom category  $\text{Adj}(\flat, \sharp)$  is the category  $\Delta_{\top}$ : the objects are the non-empty ordinals  $\mathbf{n}$  and the morphisms order-preserving functions which preserve last elements. The hom category  $\text{Adj}(\sharp, \flat)$  is the category  $\Delta_{\perp}$ : the objects are the non-empty ordinals  $\mathbf{n}$  and the morphisms order-preserving functions which preserve first elements. Composition is described in Schanuel and Street (1986); in particular, the composition functor

$$\text{Adj}(\flat, \flat) \times \text{Adj}(\flat, \flat) \longrightarrow \text{Adj}(\flat, \flat)$$

is ordinal sum, the tensor product of the monoidal category  $\Delta$ ; and composition

$$\text{Adj}(\sharp, \sharp) \times \text{Adj}(\sharp, \sharp) \longrightarrow \text{Adj}(\sharp, \sharp)$$

is the result of transporting ordinal sum across the duality  $\Delta^{\text{op}} \simeq \Delta_{\top, \perp}$ ,  $\mathbf{n} \mapsto \mathbf{n} + \mathbf{1}$ .

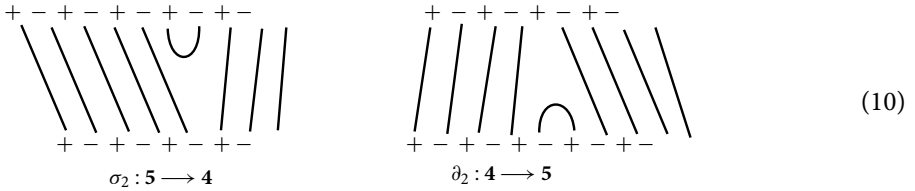
For a monoidal category  $\mathcal{A}$ , we write  $\Sigma \mathcal{A}$  for the one object bicategory whose hom category is  $\mathcal{A}$  and whose composition is the tensor product of  $\mathcal{A}$ . In particular, we have the bicategory  $\Sigma \text{Dpr}$  which contains the adjunction  $- \dashv +$ . Therefore, there is a pseudofunctor

$$\Theta : \text{Adj} \longrightarrow \Sigma \text{Dpr} \tag{9}$$

taking the generating adjunction in  $\text{Adj}$  to  $- \dashv +$ .

In order to study the image of  $\Theta$ , we distinguish some objects of  $\text{Dpr}$ . We call  $(M, S)$  *alternating* when, for all consecutive elements  $a, b \in M$ , we have  $a \in S$  if and only if  $b \notin S$ . Notice that any  $S$ -cup or  $S$ -cap in such an  $M$  can only have cardinality 0 or 2.

Let us look at the effect of  $\Theta$  on the endomorphism  $\text{hom}$  of  $\mathfrak{b} \in \text{Adj}$ . This is a strong monoidal functor  $\Theta_{\mathfrak{b}, \mathfrak{b}} : \Delta \rightarrow \text{Dpr}$ . The object  $\mathbf{n}$  of  $\Delta$  is taken to the alternating object  $(2\mathbf{n}, \text{Ev}_n)$  where  $\text{Ev}_n = \{0, 2, \dots, 2n - 2\}$  consists of the even elements of  $2\mathbf{n}$ . The surjection  $\sigma_i : \mathbf{n} + 1 \rightarrow \mathbf{n}$  in  $\Delta$ , which identifies  $i$  and  $i + 1$ , is taken to  $(\text{de}_i, 2\mathbf{n} + 2) : (2\mathbf{n} + 2, \text{Ev}_{n+1}) \rightarrow (2\mathbf{n}, \text{Ev}_n)$  where  $\text{de}_i$  is obtained from  $2\mathbf{n} + 2$  by deleting  $2i + 1$  and  $2i + 2$ . The injection  $\partial_i : \mathbf{n} \rightarrow \mathbf{n} + 1$  in  $\Delta$ , which does not have  $i$  in its image, is taken to  $(2\mathbf{n}, \text{fa}_i) : (2\mathbf{n}, \text{Ev}_n) \rightarrow (2\mathbf{n} + 2, \text{Ev}_{n+1})$  where  $\text{fa}_i$  is obtained from  $2\mathbf{n} + 2$  by deleting  $2i$  and  $2i + 1$ .



We define  $\Theta$  on general morphisms of  $\Delta$  using the following notation from Lack and Street (2014). For each morphism  $\xi : \mathbf{m} \rightarrow \mathbf{n}$  in  $\Delta$ , we put

$$\xi^\ell = \{i \in \mathbf{m} - 1 : \xi(i) = \xi(i + 1)\} \text{ and } \xi^r = \{j \in \mathbf{n} : j \notin \text{im}\xi\}.$$

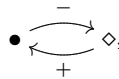
Notice that  $\xi^\ell = \emptyset$  means that  $\xi$  is injective and  $\xi^r = \emptyset$  means that  $\xi$  is surjective. In general,  $\xi$  is determined by  $(\xi^\ell, \xi^r)$ . We have that  $\Theta(\xi) = (A_\xi, B_\xi) : (2\mathbf{m}, \text{Ev}_m) \rightarrow (2\mathbf{n}, \text{Ev}_n)$  is defined by

$$A'_\xi = \{2i + 1, 2i + 2 : i \in \xi^\ell\} \text{ and } B'_\xi = \{2j, 2j + 1 : j \in \xi^r\},$$

where, as before, the primed sets denote the appropriate complements. The restricted possibility for cups and caps in alternating objects means that every morphism  $(2\mathbf{m}, \text{Ev}_m) \rightarrow (2\mathbf{n}, \text{Ev}_n)$  must be of the form  $(A_\xi, B_\xi)$  for some  $\xi : \mathbf{m} \rightarrow \mathbf{n}$  in  $\Delta$ . So the functor  $\Theta_{\mathfrak{b}, \mathfrak{b}} : \Delta \rightarrow \text{Dpr}$  is fully faithful. A similar analysis applies to the other three  $\text{hom}$  categories of  $\text{Adj}$  yielding:

**Proposition 7.** *The pseudofunctor  $\Theta$  (9) is locally fully faithful.*

We can say this differently if  $\text{Dpr}$  is given and  $\text{Adj}$  is to be obtained. Take the free category  $\mathcal{J}$  on the directed graph



regarded as a locally discrete 2-category. Then  $\text{Adj}$  is obtained by factoring the 2-functor  $\mathcal{J} \rightarrow \Sigma\text{Dpr}$ , which takes the morphisms  $-$  to  $-$  and  $+$  to  $+$ , into a 2-functor  $\mathcal{J} \rightarrow \text{Adj}$  which is bijective on both objects and 1-morphisms, and a functor  $\text{Adj} \rightarrow \Sigma\text{Dpr}$  which is locally fully faithful.

### 6. Interlude on Iterated Duals

Let  $\Lambda$  denote a set equipped with a partial endofunction  $\sigma : \Lambda \rightarrow \Lambda$  such that

- a.  $\sigma s = \sigma t$  implies  $s = t$ ;
- b.  $\sigma^n s = s$  for some  $s$  implies  $n = 0$ .

Let  $\Lambda^*$  be the set of words  $\underline{s} = s_1 s_2 \dots s_k$  in the alphabet  $\Lambda$ .

The goal in this section is to define the free monoidal category  $D(\Lambda, \sigma)$  containing the elements of  $\Lambda$  as objects in such a way that  $s \dashv \sigma s$  for all  $s \in \Lambda$  on which  $\sigma$  is defined. First we give some examples.

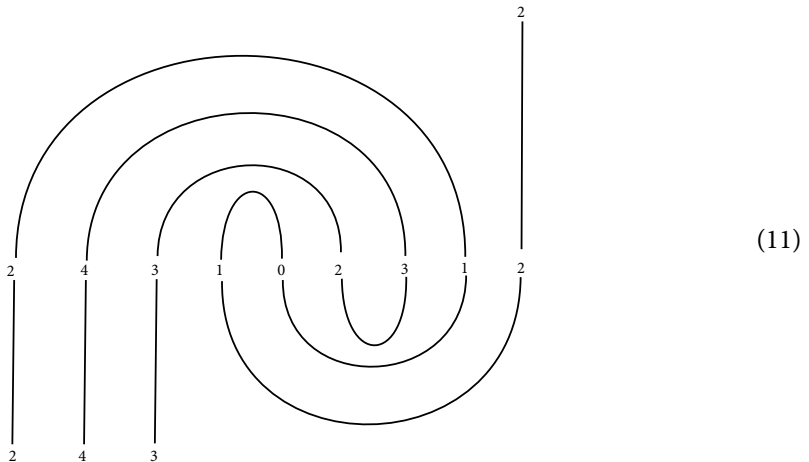


**Example.** 0.  $\Lambda$  arbitrary and  $\sigma$  with empty domain. Then  $D(\Lambda, \sigma)$  is the free monoidal category (discrete  $\Lambda^*$ ) on the set  $\Lambda$ .

1.  $\Lambda = \{-, +\}$ ,  $\sigma - = +$  and  $\sigma +$  undefined. Then  $D(\Lambda, \sigma) = \text{Dpr}$  as in Section 4.
2.  $\Lambda = \mathbb{N}$  and  $\sigma n = n + 1$  for all natural numbers  $n$ . Then  $D(\Lambda, \sigma)$  is the free monoidal category  $\text{Dseq}$  containing a duality sequence  $X_0 \dashv X_1 \dashv X_2 \dashv \dots$  of objects. Equally,  $D(\Lambda, \sigma)$  is the free right autonomous monoidal category  $\text{Dseq}$  generated by a single object. A geometric model can be derived as in Section 3. The objects are words of natural numbers and composition leads to diagrams of the form shown in (11) for morphisms

$$2 \rightarrow 2\ 4\ 3\ 1\ 0\ 2\ 3\ 1\ 2 \rightarrow 2\ 4\ 3.$$

3.  $\Lambda = \mathbb{Z}$  and  $\sigma n = n + 1$  for all integers  $n$ . Then  $D(\Lambda, \sigma)$  is the free autonomous monoidal category on a single generating object. Equally,  $D(\Lambda, \sigma)$  is the free monoidal category  $\text{Dseq}$  containing a doubly infinite string of object dualities  $\dots \dashv X_{-2} \dashv X_{-1} \dashv X_0 \dashv X_1 \dashv X_2 \dashv \dots$
4. Consider an arbitrary set  $\Lambda$  with an element  $J \in \Lambda$ , such that  $\sigma$  is only defined on  $J$ , and set  $\Pi := \Lambda \setminus \{J\}$ . Then  $D(\Lambda, \sigma)$  is  $\mathcal{C}[J^\vee]$  as defined in Section 7 for  $\mathcal{C} := \Pi^*$ .



We now describe combinatorially the monoidal category  $D(\Lambda, \sigma)$  in general. The objects are the elements of  $\Lambda^*$ . To describe morphisms we need some more notation. Let

$$| - | : \Lambda^* \rightarrow \mathbb{N}, \underline{s} \mapsto k$$

take a word to its length. Put  $\langle k \rangle = \{1, 2, \dots, k\}$ .

An interval  $I$  in  $\langle k \rangle$  is an  $\underline{s}$ -cup [respectively,  $\underline{s}$ -cap] when it has even cardinality and  $s_n = \sigma s_m$  [respectively,  $s_m = \sigma s_n$ ] whenever  $m < n$  are both in  $I$  and

$$\#\{x \in I : x \leq m\} = \#\{x \in I : n \leq x\}.$$

We put  $m^\cup = n$  [respectively,  $m^\cap = n$ ] in this situation. We will use the definition of *snaking* and *cobbers* as in Section 4.

A morphism  $(A, B) : \underline{s} \rightarrow \underline{t}$  in  $D(\Lambda, \sigma)$  consists of subsets  $A \subseteq \langle |\underline{s}| \rangle$  and  $B \subseteq \langle |\underline{t}| \rangle$  such that

- (i)  $\#A = \#B$ ,
- (ii) for all cobbers  $a \in A$  and  $b \in B$ ,  $s_a = t_b$ ,
- (iii) the complement  $A' = \langle |\underline{s}| \rangle \setminus A$  is a union of  $\underline{s}$ -cups in  $\langle |\underline{s}| \rangle$ ,
- (iv) the complement  $B' = \langle |\underline{t}| \rangle \setminus B$  is a union of  $\underline{t}$ -caps in  $\langle |\underline{t}| \rangle$ .

The identity morphism of  $\underline{s}$  is  $(\langle |\underline{s}| \rangle, \langle |\underline{s}| \rangle) : \underline{s} \rightarrow \underline{s}$ .

The composite

$$(\langle \underline{s} \rangle \xrightarrow{(E,F)} \langle \underline{t} \rangle) = (\langle \underline{s} \rangle \xrightarrow{(A,B)} \langle \underline{t} \rangle \xrightarrow{(C,D)} \langle \underline{u} \rangle)$$

is defined by

$$E = \{x \in A : \text{the cobber } y \in B \text{ of } x \text{ has } y \rightsquigarrow_{B' \cap C'} y' \text{ with } y' \in C\} \\ \cup \{x' \in A : \text{the cobber } y' \in B \text{ of } x' \text{ has } y \rightsquigarrow_{B' \cap C'} y' \text{ with } y \in C\}$$

and

$$F = \{z \in D : \text{the cobber } y \in C \text{ of } z \text{ has } y \rightsquigarrow_{B' \cap C'} y' \text{ with } y' \in B\} \\ \cup \{z' \in D : \text{the cobber } y' \in C \text{ of } z' \text{ has } y \rightsquigarrow_{B' \cap C'} y' \text{ with } y \in B\}.$$

The intersection of the two sets whose union gives the definition of  $E$  contains precisely those  $x \in A$  whose cobber  $y \in B$  is also in  $C$ . The intersection of the two sets whose union is  $F$  contains precisely those  $z \in D$  whose cobber  $y \in C$  is also in  $B$ . The sets  $E$  and  $F$  have the same cardinality: the cobber in  $F$  of an  $x \in E$  in the first set of the union is the cobber in  $D$  of the  $y' \in C$  while the cobber in  $F$  of an  $x' \in E$  in the second set of the union is the cobber in  $D$  of the  $y \in C$ .

The complement  $E'$  of  $E$  is a union of  $\underline{s}$ -cups; indeed, we have the disjoint union

$$E' = A' \cup \{x, x' \in A : \text{the cobbers } y, y' \in B \text{ of } x, x' \text{ have } y \rightsquigarrow_{B' \cap C'} y'\}.$$

Similarly,

$$F' = D' \cup \{z, z' \in D : \text{the cobbers } y, y' \in C \text{ of } z, z' \text{ have } y \rightsquigarrow_{B' \cap C'} y'\}.$$

The tensor product for  $D(\Lambda, \sigma)$  is defined by juxtaposition  $\underline{s} \otimes \underline{u} = \underline{su}$  on objects. Note that  $|\underline{su}| = |\underline{s}| + |\underline{u}|$  and so  $\langle \underline{su} \rangle$  is the ordinal sum  $\langle \underline{s} \rangle + \langle \underline{u} \rangle$ . So

$$(\underline{s} \xrightarrow{(A,B)} \underline{t}) \otimes (\underline{u} \xrightarrow{(C,D)} \underline{v}) = (\underline{su} \xrightarrow{(A+C, B+D)} \underline{tv}).$$

The unit  $\mathbb{I}$  for this tensor product is of course the word of length 0.

For each  $a \in \Lambda$  at which  $\sigma$  is defined we have a duality  $a \dashv \sigma a$  in  $D(\Lambda, \sigma)$  where we are identifying elements of  $\Lambda$  with elements of  $\Lambda^*$  as the one letter words. Put  $\varepsilon = (\emptyset, \emptyset) : a \otimes \sigma a \rightarrow \mathbb{I}$  and  $\eta = (\emptyset, \emptyset) : \mathbb{I} \rightarrow \sigma a \otimes a$ . It is an easy exercise in the definition of composition in  $D(\Lambda, \sigma)$  to see that the composites (1) (with  $X = a$  and  $Y = \sigma a$ ) are identities. So indeed  $a \dashv \sigma a$ .

**Proposition 8.** *The only invertible morphisms in  $D(\Lambda, \sigma)$  are identities.*

As in Section 4, we call *elementary* those morphisms in  $D(\Lambda, \sigma)$  of the form

$$(\emptyset, \emptyset) : \underline{s} \rightarrow \underline{u}$$

where  $|\underline{s}|$  and  $|\underline{t}|$  are not both 0. Proposition 5 generalizes.

**Proposition 9.** *Every morphism of  $D(\Lambda, \sigma)$  is uniquely of the form*

$$f_1 \otimes e_1 \otimes f_2 \otimes e_2 \otimes \dots \otimes e_{k-1} \otimes f_k$$

where  $f_1, \dots, f_k$  are identity morphisms and  $e_1, \dots, e_{k-1}$  are elementary morphisms.

**Proposition 10.** *For any strict monoidal category  $\mathcal{V}$  and any family  $(X_a)_{a \in \Lambda}$  of objects of  $\mathcal{V}$  with chosen dualities  $X_a \dashv X_{\sigma a}$ , there exists a unique strict monoidal functor  $\Phi : D(\Lambda, \sigma) \rightarrow \mathcal{V}$  such that  $\Phi a = X_a$  for all  $a \in \Lambda$  and  $\Phi$  maps the dualities  $a \dashv \sigma a$  to the chosen dualities  $X_a \dashv X_{\sigma a}$ .*

**Remark 11.** If  $a \in \Lambda$  is in the domain of  $\sigma$ , then the monoidal full subcategory of  $D(\Lambda, \sigma)$ , whose objects are the words in letters  $a$  and  $\sigma a$ , is equivalent to  $Dpr$ .

### 7. Construction of $\mathcal{C}[J^\vee]$

Let  $J$  be an object of the strict monoidal category  $\mathcal{C}$ . In this section, we define a strict monoidal category  $\mathcal{C}[J^\vee]$  obtained by freely adjoining a right dual  $J^\vee$  to  $J$ .

We start by describing the objects of  $\mathcal{C}[J^\vee]$  and their tensor product, by introducing the corresponding monoid. Every object in  $\mathcal{C}[J^\vee]$  will be a finite product of elements in  $\{J^\vee\} \cup \text{ob}\mathcal{C}$ . If we think of the occurrences of  $J^\vee$  in the product as separators between the other objects, an object in  $\mathcal{C}[J^\vee]$  will correspond to a finite ordered set of objects in  $\mathcal{C}$ , so a function from a finite ordered set to  $\text{ob}\mathcal{C}$ . This motivates our formal definition below.

We write  $[m]$  for the set  $\{0, 1, \dots, m\}$  and  $\partial_i : [m - 1] \rightarrow [m]$  for the order-preserving injective function whose image does not contain  $i$ . Let  $\mathcal{G}_0$  be the set of functions  $U : [m] \rightarrow \text{ob}\mathcal{C}$ . Objects of  $\mathcal{C}$  are identified with functions  $[0] \rightarrow \text{ob}\mathcal{C}$ . For another  $V : [n] \rightarrow \text{ob}\mathcal{C}$ , the “tensor product”  $U \otimes V : [m + n] \rightarrow \text{ob}\mathcal{C}$  is defined by

$$(U \otimes V)_i = \begin{cases} U_i & \text{for } i < m \\ U_m \otimes V_0 & \text{for } i = m \\ V_{i-m} & \text{for } i > m. \end{cases} \tag{12}$$

We thus obtain a monoid  $(\mathcal{G}_0, \otimes, \mathbb{I})$  since

$$\mathbb{I} \otimes U = U = U \otimes \mathbb{I}, (U \otimes V) \otimes W = U \otimes (V \otimes W).$$

We will already refer to elements of  $\mathcal{G}_0$  as the “objects of  $\mathcal{C}[J^\vee]$ ”.

An object of  $\mathcal{C}[J^\vee]$  of special interest is

$$J^\vee : [1] \rightarrow \text{ob}\mathcal{C}, J_0^\vee = J_1^\vee = \mathbb{I}.$$

Then we see that each object  $U$  of  $\mathcal{C}[J^\vee]$  is uniquely decomposable in the form

$$U = U_0 \otimes J^\vee \otimes U_1 \otimes J^\vee \otimes \dots \otimes J^\vee \otimes U_m \tag{13}$$

with  $U_0, U_1, \dots, U_m \in \mathcal{C}$ .

Now we will describe a directed graph  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ . We introduce two symbols

$$J \otimes J^\vee \xrightarrow{\varepsilon} \mathbb{I} \text{ and } \mathbb{I} \xrightarrow{\eta} J^\vee \otimes J.$$

For each morphism  $f : C \rightarrow D$  of  $\mathcal{C}$ , we introduce a symbol  $\bar{f} : C \rightarrow D$ . These symbols  $\varepsilon, \eta$ , and the  $\bar{f}$  will be called the *primitive edges* of  $\mathcal{G}$ . The general edges of  $\mathcal{G}$  are the whiskered primitive edges: that is, those of the form

$$U \otimes A \otimes V \xrightarrow{U \otimes p \otimes V} U \otimes B \otimes V \tag{14}$$

where  $U, V \in \mathcal{G}_0, A \xrightarrow{p} B$  is primitive, and the tensor in  $U \otimes p \otimes V$  is formal. For a general edge (14) and  $T, W \in \mathcal{G}_0$ , define

$$T \otimes (U \otimes p \otimes V) \otimes W = (T \otimes U) \otimes p \otimes (V \otimes W)$$

where  $T \otimes U$  and  $V \otimes W$  are (tensor) products in  $\mathcal{G}_0$ .

In the free category  $\mathcal{F}\mathcal{G}$  on the graph  $\mathcal{G}$ , if  $\ell = a_k \dots a_1$  with  $a_i \in \mathcal{G}_1$ , we put  $U \otimes \ell \otimes V = (U \otimes a_k \otimes V) \dots (U \otimes a_1 \otimes V)$ .

We introduce the following relations on morphisms of  $\mathcal{F}\mathcal{G}$ , where  $C, D, E, F \in \mathcal{C}$  and  $U, V, W, X, Y, Z \in \mathcal{G}_0$ :

(i) for  $f \in \mathcal{C}(C, D)$ ,

$$(E \otimes C \otimes F \xrightarrow{E \otimes \bar{f} \otimes F} E \otimes D \otimes F) \sim (E \otimes C \otimes F \xrightarrow{E \otimes \bar{f} \otimes F} E \otimes D \otimes F)$$

(ii) for  $f \in \mathcal{C}(C, D), g \in \mathcal{C}(D, E)$ ,

$$(C \xrightarrow{1_C} C) \sim (C \xrightarrow{1_C} C) \text{ and } (C \xrightarrow{\bar{g}} E) \sim (C \xrightarrow{\bar{f}} D \xrightarrow{\bar{g}} E)$$

(iii) for all edges  $U \xrightarrow{a} V, W \xrightarrow{b} X$  of  $\mathcal{G}$ ,

$$(U \otimes W \xrightarrow{a \otimes W} V \otimes W \xrightarrow{V \otimes b} V \otimes X) \sim (U \otimes W \xrightarrow{U \otimes b} U \otimes X \xrightarrow{a \otimes X} V \otimes X)$$

(iv)  $(J \xrightarrow{J \otimes \eta \otimes \mathbb{I}} J \otimes J^\vee \otimes J \xrightarrow{(\mathbb{I} \otimes \varepsilon \otimes J)} J) \sim 1_J$

(v)  $(J^\vee \xrightarrow{(\mathbb{I} \otimes \eta \otimes J^\vee)} J^\vee \otimes J \otimes J^\vee \xrightarrow{J^\vee \otimes \varepsilon \otimes \mathbb{I}} J^\vee) \sim 1_{J^\vee}$

(vi) if  $\ell \sim \ell'$  is any of the relations in (i)–(v), then

$$Y \otimes \ell \otimes Z \sim Y \otimes \ell' \otimes Z.$$

Put  $\mathcal{C}[J^\vee] = \mathcal{F}\mathcal{G} / \sim$  as a category. We write  $[\ell] : U \rightarrow V$  for the equivalence class of a morphism  $\ell : U \rightarrow V$  in  $\mathcal{F}\mathcal{G}$ .

Put  $T \otimes [\ell] \otimes W = [T \otimes \ell \otimes W]$ . We can write  $[a] \otimes [b] : U \otimes W \rightarrow V \otimes X$  for the equivalence class of both sides of relation (iii), and this extends by composition to morphisms of  $\mathcal{F}\mathcal{G}$  in place of  $a, b$ . It is because of (i), (ii), (iii), and (vi) that  $\mathcal{C}[J^\vee]$  is strict monoidal under  $\otimes$  with unit  $\mathbb{I}$  and the inclusion functor  $\Omega : \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$ , defined by  $\Omega C = C, \Omega f = [f]$ , is strict monoidal. By virtue of (iv) and (v), we have  $J \dashv J^\vee$  in  $\mathcal{C}[J^\vee]$ , and we will write the counit and unit as  $\varepsilon$  and  $\eta$  rather than  $[\varepsilon]$  and  $[\eta]$ .

### 8. Universality of $\mathcal{C}[J^\vee]$

Let us define a 2-category  $\mathfrak{s}\mathfrak{J}$ . The objects  $(\mathcal{C}, J)$  are strict monoidal categories  $\mathcal{C}$  equipped with a distinguished object  $J \in \mathcal{C}$ . The 1-morphisms  $(F, \phi) : (\mathcal{C}, J) \rightarrow (\mathcal{C}', J')$  consist of a strict monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  equipped with an isomorphism  $\phi : FJ \cong J'$  in  $\mathcal{C}'$ . The 2-morphisms  $\sigma : (F, \phi) \Rightarrow (G, \psi) : (\mathcal{C}, J) \rightarrow (\mathcal{C}', J')$  are monoidal natural transformations  $\sigma : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $\psi \circ \sigma_J = \phi$ .

Let us also define a 2-category  $\mathfrak{s}\mathfrak{J}_\vee$ . The objects  $(\mathcal{C}, J, K, \varepsilon)$  are strict monoidal categories  $\mathcal{C}$  equipped with a distinguished duality  $J \dashv K$  with its counit  $\varepsilon$ . The 1-morphisms  $(F, \phi) : (\mathcal{C}, J, K, \varepsilon) \rightarrow (\mathcal{C}', J', K', \varepsilon')$  consist of a strict monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  equipped with an isomorphism  $\phi : FJ \cong J'$  in  $\mathcal{C}'$ ; notice then that the mate of  $\phi^{-1}$  is an isomorphism  $FK \cong K'$ . The 2-morphisms  $\sigma : (F, \phi) \Rightarrow (G, \psi) : (\mathcal{C}, J, K, \varepsilon) \rightarrow (\mathcal{C}', J', K', \varepsilon')$  are monoidal natural transformations  $\sigma : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $\psi \circ \sigma_J = \phi$ .

There is a forgetful 2-functor  $\mathfrak{s}\mathfrak{L} : \mathfrak{s}\mathfrak{J}_\vee \rightarrow \mathfrak{s}\mathfrak{J}$  taking  $\sigma : (F, \phi) \Rightarrow (G, \psi) : (\mathcal{C}, J, K, \varepsilon) \rightarrow (\mathcal{C}', J', K', \varepsilon')$  to  $\sigma : (F, \phi) \Rightarrow (G, \psi) : (\mathcal{C}, J) \rightarrow (\mathcal{C}', J')$ .

**Proposition 12.** *The 2-functor  $\mathfrak{s}\mathfrak{L} : \mathfrak{s}\mathfrak{J}_\vee \rightarrow \mathfrak{s}\mathfrak{J}$  has a left biadjoint whose value at  $(\mathcal{C}, J)$  is  $(\mathcal{C}[J^\vee], J, J^\vee, \varepsilon)$ . Moreover,  $\Omega : (\mathcal{C}, J) \rightarrow (\mathcal{C}[J^\vee], J)$  is the component of the unit of the biadjunction.*

*Proof.* Using (13) and the nature of the relations (i)–(vi), we will prove that restriction along  $\Omega$  defines an equivalence of categories

$$\mathfrak{s}\mathfrak{J}_\vee((\mathcal{C}[J^\vee], J, J^\vee, \varepsilon), (\mathcal{X}, H, K, \alpha)) \simeq \mathfrak{s}\mathfrak{J}((\mathcal{C}, J), (\mathcal{X}, H)) \tag{15}$$

which is surjective on objects. To prove this surjectivity, take a strict monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{X}$  and  $\phi : FJ \cong H$ . Let  $\beta : \mathbb{I} \rightarrow K \otimes H$  be the unit corresponding to the counit  $\alpha$  for the duality  $H \dashv K$ .

Then  $\alpha' = \alpha \circ (\phi \otimes K)$  and  $\beta' = (K \otimes \phi^{-1}) \circ \beta$  are counit and unit for a duality  $FJ \dashv K$ . We can define a graph morphism  $F' : \mathcal{G} \rightarrow \mathcal{X}$  by

$$U \mapsto FU_0 \otimes K \otimes FU_1 \otimes K \otimes \dots \otimes K \otimes FU_m$$

$$\varepsilon \mapsto \alpha', \eta \mapsto \beta', f \mapsto Ff, (U, p, V) \mapsto F'U \otimes F'p \otimes F'V$$

which we can immediately extend to a functor  $\hat{F} : \mathcal{F}\mathcal{G} \rightarrow \mathcal{X}$  by the universal property of the domain free category. Relations (i)–(iii) and (vi) are preserved by  $\hat{F}$  since  $F$  is strict monoidal; relations (iv)–(v) are preserved since  $\alpha'$  and  $\beta'$  are counit and unit for  $FJ \dashv K$ . So a functor  $\bar{F} : \mathcal{C}[J^\vee] \rightarrow \mathcal{X}$  is induced. Notice that  $F'$  is a monoid morphism on the monoids of vertices of  $\mathcal{G}$  and the underlying graph of  $\mathcal{X}$  under tensor. With this, relations (iii) then imply that  $\bar{F}$  is strict monoidal. By construction, we have  $F = \bar{F}\Omega$ . We also have  $\phi : \bar{F}J = FJ \cong H$ .

It remains to prove that restriction along  $\Omega$  gives the fullness and faithfulness required for (15). Take  $(F, \phi), (G, \psi) : (\mathcal{C}[J^\vee], J, J^\vee, \varepsilon) \rightarrow (\mathcal{X}, H, K, \varepsilon)$  in  $\mathfrak{s}\mathfrak{J}_\vee$  and a monoidal natural transformation  $\sigma : F\Omega \Rightarrow G\Omega : \mathcal{C} \rightarrow \mathcal{X}$  with  $\psi \circ \sigma_J = \phi$ . We need to see that there is a unique  $\bar{\sigma} : (F, \phi) \Rightarrow (G, \psi)$  with  $\bar{\sigma}\Omega = \sigma$ .

Since  $\bar{\sigma}$  is to be monoidal and natural, we have commutativity of

$$\begin{array}{ccc} FJ \otimes FJ^\vee & \xrightarrow{F\varepsilon} & \mathbb{I} \\ \sigma_J \otimes \bar{\sigma}_{J^\vee} \downarrow & & \downarrow 1_{\mathbb{I}} \\ GJ \otimes GJ^\vee & \xrightarrow{G\varepsilon'} & \mathbb{I} \end{array}$$

which shows that  $\bar{\sigma}_{J^\vee}$  is forced to be the mate of  $\sigma_J^{-1} = \phi^{-1} \circ \psi$ . With this, we are now forced to put

$$\bar{\sigma}_U = \sigma_{U_0} \otimes \bar{\sigma}_{J^\vee} \otimes \sigma_{U_1} \otimes \bar{\sigma}_{J^\vee} \otimes \dots \otimes \bar{\sigma}_{J^\vee} \otimes \sigma_{U_m}.$$

With this as definition, the monoidality condition is obvious and naturality at equivalence classes of primitive morphisms is straightforward. We have the desired unique extension  $\bar{\sigma}$ . □

Let us define a 2-category  $\mathfrak{J}$ . The objects  $(\mathcal{C}, J)$  are monoidal categories  $\mathcal{C}$  equipped with a distinguished object  $J \in \mathcal{C}$ . The 1-morphisms  $(F, \phi) : (\mathcal{C}, J) \rightarrow (\mathcal{C}', J')$  consist of a strong monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  equipped with an isomorphism  $\phi : FJ \cong J'$  in  $\mathcal{C}'$ . The 2-morphisms  $\sigma : (F, \phi) \Rightarrow (G, \psi) : (\mathcal{C}, J) \rightarrow (\mathcal{C}', J')$  are monoidal natural transformations  $\sigma : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $\psi \circ \sigma_J = \phi$ .

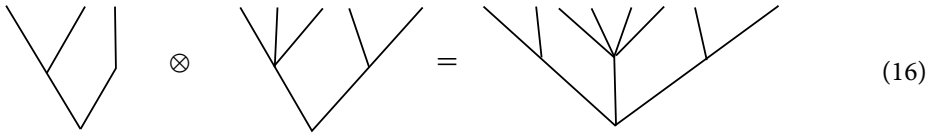
Predictably, we also define a 2-category  $\mathfrak{J}_\vee$ . It is the full sub-2-category of  $\mathfrak{J}$  consisting of those objects  $(\mathcal{C}, J)$  for which the distinguished object  $J$  has a right dual. Let  $\mathfrak{U} : \mathfrak{J}_\vee \rightarrow \mathfrak{J}$  be the inclusion 2-functor.

**Corollary 13.** *The 2-functor  $\mathfrak{U} : \mathfrak{J}_\vee \rightarrow \mathfrak{J}$  has a left biadjoint whose value at  $(\mathcal{C}, J)$  is  $(\mathcal{C}[J^\vee], J)$ . Moreover,  $\Omega : (\mathcal{C}, J) \rightarrow (\mathcal{C}[J^\vee], J)$  is the component of the unit of the biadjunction.*

*Proof.* This is a standard consequence of Proposition 12 when dealing with flexible categorical structures. □

**Remark 14.** Consider the case where  $\mathcal{C}$  is the free strict monoidal category on a single generating object; that is, it is the discrete category  $\mathbb{N}$  with addition as tensor product. Take  $J$  to be the natural number 1. We write  $\mathbb{N}[1^\vee]$  for  $\mathcal{C}[J^\vee]$  in this case. Objects of  $\mathbb{N}[1^\vee]$  can be identified with plane rooted trees of height at most 2 (in the sense of Batanin 1998) and having at least one edge attached to the root. For example, the trees in (16) correspond to the function  $[1] \rightarrow \mathbb{N}$  given by

$0 \mapsto 2, 1 \mapsto 1$ , the function  $[1] \rightarrow \mathbb{N}$  given by  $0 \mapsto 3, 1 \mapsto 2$  and the function  $[2] \rightarrow \mathbb{N}$ ,  $0 \mapsto 2, 1 \mapsto 4$  and  $2 \mapsto 2$ . Tensor product in terms of rooted trees is as shown in diagram (16).



Notice that  $\mathbb{N}[1^\vee]$  and the skeletal version of the monoidal category  $\text{Dpr}$  have the same universal property: strict monoidal functors from it into a strict monoidal category  $\mathcal{X}$  are in bijection with duality pairs in  $\mathcal{X}$ . So the monoids of objects should be isomorphic. The monoid of objects of this  $\text{Dpr}$  is the free monoid  $\{-, +\}^*$  on two symbols. It is easy to see directly that the monoid morphism  $\gamma : \{-, +\}^* \rightarrow \text{ob}\mathbb{N}[1^\vee]$  defined by

$$\gamma(-) = \left. \begin{array}{l} | \\ | \\ \backslash / \end{array} \right\} \quad \gamma(+) = \begin{array}{l} \backslash / \\ \backslash / \end{array}$$

is bijective.

**9. Fullness of  $\Omega : \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$**

**Proposition 15.** *The inclusion functor  $\Omega : \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  is full.*

*Proof.* Suppose  $\ell : C \rightarrow D$  is a morphism in  $\mathcal{C}[J^\vee]$  with  $C, D \in \mathcal{C}$ . If  $\ell$  has a representative path that passes through no vertex with a  $J^\vee$  factor, then relations (i), (ii), (iii) imply that  $\ell = \llbracket f \rrbracket$  for some  $f \in \mathcal{C}(C, D)$ . The only way that factors  $J^\vee$  can be created in vertices in the path is by using the primitive edge  $\eta$  and the only way they can be removed is by using the primitive edge  $\varepsilon$ . So suppose our path representing  $\ell$  has a vertex with a  $J^\vee$  factor. Then there must be an edge in the path of the form  $U \otimes V \xrightarrow{U \otimes \eta \otimes V} U \otimes J^\vee \otimes J \otimes V$ . In order that the created  $J^\vee$  in the target can be removed, the path must continue on as

$$U \otimes J^\vee \otimes J \otimes V \xrightarrow{a \otimes J^\vee \otimes b} X \otimes J \otimes J^\vee \otimes W \xrightarrow{X \otimes \varepsilon \otimes W} X \otimes W.$$

However, using relations (ii) and (iii), we have

$$\begin{aligned} &(X \otimes \varepsilon \otimes W)(a \otimes J^\vee \otimes b)(U \otimes \eta \otimes V) \\ &\sim (X \otimes \varepsilon \otimes W)(X \otimes J \otimes J^\vee \otimes b)(a \otimes J \otimes V)(U \otimes \eta \otimes V) \\ &\sim (X \otimes b)(X \otimes \varepsilon \otimes J \otimes V)(X \otimes J \otimes \eta \otimes V)(a \otimes V). \end{aligned}$$

Invoking relation (iv), we are left with  $U \otimes V \xrightarrow{a \otimes V} X \otimes J \otimes V \xrightarrow{X \otimes b} X \otimes W$ . By induction, any path containing edges involving an  $\eta$  is related to one not involving any. We are back to the first sentence of the proof. This proves  $\Omega$  is full. □

**Remark 16.** Relation (v) was not used in this proof.

**10. Faithfulness of  $\Omega : \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$**

Recall the bicategory  $\text{Mod}$  of categories and modules (called “bimodules” by Lawvere 2002 and “distributors” by Bénabou 1973) between them. The objects are small categories. The hom categories are presheaf categories:

$$\text{Mod}(\mathcal{A}, \mathcal{B}) = [\mathcal{B}^{\text{op}} \times \mathcal{A}, \text{Set}].$$

Composition is tensor product of modules:

$$(N \otimes_{\mathcal{B}} M)(C, A) = \int^B M(B, A) \times N(C, B).$$

The identity module of  $\mathcal{A}$  will be denoted  $H_{\mathcal{A}}$  and is the hom presheaf:  $H_{\mathcal{A}}(A', A) = \mathcal{A}(A', A)$ . All right liftings and right extensions exist in the bicategory  $\text{Mod}$ . For every functor  $F : \mathcal{A} \rightarrow \mathcal{B}$ , we obtain a module  $F_* : \mathcal{A} \rightarrow \mathcal{B}$  and a module  $F^* : \mathcal{B} \rightarrow \mathcal{A}$  with an adjunction  $F_* \dashv F^*$  in the bicategory  $\text{Mod}$ ; we have

$$F_*(B, A) = \mathcal{B}(B, FA) \text{ and } F^*(A, B) = \mathcal{B}(FA, B).$$

So then we have, for each category  $\mathcal{C}$ , a closed monoidal category  $\text{Mod}(\mathcal{C}, \mathcal{C})$ ; the tensor product is  $\otimes_{\mathcal{C}}$  with unit  $H_{\mathcal{C}}$ . If  $\mathcal{C}$  is monoidal, then each object  $C \in \mathcal{C}$  determines a functor  $X \otimes - : \mathcal{C} \rightarrow \mathcal{C}$  and so a duality

$$(X \otimes -)_* \dashv (X \otimes -)^*$$

in the monoidal category  $\text{Mod}(\mathcal{C}, \mathcal{C})$ . This leads to the *Cayley functor*

$$\Upsilon : \mathcal{C} \rightarrow \text{Mod}(\mathcal{C}, \mathcal{C}) \tag{17}$$

defined by

$$\Upsilon X(Z, Y) = (X \otimes -)_*(Z, Y) = \mathcal{C}(Z, X \otimes Y),$$

while the effect on homs is the function

$$\Upsilon : \mathcal{C}(X, X') \rightarrow \int_{Z, Y} [\mathcal{C}(Z, X \otimes Y), \mathcal{C}(Z, X' \otimes Y)]$$

taking  $X \xrightarrow{f} X'$  to the family of functions  $\mathcal{C}(Z, X \otimes Y) \xrightarrow{\mathcal{C}(Z, f \otimes Y)} \mathcal{C}(Z, X' \otimes Y)$ . So  $\Upsilon$  is *faithful and conservative*.

We also have

$$\Upsilon^\vee X(Z, Y) = (X \otimes -)^*(Z, Y) = \mathcal{C}(X \otimes Z, Y)$$

and the duality

$$\Upsilon X \dashv \Upsilon^\vee X$$

in  $\text{Mod}(\mathcal{C}, \mathcal{C})$ .

What is more,  $\Upsilon$  is strong monoidal. We have

$$\Upsilon \mathbb{I}(Z, Y) = \mathcal{C}(Z, \mathbb{I} \otimes Y) = H_{\mathcal{C}}(Z, Y)$$

and

$$\begin{aligned} (\Upsilon X \otimes_{\mathcal{C}} \Upsilon Y)(C, E) &= \int^D (\Upsilon Y)(D, E) \times (\Upsilon X)(C, D) \\ &= \int^D \mathcal{C}(D, Y \otimes E) \times \mathcal{C}(C, X \otimes D) \cong \mathcal{C}(C, X \otimes Y \otimes E) \\ &= \Upsilon(X \otimes Y). \end{aligned}$$

By Corollary 13, there exists a strong monoidal functor

$$\Gamma : \mathcal{C}[J^\vee] \rightarrow \text{Mod}(\mathcal{C}, \mathcal{C}) \tag{18}$$

such that  $\Gamma\Omega \cong \Upsilon$  and  $\Gamma$  preserves the duality  $J \dashv J^\vee$ ; so  $\Gamma J^\vee = \Upsilon^\vee J$  and  $\Gamma\eta : H_{\mathcal{C}} \Rightarrow \Upsilon^\vee J \otimes_{\mathcal{C}} \Upsilon J$  is such that triangle (19) commutes for all  $X, Z \in \text{ob } \mathcal{C}$ .

$$\begin{array}{ccc}
 \mathcal{C}(X, Z) & \xrightarrow{\Gamma\eta} & \int^Y \mathcal{C}(J \otimes X, Y) \times \mathcal{C}(Y, J \otimes Z) \\
 & \searrow^{J \otimes -} & \swarrow_{\cong} \\
 & & \mathcal{C}(J \otimes X, J \otimes Z)
 \end{array} \tag{19}$$

**Theorem 17.** *The inclusion functor  $\Omega : \mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  is fully faithful.*

*Proof.* Since  $\Gamma\Omega \cong \Upsilon$ , the fact that  $\Upsilon$  is faithful implies  $\Omega$  is too. We already know that  $\Omega$  is full by Proposition 15. □

### 11. Coends for Some Homs of $\mathcal{C}[J^\vee]$

Some of the hom sets of  $\mathcal{C}[J^\vee]$  can be expressed as iterated coends over  $\mathcal{C}$ , namely, those which hom into or out of an object of  $\mathcal{C}$ .

Take objects  $U : [m] \rightarrow \text{ob } \mathcal{C}$  and  $V : [n] \rightarrow \text{ob } \mathcal{C}$ .  
 For  $m = n = 0$ , by Theorem 17, we have

$$\mathcal{C}[J^\vee](U, V) \cong \mathcal{C}(U_0, V_0). \tag{20}$$

For  $m > 0, n = 0$ , there is a function

$$\int^{X \in \mathcal{C}} \mathcal{C}[J^\vee](U\partial_m, X \otimes J) \times \mathcal{C}(X \otimes U_m, V_0) \xrightarrow{\zeta_{m,0}} \mathcal{C}[J^\vee](U, V) \tag{21}$$

taking the equivalence class of  $(U\partial_m \xrightarrow{f} X \otimes J, X \otimes U_m \xrightarrow{g} V_0)$  to the composite

$$U = U\partial_m \otimes J^\vee \otimes U_m \xrightarrow{f \otimes 1 \otimes 1} X \otimes J \otimes J^\vee \otimes U_m \xrightarrow{1 \otimes \varepsilon \otimes 1} X \otimes U_m \xrightarrow{g} V_0 = V.$$

For  $m = 0, n > 0$ , there is a function

$$\int^{Y \in \mathcal{C}} \mathcal{C}(U_0, V_0 \otimes Y) \times \mathcal{C}[J^\vee](J \otimes Y, V\partial_0) \xrightarrow{\zeta_{0,n}} \mathcal{C}[J^\vee](U, V) \tag{22}$$

taking the equivalence class of  $(U_0 \xrightarrow{f} V_0 \otimes Y, J \otimes Y \xrightarrow{g} V\partial_0)$  to the composite

$$U = U_0 \xrightarrow{f} V_0 \otimes Y \xrightarrow{1 \otimes \eta \otimes 1} V_0 \otimes J^\vee \otimes J \otimes Y \xrightarrow{1 \otimes 1 \otimes g} V_0 \otimes J^\vee \otimes V\partial_0 = V.$$

For  $m > 0, n > 0$ , there is a function

$$\int^{X, Y \in \mathcal{C}} \mathcal{C}[J^\vee](U\partial_m, X \otimes J) \times \mathcal{C}(X \otimes U_m, V_0 \otimes Y) \times \mathcal{C}[J^\vee](J \otimes Y, V\partial_0) \xrightarrow{\zeta_{m,n}} \mathcal{C}[J^\vee](U, V) \tag{23}$$

taking the equivalence class of  $(U\partial_m \xrightarrow{f} X \otimes J, X \otimes U_m \xrightarrow{g} V_0 \otimes Y, J \otimes Y \xrightarrow{h} V\partial_0)$  to the composite

$$\begin{aligned}
 U = U\partial_m \otimes J^\vee \otimes U_m &\xrightarrow{f \otimes 1 \otimes 1} X \otimes J \otimes J^\vee \otimes U_m \xrightarrow{1 \otimes \varepsilon \otimes 1} X \otimes U_m \xrightarrow{g} V_0 \otimes Y \\
 &\xrightarrow{1 \otimes \eta \otimes 1} V_0 \otimes J^\vee \otimes J \otimes Y \xrightarrow{1 \otimes 1 \otimes h} V_0 \otimes J^\vee \otimes V\partial_0 = V.
 \end{aligned}$$



Notice that  $\zeta_{m,n}$  (23) is definitely not surjective in general. For example, in the case of  $\mathbb{N}[1^\vee]$  as in Remark 14, the coend is empty for  $U = V = 1^\vee$ . More generally, for most morphisms of the form  $f : J \otimes X \rightarrow Y \otimes J$  in  $\mathcal{C}$ , the composite

$$X \otimes J^\vee \xrightarrow{\eta \otimes 1 \otimes 1} J^\vee \otimes J \otimes X \otimes J^\vee \xrightarrow{1 \otimes f \otimes 1} J^\vee \otimes Y \otimes J \otimes J^\vee \xrightarrow{1 \otimes 1 \otimes \varepsilon} J^\vee \otimes Y$$

will not be in the image of  $\zeta_{m,n}$ .

We claim that  $\zeta_{m,n}$  is invertible when either  $m$  or  $n$  is 0.

Let us look at the case of  $\zeta_{0,n}$ . We begin by relating its domain to the functor  $\Gamma$  of (18). If  $A \in \mathcal{C}$  and  $W : [p] \rightarrow \mathcal{C}$  in  $\mathcal{C}[J^\vee]$ , then

$$\begin{aligned} &(\Gamma W)(A, \mathbb{I}) \\ &= (\Gamma W_0 \otimes_{\mathcal{C}} \Gamma J^\vee \otimes_{\mathcal{C}} \Gamma W_1 \otimes_{\mathcal{C}} \Gamma J^\vee \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} \Gamma W_p)(A, \mathbb{I}) \\ &= (\Upsilon W_0 \otimes_{\mathcal{C}} \Upsilon^\vee J \otimes_{\mathcal{C}} \Upsilon W_1 \otimes_{\mathcal{C}} \Upsilon^\vee J \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} \Upsilon W_p)(A, \mathbb{I}) \\ &\cong \int^{Y_1 \dots Y_p \in \mathcal{C}} \mathcal{C}(A, W_0 \otimes Y_1) \times \mathcal{C}(J \otimes Y_1, W_1 \otimes Y_2) \times \dots \times \mathcal{C}(J \otimes Y_p, W_p) \end{aligned} \tag{24}$$

where the last step involves the coend form of the Yoneda Lemma and  $W_p \otimes \mathbb{I} = W_p$ . Define the function

$$\zeta_{0,\cdot} : \Gamma W(A, \mathbb{I}) \longrightarrow \mathcal{C}[J^\vee](A, W) \tag{25}$$

to be the composite of the isomorphism (24) with the function which takes the equivalence class of

$$(A \xrightarrow{f_0} W_0 \otimes Y_1, J \otimes Y_1 \xrightarrow{f_1} W_1 \otimes Y_2, \dots, J \otimes Y_p \xrightarrow{f_p} W_p)$$

to the composite  $\zeta_{0,\cdot}[f_0, f_1, \dots, f_p] :=$

$$\begin{aligned} A \xrightarrow{f_0} W_0 \otimes Y_1 &\xrightarrow{1 \otimes \eta \otimes 1} W_0 \otimes J^\vee \otimes J \otimes Y_1 \xrightarrow{1 \otimes 1 \otimes f_2} W_0 \otimes J^\vee \otimes W_1 \otimes Y_2 \rightarrow \dots \\ &\xrightarrow{1 \otimes \eta \otimes 1} W \partial_p \otimes J^\vee \otimes J \otimes Y_p \xrightarrow{1 \otimes f_p} W. \end{aligned}$$

It is clear that the above is well defined, and, moreover, the functions (25) are the components of a natural transformation

$$\begin{array}{ccc} \mathcal{C}[J^\vee] & \xrightarrow{\Gamma} & \text{Mod}(\mathcal{C}, \mathcal{C}) \\ & \searrow \tilde{\Omega} & \swarrow \text{Ev}_{\mathbb{I}} \\ & & [\mathcal{C}^{\text{op}}, \text{Set}] \end{array} \quad \zeta_{0,\cdot} \tag{26}$$

where  $\tilde{\Omega}W = \mathcal{C}[J^\vee](\Omega -, W)$  and  $\text{Ev}_{\mathbb{I}}M = M(-, \mathbb{I})$ .

**Theorem 18.** *The natural transformation  $\zeta_{0,\cdot}$  in (26) is invertible.*

*Proof.* We will show that the composite  $\zeta'_{0,\cdot} :=$

$$\mathcal{C}[J^\vee](A, W) \xrightarrow{\Gamma} \int_{X,Z} [\mathcal{C}(X, A \otimes Z), \Gamma W(X, Z)] \cong \int_Z \Gamma W(A \otimes Z, Z) \xrightarrow{\text{Ev}_{\mathbb{I}}} \Gamma W(A, \mathbb{I})$$

is the inverse to  $\zeta_{0,\cdot}$ . The composite

$$\mathcal{C}[J^\vee](A, W) \xrightarrow{\zeta'_{0,\cdot}} \Gamma W(A, \mathbb{I}) \xrightarrow{\zeta_{0,\cdot}} \mathcal{C}[J^\vee](A, W)$$

is natural in  $W$  and so, by Yoneda, is determined by the value at the identity of  $W = A$  under the composite

$$\mathcal{C}[J^\vee](A, A) \xrightarrow{\zeta'_{0,\bullet}} \mathcal{C}(A, A) \xrightarrow{\zeta_{0,\bullet}} \mathcal{C}[J^\vee](A, A).$$

The composite takes  $1_A$  to itself yielding  $\zeta_{0,\bullet} \circ \zeta'_{0,\bullet} = 1_{\mathcal{C}[J^\vee](A, W)}$ .

We now look at the composite

$$\Gamma W(A, \mathbb{I}) \xrightarrow{\zeta_{0,\bullet}} \mathcal{C}[J^\vee](A, W) \xrightarrow{\zeta'_{0,\bullet}} \Gamma W(A, \mathbb{I})$$

as transported across the isomorphism (24). (Abusing notation, we use the same symbols for the transported functions.) This is performed by applying  $\Gamma$  to the morphism  $\zeta_{0,\bullet}[f_0, f_1, \dots, f_p]$  which involves morphisms in the image of  $\Omega$  and the morphism  $\eta$  in  $\mathcal{C}[J^\vee]$ . Making use of  $\Gamma\Omega = \Upsilon$ , strong monoidality of  $\Gamma$  and the value of  $\Gamma$  on  $\eta$  as given by (19), we see with some effort that  $\zeta'_{0,\bullet}\zeta_{0,\bullet}[f_0, f_1, \dots, f_p] = [f_0, f_1, \dots, f_p]$ , as required.

It may be helpful for us to give more detail in the case  $p = 1$  to indicate what is involved in general. We need to show that the composite

$$\begin{aligned} & \int^{Y \in \mathcal{C}} \mathcal{C}(A, B \otimes Y) \times \mathcal{C}(J \otimes Y, C) \xrightarrow{\zeta_{0,1}} \mathcal{C}[J^\vee](A, B \otimes J^\vee \otimes C) \xrightarrow{\Gamma} \\ & \int_{X,Z} \left[ \mathcal{C}(X, A \otimes Z), \int^{Y_1, Y_2} \mathcal{C}(X, B \otimes Y_1) \times \mathcal{C}(J \otimes Y_1, Y_2) \times \mathcal{C}(Y_2, C \otimes Z) \right] \xrightarrow{\cong} \\ & \int_Z \int^{Y_1} \mathcal{C}(A \otimes Z, B \otimes Y_1) \times \mathcal{C}(J \otimes Y_1, J \otimes C \otimes Z) \xrightarrow{\text{Ev}_1} \\ & \int^{Y \in \mathcal{C}} \mathcal{C}(A, B \otimes Y) \times \mathcal{C}(J \otimes Y, C) \end{aligned}$$

is the identity. (It is actually possible in this  $p = 1$  case to use Yoneda to check only that, taking  $A = B \otimes Y$  and  $J \otimes Y = C$ , the equivalence class of  $(1_A, 1_C)$  goes to itself. However, this possibility is not fully available for  $p > 1$ .) Take  $[A \xrightarrow{f} B \otimes Y, J \otimes Y \xrightarrow{g} C]$  in the domain. Under  $\zeta_{0,1}$  it goes to the composite  $A \xrightarrow{f} B \otimes Y \xrightarrow{1 \otimes \eta \otimes 1} B \otimes J^\vee \otimes J \otimes Y \xrightarrow{1 \otimes 1 \otimes g} B \otimes J^\vee \otimes C$ . So the component of the natural transformation  $\Gamma \zeta_{0,1}[f, g]$  at  $(X, Z)$  is the composite

$$\begin{aligned} & \mathcal{C}(X, A \otimes Z) \xrightarrow{\mathcal{C}(1, f \otimes 1)} \mathcal{C}(X, B \otimes Y \otimes Z) \xrightarrow{\Gamma(1 \otimes \eta \otimes 1)} \\ & \int^{Y_1, Y_2} \mathcal{C}(X, B \otimes Y_1) \times \mathcal{C}(J \otimes Y_1, J \otimes Y_2) \times \mathcal{C}(Y_2, Y \otimes Z) \xrightarrow{\cong} \\ & \int^{Y_1} \mathcal{C}(X, B \otimes Y_1) \times \mathcal{C}(J \otimes Y_1, J \otimes Y \otimes Z) \xrightarrow{1 \times \mathcal{C}(1, g \otimes 1)} \\ & \int^{Y_1} \mathcal{C}(X, B \otimes Y_1) \times \mathcal{C}(J \otimes Y_1, C \otimes Z) \end{aligned}$$

which, using (19), corresponds to the element

$$([A \otimes Z \xrightarrow{f \otimes 1} B \otimes Y \otimes Z, J \otimes Y \otimes Z \xrightarrow{g \otimes 1} C \otimes Z])_Z$$

of

$$\int_Z \int^{Y_1} \mathcal{C}(A \otimes Z, B \otimes Y_1) \times \mathcal{C}(J \otimes Y_1, J \otimes C \otimes Z).$$

The component at the index  $Z = \mathbb{I}$  is our original  $[A \xrightarrow{f} B \otimes Y, J \otimes Y \xrightarrow{g} C]$ . □

**Corollary 19.** *The morphism  $\zeta_{0,n}$  (22) is invertible for all  $n > 0$ .*

*Proof.* Using Theorem 18 twice, we obtain isomorphisms

$$\begin{aligned} & \int^{Y \in \mathcal{C}} \mathcal{C}(U_0, V_0 \otimes Y) \times \mathcal{C}[J^\vee](J \otimes Y, V\partial_0) \\ & \cong \int^{Y \in \mathcal{C}} \mathcal{C}(U_0, V_0 \otimes Y) \times \Gamma(V\partial_0)(J \otimes Y, \mathbb{I}) \\ & \cong \int^{Y, Y' \in \mathcal{C}} (\Gamma V_0)(U_0, Y) \times (\Gamma J^\vee)(Y, Y') \times \Gamma(V\partial_0)(Y', \mathbb{I}) \\ & \cong \Gamma(V_0 \otimes J^\vee \otimes V\partial_0)(U_0, \mathbb{I}) \cong \Gamma V(U_0, \mathbb{I}) \\ & \cong \mathcal{C}[J^\vee](U, V) \end{aligned}$$

whose composite is  $\zeta_{0,n}$ . □

Similarly, we have functions  $\Gamma U(\mathbb{I}, A) \rightarrow \mathcal{C}[J^\vee](U, \Omega A)$  which are the components of a natural isomorphism

$$\begin{array}{ccc} \mathcal{C}[J^\vee] & \xrightarrow{\Gamma} & \text{Mod}(\mathcal{C}, \mathcal{C}) \\ & \searrow \Omega^\dagger & \swarrow \text{Ev}'_{\mathbb{I}} \\ & & [\mathcal{C}, \text{Set}]^{\text{op}} \end{array} \quad \begin{array}{c} \xleftarrow{\zeta_{\cdot, 0}} \\ \xleftarrow{\zeta_{\cdot, 0}} \end{array} \quad (27)$$

where  $\Omega^\dagger W = \mathcal{C}[J^\vee](W, \Omega -)$  and  $\text{Ev}'_{\mathbb{I}} M = M(\mathbb{I}, -)$ .

**12.  $\mathcal{C}[J^\vee]$  from  $\mathbb{N}[1^\vee] \simeq \mathbf{Dpr}$**

We use the notation of Remark 14. The (bicategorical) initial object of  $\mathfrak{J}$  is  $(\mathbb{N}, 1)$  while that of  $\mathfrak{J}_\vee$  is  $(\mathbb{N}[1^\vee], 1)$ . For any  $(\mathcal{C}, J) \in \mathfrak{J}_\vee$ , we write  $J^\vee$  for a choice of right dual for  $J$ .

Binary (bicategorical) coproduct in  $\mathfrak{J}$  will be denoted by

$$(\mathcal{C}, J) \xrightarrow{\text{in}_1} (\mathcal{C} +_{\mathbb{N}} \mathcal{D}, J) \xleftarrow{\text{in}_2} (\mathcal{D}, K).$$

We point out that, if either  $J \in \mathcal{C}$  or  $K \in \mathcal{D}$  has a right dual, then  $J \in \mathcal{C} +_{\mathbb{N}} \mathcal{D}$  has a right dual.

**Proposition 20.** *For each object  $(\mathcal{C}, J) \in \mathfrak{J}$ , there is an equivalence*

$$(\mathcal{C}[J^\vee], J) \simeq (\mathbb{N}[1^\vee] +_{\mathbb{N}} \mathcal{C}, 1)$$

in  $\mathfrak{J}_\vee$ .

*Proof.* Take  $(\mathcal{X}, K) \in \mathfrak{J}_\vee$ . We have pseudonatural equivalences

$$\begin{aligned} & \mathfrak{J}_\vee((\mathbb{N}[1^\vee] +_{\mathbb{N}} \mathcal{C}, 1), (\mathcal{X}, K)) \\ & \simeq \mathfrak{J}((\mathbb{N}[1^\vee] +_{\mathbb{N}} \mathcal{C}, 1), (\mathcal{X}, K)) \\ & \simeq \mathfrak{J}((\mathbb{N}[1^\vee], 1), (\mathcal{X}, K)) \times \mathfrak{J}((\mathcal{C}, J), (\mathcal{X}, K)) \\ & \simeq \mathbf{1} \times \mathfrak{J}((\mathcal{C}, J), (\mathcal{X}, K)) \simeq \mathfrak{J}((\mathcal{C}, J), (\mathcal{X}, K)) \\ & \simeq \mathfrak{J}_\vee((\mathcal{C}[J^\vee], J), (\mathcal{X}, K)). \end{aligned}$$

The result now follows using the bicategorical Yoneda lemma (Street 1980). □

**13. A Comment on Enriched Versions**

For brevity we did not pursue the enriched version of our results. In this section, we briefly comment on the  $K$ -linear setting, for a commutative ring  $K$ . We say that a category is monoidal  $K$ -linear if it comes equipped with a  $K$ -linear and a monoidal structure where  $- \otimes -$  is  $K$ -linear in each variable.

Starting from a  $K$ -linear monoidal category  $\mathcal{C}$  and  $J \in \mathcal{C}$ , Section 7 still gives us a monoid  $(\mathcal{G}_0, \otimes, \mathbb{I})$  and graph  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ . Then we can form the free  $K$ -linear category  $\mathcal{F}_K\mathcal{G}$  on the graph. Morphisms in  $\mathcal{F}_K\mathcal{G}$  are formal  $K$ -linear combinations of morphisms in  $\mathcal{F}\mathcal{G}$ . We then consider the same relations (i)–(vi) from Section 7 on  $\mathcal{F}_K\mathcal{G}$  as we did on  $\mathcal{F}\mathcal{G}$  with only modification that (i) needs to be extended to

$$\overline{E \otimes (\lambda f + \mu g) \otimes F} \sim \lambda E \otimes \bar{f} \otimes F + \mu E \otimes \bar{g} \otimes F,$$

for  $\lambda, \mu \in K$  and  $f, g \in \mathcal{C}(C, D)$ . We then denote again by  $\mathcal{C}[J^\vee]$  the corresponding quotient category of  $\mathcal{F}_K\mathcal{G}$ . That  $\mathcal{C}[J^\vee]$  is  $K$ -linear monoidal such that the canonical strict monoidal functor  $\mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  (which now is  $K$ -linear) satisfies the corresponding universal property follows exactly as before. Also the proof of the fullness of  $\mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  carries over verbatim. The constructions in Section 10 are written such that the enriched version is obtained by trivial replacement of the category  $\text{Set}$ . In particular,  $\text{Mod}(\mathcal{C}, \mathcal{C})$  can be defined as the category of functors from  $\mathcal{C}^{\text{op}} \times \mathcal{C}$  to the category of  $K$ -modules which are  $K$ -linear in each variable, or as  $K$ -linear functors from  $\mathcal{C}^{\text{op}} \otimes_K \mathcal{C}$ . Consequently,  $\mathcal{C} \rightarrow \mathcal{C}[J^\vee]$  is still fully faithful. Furthermore, the technique in Section 11 shows for instance that we get an isomorphism of  $K$ -modules

$$\int^{X \in \mathcal{C}} \mathcal{C}[J^\vee](U \partial_m, X \otimes J) \otimes_K \mathcal{C}(X \otimes U_m, V) \xrightarrow{\cong} \mathcal{C}[J^\vee](U, V),$$

for  $V \in \text{ob}\mathcal{C}$  and  $U : [m] \rightarrow \text{ob}\mathcal{C}$ .

Note that the definition of  $\mathcal{C}[J^\vee]$  in general actually depends on whether we view  $\mathcal{C}$  as a  $K$ -linear category or not. For instance, if  $J$  has an endomorphism which is not a scalar multiple of the identity, the set of morphisms  $\mathbb{I} \rightarrow J^\vee \otimes J$  depends on which definition we use.

**14. A Comment on Autonomization**

We briefly justify our focus on the adjoining of *one* (right) dual. First, we can define  $\mathcal{C}[\vee J]$  similarly to  $\mathcal{C}[J^\vee]$  by adjoining a left dual. In particular, we can set  $\mathcal{C}[\vee J] = (\mathcal{C}^{\text{rev}}[J^\vee])^{\text{rev}}$ , where  $-^{\text{rev}}$  denotes the reverse of a monoidal category, which switches the order of the tensor product. Using the explicit model of our categories  $\mathcal{C}[J^\vee]$  and via the principle of transfinite induction, we can then construct an autonomous category from any strict monoidal category as a direct limit by iteratively adjoining left and right duals.

It is instructive to break up the above procedure in two steps. In a first step, we can adjoin all iterated left and right duals of a given object in a strict monoidal category. Just as in Section 12 one observes that this is actually a pseudopushout with respect to the free autonomous monoidal category on a single object of Example 3 in Section 6. Now we can construct an autonomous category from a small strict monoidal category  $\mathcal{C}$  by iteratively adjoining (at once) all duals of a given object along a well order on  $\text{ob}\mathcal{C}$ .

We denote the category obtained in the above procedure by  $\text{Auton}\mathcal{C}$ . It follows from Theorem 17 and standard techniques that the canonical monoidal functor  $\mathcal{C} \rightarrow \text{Auton}\mathcal{C}$  is fully faithful. Furthermore, by construction and Corollary 13, the functor  $\mathcal{C} \rightarrow \text{Auton}\mathcal{C}$  is universal in the sense that it yields an equivalence between the categories of strong monoidal functors from  $\mathcal{C}$  and  $\text{Auton}\mathcal{C}$  to any autonomous monoidal category. All of the above remains true in the  $K$ -linear setting.

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