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RESEARCH ARTICLE

Stereographic compactification and affine bi-Lipschitz homeomorphisms

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Abstract
Let $\sigma_q : \mathbb{R}^q \to \mathbf{S}^q \setminus N_q$ be the inverse of the stereographic projection with center the north pole N_q. Let W_i be a closed subset of $\mathbb{R}^{q_i}$, for $i = 1, 2$. Let $\Phi : W_1 \to W_2$ be a bi-Lipschitz homeomorphism. The main result states that the homeomorphism $\sigma_{q_2} \circ \Phi \circ \sigma_{q_1}^{-1}$ is a bi-Lipschitz homeomorphism, extending bi-Lipschitz-ly at N_{q_1} with value N_{q_2} whenever W_1 is unbounded.
As two straightforward applications in the polynomially bounded o-minimal context over the real numbers, we obtain for free a version at infinity of: (1) Sampaio's tangent cone result and (2) links preserving re-parametrization of definable bi-Lipschitz homeomorphisms of Valette.

1. Introduction

Any subset S of $\mathbb{R}^q$ is equipped with the *outer metric structure*, where the distance between points of S is their distance in $\mathbb{R}^q$. Thus, (outer) Lipschitz mappings $S_1 \to S_2$, for S_i a subset of $\mathbb{R}^{q_i}$, are well defined.

The inverse of the *stereographic projection of the unit sphere* $\mathbf{S}^q$ *of* $\mathbb{R}^{q+1}$ *onto* $\mathbb{R}^q$ *with center the north pole* $N_q = (0, \ldots, 0, 1) \in \mathbb{R}^q \times \mathbb{R}$ is denoted $\sigma_q : \mathbb{R}^q \to \mathbf{S}^q \setminus N_q$. Let $\widetilde{S}$ be the closure in $\mathbf{S}^q$ of $\sigma_q(S)$ where S is a closed subset of $\mathbb{R}^q$. For $i = 1, 2$, let W_i be a closed subset of $\mathbb{R}^{q_i}$. If $\Phi : W_1 \to W_2$ is a homeomorphism, the *stereographic pre-compactification of* Φ is the following homeomorphism:

$$\sigma_{q_2} \circ \Phi \circ \sigma_{q_1}^{-1} : \sigma_{q_1}(W_1) \to \sigma_{q_2}(W_2).$$

Since W_1, W_2 are closed, the stereographic pre-compactification of Φ extends as a homeomorphism $\widetilde{\Phi} : \widetilde{W}_1 \to \widetilde{W}_2$ mapping N_{q_1} onto N_{q_2} whenever W_1 is unbounded. We call this extension the *stereographic compactification of* Φ.

The main result of this note is the following:

Theorem 10. *The mapping* Φ *is bi-Lipschitz if and only if its stereographic compactification* $\widetilde{\Phi}$ *is bi-Lipschitz.*

The main result is a consequence of Lemma 9 presented below. We recall that the *Euclidean inversion of* $\mathbb{R}^q$ is the following mapping:

$$\iota_q : \mathbb{R}^q \setminus \mathbf{0} \to \mathbb{R}^q \setminus \mathbf{0}, \;\; \mathbf{x} \mapsto \frac{\mathbf{x}}{|\mathbf{x}|^2}.$$

Let $\Phi : W_1 \to W_2$ be a homeomorphism between the closed subsets W_i of $\mathbb{R}^{q_i}$, $i = 1, 2$. The *inversion of the mapping* $\Phi : W_1 \to W_2$ is defined as follows:

$$\iota(\Phi) : \iota_{q_1}(W_1 \setminus \mathbf{0}) \to \iota_{q_2}(W_2 \setminus \mathbf{0}), \;\; \mathbf{x} \mapsto \iota(\Phi)(\mathbf{x}) := \iota_{q_2} \circ \Phi \circ \iota_{q_1}^{-1}(\mathbf{x}).$$

The next result (so called *the inversion lemma*) is the main tool we use to obtain the main result. It is of interest on its own and can be applied in many different contexts.

Lemma 9. *Assume furthermore that, either W_i contains the origin $\mathbf{0} \in \mathbb{R}^{q_i}$ for $i = 1, 2$ and $\Phi(\mathbf{0}) = \mathbf{0}$, or $\mathbf{0} \notin W_i$ for $i = 1, 2$. The homeomorphism $\iota(\Phi)$ is bi-Lipschitz if and only Φ is. Moreover, if W_1 is unbounded, then $\iota(\Phi)$ extends bi-Lipschitz-ly at $\mathbf{0}$ taking the value $\mathbf{0}$.*

Our interest in this problem arose from results of the recent PhD Thesis of the second named author [10], where a bi-Lipschitz classification of local plane objects germs, at the origin, respectively, at infinity and in correspondence by the Euclidean inversion, presented strikingly similar properties, now explained by Lemma 9. The main result is mostly a convenient reformulation of the inversion lemma. It is also in tune with the joint works of the first named author [2–4] expanding the results of the recent PhD Thesis [1]. Last, we want to point out that the proofs of the inversion lemma and of the main result, presented here, are self-contained.

The paper is organized as follows: Section 2 introduces preliminary materials and notations. Section 3 presents the special case of a global bi-Lipschitz homeomorphism of $\mathbb{R}^q$. Sections 4 and 5, respectively, show germ-ified versions of the inversion lemma, namely Lemma 7 at ∞ and, respectively Lemma 8 at $\mathbf{0}$. Section 6 is the short proof of our main tool, the inversion Lemma 9. The main result is dealt with in Section 7. The last section presents two immediate applications, versions at infinity of two results about germs of definable subsets at the origin: Proposition 15 (tangent cone result [12]) and Proposition 17 (links preserving reparametrization of definable bi-Lipschitz homeomorphism [16]).

A few days after making public this result on ArXiv [7], the preprint [13] found independently (among other results) what we present here.

2. Preliminaries

2.1. Notations

The Euclidean space $\mathbb{R}^q$ is equipped with the Euclidean distance, denoted $|-|$. We denote by B_r^q the open ball of $\mathbb{R}^q$ of radius r and centered at the origin $\mathbf{0}$, by $\mathbf{B}_r^q$ its closure and by $\mathbf{S}_r^{q-1}$ its boundary. The open ball of radius r and with center $\mathbf{x}_0 \in \mathbb{R}^q$ is $B^q(\mathbf{x}_0, r)$, its closure is $\mathbf{B}^q(\mathbf{x}_0, r)$, and $\mathbf{S}^{q-1}(\mathbf{x}_0, r)$ is its boundary. The unit sphere $\mathbf{S}_1^{q-1}$ is simply denoted by $\mathbf{S}^{q-1}$.

If S is any subset of $\mathbb{R}^q$, its closure in $\mathbb{R}^q$ is $\mathrm{clos}(S)$, and S^* is $S \setminus \mathbf{0}$.

Let $\mathbf{U}_q$ be the punctured affine space $\mathbb{R}^{q*}$.

Compactifying the space $\mathbb{R}^q$ with the point ∞ at infinity as:

$$\overline{\mathbb{R}^q} := \mathbb{R}^q \sqcup \infty = \mathbf{0} \sqcup \mathbf{U}_q \sqcup \infty$$

yields a space that is smoothly diffeomorphic to the unit sphere $\mathbf{S}^q$ of $\mathbb{R}^{q+1}$, using the stereographic projections centered at the “north” and “south” poles of $\mathbf{S}^q$. Under this correspondence, the points $\mathbf{0}$ and ∞ are antipodal.

If S is any subset of $\mathbb{R}^q$ its closure in $\overline{\mathbb{R}^q}$ is $\overline{S}$. Thus, S is unbounded if and only if $\overline{S} = \mathrm{clos}(S) \cup \infty$.

The germ $(\mathbb{R}^q, \infty)$ of $\mathbb{R}^q$ at infinity is well defined and can be considered as a germ in $\mathbb{R}^q$ and in $\overline{\mathbb{R}^q}$.

Let γ be a point of $\overline{\mathbb{R}^q}$. Let $(\mathbf{x}_n)_n$, $(\mathbf{y}_n)_n$ be two sequences of $\mathbb{R}^q$ converging to γ in $\overline{\mathbb{R}^q}$. Let $\mathbf{z}_n$ be $\mathbf{x}_n$ or $\mathbf{y}_n$ and let

$$z_n := \begin{cases} |\mathbf{z}_n - \gamma| & \text{if} \quad \gamma \in \mathbb{R}^q \\ |\mathbf{z}_n| & \text{if} \quad \gamma = \infty \end{cases}$$

We will use the following notation:

$$\mathbf{x}_n \sim \mathbf{y}_n \iff \frac{x_n}{y_n} \in [a,b] \text{ for } a,b>0 \text{ whenever } n \gg 1,$$

as well as the next one

$$\mathbf{x}_n \lesssim \mathbf{y}_n \iff \frac{x_n}{y_n} \in [0,a] \text{ for } a>0 \text{ whenever } n \gg 1,$$

and the last one

$$\mathbf{x}_n = o(\mathbf{y}_n) \iff \lim_n \frac{x_n}{y_n} = 0.$$

2.2. *On affine subsets*

Any non-empty subset S of $\mathbb{R}^q$ inherits from the ambient Euclidean structure of $\mathbb{R}^q$ the *outer metric space structure* (S, d_S), where

$$d_S(\mathbf{x}, \mathbf{x}') := |\mathbf{x} - \mathbf{x}'|$$

for any pair of points $\mathbf{x}, \mathbf{x}'$ of S. We recall that if a mapping $\varphi : (S, d_S) \to \mathbb{R}^p$ is Lipschitz with Lipschitz constant C, it extends as a Lipschitz mapping $(\mathrm{clos}(S), d_{\mathrm{clos}(S)}) \to \mathbb{R}^p$ with the same Lipschitz constant C. In practice, we can assume that S is closed in $\mathbb{R}^q$.

In order to ease the accumulation of hypotheses and notations, we introduce the following:

Definition 1. *A q-affine subset is a non-empty closed subset of $\mathbb{R}^q$ with $q \geq 1$.*
An affine subset *is a q-affine subset for some positive integer q.*

Since any affine subset S is equipped with the outer metric space structure (S, d_S) described above, we introduce the following

Definition 2. *A Lipschitz mapping $S \to T$ between the affine subsets S, T is a Lipschitz mapping $(S, d_S) \to (T, d_T)$.*

2.3. *On the inversion*

The inversion of the (punctured) affine space $\mathbb{R}^q$, defined as:

$$\iota_q : \mathbf{U}_q \to \mathbf{U}_q, \quad \mathbf{x} \mapsto \frac{\mathbf{x}}{|\mathbf{x}|^2}$$

is a C^∞ (semi-algebraic) diffeomorphism and extends as a (semi-algebraic) homeomorphism (C^∞ actually) over $\overline{\mathbb{R}^q}$ exchanging the origin $\mathbf{0}$ and the point at infinity ∞.

Let $\mathbf{x}$ be any point of $\mathbf{U}_q$. Let

$$R(\mathbf{x}) := \mathbb{R}\mathbf{x}$$

be the real vector line through $\mathbf{x}$. The tangent space of $\mathbf{U}_q$ at $\mathbf{x}$ decomposes as the Euclidean orthogonal sum:

$$T_\mathbf{x}\mathbf{U}_q = R(\mathbf{x}) \oplus \mathbf{S}(\mathbf{x}) \text{ where } \mathbf{S}(\mathbf{x}) := T_\mathbf{x}\mathbf{S}_R^{q-1}.$$

Observe that $\mathbf{S}(s\mathbf{x}) = \mathbf{S}(\mathbf{x})$ and $R(s\mathbf{x}) = R(\mathbf{x})$, as vector subspaces of $\mathbb{R}^q$, whenever $s \neq 0$. An elementary computation shows that in the previous orthogonal basis of $T_\mathbf{x}\mathbf{U}_q$ we obtain

$$D_\mathbf{x}\iota_q := -\frac{1}{|\mathbf{x}|^2} Id_{R(\mathbf{x})} \oplus \frac{1}{|\mathbf{x}|^2} Id_{\mathbf{S}(\mathbf{x})} = \frac{1}{|\mathbf{x}|^2}\left[-Id_{R(\mathbf{x})} \oplus Id_{\mathbf{S}(\mathbf{x})}\right]$$

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In particular, $D_{\mathbf{x}}\iota_q$ is an orthogonal mapping, since $|\mathbf{x}|^2 D_{\mathbf{x}}\iota_q$ is simply the orthogonal symmetry w.r.t. the hyperplane $\mathbf{S}(\mathbf{x})$. We thus deduce the (Euclidean) norm of $D_{\mathbf{x}}\iota_q$:

$$\|D_{\mathbf{x}}\iota_q\| = \frac{1}{|\mathbf{x}|^2}. \tag{2.1}$$

2.4. Elementary, yet very useful, identities

We recall the following known estimates

Claim 3. *Let* $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^q$ *and* $C > 0$ *such that* $|\mathbf{x}'| \geq (1+C)|\mathbf{x}|$. *Then,*

$$\frac{C}{1+C}|\mathbf{x}'| \leq |\mathbf{x}' - \mathbf{x}| \leq \frac{2+C}{1+C}|\mathbf{x}'|.$$

Given $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{U}_q$, we define

$$e := |\mathbf{x}_1 - \mathbf{x}_2|, \ r_i := |\mathbf{x}_i|, \ \mathbf{y}_i := \iota_q(\mathbf{x}_i), \ E := |\mathbf{y}_1 - \mathbf{y}_2| \text{ and } R_i := |\mathbf{y}_i|, \ i = 1, 2.$$

We assume that $r_1 = (1 + C)r_2$ for some $C \geq 0$. Let $2\theta \in [0, \pi]$ be the angle between $\mathbf{x}_1, \mathbf{x}_2$ (thus between $\mathbf{y}_1, \mathbf{y}_2$ as well). Let $r_1 - r_2 = Cr_2$ and $R_2 - R_1 = CR_1$. We recall that *the law of cosines* is the following identity:

$$e^2 = (r_1 - r_2)^2 \cos^2\theta + (r_1 + r_2)^2 \sin^2\theta = r_2^2[C^2 + 4(1 + C)\sin^2\theta]. \tag{2.2}$$

The inversion and the law of cosines give the following identity:

$$E = R_1R_2 \cdot e \quad \text{and} \quad e = r_1r_2 \cdot E. \tag{2.3}$$

3. Inversion mapping and global bi-Lipschitz homeomorphisms

We present a special occurrence of the inversion lemma. Although it is likely that it has already been written in a few books, we give a proof, following from elementary Lipschitz analysis.

Let $\mathcal{L}(a, b)$ be the space of $\mathbb{R}$-linear mappings $\mathbb{R}^a \to \mathbb{R}^b$.

Let $\varphi : \mathbb{R}^p \to \mathbb{R}^q$ be a Lipschitz mapping with Lipschitz constant A_φ:

$$\mathbf{x}, \mathbf{x}' \in \mathbb{R}^p \implies |\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| \leq A_\varphi \cdot |\mathbf{x} - \mathbf{x}'|.$$

Let $\mathcal{D}(\varphi)$ be the set of points where φ is differentiable. Rademacher Theorem states that the complement $\mathbb{R}^p \setminus \mathcal{D}(\varphi)$ is of null measure [8, 11]. We consider the following closed subset:

$$\Delta(\varphi) := \text{clos}\,(\{(\mathbf{x}, D_{\mathbf{x}}\varphi) \in \mathcal{D}(\varphi) \times \mathcal{L}(q, p)\}) \subset \mathbb{R}^p \times \mathcal{L}(p, q).$$

Let $\pi_{\mathcal{L}} : \mathbb{R}^p \times \mathcal{L}(p, q) \to \mathcal{L}(p, q)$ be the projection onto the second factor. Let

$$\mathcal{L}(\varphi) := \pi_{\mathcal{L}}(\Delta).$$

For any $\mathbf{x} \in \mathcal{D}(\varphi)$, the Lipschitz condition on φ yields the following estimate about the norm of $D_{\mathbf{x}}\varphi$:

$$\|D_{\mathbf{x}}\varphi\| \leq A_\varphi.$$

Since the norm is continuous over $\mathcal{L}(q, p)$, we deduce that

$$L \in \mathcal{L}(\varphi) \implies \|L\| \leq A_\varphi. \tag{3.1}$$

Let $H : \mathbb{R}^q \to \mathbb{R}^q$ be a bi-Lipschitz homeomorphism mapping the origin onto itself, with Lipschitz constant $A_H > 0$:

$$\mathbf{x}, \mathbf{x}' \in \mathbb{R}^q \implies \frac{1}{A_H} \cdot |\mathbf{x} - \mathbf{x}'| \leq |H(\mathbf{x}) - H(\mathbf{x}')| \leq A_H \cdot |\mathbf{x} - \mathbf{x}'|.$$

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Therefore from Estimate (3.1), we get

$$L \in \mathcal{L}(H) \implies \frac{1}{A_H} \leq \|L\| \leq A_H.$$

The mapping H extends as a homeomorphism of $\overline{\mathbb{R}^q}$ mapping ∞ onto ∞.

The *inversion of* H is the mapping $\iota_q \circ H \circ \iota_q^{-1}$. It is a homeomorphism of $\mathbf{U}_q$ which extends continuously to $\mathbf{0}$, taking the value $\mathbf{0}$, as the homeomorphism $\iota(H) : \mathbb{R}^q \to \mathbb{R}^q$. More precisely, the following holds true:

Proposition 4. *The inversion $\iota(H)$ of H is bi-Lipschitz.*

Proof. It is enough to show that $\iota(H)$ is Lipschitz, since $\iota(H^{-1}) = \iota(H)^{-1}$. Let $\mathbf{y} \in \mathbf{U}_q$ and let $\mathbf{x} := \iota_q^{-1}(\mathbf{y})$ and $\mathbf{z} := H(\mathbf{x})$. Since $H(\mathbf{0}) = \mathbf{0}$, we find

$$\frac{1}{A_H \cdot |\mathbf{y}|} \leq |\mathbf{z}| \leq \frac{A_H}{|\mathbf{y}|}.$$

If $\mathbf{x}$ is a point of $\mathbf{U}_q$ at which H is differentiable, we find the following estimate:

$$\|D_\mathbf{y}\iota(H)\| \leq \frac{1}{|\mathbf{z}|^2} \cdot A_H \cdot \frac{1}{|\mathbf{y}|^2} \leq A_H^3.$$

Since ι_q is a C^∞ diffeomorphism, the subset $\iota_q(\mathbb{R}^q \setminus \mathcal{D}(H))$ has null measure. Thus, $\iota(H)$ is differentiable outside a subset of zero measure with uniformly bounded first derivatives. Thus, it is Lipschitz. □

Let ω be either $\mathbf{0}$ or ∞. Let $h : \left(\overline{\mathbb{R}^q}, \omega\right) \to \left(\overline{\mathbb{R}^q}, \omega\right)$ be a germ of homeomorphism which is bi-Lipschitz over $(\mathbf{U}_q, \omega)$. Let ω^* be the point of $\overline{\mathbb{R}^q}$ antipodal to ω, that is,

$$\{\omega, \omega^*\} = \{\mathbf{0}, \infty\}.$$

The map germ $\iota_q \circ h \circ \iota_q^{-1} : \left(\mathbf{U}_q, \omega^*\right) \to \left(\mathbf{U}_q, \omega^*\right)$ extends as a homeomorphism germ $\iota(h) : \left(\overline{\mathbb{R}^q}, \omega^*\right) \to (\overline{\mathbb{R}^q}, \omega^*)$. A consequence of Proposition 4 is the (now expected) following result, initial motivation of the paper:

Corollary 5. *The germ of homeomorphism $\iota(h)$ is bi-Lipschitz over $\left(\mathbf{U}_q, \omega^*\right)$.*

Remark 6. *The proof of Proposition 4 we gave uses Rademacher Theorem and is a direct proof. But this result is a special case of Lemma 9, whose demonstration, although longer and mostly by absurd, uses even more elementary arguments.*

4. Inversion and germs of bi-Lipschitz homeomorphisms at infinity

Let σ be a point of $\overline{\mathbb{R}^q}$. Following Definition 1, the notion of germ of q-affine subset at σ is well defined. If τ is a point of $\overline{\mathbb{R}^p}$, the notion of Lipschitz mapping of affine germs $(S, \sigma) \to (T, \tau)$ is also well defined by Definition 2.

Let $\phi : (Y_1, \infty) \to (Y_2, \infty)$ be a germ of bi-Lipschitz homeomorphism between q_i-affine subsets germs (Y_i, ∞) with $i = 1, 2$. There exists a positive constant A_ϕ such that

$$\mathbf{y}, \mathbf{y}' \in Y_1 \implies \frac{1}{A_\phi} \cdot |\mathbf{y}' - \mathbf{y}| \leq |\phi(\mathbf{y}') - \phi(\mathbf{y})| \leq A_\phi \cdot |\mathbf{y}' - \mathbf{y}|.$$

Thus, we can assume that the Lipschitz constant A_ϕ is such that the following estimates are also satisfied:

$$\mathbf{y} \in Y_1 \implies \frac{1}{A_\phi} \cdot |\mathbf{y}| \leq |\phi(\mathbf{y})| \leq A_\phi \cdot |\mathbf{y}|.$$

With the previous notation, we deduce $\phi(\mathbf{y}) \sim \mathbf{y}$.

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For $i = 1, 2$, we denote by X_i the closure $\operatorname{clos}(\iota_{q_i}(Y_i))$ of $\iota_{q_i}(Y_i)$ in $\mathbb{R}^q$. The *inversion of* ϕ is the mapping defined as follows:

$$\iota(\phi) : (X_1, \mathbf{0}) \to (X_2, \mathbf{0}), \quad \mathbf{x} \to \iota(\phi)(\mathbf{x}) := \begin{cases} \iota_{q_2} \circ \phi \circ \iota_{q_1}^{-1}(\mathbf{x}) & \text{if} \quad \mathbf{x} \in X_1^* \\ \mathbf{0} & \text{if} \quad \mathbf{x} = \mathbf{0} \end{cases}$$

It is a germ of homeomorphism which extends continuously at $\mathbf{0}$ taking the value $\mathbf{0}$ at $\mathbf{0}$. The homogeneity of the Euclidean metric as well as the existence of the inversion mapping yield the following result.

Lemma 7. *If ϕ:$(Y_1, \infty) \to (Y_2, \infty)$ is a bi-Lipschitz homeomorphism germ between q_i-affine subsets germs (Y_i, ∞), for $i = 1, 2$, then its inversion $\iota(\phi) : (X_1, \mathbf{0}) \to (X_2, \mathbf{0})$ is bi-Lipschitz homeomorphism germ.*

Proof. First, let us denote $h := \phi \circ \iota_{q_1}^{-1}$, that is

$$h(\mathbf{x}) = \phi\left(\frac{\mathbf{x}}{|\mathbf{x}|^2}\right).$$

Therefore, we get that

$$\iota(\phi)(\mathbf{x}) = \frac{h(\mathbf{x})}{|h(\mathbf{x})|^2}.$$

Since $|h(\mathbf{x})| \sim |\mathbf{x}^{-1}|$, we observe that $\iota(\phi)(\mathbf{x}) \sim \mathbf{x}$, more precisely:

$$\frac{1}{A_\phi^3} \cdot |\mathbf{x}| \le |\iota(\phi)(\mathbf{x})| \le A_\phi^3 \cdot |\mathbf{x}|.$$

It is sufficient to show that $\iota(\phi)$ is Lipschitz.

Assume that $\iota(\phi)$ is not Lipschitz. Therefore, there exist two sequences $(\mathbf{x}_n)_n$ and $(\mathbf{x}'_n)_n$ of $\mathbf{U}_{q_1}$ such that

$$\lim_n \frac{|\iota(\phi)(\mathbf{x}_n) - \iota(\phi)(\mathbf{x}'_n)|}{|\mathbf{x}_n - \mathbf{x}'_n|} = \infty. \tag{4.1}$$

We work with a representative of ϕ outside a compact subset C_1 of $\mathbb{R}^{q_1}$ containing $\mathbf{0}$ and with the representative of $\iota(\phi)$ over X_1, the closure $\operatorname{clos}(\iota_{q_1}(Y_1 \setminus C_1))$. Thus, X_1 is compact. For convenience sake let

$$\mathbf{y}_n := \iota_{q_1}(\mathbf{x}_n) \text{ and } \mathbf{y}'_n := \iota_{q_1}(\mathbf{x}'_n).$$

We further define the following numbers:

$$e_n := |\mathbf{x}_n - \mathbf{x}'_n|, \;\; \iota(\phi)_n := |\iota(\phi)(\mathbf{x}_n) - \iota(\phi)(\mathbf{x}'_n)|, \;\; t_n := |\mathbf{x}_n|, \;\; t'_n := |\mathbf{x}'_n|, \;\; s_n := |\mathbf{y}_n| \text{ and } s'_n := |\mathbf{y}'_n|.$$

Of course we have $s_n t_n = s'_n t'_n = 1$.

Without the loss of generality, we can assume that the sequence $(\mathbf{x}_n)_n$ converges to $\chi \in X_1$ and $(\mathbf{x}'_n)_n$ converges to $\chi' \in X_1$.

- **Case 1.** $\chi \neq \mathbf{0}$ *and* $\chi' \neq \mathbf{0}$.

In other words, there exists a compact subset K_1 of $\mathbf{U}_{q_1}$ which contains $\mathbf{x}_n, \mathbf{x}'_n$ for all n. Since the inversion ι_{q_1} is bi-Lipschitz over K_1, so is the mapping $\iota(\phi)$, contradicting the estimate (4.1). Therefore, this case cannot happen and we can assume that $\chi = \mathbf{0}$.

- **Case 2.** $\chi = \mathbf{0}$ *and* $\chi' \neq \mathbf{0}$.

Observe that the following estimates hold true

$$e_n = t'_n + o(t'_n), \;\; |\iota(\phi)(\mathbf{x}_n)| \sim t_n \to 0, \quad \text{and} \quad |\iota(\phi)(\mathbf{x}'_n)| \sim t'_n \in [a, b]$$

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for positive real numbers $b > a$. Therefore, we deduce that

$$\iota(\phi)_n \sim t'_n$$

contradicting the estimate (4.1). Therefore, this case cannot happen and thus $\chi' = \mathbf{0}$ as well.

- **Case 3.** $\chi = \chi' = \mathbf{0}$ *and there exists* $B > 1$ *such that* $|\mathbf{x}'_n| \geq B|\mathbf{x}_n|$ *for n large enough.*
  For n large enough, Claim 3 yields

$$\frac{e_n}{t'_n} \in \left[\frac{B-1}{B}, \frac{B+1}{B}\right].$$

Since $\iota(\phi)_n \leq (1+B)A_\phi^3 t'_n$ for large n, we produce again a contradiction to the estimate (4.1). This case does not occur and we can assume, up to taking a subsequence that $\frac{t_n}{t'_n} \to 1$ as n goes to ∞.

Let $2\theta_n \in [0, \pi]$ be the angle between the vectors $\mathbf{x}_n$ and $\mathbf{x}'_n$.

- **Case 4.** $\lim_n \frac{|\mathbf{x}_n|}{|\mathbf{x}'_n|} = 1$ *and* $\liminf_n 2\theta_n \in]0, \pi]$.
  We can assume that $2\theta_n \geq 2\theta \in]0, \pi]$. From Identity (2.2), we deduce that

$$\lim_n \frac{e_n}{t_n} \geq 2\sin\theta > 0.$$

Since $\iota(\phi)_n \leq A_\phi^3(t_n + t'_n)$ for n large enough, estimate (4.1) cannot be satisfied and thus $\theta = 0$.

Up to passing to subsequences, we can assume that $(\theta_n)_n$ converges to 0.

- **Case 5.** $\lim_n \frac{|\mathbf{x}_n|}{|\mathbf{x}'_n|} = 1$ *and* $\lim_n \theta_n = 0$.
  We can assume that $t'_n \geq t_n$ and $\mathbf{x}'_n = \mathbf{x}_n + \mathbf{z}_n$ so that

$$\upsilon_n := \frac{|\mathbf{z}_n|}{t'_n} = s'_n e_n \to 0 \text{ as } n \to \infty.$$

Let $a_n := \cos(\theta_n)$, $b_n := \sin(\theta_n)$, and $\delta_n t_n := t'_n - t_n$. We get

$$|\mathbf{z}_n|^2 = e_n^2 = (2b_n t_n + \delta_n t_n b_n)^2 + (\delta_n t_n a_n)^2 = t_n^2 \cdot [\delta_n^2 + 4b_n^2 + o\big(b_n^2\big)].$$

Since we can write $\mathbf{y}_n = \mathbf{y}'_n + \mathbf{w}_n$, equation (2.3) yields

$$|\mathbf{w}_n| = s_n s'_n \cdot |\mathbf{z}_n|$$

and thus we deduce that

$$\frac{|\mathbf{w}_n|}{s_n} = \frac{|\mathbf{z}_n|}{t'_n} = \upsilon_n \to 0 \text{ as } n \to \infty.$$

Since ϕ is bi-Lipschitz, we obtain the following estimate:

$$h(\mathbf{x}_n) = h(\mathbf{x}'_n) + s_n \mathbf{u}_n, \text{ where } |\mathbf{u}_n| \sim \upsilon_n,$$

from which we deduce

$$\frac{|h(\mathbf{x}'_n)|^2}{|h(\mathbf{x}_n)|^2} = 1 + r_n \text{ where } |r_n| \lesssim \upsilon_n.$$

Combining the various previous estimates yields the following one:

$$\iota(\phi)_n = \frac{1}{|h(\mathbf{x}'_n)|^2} \big|h(\mathbf{x}'_n) - (1+r_n)[h(\mathbf{x}'_n) + s_n \mathbf{u}_n]\big| \lesssim \frac{|r_n| + (1+r_n)|\mathbf{u}_n|}{s_n} \sim t'_n \upsilon_n = e_n,$$

contradicting estimate (4.1).

5. Inversion and germs of bi-Lipschitz homeomorphism at 0

This section is about the counterpart at the origin of the previous result at infinity Lemma 7, more precisely its converse.

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Let $\psi : (X_1, \mathbf{0}) \to (X_2, \mathbf{0})$ be a germ of bi-Lipschitz homeomorphism between q_i-affine subsets X_i, where $i = 1, 2$. Thus, there exists a positive constant A_ψ such that

$$\mathbf{x}, \mathbf{x}' \implies \frac{1}{A_\psi} \cdot |\mathbf{x}' - \mathbf{x}| \le |\psi(\mathbf{x}') - \psi(\mathbf{x})| \le A_\psi \cdot |\mathbf{x}' - \mathbf{x}|.$$

Thus, the following estimates are also satisfied:

$$\mathbf{x} \in X_1 \implies \frac{1}{A_\psi} \cdot |\mathbf{x}| \le |\psi(\mathbf{x})| \le A_\psi \cdot |\mathbf{x}|,$$

that is $\psi(\mathbf{x}) \sim \mathbf{x}$, with the previous notation. Denoting $Y_i := \iota_{q_i}(X_i^*)$ for $i = 1, 2$, the *inversion of* ψ is the germ of mapping defined as follows:

$$\iota(\psi) : (Y_1, \infty) \to (Y_2, \infty), \ \ \mathbf{y} \to \iota(\psi)(\mathbf{y}) := \iota_{q_2} \circ \psi \circ \iota_{q_1}^{-1}(\mathbf{y}).$$

It clearly extends as a germ of homeomorphism $(\overline{Y}_1, \infty) \to (\overline{Y}_2, \infty)$.

The converse of Lemma 7 is

Lemma 8. *If $\psi : (X_1, \mathbf{0}) \to (X_2, \mathbf{0})$ is a bi-Lipschitz homeomorphism germ between q_i-affine subset germs $(X_i, \mathbf{0})$, for $i = 1, 2$, then its inversion $\iota(\psi) : (Y_1, \infty) \to (Y_2, \infty)$ is a bi-Lipschitz homeomorphism germ.*

The proof will be symmetric to that of Lemma 7 in the sense that arguments at $\mathbf{0}$ are replaced by their exact analogs at ∞, as expected from such a statement.

Proof. First, let $g := \psi \circ \iota_{q_1}^{-1}$, that is,

$$g(\mathbf{y}) = \psi\left(\frac{\mathbf{y}}{|\mathbf{y}|^2}\right).$$

Therefore, we get that

$$\iota(\psi)(\mathbf{y}) = \frac{g(\mathbf{y})}{|g(\mathbf{y})|^2},$$

and since $|g(\mathbf{y})| \sim |\mathbf{y}|^{-1}$ we find that $\iota(\psi)(\mathbf{y}) \sim \mathbf{y}$, more precisely

$$\frac{1}{A_\psi^3} \cdot |\mathbf{y}| \le |\iota(\psi)(\mathbf{y})| \le A_\psi^3 \cdot |\mathbf{y}|.$$

As in the previous section, it is enough to show that $\iota(\psi)$ is Lipschitz.

Assume that $\iota(\psi)$ is not Lipschitz. Therefore, there exist two sequences $(\mathbf{y}_n)_n$ and $(\mathbf{y}'_n)_n$ of $\mathbf{U}_{q_1}$ such that

$$\lim_n \frac{|\iota(\psi)(\mathbf{y}_n) - \iota(\psi)(\mathbf{y}'_n)|}{|\mathbf{y}_n - \mathbf{y}'_n|} = \infty. \tag{5.1}$$

We work with a representative of ψ within a compact subset K_1 of $\mathbb{R}^{q_1}$ containing $\mathbf{0}$. Let

$$\mathbf{x}_n := \iota_{q_1}(\mathbf{y}_n) \text{ and } \mathbf{x}'_n := \iota_{q_1}(\mathbf{y}'_n).$$

In order to ease computations, we further define the following numbers:

$$E_n := |\mathbf{y}_n - \mathbf{y}'_n|, \ \ \iota(\psi)_n := |\iota(\psi)(\mathbf{y}_n) - \iota(\psi)(\mathbf{y}'_n)|, \ \ s_n := |\mathbf{y}_n|, \ \ s'_n := |\mathbf{y}'_n|, \ \ t_n := |\mathbf{x}_n| \text{ and } t'_n := |\mathbf{x}'_n|.$$

Of course we find again that $s_n t_n = s'_n t'_n = 1$.

- **Case 1.** $\limsup_n \max(|\mathbf{y}_n|, |\mathbf{y}'_n|) < \infty$.

In other words, there exists a compact subset C_1 of $\mathbf{U}_{q_1}$ which contains $\mathbf{y}_n, \mathbf{y}'_n$ for all n. Since the inversion ι_{q_1} is bi-Lipschitz over C_1, so is the mapping $\iota(\psi)$, contradicting the estimate (5.1). Therefore, this case cannot happen and we can assume, after taking a subsequence, that $(\mathbf{y}'_n)_n$ converges to ∞.

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- **Case 2.** $\mathbf{y}'_n \to \infty$ *and* $\limsup_n |\mathbf{y}_n| < \infty$.
  Observe that the following estimates hold true

$$E_n = s'_n + o\big(s'_n\big),\ \ |\phi(\mathbf{y}'_n)| \sim s'_n \to \infty,\ \text{ and }\ \iota(\psi)(\mathbf{y}_n)| \sim s_n \in [a, b]$$

for positive real numbers $b > a$. Therefore, we deduce that

$$\iota(\psi)_n \sim s'_n$$

contradicting the estimate (5.1). This case cannot happen and therefore a subsequence of $(\mathbf{y})_n$ converges to ∞ as well.

- **Case 3.** $\mathbf{y}_n, \mathbf{y}'_n \to \infty$ *and there exists* $B > 1$ *such that* $|\mathbf{y}'_n| \ge B|\mathbf{y}_n|$ for *n large enough.*
  For n large enough, Claim 3 yields

$$\frac{E_n}{s'_n} \in \left[\frac{B-1}{B}, \frac{B+1}{B}\right].$$

Since $\iota(\psi)_n \le (1+B)A_\psi^3 s'_n$, we produce again a contradiction to the estimate (5.1). This case does not occur and we can assume up to taking subsequences that $\frac{s_n}{s'_n} \to 1$ as n goes to ∞.
  Let $2\theta_n \in [0, \pi]$ be the angle between the vectors $\mathbf{y}_n$ and $\mathbf{y}'_n$.

- **Case 4.** $\lim_n \frac{|\mathbf{y}_n|}{|\mathbf{y}'_n|} = 1$ *and* $\liminf_n 2\theta_n \in]0, \pi]$.
  We can assume that $2\theta_n$ converges to $2\theta \in]0, \pi]$. We check that

$$\lim_n \frac{E_n}{s_n} \ge 2\sin\theta > 0.$$

Since $\iota(\psi)_n \le A_\psi^3(s_n + s'_n)$ for n large enough, estimate (5.1) cannot be satisfied and thus $\theta = 0$.

- **Case 5.** $\lim_n \frac{|\mathbf{y}_n|}{|\mathbf{y}'_n|} = 1$ *and* $\lim_n \theta_n = 0$.
  We can assume that $s_n \ge s'_n$ and $\mathbf{y}_n = \mathbf{y}'_n + \mathbf{w}_n$ so that

$$\upsilon_n := \frac{|\mathbf{w}_n|}{s_n} = t_n E_n \to 0 \ \text{ as } \ n \to \infty.$$

Let $a_n :=$ cos (θ_n), $b_n :=$ sin (θ_n), and $\delta_n s'_n := s_n - s'_n$. We get

$$|\mathbf{w}_n|^2 = E_n^2 = \big(2b_n s'_n + \delta_n s'_n b_n\big)^2 + \big(\delta_n s'_n a_n\big)^2 = \big[\delta_n^2 + 4b_n^2 + o\big(b_n^2\big)\big]\,\big(s'_n\big)^2.$$

Since we can write $\mathbf{x}'_n = \mathbf{x}_n + \mathbf{z}_n$, equation (2.3) yields

$$|\mathbf{z}_n| = t_n t'_n \cdot |\mathbf{w}_n|$$

and thus we deduce that

$$\frac{|\mathbf{z}_n|}{t'_n} = \frac{|\mathbf{w}_n|}{s_n} = \upsilon_n \to 0 \ \text{ as } \ n \to \infty.$$

Since ψ is bi-Lipschitz, we obtain the following estimate:

$$g(\mathbf{y}_n) = g(\mathbf{y}'_n) + t'_n \mathbf{u}_n, \ \text{ where } \ |\mathbf{u}_n| \sim \upsilon_n,$$

from which we deduce

$$\frac{|g(\mathbf{y}'_n)|^2}{|g(\mathbf{y}_n)|^2} = 1 + r_n \ \text{ where } \ |r_n| \lesssim \upsilon_n.$$

Combining the various previous estimates yields the following one:

$$\iota(\psi)_n = \frac{1}{|g(\mathbf{y}'_n)|^2}|g(\mathbf{y}'_n) - (1+r_n)(g(\mathbf{y}'_n) + t'_n\mathbf{u}_n)| \lesssim \frac{|r_n| + (1+r_n)|\mathbf{u}_n|}{t'_n} \sim s_n\upsilon_n = E_n,$$

contradicting estimate (5.1). □

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6. Inversion and bi-Lipschitz homeomorphisms

This section presents the inversion lemma, that is, the main tool of this note. It is a rather straightforward consequence of Lemmas 8 and 7.

Lemma 9. *Let $\Phi : W_1 \to W_2$ be a homeomorphism between q_i-affine subsets W_i, for $i = 1, 2$. Assume furthermore that, either W_i contains the origin $\mathbf{0} \in \mathbb{R}^{q_i}$ for $i = 1, 2$ and $\Phi(\mathbf{0}) = \mathbf{0}$, or $\mathbf{0} \notin W_i$ for $i = 1, 2$. The mapping defined as*

$$\iota(\Phi) : \iota_{q_1}\left(W_1^*\right) \to \iota_{q_2}(W_2^*),\ \ \mathbf{x} \to \iota(\Phi)(\mathbf{x}) := \iota_{q_2} \circ \Phi \circ \iota_{q_1}^{-1}(\mathbf{x}).$$

is bi-Lipschitz if and only if Φ is. Moreover if W_1 is unbounded, the mapping $\iota(\Phi)$ extends bi-Lipschitz-ly at $\mathbf{0}$, taking the value $\mathbf{0}$.

Proof. Since $\iota(\iota(\Phi)) = \Phi$, it is enough to show the result when Φ is bi-Lipschitz. Since $\iota(\Phi)^{-1} = \iota_{q_1} \circ \Phi^{-1} \circ \iota_{q_2}^{-1}$, we only need to show that $\iota(\Phi)$ is Lipschitz.

By construction, we already know that $\iota(\Phi)$ is a homeomorphism of $\iota_{q_1}\left(W_1^*\right) \to \iota_{q_2}(W_2^*)$ which extends homeomorphically to $\overline{\iota_{q_1}\left(W_1^*\right)} \to \overline{\iota_{q_2}(W_2^*)}$, mapping $\mathbf{0}$ to $\mathbf{0}$ whence W_1 is unbounded.

Let $0 < r < R < \infty$ be radii. We define the following subsets:

$$\begin{aligned} C_r^* &:= \left\{\mathbf{y} \in \iota_{q_1}\left(W_1^*\right) : 0 < |\mathbf{y}| \le r\right\} \\ C_r^R &:= \left\{\mathbf{y} \in \iota_{q_1}\left(W_1^*\right) : r \le |\mathbf{y}| \le R\right\} \\ C^R &:= \left\{\mathbf{y} \in \iota_{q_1}\left(W_1^*\right) : R \le |\mathbf{y}|\right\}. \end{aligned}$$

The "annulus" C_r^R is compact and does not contain the origin. We recall that the mapping ι_q induces a bi-Lipschitz homeomorphism $K \to \iota_q(K)$ over any compact subset K of $\mathbf{U}_q$. Thus, $\iota(\Phi)$ induces a bi-Lipschitz homeomorphism from C_r^R onto its image. By Lemma 8, $\iota(\Phi)$ is a bi-Lipschitz homeomorphism from C^R onto its image. If W_1 is compact, we can take r small enough so that C_r^* is empty. If W_1 is unbounded, Lemma 7 implies that $\iota(\Phi)$ is a bi-Lipschitz homeomorphism from C_r^* onto its image and extends bi-Lipschitz-ly at $\mathbf{0}$.

Let $A, B \in \{C^R, C_r^R, C_r^*\}$ with $A \ne B$. If $\iota(\Phi)$ is not Lipschitz in $A \cup B$, then there exist a pair of sequences $(\mathbf{a}_n)_n$ of A and $(\mathbf{b}_n)_n$ of B such that

$$\lim_{n\to\infty} \frac{|\iota(\Phi)(\mathbf{a}_n) - \iota(\Phi)(\mathbf{b}_n)|}{|\mathbf{a}_n - \mathbf{b}_n|} = \infty.$$

Up to taking some subsequences, we can further assume that each sequence $(\mathbf{a}_n)_n$ and $(\mathbf{b}_n)_n$ either converges or goes to infinity. We check that the only scenario where such a pair of sequences could exist satisfying the required limit condition above is when both converge to a point $\mathbf{c} \in A \cap B$, thus $\mathbf{c} \ne \mathbf{0}$. Which is absurd since nearby the point $\mathbf{c}$ the inversion $\iota_{q_1}^{-1}$ is Lipschitz as is ι_{q_2} near the point $\Phi(\iota_{q_1}^{-1}(\mathbf{c}))$.

Thus, $\iota(\Phi)$ is Lipschitz over each $A \cup B$ with $A, B \in \{C^R, C_r^R, C_r^*\}$ with $A \ne B$. Let C be the remaining subset among $\{C^R, C_r^R, C_r^*\}$ so that $A \cup B \cup C = \iota_{q_1}(W_1)$. Working with $A' = A \cup B$ and $B' = C$ instead of A, B as we did in the previous case, we conclude that $\iota(\Phi)$ is Lipschitz. ☐

7. Main result

Let $N_q = (0, \ldots, 0, 1) \in \mathbb{R}^{q+1}$ be the north pole of the unit sphere $\mathbf{S}^q$. Let

$$\sigma_q : \mathbb{R}^q \to \mathbf{S}^q \setminus N_q,\ \ \mathbf{x} \mapsto \left(\frac{2\mathbf{x}}{1 + |\mathbf{x}|^2}, \frac{|\mathbf{x}|^2 - 1}{|\mathbf{x}|^2 + 1}\right)$$

be the inverse of the stereographic projection with center N_q. Given a subset of S of $\mathbb{R}^q$, let

$$\widetilde{S} := \operatorname{clos}(\sigma_q(S)).$$

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If $\Phi : W_1 \to W_2$ is a homeomorphism, where each subset W_i is q_i-affine, $i = 1, 2$, its stereographic pre-compactification is the homeomorphism $\sigma_{q_2} \circ \Phi \circ \sigma_{q_1}^{-1}$. If W_1 is unbounded, the stereographic compactification of Φ is the mapping $\widetilde{\Phi}$, extension of $\sigma_{q_2} \circ \Phi \circ \sigma_{q_1}^{-1}$ to $\widetilde{W}_1$.

Theorem 10. *Let W_i be q_i-affine subsets, $i = 1, 2$. A mapping $\Phi : W_1 \to W_2$ is bi-Lipschitz, if and only if its stereographic compactification $\widetilde{\Phi} : \widetilde{W}_1 \to \widetilde{W}_2$ is bi-Lipschitz*

The rest of this section is devoted to the proof of this result.

We recall that the quotient space obtained from gluing two copies of $\mathbb{R}^q$, when both $\mathbf{U}_q$ are identified by the inversion ι_q is $\mathbf{S}^q$. Therefore, the next result, somehow tuned with [3, Lemma 7.2], should not come as a surprise.

Lemma 11. *Suppose that W_1 is unbounded. The mapping germ $\Phi : (W_1, \infty) \to (W_2, \infty)$ is bi-Lipschitz if and only if $\widetilde{\Phi} : \left(\widetilde{W}_1, N_{q_1}\right) \to \left(\widetilde{W}_2, N_{q_2}\right)$ is bi-Lipschitz.*

Proof. Let $\mathbf{z} = (\mathbf{z}', t)$ be Euclidean coordinates on $\mathbb{R}^{q+1} = \mathbb{R}^q \times \mathbb{R}$. The following mapping

$$\beta_q : \mathbf{B}^q_{\frac{1}{2}} \to \mathbf{S}^q \cap \left\{ t \geq \frac{3}{5} \right\}, \ \ \mathbf{y} \to \left(\frac{1}{1 + |\mathbf{y}|^2} \cdot \mathbf{y}, \frac{1 - |\mathbf{y}|}{1 + |\mathbf{y}|^2} \right)$$

is a C^∞ diffeomorphism; thus, it is a bi-Lipschitz homeomorphism mapping $\mathbf{0}$ onto N_q. We also check that

$$|\mathbf{x}| \geq 2 \implies \beta_q \circ \iota_q(\mathbf{x}) = \sigma_q(\mathbf{x}).$$

The lemma follows from Lemmas 8 and 7, and the fact that β_q is bi-Lipschitz. □

Proof of Theorem 10. We recall that σ_q is bi-Lipschitz over any given compact subset of $\mathbb{R}^q$. If W_1 is compact, the result is thus obvious.

Assume that W_1 is unbounded. Let $K_1 = \mathbf{B}^{q_1}_{R_1} \cap W_1$ and $K_2 = \Phi(K_1)$ with $R_1 \geq 2$ chosen so that $W_i \setminus K_i$ is contained in $\mathbb{R}^{q_i} \setminus \mathbf{B}^{q_i}_2$, where $i = 1, 2$. Thus, the mapping

$$\Phi_b := \ \Phi|_{K_1} : K_1 \to K_2$$

is bi-Lipschitz, if and only if $\widetilde{\Phi}_b := \ \widetilde{\Phi}|_{\widetilde{K}_1} : \widetilde{K}_1 \to \widetilde{K}_2$ is bi-Lipschitz.

Up to increasing R_1, following the proof of Lemma 11, we deduce that the mapping

$$\Phi_u := \ \Phi|_{W_1 \setminus K_1} : W_1 \setminus K_1 \to W_2 \setminus K_2$$

is bi-Lipschitz if and only if the mapping $\widetilde{\Phi}_u := \ \widetilde{\Phi}|_{\widetilde{W}_1 \setminus \widetilde{K}_1} : \widetilde{W}_1 \setminus \widetilde{K}_1 \to \widetilde{W}_2 \setminus \widetilde{K}_2$ is observe that Φ_u and $\widetilde{\Phi}_u$, respectively, extend bi-Lipschitz-ly on the closure of their domains when Φ and $\widetilde{\Phi}$, respectively, are bi-Lipschitz.

- Assume that Φ is bi-Lipschitz. Thus, $\widetilde{\Phi}$ is a homeomorphism and both $\widetilde{\Phi}_b$ and $\widetilde{\Phi}_u$ are bi-Lipschitz. If $\widetilde{\Phi}$ were not Lipschitz, there would exist two sequences $(\mathbf{z}_n)_n$, $(\mathbf{z}'_n)_n$ of $\widetilde{W}_1$ such that

$$\lim_{n \to \infty} \frac{\left|\widetilde{\Phi}(\mathbf{z}'_n) - \widetilde{\Phi}(\mathbf{z}_n)\right|}{|\mathbf{z}'_n - \mathbf{z}_n|} = \infty.$$

  Since $\widetilde{W}_1$ is compact, up to passing to subsequences we can assume that both sequences converge to ω_1. Since Φ_u, Φ_b are bi-Lipschitz and Φ_u extends bi-Lipschitz-ly onto $\text{clos}(\widetilde{W}_1 \setminus \widetilde{K}_1)$, necessarily one of the sequences is contained in $\widetilde{K}_1$ and the other one in $\widetilde{W}_1 \setminus \widetilde{K}_1$. Thus, $\omega_1 \in \widetilde{K}_1$ and $\omega_2 := \widetilde{\Phi}(\omega_1) \in \widetilde{K}_2$. Since $\sigma_{q_1}^{-1}$ is bi-Lipschitz nearby ω_1 and $\sigma_{q_2}^{-1}$ is bi-Lipschitz nearby ω_2, the mapping $\widetilde{\Phi}$ is Lipschitz nearby ω_1, yielding a contradiction.
- Assume that $\widetilde{\Phi}$ is bi-Lipschitz. Thus, Φ is a homeomorphism and both Φ_b and Φ_u are bi-Lipschitz. Moreover, Φ_u extends bi-Lipschitz-ly to $\text{clos}(W_1 \setminus K_1)$. If Φ were not Lipschitz, there would exist

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two sequences $(\mathbf{x}_n)_n$ and $(\mathbf{x}'_n)_n$ of $\widetilde{W}_1$ such that

$$\lim_{n\to\infty} \frac{\left|\Phi(\mathbf{x}'_n) - \widetilde{\Phi}(\mathbf{x}_n)\right|}{|\mathbf{x}'_n - \mathbf{x}_n|} = \infty.$$

Necessarily one sequence belongs to K_1 and the other one to $\operatorname{clos}(W_1 \setminus K_1)$. Assume that $(\mathbf{x}_n)_n$ is contained in K_1. So, we can assume it converges to $\mathbf{y}_1$. If $\mathbf{y}_1$ does not lie in the compact set $L_1 = K_1 \cap \operatorname{clos}(W_1 \setminus K_1)$, thus

$$\liminf_n |\mathbf{x}'_n - \mathbf{x}_n| \in (0, \infty]$$

therefore $\Phi(\mathbf{x}'_n)$ goes to ∞, and using $\widetilde{\Phi}$, we conclude that $\mathbf{x}'_n \to \infty$. Let M be a Lipschitz constant common to Φ_b and Φ_u. Let $\mathbf{y}'_1$ be a point of L_1. Thus,

$$|\Phi(\mathbf{x}'_n) - \Phi(\mathbf{x}_n)| \leq |\Phi_u(\mathbf{x}_n) - \Phi_u(\mathbf{y}'_1)| + |\Phi_b(\mathbf{y}'_1) - \Phi_b(\mathbf{x}_n)| \leq M|\mathbf{x}_n - \mathbf{y}'_1| + M|\mathbf{y}'_1 - \mathbf{x}_n|$$

yielding a contradiction since $|\mathbf{x}_n - \mathbf{y}'_1| \to \infty$. Thus, $\mathbf{y}_1$ lies in L_1.

The same argument involving the point $\mathbf{y}'_1 = \mathbf{y}_1$ implies that $\liminf_n |\mathbf{x}'_n - \mathbf{x}_n| = 0$, so we can assume that $(\mathbf{x}'_n)_n$ converges to $\mathbf{y}_1$ as well. Since $\sigma_{q_1}^{-1}$ is bi-Lipschitz nearby $\mathbf{y}_1$ and σ_{q_2} is bi-Lipschitz nearby $\Phi(\mathbf{y}_1)$, the mapping Φ is Lipschitz nearby $\mathbf{y}_1$, yielding a contradiction. □

8. Geometry at infinity of tame sets

There are many possible applications of the inversion Lemma 9. In particular, any bi-Lipschitz classification problem of subsets at infinity is equivalent to a bi-Lipschitz classification problem at the origin.

There are quite a few questions of bi-Lipschitz definable geometry at infinity which now reduce to a problem at the origin by our main result. Many of them would require some specific preparations, that is why we present here only two such applications, which are immediate consequences of Lemma 9.

8.1. Bi-Lipschitz definable sets at infinity and their tangent cones

A non-negative cone C of $\mathbb{R}^q$ is any subset of $\mathbb{R}^q$ stable by non-negative rescaling:

$$\mathbf{x} \in C \implies t \cdot \mathbf{x} \in C, \ \forall t \geq 0.$$

For a given non-negative cone C, the *link of* C is defined as;

$$\mathbf{S}(C) := C \cap \mathbf{S}^{q-1}.$$

Let S be a non-empty subset of $\mathbb{R}^q$. The non-negative cone over S with vertex the origin $\mathbf{0}$ is the subset of $\mathbb{R}^q$ defined as:

$$\widehat{S}^+ := \{t\mathbf{u} \in \mathbb{R}^q : \mathbf{u} \in S, \ t \geq 0\}.$$

In particular, a subset C is a non-negative cone of $\mathbb{R}^q$ if and only if it is the non-negative cone over its link:

$$C = \widehat{\mathbf{S}(C)}^+.$$

Definition 12. *Let S be a subset of $\mathbb{R}^q$.*

(i) The asymptotic set of S at $\mathbf{0}$ is the closed subset of the unit sphere $\mathbf{S}^{q-1}$

$$S^{\mathbf{0}} := \left\{\mathbf{u} \in \mathbf{S}^{q-1} : \exists\, (\mathbf{x}_k)_k \in S^* \text{ such that } \mathbf{x}_k \to \mathbf{0} \text{ and } \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \to \mathbf{u}\right\}.$$

(ii) The asymptotic set of S at ∞ is the closed subset of the unit sphere $\mathbf{S}^{q-1}$

$$S^\infty := \left\{ \mathbf{u} \in \mathbf{S}^{q-1} : \exists\, (\mathbf{x}_k)_k \in S \text{ such that } |\mathbf{x}_k| \to \infty \text{ and } \frac{\mathbf{x}_k}{|\mathbf{x}_k|} \to \mathbf{u} \right\}.$$

The subsets $S^{\mathbf{0}}$ and S^∞ are classical objects, with various names. We decided for a common denomination. The subset S^ω is often called, misleadingly, the tangent cone at ω. Note that $S^{\mathbf{0}}$ is call set of directions in [9]. The subset S^ω is not empty if and only if $\overline{S}$ contains ω, and observe that

$$\operatorname{clos}(S)^\omega = S^\omega$$

where $\omega = \mathbf{0}$ or ∞. Since we are interested in the non-negative cones $\widehat{S^{\mathbf{0}}}^+$ and $\widehat{S^\infty}^+$, we will work only with closed subsets. The non-negative cone $\widehat{S^\omega}^+$ is also known as the *tangent cone of S at ω*, for $\omega = \mathbf{0}$ or ∞.

Given $\mathbf{x} \in \mathbf{U}_q$, observe the following obvious fact

$$\frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\iota_q(\mathbf{x})}{|\iota_q(\mathbf{x})|}.$$

Let X be a closed subset of $\mathbb{R}^q$ and let $\iota(X)$ be the closure $\operatorname{clos}(\iota_q(X^*))$. The following result is obvious from the definitions of asymptotic sets and the inversion.

Lemma 13. *The following identities hold true:*

$$X^\infty = \iota(X)^{\mathbf{0}} \text{ and } X^{\mathbf{0}} = \iota(X)^\infty.$$

From this lemma, we deduce

$$\widehat{X^\infty}^+ = \widehat{\iota(X)^{\mathbf{0}}}^+ \text{ and } \widehat{X^{\mathbf{0}}}^+ = \widehat{\iota(X)^\infty}^+. \tag{8.1}$$

Let $\mathcal{M}$ be a polynomially bounded o-minimal structure expanding the real field $(\mathbb{R}, +, ., \geq)$ (see [5]). A subset of an Euclidean space $\mathbb{R}^q$ is *definable* if it is definable in $\mathcal{M}$. Let S be a subset of $\mathbb{R}^q$. A mapping $S \to \mathbb{R}^p$ is *definable* if its graph is definable.

We recall the following result of Sampaio about tangent cones:

Theorem 14 ([12]). *Let $(X_i, \mathbf{0})$ be the germ of a definable set of $\mathbb{R}^{q_i}$ at the origin, $i = 1,2$. If there exists a bi-Lipschitz homeomorphism $(X_1, \mathbf{0}) \to (X_2, \mathbf{0})$, then there exists a bi-Lipschitz homeomorphism $\widehat{X_1^{\mathbf{0}}}^+ \to \widehat{X_2^{\mathbf{0}}}^+$ mapping $\mathbf{0}$ onto $\mathbf{0}$.*

In truth [12] deals only with sub-analytic subsets, but the part of the demonstration using sub-analyticity goes through the definable context readily.

We recall that the inversion ι_q is a rational mapping, thus semi-algebraic, therefore definable in $\mathcal{M}$. As a corollary of this latter fact, of the inversion Lemma 9 and of identity (8.1), we deduce the following ([6, Theorem 2.19], [14, Theorem 3.1])

Proposition 15. *Let (W_i, ∞) be the germ of a closed definable set of $\mathbb{R}^{q_i}$ at infinity, $i = 1,2$. If there exists a bi-Lipschitz homeomorphism $(W_1, \infty) \to (W_2, \infty)$, then there exists a bi-Lipschitz homeomorphism $\widehat{W_1^\infty}^+ \to \widehat{W_2^\infty}^+$ mapping $\mathbf{0}$ onto $\mathbf{0}$.*

8.2. On the link at infinity

Let S be a subset of $\mathbb{R}^q$. For any positive radius R, we define the following subsets:

$$S_R := S \cap \mathbf{S}_R^{q-1}, \quad S_{\leq R} := S \cap \mathbf{B}_R^q \text{ and } S_{\geq R} := S \setminus B_R^q.$$

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Let again denote $\iota(S)$ the closure of $\iota_q(S^*)$. Thus, we get the obvious identifications:

$$\iota_q(S_R) = \iota(S)_{\frac{1}{R}}, \quad \iota_q(S^*_{\leq R}) = \iota_q(S)_{\geq \frac{1}{R}} \text{ and } \iota_q(S_{\geq R}) = \iota_q(S)^*_{\leq \frac{1}{R}}.$$

When X is definable and contains the origin $\mathbf{0}$, the local conic structure theorem states that *there exists r_0 such that for any radius $r_0 \geq r > 0$, the definable subset $X_{\leq r}$ is definably homeomorphic with $(\widehat{X_r}^+)_{\leq r}$, the "non-negative cone over X_r"* [5]. Moreover, such a definable homeomorphism can be found so that it preserves the distance to $\mathbf{0}$ [15]. In particular, X_r has constant topological type for $r \leq r_0$. It can also be shown that for any pair of radii $0 < r < r' \leq r_0$, the links X_r and $X_{r'}$ are bi-Lipschitz definably homeomorphic [15, 17, 18], although the Lipschitz constant in general cannot be uniform over $]0, r_0]$.

Let W be a definable subset of $\mathbb{R}^q$. Using the inversion and the local conicstructure theorem yield the locally conic structure theorem at infinity: *there exists a positive radius R_0 such that for any $R \geq R_0$, the subset $W_{\geq R}$ is definably homeomorphic to $\left(\widehat{W_R}^+\right)_{\geq R}$, the "non-negative cone over W_R" at infinity.* Moreover, such a definable homeomorphism can be found so that it also preserves the distance to the origin. Last given any pair of radii $R, R' \geq R_0$, the links W_R and $W_{R'}$ are definable and bi-Lipschitz homeomorphic.

We have mentioned the local conic structure theorems and the bi-Lipschitz constancy of the links in light of the following result about links preserving reparametrization of definable bi-Lipschitz homeomorphism of Valette:

Theorem 16 ([16–18]). *Let $(X_i, \mathbf{0})$ be a closed definable germ of $\mathbb{R}^{q_i}$, $i = 1, 2$. If there exists a definable bi-Lipschitz homeomorphism $(X_1, \mathbf{0}) \to (X_2, \mathbf{0})$, then there exists a definable bi-Lipschitz homeomorphism $(X_1, \mathbf{0}) \to (X_2, \mathbf{0})$ preserving the distance to the origin.*

Again as a corollary of our main result of Theorems 10 and 16 and the semi-algebraicity of the inversion, thus definable in $\mathcal{M}$, we find the following

Proposition 17. *Let (W_i, ∞) be a closed definable germ of $\mathbb{R}^{q_i}$, $i = 1, 2$. If there exists a definable bi-Lipschitz homeomorphism $(W_1, \infty) \to (W_2, \infty)$, then there exists a definable bi-Lipschitz homeomorphism $(W_1, \infty) \to (W_2, \infty)$ preserving the distance to the origin.*

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