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ON A GENERALISATION OF A RESTRICTED SUM FORMULA FOR MULTIPLE ZETA VALUES AND FINITE MULTIPLE ZETA VALUES

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Abstract

We prove a new linear relation for multiple zeta values. This is a natural generalisation of the restricted sum formula proved by Eie, Liaw and Ong. We also present an analogous result for finite multiple zeta values.

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1. Main results

1.1. Main result for multiple zeta values. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ with $k_1 \geq 2$, the multiple zeta values (MZVs) and the multiple zeta-star values (MZSVs) are defined respectively by

$$\zeta(k_1, \dots, k_r) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

and

$$\zeta^{\star}(k_1, \dots, k_r) := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

They are both generalisations of the Riemann zeta values $\zeta(k)$ at positive integers.

For an index $\mathbf{k} = (k_1, \dots, k_r)$, we call $|\mathbf{k}| := k_1 + \cdots + k_r$ the weight and r the depth. We write $\zeta^{+}(k_1, \dots, k_r) := \zeta(k_1 + 1, k_2, \dots, k_r)$. For two indices $\mathbf{k}$ and $\mathbf{l}$, we denote by $\mathbf{k} + \mathbf{l}$ the index obtained by componentwise addition, and always assume implicitly the depths of both $\mathbf{k}$ and $\mathbf{l}$ are equal. We also write $\mathbf{l} \geq 0$ if every component of $\mathbf{l}$ is a nonnegative integer. Our first main result is the following formula.

Theorem 1.1. *For* $(k_1,\dots,k_r)\in\mathbb{Z}_{\geq 1}^r$ *and* $t\in\mathbb{Z}_{\geq 0}$,

$$\sum_{\substack{m_1+\cdots+m_r=r+t\\ m_i\geq 1\,(1\leq i\leq r)}}\ \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq r)}}\zeta^+(\mathbf{a}_{m_1},\dots,\mathbf{a}_{m_r})$$
$$=\sum_{l=0}^{t}\ \sum_{\substack{m_1+\cdots+m_{r-1}=t-l\\ m_i\geq 0\,(1\leq i\leq r-1)}}\ \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}}\zeta^+((k_1,\{1\}^{m_1},\dots,k_{r-1},\{1\}^{m_{r-1}},k_r)+\mathbf{e}).$$

Here and hereafter, each $\mathbf{a}_{m_i}$ *denotes an* m_i*-tuple of positive integers. When* $r=1$*, we understand the right-hand side as* $\zeta^+(k_1+t)$.

Remark 1.2. Theorem 1.1 is equivalent to the derivation relation which was obtained by Ihara *et al.* [3]. This equivalence will be explained in Section 3.

Remark 1.3. We can deduce the sum formula

$$\sum_{\substack{s_1+\cdots+s_u=k\\ s_1\geq 2,s_i\geq 1(2\leq i\leq u)}}\zeta(s_1,\dots,s_u)=\zeta(k)$$

from Theorem 1.1 by taking $r=1, k_1=k-u$ and $t=u-1$ for any positive integers k and u with $k-u\geq 1$.

Example 1.4. For $(k_1,k_2)=(1,2)$ and $t=1$,

$$2\zeta(2,1,2)+\zeta(2,2,1)=\zeta(2,3)+\zeta(3,2)+\zeta(2,1,2).$$

Theorem 1.1 is also equivalent to the following result.

Theorem 1.5. *For* $(k_1,\dots,k_r)\in\mathbb{Z}_{\geq 1}^r$ *and* $s,t\in\mathbb{Z}_{\geq 0}$,

$$\sum_{\substack{m_1+\cdots+m_r=r+t\\ m_i\geq 1\,(1\leq i\leq r)}}\ \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq r)}}\zeta^+(\mathbf{a}_{m_1},\dots,\mathbf{a}_{m_r},\{1\}^s)$$
$$=\sum_{l=0}^{t}\ \sum_{\substack{m_1+\cdots+m_{r-1}=t-l\\ m_i\geq 0\,(1\leq i\leq r-1)}}\ \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}}\zeta^+((k_1,\{1\}^{m_1},\dots,k_{r-1},\{1\}^{m_{r-1}},k_r,\{1\}^s)+\mathbf{e}).$$

When $r=1$*, we understand the right-hand side as* $\sum_{\substack{|\mathbf{e}|=t\\ \mathbf{e}\geq 0}}\zeta^+((k_1,\{1\}^s)+\mathbf{e})$.

This is a generalisation of the restricted sum formula obtained by Eie *et al.* [1]. The case $r=1$ gives the original formula.

The proof of Theorem 1.5 will be given in Section 2. Here, we prove the equivalence of Theorems 1.1 and 1.5.

Proof of the equivalence of Theorems 1.1 and 1.5. It is clear that Theorem 1.5 implies Theorem 1.1 by setting $s=0$. So, we prove that Theorem 1.1 implies Theorem 1.5. Write $G(\mathbf{k},s,t)$ (respectively $H(\mathbf{k},s,t)$) for the left-hand side (respectively the right-hand side) of Theorem 1.5 and let $F(\mathbf{k},s,t):=G(\mathbf{k},s,t)-H(\mathbf{k},s,t)$. We prove

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$F(\mathbf{k},s,t)=0$ for $\mathbf{k}\in\mathbb{Z}_{\geq 1}^r$, $s,t\in\mathbb{Z}_{\geq 0}$ by induction on s. If $s=0$, then $F(\mathbf{k},0,t)=0$ by Theorem 1.1. We assume $F(\mathbf{k},s,t)=0$ for some $s\in\mathbb{Z}_{\geq 0}$ and show $F(\mathbf{k},s+1,t)=0$.

$$\begin{aligned}
G((\mathbf{k},1),s,t) &= \sum_{\substack{m_1+\cdots+m_{r+1}=r+t+1\\ m_i\geq 1\,(1\leq i\leq r+1)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq r)}} \zeta^+(\mathbf{a}_{m_1},\ldots,\mathbf{a}_{m_r},\{1\}^{m_{r+1}+s})\\
&= \sum_{m_{r+1}=1}^{t+1} \sum_{\substack{m_1+\cdots+m_r=r+t-m_{r+1}+1\\ m_i\geq 1\,(1\leq i\leq r)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq r)}} \zeta^+(\mathbf{a}_{m_1},\ldots,\mathbf{a}_{m_r},\{1\}^{m_{r+1}+s})\\
&= \sum_{m_{r+1}=1}^{t+1} G(\mathbf{k},s+m_{r+1},t-m_{r+1}+1)\\
&= \sum_{u=0}^{t} G(\mathbf{k},s+t-u+1,u),
\end{aligned}$$

$$\begin{aligned}
H((\mathbf{k},1),s,t) &= \sum_{l=0}^{t} \sum_{\substack{m_1+\cdots+m_r=t-l\\ m_i\geq 0\,(1\leq i\leq r)}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+((k_1,\{1\}^{m_1},\ldots,\{1\}^{m_{r-1}},k_r,\{1\}^{m_r+s+1})+\mathbf{e})\\
&= \sum_{l=0}^{t}\sum_{m_r=0}^{t-l} \sum_{\substack{m_1+\cdots+m_{r-1}=t-l-m_r\\ m_i\geq 0\,(1\leq i\leq r-1)}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+((k_1,\{1\}^{m_1},\ldots,\{1\}^{m_{r-1}},k_r,\{1\}^{m_r+s+1})+\mathbf{e})\\
&= \sum_{m_r=0}^{t}\sum_{l=0}^{t-m_r} \sum_{\substack{m_1+\cdots+m_{r-1}=t-l-m_r\\ m_i\geq 0\,(1\leq i\leq r-1)}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+((k_1,\{1\}^{m_1},\ldots,\{1\}^{m_{r-1}},k_r,\{1\}^{m_r+s+1})+\mathbf{e})\\
&= \sum_{m_r=0}^{t} H(\mathbf{k},s+m_r+1,t-m_r)\\
&= \sum_{u=0}^{t} H(\mathbf{k},s+t-u+1,u).
\end{aligned}$$

Therefore,

$$F((\mathbf{k},1),s,t)=\sum_{u=0}^{t}F(\mathbf{k},s+t-u+1,u).$$

By replacing s by $s+1$ and t by $t-1$,

$$F((\mathbf{k},1),s+1,t-1)=\sum_{u=0}^{t-1}F(\mathbf{k},s+t-u+1,u).$$

Subtracting the two previous equations,

$$F(\mathbf{k},s+1,t)=F((\mathbf{k},1),s,t)-F((\mathbf{k},1),s+1,t-1).$$

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By applying this equation repeatedly and $F(\mathbf{k}, s, 0) = 0$ for arbitrary index $\mathbf{k}$ and $s \in \mathbb{Z}_{\geq 0}$, we obtain

$$F(\mathbf{k}, s+1, t) = \sum_{t'=1}^{t} (-1)^{t'-1} F((\mathbf{k}, \{1\}^{t'}), s, t-t'+1),$$

which gives the desired result. □

1.2. Main result for finite multiple zeta values. There are two types of finite multiple zeta value (FMZV): $\mathcal{A}$-finite multiple zeta(-star) values ($\mathcal{A}$-FMZ(S)Vs) and symmetric multiple zeta(-star) values (SMZ(S)Vs).

We consider the collection of truncated sums

$$\zeta_p(k_1, \ldots, k_r) = \sum_{p > n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

modulo all primes p in the quotient ring $\mathcal{A} = (\prod_p \mathbb{Z}/p\mathbb{Z})/(\bigoplus_p \mathbb{Z}/p\mathbb{Z})$, which is a $\mathbb{Q}$-algebra. Elements of $\mathcal{A}$ are represented by $(a_p)_p$, where $a_p \in \mathbb{Z}/p\mathbb{Z}$, and two elements $(a_p)_p$ and $(b_p)_p$ are identified if and only if $a_p = b_p$ for all but finitely many primes p. For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$, the $\mathcal{A}$-FMZVs and the $\mathcal{A}$-FMZSVs are defined by

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r) := \Big(\sum_{p > n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \pmod{p} \Big)_p,$$

$$\zeta^{\star}_{\mathcal{A}}(k_1, \ldots, k_r) := \Big(\sum_{p > n_1 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \pmod{p} \Big)_p.$$

SMZ(S)Vs were first introduced by Kaneko and Zagier [4, 6]. For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$,

$$\zeta^{*}_{\mathcal{S}}(k_1, \ldots, k_r) := \sum_{i=0}^{r} (-1)^{k_1+\cdots+k_i} \zeta^{*}(k_i, \ldots, k_1) \zeta^{*}(k_{i+1}, \ldots, k_r).$$

Here, the symbol ζ^* on the right-hand side stands for the regularised value coming from harmonic regularisation, that is, a real value obtained by taking constant terms of harmonic regularisation as explained in [3]. In the sum, we understand $\zeta^*(\emptyset) = 1$. Let $\mathcal{Z}_{\mathbb{R}}$ be the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by 1 and all MZVs, which is a $\mathbb{Q}$-algebra. Then, the SMZVs are defined as elements in the quotient ring $\mathcal{Z}_{\mathbb{R}}/(\zeta(2))$ by

$$\zeta_{\mathcal{S}}(k_1, \ldots, k_r) := \zeta^{*}_{\mathcal{S}}(k_1, \ldots, k_r) \pmod{\zeta(2)}.$$

For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$, we also define the SMZSVs in $\mathcal{Z}_{\mathbb{R}}/(\zeta(2))$ by

$$\zeta^{\star}_{\mathcal{S}}(k_1, \ldots, k_r) := \sum_{\substack{\square \text{ is either a comma `,'} \\ \text{or a plus `+'}}} \zeta^{*}_{\mathcal{S}}(k_1 \square \cdots \square k_r) \pmod{\zeta(2)}.$$

Denoting by $\mathcal{Z}_{\mathcal{A}}$ the $\mathbb{Q}$-vector subspace of $\mathcal{A}$ spanned by 1 and all $\mathcal{A}$-FMZVs, Kaneko and Zagier conjectured that there is an isomorphism between $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}_{\mathbb{R}}/\zeta(2)$ as $\mathbb{Q}$-algebras such that $\zeta_{\mathcal{A}}(k_1, \ldots, k_r)$ and $\zeta_{\mathcal{S}}(k_1, \ldots, k_r)$ correspond with each other. (For more details, see [4, 6].) In the following, the letter '$\mathcal{F}$' stands either for '$\mathcal{A}$' or '$\mathcal{S}$'. Now, we state our second main result.

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THEOREM 1.6. *For* $(k_1, \dots, k_r) \in \mathbb{Z}^r_{\geq 1}$ *and* $t \in \mathbb{Z}_{\geq 0}$,

$$\sum_{\substack{m_1+\cdots+m_r=r+t\\ m_i\geq 1\,(1\leq i\leq r)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1\\ (1\leq i\leq r)}} \zeta_{\mathcal{F}}(\mathbf{a}_{m_1}, \dots, \mathbf{a}_{m_r}) = \sum_{l=0}^{t} \sum_{\substack{m_1+\cdots+m_{r-1}=t-l\\ m_i\geq 0\,(1\leq i\leq r-1)}} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta_{\mathcal{F}}((k_1, \{1\}^{m_1}, \dots, k_{r-1}, \{1\}^{m_{r-1}}, k_r) + \mathbf{e}).$$

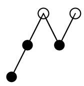
When $r = 1$, *we understand the right-hand side as* $\zeta_{\mathcal{F}}(k_1 + t)$.

REMARK 1.7. We can also obtain the FMZVs version of the restricted sum formula by replacing ζ^+ with $\zeta_{\mathcal{F}}$ in Theorem 1.5.

2. Proof of Theorem 1.5

2.1. Integral series identity. A 2-poset is a pair (X, δ_X), where $X = (X, \leq)$ is a finite partially ordered set and δ_X is a label map from X to $\{0, 1\}$. A 2-poset (X, δ_X) is called admissible if $\delta_X(x) = 0$ for all maximal elements $x \in X$ and $\delta_X(x) = 1$ for all minimal elements $x \in X$.

A 2-poset (X, δ_X) is depicted as a Hasse diagram in which an element x with $\delta(x) = 0$ (respectively $\delta(x) = 1$) is represented by $\circ$ (respectively $\bullet$). For example, the diagram



represents the 2-poset $X = \{x_1, x_2, x_3, x_4, x_5\}$ with order $x_1 < x_2 < x_3 > x_4 < x_5$ and label $(\delta_X(x_1), \dots, \delta_X(x_5)) = (1, 1, 0, 1, 0)$. This 2-poset is admissible.

For an admissible 2-poset X, we define the associated integral

$$I(X) := \int_{\Delta_X} \prod_{x\in X} \omega_{\delta_X(x)}(t_x),$$

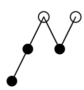
where

$$\Delta_X := \{(t_x)_x \in [0, 1]^X | t_x < t_y \text{ if } x < y\}$$

and

$$\omega_0(t) := \frac{dt}{t}, \quad \omega_1(t) := \frac{dt}{1-t}.$$

For example,

$$I\Bigg($$

$$\Bigg) = \int_{t_1<t_2<t_3>t_4<t_5} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} \frac{dt_4}{1-t_4} \frac{dt_5}{t_5}.$$

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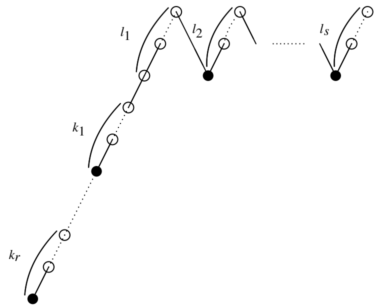
For indices $\mathbf{k} = (k_1, \ldots, k_r)$ and $\mathbf{l} = (l_1, \ldots, l_s)$, we define $\mu(\mathbf{k}, \mathbf{l})$ as a 2-poset corresponding to the following diagram:



For an index $\mathbf{k} = (k_1, \ldots, k_r)$, let $\mathbf{k}^\star$ be the formal sum of 2^{r-1} indices of the form $(k_1 \square \cdots \square k_r)$, where each $\square$ is replaced by ',' or '+'.

We also define the $\mathbb{Q}$-bilinear 'circled harmonic product' $\circledast$ by

$$(k_1, \ldots, k_r) \circledast (l_1, \ldots, l_s) := (k_1 + l_1, (k_2, \ldots, k_r) * (l_2, \ldots, l_s)),$$

where product '$*$' is the harmonic product defined inductively by

$$\varnothing * \mathbf{k} = \mathbf{k} * \varnothing = \mathbf{k},$$
$$\mathbf{k} * \mathbf{l} = (k_1, \mathbf{k}' * \mathbf{l}) + (l_1, \mathbf{k} * \mathbf{l}') + (k_1 + l_1, \mathbf{k}' * \mathbf{l}')$$

for any indices $\mathbf{k} = (k_1, \mathbf{k}')$ and $\mathbf{l} = (l_1, \mathbf{l}')$.

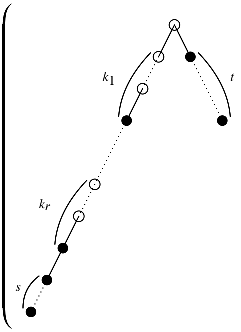
Kaneko and Yamamoto proved the following formula for MZVs.

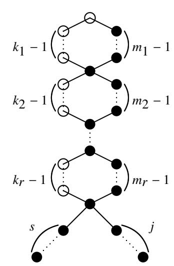
THEOREM 2.1 (Kaneko–Yamamoto [5]). *For any nonempty indices* $\mathbf{k}$ *and* $\mathbf{l}$*,*

$$\zeta(\mu(\mathbf{k}, \mathbf{l})) = \zeta(\mathbf{k} \circledast \mathbf{l}^\star).$$

2.2. Proof of Theorem 1.5. For $\mathbf{k} = (k_1, \ldots, k_r, \{1\}^s)$ and $\mathbf{l} = (\{1\}^{t+1})$,

$$\zeta(\mu(\mathbf{k}, \mathbf{l})) = I\left(\ \right) = \sum_{\substack{m_1+\cdots+m_r+j=r+t \\ (m_i \geq 1, j \geq 0)}} I\left(\ \right)$$





$$= \sum_{j=0}^{t} \binom{s+j}{s} \sum_{\substack{m_1+\cdots+m_r=r+t-j \\ m_i \geq 1\ (1 \leq i \leq r)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1 \\ (1 \leq i \leq r)}} \zeta^+(\mathbf{a}_{m_1}, \ldots, \mathbf{a}_{m_r}, \{1\}^{s+j}).$$

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In general, for $\mathbf{k}' = (k_1, \ldots, k_{r+s})$ and $\mathbf{l} = (\{1\}^{t+1})$,

$$\zeta(\mathbf{k}' \circledast \mathbf{l}^\star) = \sum_{l=0}^{t} \sum_{\substack{m_1+\cdots+m_{r+s}=r+s+t-l \\ m_i \geq 1\,(1\leq i\leq r)}} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+((k_1, \{1\}^{m_1-1}, \ldots, k_{r+s}, \{1\}^{m_{r+s}-1}) + \mathbf{e})$$

because the index $(\{1\}^{t+1})^\star$ is equal to the formal sum of all indices of weight $t+1$. Now, we put $k_{r+1} = \cdots = k_{r+s} = 1$ here. Then, the index $(k_1, \{1\}^{m_1-1}, \ldots, k_{r+s}, \{1\}^{m_{r+s}-1})$ on the right becomes $(k_1, \{1\}^{m_1-1}, \ldots, k_{r-1}, \{1\}^{m_{r-1}-1}, k_r, \{1\}^{u-1})$ with $u = m_r + \cdots + m_{r+s}$. For a fixed u, the number of $(s+1)$-tuples $(m_r, \ldots, m_{r+s})$ giving $u = m_r + \cdots + m_{r+s}$ is $\binom{u-1}{s}$. Thus,

$$\zeta(\mathbf{k} \circledast \mathbf{l}^\star) = \sum_{l=0}^{t} \sum_{u=s+j}^{s+t-l+1} \binom{u-1}{s} \sum_{\substack{m_1+\cdots+m_{r-1}=r+t+s-l-u \\ m_i\geq 1}} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+((k_1, \{1\}^{m_1-1}, \ldots, k_r, \{1\}^{u-1}) + \mathbf{e}).$$

By writing $u = s + j + 1$,

$$\begin{aligned}\zeta(\mathbf{k} \circledast \mathbf{l}^\star) &= \sum_{l=0}^{t} \sum_{j=0}^{t-l} \binom{s+j}{s} \sum_{\substack{m_1+\cdots+m_{r-1}=r+t-l-j-1 \\ m_i\geq 1}} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+((k_1, \{1\}^{m_1-1}, \ldots, k_r, \{1\}^{s+j}) + \mathbf{e}) \\ &= \sum_{j=0}^{t} \binom{s+j}{s} \sum_{l=0}^{t-j} \sum_{\substack{m_1+\cdots+m_{r-1}=r+t-l-j-1 \\ m_i\geq 1}} \sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+((k_1, \{1\}^{m_1-1}, \ldots, k_r, \{1\}^{s+j}) + \mathbf{e}).\end{aligned}$$

By the integral-series identity and by induction on t, Theorem 1.5 follows.

3. Alternative proof of Theorem 1.1 and proof of Theorem 1.6

3.1. Alternative proof of Theorem 1.1. The derivation relation for MZVs was first proved by Ihara *et al.* [3]. Horikawa *et al.* [2] showed the equivalence of the derivation relation and Theorem 3.2.

Definition 3.1. For $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$, we define Hoffman's dual index of $\mathbf{k}$ by

$$\mathbf{k}^\vee = (\underbrace{1, \ldots, 1}_{k_1} + \underbrace{1, \ldots, 1}_{k_2} + 1, \ldots, 1 + \underbrace{1, \ldots, 1}_{k_r}).$$

Theorem 3.2 (Horikawa *et al.* [2]). *For* $\mathbf{k} \in \mathbb{Z}_{\geq 1}^r$ *and* $l \in \mathbb{Z}_{\geq 0}$,

$$\sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta^+(\mathbf{k} + \mathbf{e}) = \sum_{\substack{|\mathbf{e}'|=l \\ \mathbf{e}'\geq 0}} \zeta^+((\mathbf{k}^\vee + \mathbf{e}')^\vee).$$

In this subsection, we prove Theorem 1.1 by showing that it is equivalent to Theorem 3.2, that is, we will show the following result.

Theorem 3.3. *Theorem 1.1 and Theorem 3.2 are equivalent.*

REMARK 3.4. Tanaka [9] showed that the restricted sum formula in [1] can be written as the linear combination of the derivation relation. Theorem 3.3 is a generalisation of this result.

The implications among ‘Generalised restricted sum formula’, ‘Restricted sum formula’, ‘Ohno-type relation’ and ‘Derivation relation’ for MZVs can be summarised as follows:

Generalised restricted sum formula (Theorem 1.1, Theorem 1.5) $\supset$ Restricted sum formula [1]

$\Updownarrow$ (This paper)

Ohno-type relation ([2], Theorem 3.2)

$\Updownarrow$ [2]

Derivation relation [3]

PROOF OF THEOREM 3.3. The case $r = 1$ is obvious. For $r \geq 2$, the following Lemma 3.5 gives Theorem 3.3. $\square$

We denote the naive shuffle of two indices $(k_1, \dots, k_r)$ and $(l_1, \dots, l_s)$ by

$$(k_1, \dots, k_r) \sqcup\!\sqcup (l_1, \dots, l_s)$$

and we extend ζ linearly. For example, $(3, 1) \sqcup\!\sqcup (2) = (2, 3, 1) + (3, 2, 1) + (3, 1, 2)$. For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{\geq 1}$ with $r \geq 2$ and $u \in \mathbb{Z}_{\geq 0}$, set $\mathbf{k}_u := (k_1, ((k_2, \dots, k_{r-1}) \sqcup\!\sqcup (\{1\}^u)), k_r)$. We also let

$$f_L(\mathbf{k}, t) :=(\text{L.H.S. of Theorem 1.1 for } \mathbf{k}, t), \quad f_R(\mathbf{k}, t) :=(\text{R.H.S. of Theorem 1.1 for } \mathbf{k}, t),$$
$$g_L(\mathbf{k}, t) := \sum_{\substack{|\mathbf{e}|=t \\ \mathbf{e}\geq 0}} \zeta^+(\mathbf{k} + \mathbf{e}), \qquad g_R(\mathbf{k}, t) := \sum_{\substack{|\mathbf{e}'|=t \\ \mathbf{e}'\geq 0}} \zeta^+((\mathbf{k}^\vee + \mathbf{e}')^\vee),$$
$$f(\mathbf{k}, t) := f_L(\mathbf{k}, t) - f_R(\mathbf{k}, t), \qquad g(\mathbf{k}, t) := g_L(\mathbf{k}, t) - g_R(\mathbf{k}, t)$$

and we extend them linearly with respect to the indices. Under these settings, we have the following result.

LEMMA 3.5. *For $\mathbf{k} \in \mathbb{Z}^r_{\geq 1}$ with $r \geq 2$ and $t \in \mathbb{Z}_{\geq 0}$,*

$$f(\mathbf{k}, t) = -\sum_{u=0}^{t} g(\mathbf{k}_u, t - u), \quad g(\mathbf{k}, t) = -\sum_{u=0}^{t} (-1)^u f(\mathbf{k}_u, t - u).$$

PROOF. To prove the first equation, it is sufficient to show $f_R(\mathbf{k}, t) = \sum_{u=0}^{t} g_L(\mathbf{k}_u, t - u)$ and $f_L(\mathbf{k}, t) = \sum_{u=0}^{t} g_R(\mathbf{k}_u, t - u)$. The proof of the former is obvious as follows:

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$$\begin{aligned} f_R(\mathbf{k},t) &= \sum_{l=0}^{t} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+((k_1,((k_2,\dots,k_{r-1}) ш (\{1\}^{t-l})),k_r)+\mathbf{e}) \\ &= \sum_{l=0}^{t} \sum_{\substack{|\mathbf{e}|=l\\ \mathbf{e}\geq 0}} \zeta^+(\mathbf{k}_{t-l}+\mathbf{e}) = \sum_{l=0}^{t} g_L(\mathbf{k}_{t-l},l). \end{aligned}$$

To prove the latter, we denote the m-fold repetition of '1+' (respectively '1,') by $\boxed{1+}^{m}$ (respectively $\boxed{1,}^{m}$), and 1 by $\boxed{1}$. For example,

$$\zeta(\boxed{1+}^{3}\boxed{1,}^{2}\boxed{1+}^{0}\boxed{1}) = \zeta(1+1+1+1,1,1) = \zeta(4,1,1).$$

Then

$$\begin{aligned} g_R(\mathbf{k}_u,t-u) &= \sum_{\substack{|\mathbf{e}'|=t-u\\ \mathbf{e}'\geq 0}} \zeta^+((\mathbf{k}_u^\vee+\mathbf{e}')^\vee) \\ &= \sum_{\substack{|\mathbf{e}'|=t-u\\ \mathbf{e}'\geq 0}} \zeta^+\Big(((k_1,((k_2,\dots,k_{r-1}) ш (\{1\}^{u})),k_r)^\vee+\mathbf{e}')^\vee\Big) \\ &= \sum_{\substack{\alpha_1+\cdots+\alpha_{r-1}=u,\,\alpha_i\geq 0\\ |\mathbf{e}'|=t-u,\,\mathbf{e}'\geq 0}} \zeta^+\Big(((k_1,\{1\}^{\alpha_1},k_2,\{1\}^{\alpha_2},\dots,k_{r-1},\{1\}^{\alpha_{r-1}},k_r)^\vee+\mathbf{e}')^\vee\Big) \\ &= \sum_{\substack{\alpha_1+\cdots+\alpha_{r-1}=u,\,\alpha_i\geq 0\\ |\mathbf{e}'|=t-u,\,\mathbf{e}'\geq 0}} \zeta^+\Big(\Big((\boxed{1+}^{k_1-1}\boxed{1,}^{\alpha_1+1}\boxed{1+}^{k_2-1}\boxed{1,}^{\alpha_2+1}\cdots \\ &\qquad\qquad \cdots\boxed{1+}^{k_{r-1}-1}\boxed{1,}^{\alpha_{r-1}+1}\boxed{1+}^{k_r-1}\boxed{1})^\vee+\mathbf{e}'\Big)^\vee\Big) \\ &= \sum_{\substack{\alpha_1+\cdots+\alpha_{r-1}=u,\,\alpha_i\geq 0\\ |\mathbf{e}'|=t-u,\,\mathbf{e}'\geq 0}} \zeta^+\Big(\Big((\boxed{1,}^{k_1-1}\boxed{1+}^{\alpha_1+1}\boxed{1,}^{k_2-1}\boxed{1+}^{\alpha_2+1}\cdots \\ &\qquad\qquad \cdots\boxed{1,}^{k_{r-1}-1}\boxed{1+}^{\alpha_{r-1}+1}\boxed{1,}^{k_r-1}\boxed{1})+\mathbf{e}'\Big)^\vee\Big) \\ &= \sum_{\substack{\alpha_1+\cdots+\alpha_{r-1}=u\\ e_{1,1}+\cdots+e_{r,k_r-1}=t-u\\ \alpha_i\geq 0,\,e_{i,j}\geq 0}} \zeta^+\Big(\Big(\boxed{1+}^{e_{1,1}}\boxed{1,}\cdots\boxed{1+}^{e_{1,k_1-1}}\boxed{1,}\boxed{1+}^{e_{1,k_1}+1} \\ &\qquad\qquad \boxed{1+}^{\alpha_1}\boxed{1,}\boxed{1+}^{e_{2,1}}\boxed{1,}\cdots\boxed{1+}^{e_{2,k_2-2}}\boxed{1,}\boxed{1+}^{e_{2,k_2-1}+1} \\ &\qquad\qquad \cdots\cdots \\ &\qquad\qquad \boxed{1+}^{\alpha_{r-1}}\boxed{1,}\boxed{1+}^{e_{r,1}}\boxed{1,}\cdots\boxed{1+}^{e_{r,k_r-2}}\boxed{1,}\boxed{1+}^{e_{r,k_r-1}}\boxed{1}\Big)^\vee\Big) \end{aligned}$$

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$$= \sum_{\substack{\alpha_1+\cdots+\alpha_{r-1}=u \\ e_{1,1}+\cdots+e_{r,k_r-1}=t-u \\ \alpha_i\geq 0,\, e_{i,j}\geq 0}} \zeta^+\Big(\boxed{1,}^{e_{1,1}}\boxed{1+}\cdots\boxed{1,}^{e_{1,k_1-1}}\boxed{1+}\boxed{1,}^{e_{1,k_1}+1}$$

$$\boxed{1,}^{\alpha_1}\boxed{1+}\boxed{1,}^{e_{2,1}}\boxed{1+}\cdots\boxed{1,}^{e_{2,k_2-2}}\boxed{1+}\boxed{1,}^{e_{2,k_2-1}+1}$$

$$\cdots\cdots$$

$$\boxed{1,}^{\alpha_{r-1}}\boxed{1+}\boxed{1,}^{e_{r,1}}\boxed{1+}\cdots\boxed{1,}^{e_{r,k_r-2}}\boxed{1+}\boxed{1,}^{e_{r,k_r-1}}\boxed{1}\Big).$$

Taking the sum over $u = 0, \ldots, t$,

$$\sum_{u=0}^{t} g_R(\mathbf{k}_u, t-u) = \sum_{\substack{\alpha_1+\cdots+\alpha_{r-1}+e_{1,1}+\cdots+e_{r,k_r-1}=t \\ \alpha_i\geq 0, e_{i,j}\geq 0}} \zeta^+\Big(\underbrace{\boxed{1,}^{e_{1,1}}\boxed{1+}\cdots\boxed{1,}^{e_{1,k_1-1}}\boxed{1+}\boxed{1,}^{e_{1,k_1}+1}}_{\substack{\text{weight}=e_{1,1}+\cdots+e_{1,k_1}+k_1 \\ \text{depth}=e_{1,1}+\cdots+e_{1,k_1}+1}}$$

$$\underbrace{\boxed{1,}^{\alpha_1}\boxed{1+}\boxed{1,}^{e_{2,1}}\boxed{1+}\cdots\boxed{1,}^{e_{2,k_2-2}}\boxed{1+}\boxed{1,}^{e_{2,k_2-1}+1}}_{\substack{\text{weight}=\alpha_1+e_{2,1}+\cdots+e_{2,k_2-1}+k_2 \\ \text{depth}=\alpha_1+e_{2,1}+\cdots+e_{2,k_2-1}+1}}$$

$$\cdots\cdots$$

$$\underbrace{\boxed{1,}^{\alpha_{r-1}}\boxed{1+}\boxed{1,}^{e_{r,1}}\boxed{1+}\cdots\boxed{1,}^{e_{r,k_r-2}}\boxed{1+}\boxed{1,}^{e_{r,k_r-1}}\boxed{1}}_{\substack{\text{weight}=\alpha_{r-1}+e_{r,1}+\cdots+e_{r,k_r-1}+k_r \\ \text{depth}=\alpha_{r-1}+e_{r,1}+\cdots+e_{r,k_r-1}+1}}\Big)$$

$$= \sum_{\substack{m_1+\cdots+m_r=r+t \\ m_i\geq 1\,(1\leq i\leq r)}} \sum_{\substack{|\mathbf{a}_{m_i}|=k_i+m_i-1 \\ (1\leq i\leq r)}} \zeta^+(\mathbf{a}_{m_1}, \ldots, \mathbf{a}_{m_r}) = f_L(\mathbf{k}, t).$$

We assume the first equation in the lemma and prove the second by induction on t. The case $t = 0$ is clear. Let $t > 0$ and assume $g(\mathbf{k}, t') = -\sum_{u=0}^{t'}(-1)^u f(\mathbf{k}_u, t'-u)$ for all integers t' with $0 \leq t' < t$. From the first equation,

$$g(\mathbf{k}, t) = -f(\mathbf{k}, t) - \sum_{u=1}^{t} g(\mathbf{k}_u, t-u) = -f(\mathbf{k}, t) + \sum_{u=1}^{t}\sum_{u'=0}^{t-u}(-1)^{u'} f((\mathbf{k}_u)_{u'}, t-u-u').$$

Since $(\mathbf{k}_u)_{u'} = \binom{u+u'}{u}\mathbf{k}_{u+u'}$, by writing $v = u + u'$,

$$\begin{aligned} g(\mathbf{k}, t) &= -f(\mathbf{k}, t) + \sum_{u=1}^{t}\sum_{v=u}^{t}(-1)^{v-u}\binom{v}{u} f(\mathbf{k}_v, t-v) \\ &= -f(\mathbf{k}, t) + \sum_{v=1}^{t}(-1)^v \sum_{u=1}^{v}(-1)^u\binom{v}{u} f(\mathbf{k}_v, t-v) \\ &= -f(\mathbf{k}, t) - \sum_{v=1}^{t}(-1)^v f(\mathbf{k}_v, t-v) = -\sum_{u=0}^{t}(-1)^u f(\mathbf{k}_u, t-u). \end{aligned}$$

This completes the proof. □

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3.2. Proof of Theorem 1.6. The following theorem is called an Ohno-type relation for FMZVs. It was conjectured by Kaneko [4] and proved by Oyama [8].

THEOREM 3.6 (Oyama [8]). *For* $\mathbf{k} \in \mathbb{Z}_{\geq 1}^r$ *and* $l \in \mathbb{Z}_{\geq 0}$,

$$\sum_{\substack{|\mathbf{e}|=l \\ \mathbf{e}\geq 0}} \zeta_{\mathcal{F}}(\mathbf{k}+\mathbf{e}) = \sum_{\substack{|\mathbf{e}'|=l \\ \mathbf{e}'\geq 0}} \zeta_{\mathcal{F}}((\mathbf{k}^\vee+\mathbf{e}')^\vee).$$

REMARK 3.7. The derivation relation for FMZVs was conjectured by Oyama and proved by the second author [7]. Horikawa *et al.* [2] showed the equivalence of the derivation relation and the above theorem for FMZVs.

By Theorem 3.6, we can prove Theorem 1.6 in exactly the same manner as in the previous subsection. The relations among 'Generalised restricted sum formula', 'Ohno-type relation' and 'Derivation relation' for FMZVs can be summarised as follows.

$$\boxed{\text{Generalised restricted sum formula (Theorem 1.6)}}$$
$$\Updownarrow \text{ (This paper)}$$
$$\boxed{\text{Ohno-type relation ([8], Theorem 3.6)}}$$
$$\Updownarrow \text{ [2]}$$
$$\boxed{\text{Derivation relation [7]}}$$

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