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Geodesics on the unit tangent bundle

J. Berndt
Department of Mathematics, University of Hull,
Hull HU6 7RX, UK (j.berndt@maths.hull.ac.uk)

E. Boeckx
Department of Mathematics, Katholieke Universiteit Leuven,
Celestijnenlaan 200 B, 3001 Leuven, Belgium
(eric.boeckx@wis.kuleuven.ac.be)

P. T. Nagy
Institute of Mathematics and Informatics, University of Debrecen,
PO Box 12, H-4010, Hungary (nagypeti@math.klte.hu)

L. Vanhecke
Department of Mathematics, Katholieke Universiteit Leuven,
Celestijnenlaan 200 B, 3001 Leuven, Belgium
(lieven.vanhecke@wis.kuleuven.ac.be)

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A geodesic γ on the unit tangent sphere bundle T_1M of a Riemannian manifold (M,g), equipped with the Sasaki metric g_S, can be considered as a curve x on M together with a unit vector field V along it. We study the curves x. In particular, we investigate for which manifolds (M,g) all these curves have constant first curvature κ_1 or have vanishing curvature κ_i for some $i=1,2$ or 3.

1. Introduction

Let (M,g) be a Riemannian manifold and (T_1M,g_S) its unit tangent sphere bundle equipped with the Sasaki metric. Special geometric properties of the base manifold (M,g) will be reflected in the geometry of (T_1M,g_S) and, conversely, the geometry of (T_1M,g_S) will influence, or even determine, that of (M,g). This (indirect) approach to the study of the geometry of (M,g) has been successfully exploited by the second and the fourth author in a series of papers where they study the curvature of (T_1M,g_S) in relation to that of (M,g) (see the survey [12] for an overview of their results and for further references).

The study of the tangent bundle TM and the unit tangent sphere bundle T_1M as Riemannian manifolds was initiated in the late fifties and early sixties by Sasaki [34, 35]. He introduced a rather simple Riemannian metric g_S on these bundles, now known as the *Sasaki metric*, which is completely determined by the metric structure g on the base manifold M. Every geodesic γ on (TM,g_S) (or on (T_1M,g_S)) can be considered as a curve x on M with a (unit) vector field V along it: $\gamma=(x,V)$.

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Sasaki derived the geodesic equation for γ in terms of x and V and proved some first results concerning the geodesics of (TM, g_S) (or (T_1M, g_S)).

About a decade later, Chavel constructed Riemannian metrics on the three-dimensional real projective space $\mathbb{R}P^3$, which is diffeomorphic to the unit tangent sphere bundle of the two-sphere S^2. These metrics are precisely the Sasaki metrics associated to *arbitrary* Riemannian metrics on S^2. He investigated some global properties of the geodesics of $\mathbb{R}P^3$ with these metrics [15]. Klingenberg and Sasaki, on the other hand, gave an explicit description of the curves x and the unit vector fields V along x, which together form a geodesic of (T_1M, g_S) in the case when (M, g) is S^2 with its *standard* metric [24]. Using the same methods, Sasaki then extended these results to base manifolds that are spaces of constant curvature of any dimension [36]. Later, Gluck also determined the geodesics of the unit tangent sphere bundle of a round sphere of any dimension in a very nice paper, using a completely different approach [21].

For general base manifolds, it is much harder to obtain an explicit description for all geodesics on the unit tangent sphere bundle. Only in the case when the base manifold is two dimensional do we have a complete description, due to the third author in [28]. In later work, he proved some interesting results for locally symmetric base spaces [29], obtaining also Sasaki's classification for the geodesics of the tangent bundle of a space of constant curvature.

The inspiration for the present article comes from two results about the curves x on M that are projections of geodesics $\gamma = (x, V)$ on (T_1M, g_S). The first result, theorem 2.1 in this paper, is due to the third author [29] and says that the curve x has constant curvatures if the base space (M, g) is locally symmetric. The second, theorem 4.1, is due to Sasaki and states that the curve x has vanishing curvatures $\kappa_3, \kappa_4, \dots, \kappa_{n-1}$ if the base space (M, g) has constant curvature.

Both results are *direct* results: from special properties of the base manifold, one deduces information about the geometry (in casu, the geodesics) of the unit tangent sphere bundle. Now we are interested in possible converses, in *indirect* results. Therefore, in § 3, we consider Riemannian manifolds (M, g) such that all the projected geodesics of (T_1M, g_S) have constant first curvature. We cannot deduce from this that (M, g) must be locally symmetric, but we do obtain some interesting partial answers. In § 4, we investigate those Riemannian manifolds (M, g) for which all the projected geodesics of (T_1M, g_S) have vanishing first, second or third curvature.

2. Geodesics on T_1M and the projected curves

Let (M, g) be an n-dimensional Riemannian manifold, which we suppose to be smooth and connected. By ∇ we denote its Levi-Civita connection and by R its Riemann curvature tensor, where we use the sign convention

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Any curve $\gamma(t) = (x(t), V(t))$ in the unit tangent sphere bundle can be considered as a curve $x(t)$ in the base manifold M together with a unit vector field $V(t)$ along it. If we equip T_1M with the Sasaki metric g_S, then a curve $\gamma(t) = (x(t), V(t))$ is a geodesic of (T_1M, g_S) if and only if

$$\left.\begin{aligned} \nabla_{\dot{x}}\dot{x} &= -R(V, \nabla_{\dot{x}}V)\dot{x}, \\ \nabla_{\dot{x}}\nabla_{\dot{x}}V &= -c^2V, \end{aligned}\right\} \tag{2.1}$$

where $c^2 = g(\nabla_{\dot{x}}V, \nabla_{\dot{x}}V)$ is a constant along $x(t)$. (see, for example, [35]). For a precise definition of the Sasaki metric g_S and for a convenient mechanism for computing on T_1M, the reader may consult [11].

In what follows, we are interested in the projections on the base manifold M of the geodesics of (T_1M, g_S), i.e. in the curves $x(t) = (\pi \circ \gamma)(t)$ satisfying (2.1) for some unit vector field $V(t)$ along $x(t)$. At a point $p \in M$ and for given tangent vectors v, w and y at p such that $|v| = 1$ and $g_p(v, w) = 0$, there is a unique curve $x(t)$ satisfying (2.1) with $x(0) = p$, $\dot{x}(0) = y$, $V(0) = v$ and $\nabla_y V = w$. This curve is the projection of the unique geodesic $\gamma(t)$ of (T_1M, g_S) with initial conditions $\gamma(0) = (p, v)$ and $\dot{\gamma}(0) = y^h + w^v$. We can consider three types of geodesics.

(i) If $y = 0$, we obtain a great circle on the fibre $\pi^{-1}(p) = S^{n-1}$ and $x(t) = p$: a *vertical* geodesic.

(ii) If $w = 0$, we obtain a curve $\gamma(t) = (x(t), V(t))$, where $x(t)$ is a geodesic of (M, g) with a parallel vector field V along it, obtained by parallel translation of v along $x(t)$: a *horizontal* geodesic.

(iii) If both y and w are non-zero, we obtain a geodesic of *oblique* type. It is this family of geodesics (and their projections) that we are most interested in. For spaces of constant curvature, these were described explicitly in [36].

Now we consider a curve $x(t)$ satisfying (2.1), where $\dot{x} \neq 0$ and $\nabla_{\dot{x}}V \neq 0$. Let $\gamma(t) = (x(t), V(t))$ be the corresponding geodesic of (T_1M, g_S). As both

$$|\dot{\gamma}|^2 = |\dot{x}|^2 + |\nabla_{\dot{x}}V|^2 \quad \text{and} \quad |\nabla_{\dot{x}}V|^2 = c^2$$

are constant, we can reparametrize $\gamma(t)$ (and $x(t)$) so that $|\dot{x}| = 1$. Hence we can take $T = \dot{x}$ as the first vector in the Frenet frame $\{T, N_1, \ldots, N_{n-1}\}$ along x and we have, for the first three covariant derivatives of $\dot{x}$,

$$\left.\begin{aligned} \dot{x}^{(1)} &= \nabla_{\dot{x}}\dot{x} = \kappa_1 N_1, \\ \dot{x}^{(2)} &= \nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} = -{\kappa_1}^2 T + \kappa_1' N_1 + \kappa_1\kappa_2 N_2, \\ \dot{x}^{(3)} &= \nabla_{\dot{x}}\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} \\ &= -3\kappa_1\kappa_1' T + (\kappa_1'' - \kappa_1({\kappa_1}^2 + {\kappa_2}^2))N_1 \\ &\qquad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')N_2 + \kappa_1\kappa_2\kappa_3 N_3. \end{aligned}\right\} \tag{2.2}$$

On the other hand, using (2.1), we can calculate

$$\left.\begin{aligned} \dot{x}^{(1)} &= -R(V, \dot{V})\dot{x}, \\ \dot{x}^{(2)} &= -(\nabla_{\dot{x}}R)(V, \dot{V})\dot{x} + R(V, \dot{V})^2\dot{x}, \\ \dot{x}^{(3)} &= -(\nabla^{(2)}_{\dot{x}\dot{x}}R)(V, \dot{V})\dot{x} + (\nabla_{R(V,\dot{V})\dot{x}}R)(V, \dot{V})\dot{x} \\ &\qquad + 2(\nabla_{\dot{x}}R)(V, \dot{V})R(V, \dot{V})\dot{x} + R(V, \dot{V})(\nabla_{\dot{x}}R)(V, \dot{V})\dot{x} - R(V, \dot{V})^3\dot{x}, \end{aligned}\right\} \tag{2.3}$$

where we have put $\dot{V} = \nabla_{\dot{x}}V$ for simplicity.

Comparing (2.2) and (2.3), we find explicit expressions for the curvatures κ_1, κ_2 and κ_3 of $x(t)$, as well as for the normals N_1, N_2 and N_3 (whenever they are defined) in terms of $\dot{x}$ and V. We will use these in the following sections. For future use, we recall, from [29], the following result.

THEOREM 2.1. *Let (M, g) be a locally symmetric space and γ a geodesic of the unit tangent bundle (T_1M, g_S). Then the projected curve $x = \pi \circ \gamma$ has constant curvatures.*

Indeed, for a symmetric space, formulae (2.3) reduce to

$$\dot{x}^{(k)} = (-1)^k R(V, \dot{V})^k \dot{x}.$$

It is easy to see from this formula that $\dot{x}^{(k)}$ has constant length for all k. Using this and the corresponding formulae (2.2) for arbitrary $\dot{x}^{(k)}$, $k = 1, \dots, n-1$, one proves by induction that all curvatures κ_i are constant.

3. Projected geodesics with constant first curvature

In this section we investigate whether the converse of theorem 2.1 holds true. Expressing that the curvatures κ_i of the projected geodesics are constant quickly becomes rather complicated and of little practical use. For now, therefore, we only consider the case when the first curvature κ_1 is constant. We are not able to deduce from this the local symmetry of the manifold (M, g) in the generic case, but we do obtain a number of partial results.

We begin by relating the constancy of the first curvature κ_1 to a curvature condition on (M, g).

PROPOSITION 3.1. *Let (M, g) be a Riemannian manifold. Then, for any geodesic γ of (T_1M, g_S), the projected curve $x = \pi \circ \gamma$ has constant first curvature κ_1 if and only if the curvature condition*

$$g((\nabla_Y R)(V, W)Y, R(V, W)Y) = 0 \tag{3.1}$$

is satisfied for all vector fields Y, V and W on M.

Proof. Comparing the expressions for $\dot{x}^{(1)}$ in (2.2) and (2.3), we see that the first curvature κ_1 is given by

$$\kappa_1^2 = g(R(V, \dot{V})\dot{x}, R(V, \dot{V})\dot{x}),$$

and hence, using (2.1), we have

$$\begin{aligned}\frac{\mathrm{d}\kappa_1^2}{\mathrm{d}t} &= 2g((\nabla_{\dot{x}} R)(V, \dot{V})\dot{x} - R(V, \dot{V})^2\dot{x}, R(V, \dot{V})\dot{x}) \\ &= 2g((\nabla_{\dot{x}} R)(V, \dot{V})\dot{x}, R(V, \dot{V})\dot{x}).\end{aligned}$$

At $t = 0$, we have $\mathrm{d}\kappa_1^2/\mathrm{d}t = 2g((\nabla_y R)(v, w)y, R(v, w)y)$. As $y, v, w \in T_pM$ can be chosen arbitrarily, subject to the conditions $|v| = 1$ and $g_p(v, w) = 0$, the constancy of κ_1 for all projected geodesics clearly implies (3.1). The converse is immediate from the expression for $\mathrm{d}\kappa_1^2/\mathrm{d}t$ above. □

As an immediate corollary, we see that it suffices to deal with locally irreducible spaces.

Corollary 3.2. *Let* (M, g) *be a (locally) reducible Riemannian manifold. Then any geodesic of* (T_1M, g_S) *projects down to a curve with constant first curvature* κ_1 *if and only if the same property holds for each of the factors.*

We would like to prove the converse of theorem 2.1 or, even stronger, to show that the condition 'κ_1 is constant for every projected geodesic' already implies local symmetry. As a first step in that direction, we have the following result.

Proposition 3.3. *Let* (M, g) *be a Riemannian manifold and assume that the curvature condition (3.1) is fulfilled. Then* (M, g) *is a* $\mathfrak{C}$*-space.*

We recall that a $\mathfrak{C}$-*space* is a Riemannian manifold (M, g) such that, for any geodesic σ in (M, g), the eigenvalues of the Jacobi operator $R_\sigma := R(\cdot, \dot\sigma)\dot\sigma$ are constant along σ. Any locally symmetric space is a $\mathfrak{C}$-space, but the converse does not hold (see [2] and [26] for examples and more information).

Proof. Let σ be a geodesic in (M, g). On a dense open subset of σ, one can choose smooth eigenfunctions λ_i, $i = 1, \dots, n$, for R_σ and smooth unit vector fields E_i, $i = 1, \dots, n$, such that $R_\sigma E_i = \lambda_i E_i$. Now, put $Y = W = \dot\sigma$ and $V = E_i$ in condition (3.1). Then we find

$$\begin{aligned} 0 &= g((\nabla_{\dot\sigma} R_\sigma)E_i, R_\sigma E_i) \\ &= g(\nabla_{\dot\sigma}(\lambda_i E_i) - R_\sigma(\nabla_{\dot\sigma} E_i), \lambda_i E_i) \\ &= \lambda_i \left(\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} - \lambda_i g(\nabla_{\dot\sigma} E_i, E_i) - g(R_\sigma E_i, \nabla_{\dot\sigma} E_i) \right) \\ &= \frac{1}{2} \frac{\mathrm{d}\lambda_i^2}{\mathrm{d}t}. \end{aligned}$$

So the eigenvalues of R_σ are constant along σ and (M, g) is a $\mathfrak{C}$-space. □

Remark 3.4. Any $\mathfrak{C}$-space has a cyclic-parallel Ricci tensor [2],

$$\mathrm{L}_3 : (\nabla_X \rho)(X, X) = 0.$$

Moreover, from (3.1), it follows immediately,

$$\mathrm{L}_5 : \sum_{i=1}^{n} g((\nabla_X R)(E_i, X)X, R(E_i, X)X) = 0,$$

where $\{E_1, \dots, E_n\}$ is a local orthonormal frame. These conditions are known as the odd Ledger conditions L_3 and L_5. They appear in the study of *D'Atri spaces* (i.e. spaces with volume-preserving geodesic symmetries up to sign) and, in particular, of harmonic spaces. For that reason, they have been studied extensively (see [26] for more information).

Corollary 3.5. *Let* (M, g) *be a Riemannian manifold such that, for any geodesic* γ *of* (T_1M, g_S)*, the projected curve* $x = \pi \circ \gamma$ *has constant first curvature* κ_1*. Then* (M, g) *is locally symmetric under each of the following additional assumptions.*

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(a) *The dimension of M is two or three.*

(b) *The dimension of M is four and (M, g) is a Hermitian Einstein, or a 2-stein, or a Kähler space.*

(c) *(M, g) is a four-dimensional locally homogeneous space.*

(d) *(M, g) is semi-symmetric.*

(e) *(M, g) is conformally flat.*

(f) *(M, g) is a Damek–Ricci space or, more generally, (M, g) is an irreducible non-flat homogeneous space with non-positive curvature and algebraic rank one.*

Proof. (a) In [2], the classification of two- and three-dimensional $\mathfrak{C}$-spaces is given. A two-dimensional $\mathfrak{C}$-space has constant curvature, and hence is locally symmetric. Any three-dimensional $\mathfrak{C}$-space is locally isometric to a naturally reductive space. Hence, by a result in [41] and [25], it is either locally symmetric or locally isometric to

(i) $SU(2)$ with a special left-invariant metric;

(ii) $SL(2, \mathbb{R})$ with a special left-invariant metric;

(iii) the three-dimensional Heisenberg group with any left-invariant metric.

Each of these can be equipped with a Sasakian structure compatible with the metric, possibly up to a homothety, which leaves the condition (3.1) invariant (see, for example, [6]). The result then follows from proposition 3.6 further on.

(b) A four-dimensional Hermitian Einstein space, which is at the same time a $\mathfrak{C}$-space, is locally symmetric (see [13, corollary 30]).

For a four-dimensional manifold, the property of being 2-stein is equivalent to being *pointwise Osserman* [20,37], i.e. at each point p, the eigenvalues of the Jacobi operators R_y do not depend on the choice of unit vector $y \in T_pM$, though they may depend on p. Combining this with the $\mathfrak{C}$-property, we see that the manifold must be *globally Osserman*, i.e. the eigenvalues of R_y are independent of both the unit vector $y \in T_pM$ and $p \in M$. But then, using a result of Chi in [17], the manifold is either flat or locally isometric to a rank-one symmetric space.

A Kähler manifold whose Ricci tensor is cyclic-parallel has parallel Ricci tensor (see [38]), and hence it is either Einstein or a local product of Einstein spaces. If we start from a four-dimensional Kähler manifold, which is also a $\mathfrak{C}$-space, it is either (Hermitian) Einstein or a local product of Einstein spaces of dimensions at most three. In both cases, the manifold must be locally symmetric.

(c) Four-dimensional locally homogeneous spaces with cyclic-parallel Ricci tensor have been completely classified in [14], after partial results in [33]. We distinguish three cases.

(i) If all Ricci eigenvalues are equal, the manifold is Einstein. From a theorem of Jensen [23], the manifold must be locally symmetric.

(ii) If at most three Ricci eigenvalues are distinct, then three cases can occur.

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(a) (M^4, g) is locally symmetric.

(b) (M^4, g) is a local product of $\mathbb{R}$ and a specific three-dimensional manifold. If we impose the condition (3.1), then, from corollaries 3.2 and 3.5 (a), we find that (M^4, g) must again be locally symmetric.

(c) (M^4, g) is locally isometric to a specific four-dimensional Lie group. However, this Lie group does not satisfy condition (3.1), since, as pointed out in [14, p. 134], it does not even satisfy the weaker Ledger condition L_5.

(iii) If all four Ricci eigenvalues are distinct, then the resulting spaces do not satisfy the Ledger condition L_5 either (see again [14, p. 134]).

(d) A semi-symmetric space which is also a $\mathfrak{C}$-space is locally symmetric (see [8]).

(e) As the Ricci tensor ρ of a $\mathfrak{C}$-space is cyclic-parallel, its scalar curvature τ is constant. As (M, g) is also conformally flat, it follows that the Ricci tensor is a Codazzi tensor, i.e. $(\nabla_X \rho)(Y, Z) = (\nabla_Y \rho)(X, Z)$. Hence the Ricci tensor is parallel and the same holds for the Riemann curvature tensor R.

(f) In both cases, the $\mathfrak{C}$-space property implies symmetry (see [3, 18]). □

To finish the above proof, we still need the following result.

PROPOSITION 3.6. *Let (M, g) be a Sasakian space and suppose that, for any geodesic γ of (T_1M, g_S), the projected curve $x = \pi \circ \gamma$ has constant first curvature κ_1. Then (M, g) has constant curvature* 1.

Proof. We refer to [5] for the basic definitions and formulae in contact and Sasakian geometry. Here we recall only that a Sasakian space (M, g) is equipped with a unit Killing vector field ξ such that $R(X, Y)\xi = g(\xi, Y)X - g(\xi, X)Y$. The $(1, 1)$-tensor field φ given by $\varphi X = -\nabla_X \xi$ satisfies $\varphi^2 = -\operatorname{Id} + \eta \otimes \xi$, where η is the one-form dual to ξ. Then

$$
\begin{aligned}
(\nabla_Y R)(\xi, W)Y &= \nabla_Y (g(Y, W)\xi - \eta(Y)W) - R(\nabla_Y \xi, W)Y \\
&= -g(Y, W)\varphi Y + g(\varphi Y, Y)W + R(\varphi Y, W)Y \\
&= -g(Y, W)\varphi Y + R(\varphi Y, W)Y.
\end{aligned}
$$

The condition (3.1) for $V = \xi$ then yields

$$
\begin{aligned}
0 &= g((\nabla_Y R)(\xi, W)Y, R(\xi, W)Y) \\
&= -g(Y, W)g(R(\xi, W)Y, \varphi Y) + g(R(\varphi Y, W)Y, R(\xi, W)Y) \\
&= g(Y, W)\eta(Y)g(W, \varphi Y) + g(Y, W)g(R(\varphi Y, W)Y, \xi) - \eta(Y)g(R(\varphi Y, W)Y, W) \\
&= \eta(Y)(g(Y, W)g(\varphi Y, W) - g(R(\varphi Y, W)Y, W)).
\end{aligned}
$$

Now, take a unit vector W orthogonal to ξ and put $Y = \xi + W + \varphi W$. Then the above equality reduces to

$$
0 = -1 + g(R(\varphi W, W)W, \varphi W).
$$

So (M, g) is a Sasakian space with constant φ-sectional curvature 1, and hence is locally isometric to a unit sphere. □

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REMARK 3.7. Sasakian spaces belong to the larger class of *contact metric* (k,μ)-*spaces* defined in [7]. These contact metric spaces are characterized by the curvature condition

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{3.2}$$

for all vector fields X and Y, where $h = (\frac{1}{2})\mathcal{L}_\xi\varphi$, $\mathcal{L}_\xi$ being the Lie derivative in the direction of ξ. The geometry of these spaces has been studied in [7] and [9], and the non-Sasakian (k,μ)-spaces have been explicitly classified in [10]. In particular, the curvature tensor R and its covariant derivatives are completely determined by condition (3.2). By a straightforward calculation using the explicit expressions for R and ∇R, one can verify that a non-Sasakian contact metric (k,μ)-space of dimension $2n+1$ satisfying (3.1) has $k=\mu=0$ and, hence, is locally isometric to $\mathbb{R}^{n+1}\times S^n(4)$. In particular, it is locally symmetric.

From the condition (3.1), we can deduce a second condition on the Jacobi operators.

PROPOSITION 3.8. *Let (M,g) be a Riemannian manifold and suppose that, for any geodesic γ of (T_1M, g_S), the projected curve $x=\pi\circ\gamma$ has constant first curvature κ_1. Then, for any vector field Y, it holds that $R'_Y\circ R_Y + R_Y\circ R'_Y = 0$, where $R_Y = R(\cdot,Y)Y$ and $R'_Y = (\nabla_Y R)(\cdot,Y)Y$, or, equivalently, along any geodesic σ of (M,g) the operator R^2_σ is parallel.*

Proof. Put $W=Y$ in (3.1) and polarize with respect to V. In this way, we get

$$0 = g(R'_YU, R_YV) + g(R'_YV, R_YU) = g((R_Y\circ R'_Y + R'_Y\circ R_Y)U, V)$$

and the proposition follows. □

COROLLARY 3.9. *Let (M,g) be a Riemannian manifold and suppose that, for any geodesic γ of (T_1M,g_S), the projected curve $x=\pi\circ\gamma$ has constant first curvature κ_1. If the sectional curvature K of (M,g) is either strictly positive or strictly negative, then (M,g) is locally symmetric.*

Proof. Take any unit vector y and a unit vector v orthogonal to y such that $R_yv = \lambda v$. Then $\lambda = K(y,v)$ and, from the assumption about the sectional curvature of (M,g), it follows that λ is non-zero. Furthermore, we have

$$R_y(R'_yv) = -R'_y(R_yv) = -\lambda R'_yv.$$

If R'_yv is non-zero, then $K(y,R'_yv) = -\lambda$. But this is impossible, since all sectional curvatures must have the same sign. Hence $R'_yv = 0$ for all y and v as above, and we must have $\nabla R = 0$. □

4. Projected geodesics with vanishing curvature κ_i

The starting point for this section is the following result by Sasaki [36].

THEOREM 4.1. *Let (M,g) be a space of constant curvature. Then the projection $x=\pi\circ\gamma$ of any geodesic γ of (T_1M,g_S) has constant curvatures κ_1 and κ_2 and vanishing third curvature κ_3.*

REMARK 4.2. In [30, theorem 4], it was claimed, incorrectly, that this property holds for the larger class of two-point homogeneous spaces (see also the proof of theorem 4.17 further on).

In what follows, we investigate what spaces (M, g) are such that all projected geodesics of (T_1M, g_S) have vanishing curvature κ_i for some $i = 1, 2, 3$.

4.1. $\kappa_1 \equiv 0$

Comparing (2.2) and (2.3), we see that $\kappa_1 N_1 = -R(V, \dot{V})\dot{x}$. Hence we have the following result.

PROPOSITION 4.3. *Let (M, g) be a Riemannian space for which every projected geodesic of (T_1M, g_S) is a geodesic of (M, g). Then (M, g) is flat.*

4.2. $\kappa_2 \equiv 0$

Next, take a non-flat space (M, g) and a geodesic $\gamma(t) = (x(t), V(t))$ of (T_1M, g_S) for which the projected curve $x(t)$ has non-zero first curvature. Suppose that $\kappa_2 \equiv 0$. Then we get from (2.2) and (2.3)

$$(\nabla_{\dot{x}}R)(V, \dot{V})\dot{x} - \frac{\kappa_1'}{\kappa_1}R(V, \dot{V})\dot{x} = R(V, \dot{V})^2\dot{x} + {\kappa_1}^2\dot{x} \tag{4.1}$$

or, equivalently,

$$\begin{aligned}|R(V, \dot{V})\dot{x}|^2(\nabla_{\dot{x}}R)(V, \dot{V})\dot{x} - g((\nabla_{\dot{x}}R)(V, \dot{V})\dot{x}, R(V, \dot{V})\dot{x})R(V, \dot{V})\dot{x} \\ = |R(V, \dot{V})\dot{x}|^2(R(V, \dot{V})^2\dot{x} + |R(V, \dot{V})\dot{x}|^2\dot{x}).\end{aligned} \tag{4.2}$$

Note that this last expression also holds for geodesics $\gamma(t) = (x(t), V(t))$ such that $x(t)$ has $\kappa_1 = 0$.

PROPOSITION 4.4. *Let (M, g) be a Riemannian space. Then any geodesic γ of (T_1M, g_S) projects to a curve x of M for which $\kappa_2 \equiv 0$ if and only if*

$$R(V, W)^2Y = -|R(V, W)Y|^2Y, \tag{4.3}$$

$$|R(V, W)Y|^2(\nabla_Y R)(V, W)Y = g((\nabla_Y R)(V, W)Y, R(V, W)Y)R(V, W)Y \tag{4.4}$$

for all vector fields V, W and Y on M with $|Y| = 1$.

Proof. Put $t = 0$ in (4.2) to obtain

$$\begin{aligned}|R(v, w)y|^2(\nabla_y R)(v, w)y - g((\nabla_y R)(v, w)y, R(v, w)y)R(v, w)y \\ = |R(v, w)y|^2(R(v, w)^2y + |R(v, w)y|^2y).\end{aligned}$$

Because we suppose that $\kappa_2 \equiv 0$ for any projected geodesic, we can take y, v and w arbitrarily, as long as $|y| = 1$. Replacing y by $-y$ and comparing the result with the above, we obtain the conditions (4.3) and (4.4). □

Using the condition (4.3), we can deal at once with locally reducible spaces and odd-dimensional manifolds.

Proposition 4.5. *Let (M, g) be a locally reducible manifold and suppose that condition (4.3) holds. Then (M, g) is flat.*

Proof. Let v and w be arbitrary tangent vectors at $p \in M$. Condition (4.3) then says that *any* tangent vector y at p is an eigenvector of $R(v, w)^2$. Hence

$$R(v, w)^2 = -\lambda^2 \,\mathrm{Id}, \quad \text{where } \lambda \text{ depends only on } v \text{ and } w. \tag{4.5}$$

Moreover, if $\lambda = 0$, then $R(v, w) = 0$ by the skew-symmetry of $R(v, w)$.

Now decompose (M, g) in factors $(M, g) = (M_1, g_1) \times (M_2, g_2)$. Take any v_1, w_1 tangent to M_1 and y_2 tangent to M_2. Obviously, $R(v_1, w_1)^2 y_2 = 0$, and hence $\lambda(v_1, w_1) = 0$ and $R(v_1, w_1) = R_1(v_1, w_1) = 0$. Consequently, (M_1, g_1) must be flat. Similarly, (M_2, g_2) is flat. □

Proposition 4.6. *Let (M, g) be an odd-dimensional manifold and suppose that condition (4.3) holds. Then (M, g) is flat.*

Proof. Take arbitrary tangent vectors v and w to M at a point p. Because $R(v, w)$ is a skew-symmetric operator on an odd-dimensional vector space, there exists a non-zero tangent vector y at p such that $R(v, w)y = 0$. Hence, from (4.5), it follows that $\lambda(v, w) = 0$ and also $R(v, w) = 0$. As v and w were arbitrary, as well as the point p, we have $R \equiv 0$. □

Condition (4.4) implies a nice geometric property.

Proposition 4.7. *Let (M, g) be a Riemannian space satisfying condition (4.4). Then (M, g) is a $\mathfrak{P}$-space.*

We recall that a $\mathfrak{P}$-*space* is a Riemannian manifold (M, g) such that, for any geodesic σ in (M, g), the eigenspaces of the Jacobi operator R_σ are parallel along σ. Again, any locally symmetric space is a $\mathfrak{P}$-space, but many examples exist that are not locally symmetric. Actually, the locally symmetric spaces are precisely those Riemannian manifolds that are at the same time $\mathfrak{C}$- and $\mathfrak{P}$-spaces. (Again, we refer to [2] for the proof of this statement and for further information and examples.)

Proof. Let σ be a geodesic in (M, g) and let λ be a smooth non-zero eigenfunction for R_σ, defined on an open dense subset of σ. Choose a corresponding unit eigenvector field V: $R_\sigma V = \lambda V$. If we put $Y = W = \dot\sigma$ in (4.4), we can write

$$R'_\sigma V = \alpha R_\sigma V = \alpha\lambda V,$$

where α is some function along σ which we need not specify. On the other hand, the left-hand side of this equation can be rewritten as

$$R'_\sigma V = \nabla_{\dot\sigma}(R_\sigma V) - R_\sigma(\nabla_{\dot\sigma} V) = \frac{\mathrm{d}\lambda}{\mathrm{d}t} V + \lambda \nabla_{\dot\sigma} V - R_\sigma(\nabla_{\dot\sigma} V).$$

Combining these last two formulae, we obtain

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t} - \alpha\lambda\right) V + (\lambda \nabla_{\dot\sigma} V - R_\sigma(\nabla_{\dot\sigma} V)) = 0.$$

Taking the interior product with V, we get that $\mathrm{d}\lambda/\mathrm{d}t = \alpha\lambda$ and, consequently, $R_\sigma(\nabla_{\dot\sigma} V) = \lambda \nabla_{\dot\sigma} V$. This means that $\nabla_{\dot\sigma} V$ is also an eigenvector field corresponding to the eigenfunction λ, so this eigenspace is parallel along σ. Since this holds for all the eigenspaces corresponding to non-zero eigenfunctions, it must also be true for the eigenspace corresponding to the zero eigenfunction. Hence (M, g) is a $\mathfrak{P}$-space. □

Combining proposition 4.7 with proposition 3.3, we get the following.

PROPOSITION 4.8. *Let (M, g) be a Riemannian space and suppose that any geodesic γ of (T_1M, g_S) projects to a curve x of M with* constant κ_1 *and vanishing κ_2. Then (M, g) is locally symmetric.*

REMARK 4.9. In [31], Nomizu and Yano defined a *circle* in a general Riemannian manifold (M, g) as a curve with constant non-vanishing first curvature κ_1 and vanishing second curvature κ_2. The projected curves x in proposition 4.8 can therefore be described more geometrically as being either geodesics ($\kappa_1 = 0$) or circles ($\kappa_1 \neq 0$).

In the rest of this subsection, we consider locally symmetric spaces (M, g) for which all projected geodesics of (T_1M, g_S) are curves with vanishing second curvature κ_2, i.e. for which the curvature equality (4.3) holds. Clearly, all two-dimensional locally symmetric spaces satisfy this requirement. For higher-dimensional locally symmetric spaces, we expect to find only flat spaces. This is based partly on the results about reducible manifolds and odd-dimensional spaces above, but also on the explicit description of the geodesics of the unit tangent sphere bundles of spaces of constant curvature in [36], in view of the fact that these spaces are the 'simplest' locally symmetric manifolds. As a first step, we show that the curvature condition (4.3) has profound implications for the rank of a symmetric space.

PROPOSITION 4.10. *Let (M, g) be an irreducible locally symmetric space and suppose that (4.3) holds. Then the rank of its universal covering equals one.*

Proof. Let $(\tilde{M}, \tilde{g})$ be the connected and simply connected globally symmetric space that is locally isometric to (M, g). If (M, g) satisfies (4.3), then the same holds for $(\tilde{M}, \tilde{g})$. We can consider $\tilde{M}$ as the quotient manifold G/H, where $G = I_0(M)$ is the connected component of the isometry group of $\tilde{M}$ and H is the isotropy subgroup of G at some fixed point $o \in \tilde{M}$.

If we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the groups G and H, respectively, then we have a decomposition of the vector space $\mathfrak{g}$ in the form $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where the subspace $\mathfrak{m}$ satisfies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. This subspace can be identified with $T_o\tilde{M}$ and, under this identification, the curvature tensor of $(\tilde{M}, \tilde{g})$ at o is given by $\tilde{R}_o(X, Y)Z = -[[X, Y], Z]$. Furthermore, fixing a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{m}$, the *rank* of $\tilde{M}$, $\operatorname{rk} \tilde{M}$, equals the dimension of $\mathfrak{a}$.

Assume first that $(\tilde{M}, \tilde{g})$ is an irreducible symmetric space of *non-compact* type. We work with the *root space decomposition* of $\mathfrak{g}$ with respect to the maximal Abelian subalgebra $\mathfrak{a}$ (see, for example, [19,22,27]). For every linear function $\alpha\colon \mathfrak{a} \to \mathbb{R}$, we define the vector subspace

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [A, X] = \alpha(A)X \text{ for all } A \in \mathfrak{a}\}.$$

A linear function $\alpha \neq 0$ is called a *root* of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ the corresponding *root space*, if $\mathfrak{g}_\alpha \neq \{0\}$. If we denote by Λ the set of all roots, then we have the direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha.$$

Since $\mathfrak{a}$ is Abelian, we have $\mathfrak{a} \subset \mathfrak{g}_0$. Moreover, if $\alpha \in \Lambda$, then also $-\alpha \in \Lambda$, and the corresponding root spaces are isomorphic.

Now, let $\mathfrak{m}_\alpha$, $\alpha \in \Lambda$, be the image of $\mathfrak{g}_\alpha$ with respect to the projection map of $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ onto $\mathfrak{m}$. The following assertions are immediate consequences of [19, proposition 2.14.2]:

(i) $\dim \mathfrak{m}_\alpha = \dim \mathfrak{g}_\alpha$;

(ii) $\mathfrak{m}_\alpha = \mathfrak{m}_{-\alpha} = \mathfrak{m} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$;

(iii) $\mathfrak{m} = \mathfrak{a} \oplus \sum_{\alpha \in \Lambda^+} \mathfrak{m}_\alpha$ where Λ^+ is a subset of Λ such that $\Lambda^+ \cap (-\Lambda^+) = \emptyset$ and $\Lambda = \Lambda^+ \cup (-\Lambda^+)$.

Next, we assume that the rank of $\tilde{M}$ is at least two, i.e. $\dim \mathfrak{a} \geqslant 2$. Fix a root $\alpha_0 \in \Lambda^+$ and choose two non-zero vectors $U, V \in \mathfrak{a}$ such that $\alpha_0(U) = 1$ and $\alpha_0(V) = 0$. Take a non-zero element

$$Y \in \mathfrak{m}_{\alpha_0} : Y = Y^+ + Y^-, \quad Y^+ \in \mathfrak{g}_{\alpha_0}, \quad Y^- \in \mathfrak{g}_{-\alpha_0}.$$

Then $[U, Y] = Y^+ - Y^-$ is a non-zero vector in $\mathfrak{h} \cap (\mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{-\alpha_0})$ and $[V, Y^\pm] = 0$. Hence

$$\tilde{R}_o(U, Y)V = -[[U, Y], V] = 0.$$

But the curvature operator $\tilde{R}_o(U, Y)$ satisfies condition (4.5) and hence $\lambda(U, Y) = 0$. So, $\tilde{R}_o(U, Y)Z = -[[U, Y], Z] = 0$ for *any* $Z \in \mathfrak{m}$.

Since $[U, Y]$ is non-zero, the linear subspace $\mathfrak{k} = \{W \in \mathfrak{h} \mid [W, \mathfrak{m}] = \{0\}\}$ is non-trivial. But $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. Indeed, for $P \in \mathfrak{m}$, $Q \in \mathfrak{h}$ and $W \in \mathfrak{k}$, we have

$$[[P + Q, W], \mathfrak{m}] = [[P, W], \mathfrak{m}] - [[W, \mathfrak{m}], Q] + [[Q, \mathfrak{m}], W]$$

and each of the terms evaluates to 0 by the definition of $\mathfrak{k}$. According to [22, theorem 3.3, p. 173], however, the isotropy subgroup H contains no non-trivial normal subgroups of G. This gives a contradiction, hence $\dim \mathfrak{a} = 1$ and the proposition is proved for symmetric spaces of non-compact type.

Next we assume that $\tilde{M} = \tilde{G}/\tilde{H}$ is a symmetric space of compact type and rank greater than or equal to two. Let G/H be its dual symmetric space of non-compact type and denote by $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ the Lie algebras corresponding to $\tilde{G}$ and G, respectively. Suppose that a Cartan decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is given as before. It is well known (see, form example, [22]) that $\tilde{\mathfrak{g}}$ can be identified with the subalgebra $\mathrm{i}\mathfrak{m} \oplus \mathfrak{h}$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ and we obtain a Cartan decomposition for $\tilde{\mathfrak{g}}$: $\tilde{\mathfrak{g}} = \mathrm{i}\mathfrak{m} \oplus \mathfrak{h}$. Moreover, if $\mathfrak{a}$ is a maximal Abelian subalgebra of $\mathfrak{m}$, then $\mathrm{i}\mathfrak{a}$ is a maximal Abelian subalgebra of $\mathrm{i}\mathfrak{m}$. In particular, $\mathrm{rk}(G/H) = \mathrm{rk}(\tilde{G}/\tilde{H})$. Putting $\tilde{U} = \mathrm{i}U$, $\tilde{V} = \mathrm{i}V$ and $\tilde{Y} = \mathrm{i}Y$ in the previous computation, we obtain a contradiction in this case too. □

Before dealing with rank-one symmetric spaces of arbitrary dimensions, we first look at the four-dimensional case.

PROPOSITION 4.11. *There do not exist four-dimensional locally irreducible symmetric spaces for which (4.3) holds.*

Proof. From propositions 4.5 and 4.10, we know that we only have to consider the four-dimensional spaces of non-zero constant curvature and the four-dimensional Kähler spaces of non-zero constant holomorphic sectional curvature. For these, we have explicit expressions for the curvature tensor, and we can check condition (4.3) directly.

For a space of constant curvature c, $c \neq 0$, we take three linearly independent vectors x, y, z such that $g(x, y) = 0$ and calculate

$$R(x, y)^2 z = -c^2(g(x, x)g(y, z)y + g(y, y)g(x, z)x),$$

which is not proportional to z if z is not orthogonal to both x and y.

For a space of constant holomorphic sectional curvature c, $c \neq 0$, take tangent vectors x and z such that x, Jx and z are linearly independent. Then

$$R(x, Jx)^2 z = -\tfrac{1}{4}c^2 g(x, x)(3g(x, z)x + 3g(Jx, z)Jx + g(x, x)z),$$

which is not proportional to z if z is not orthogonal to both x and Jx. □

By means of submanifold theory, we can now prove the following definitive result.

THEOREM 4.12. *Let (M, g) be a non-flat locally symmetric space and suppose that any geodesic γ of (T_1M, g_S) projects to a curve x in M with vanishing second curvature κ_2. Then (M, g) is two dimensional.*

Proof. From proposition 4.5, it follows that (M, g) is locally irreducible and proposition 4.10 tells us that the rank of $(\tilde{M}, \tilde{g})$ must be one. Because of propositions 4.6 and 4.11, we only have to deal with rank-one symmetric spaces of dimension at least six. In [16], Chen and Nagano have described all maximal totally geodesic submanifolds of these spaces. It follows at once from their list that there always exists a four-dimensional totally geodesic submanifold. But then, clearly, this submanifold is itself locally symmetric and also satisfies condition (4.3). Hence it must be flat by the previous proposition. This implies that the rank of the ambient symmetric space is at least four, which gives a contradiction. □

REMARK 4.13. In the appendix we give an alternative proof for the above theorem, which uses the Hurwitz function on the one hand and a simple relation between the rank and the dimension of an irreducible symmetric space on the other.

Combining proposition 4.8 with theorem 4.12, we obtain the final result.

THEOREM 4.14. *Let (M, g) be a Riemannian manifold. Then any geodesic γ of (T_1M, g_S) projects to a curve x of M which is either a geodesic or a circle if and only if (M, g) is either flat or a two-dimensional space of constant curvature.*

4.3. $\kappa_3 \equiv 0$

In this subsection, we investigate a possible converse to theorem 4.1. Again, we start by relating the vanishing of κ_3 to a curvature condition on the base manifold. Contrary to the previous discussions, we start at once with locally symmetric spaces.

Proposition 4.15. *Let (M,g) be a locally symmetric space. Then any geodesic γ of (T_1M, g_S) projects to a curve x in M for which $\kappa_3 \equiv 0$ if and only if*

$$R(V,W)^3Y + (\kappa_1^2 + \kappa_2^2)R(V,W)Y = 0 \tag{4.6}$$

for all vector fields V, W and Y on M. The coefficient $\kappa_1^2 + \kappa_2^2$ only depends on V and W, not on Y. Its value is given by

$${\kappa_1}^2 + {\kappa_2}^2 = \frac{|R(V,W)^2Y|^2}{|R(V,W)Y|^2}$$

for any Y such that $R(V,W)Y \neq 0$.

Proof. Condition (4.6) follows as before by comparing the expressions for $\dot{x}^{(3)}$ in (2.2) and (2.3), taking into account this time the local symmetry of (M,g).

In order to show that $\kappa_1^2 + \kappa_2^2$ does not depend on Y, take arbitrary vectors v and w at a point p in M and two vectors y_1 and y_2 for which $R(v,w)y_i \neq 0$, $i = 1,2$. Put $\lambda_i = (\kappa_1^2 + \kappa_2^2)(v,w,y_i)$. Then

$$R(v,w)^3(\cos\theta y_1 + \sin\theta y_2) = -\lambda_1 R(v,w)(\cos\theta y_1) - \lambda_2 R(v,w)(\sin\theta y_2).$$

This should be proportional to $R(v,w)(\cos\theta y_1 + \sin\theta y_2)$ for all values of θ. Hence $\lambda_1 = \lambda_2$.

The expression for $\kappa_1^2 + \kappa_2^2$ follows at once from (4.6). □

Again, the reducible case is easy to deal with.

Proposition 4.16. *Let (M,g) be a reducible locally symmetric space and suppose that (4.6) holds. Then (M,g) factors into a flat component and an irreducible locally symmetric space satisfying (4.6). Conversely, for any such product, the condition (4.6) holds.*

Proof. The only non-trivial statement is that one of the factors in the decomposition of (M,g) must be flat. So, suppose $(M,g) = (M_1,g_1) \times (M_2,g_2)$ with both factors non-flat. Then take Y_1, V_1, W_1 tangent to M_1 and Y_2, V_2, W_2 tangent to M_2 such that $R_i(V_i,W_i)Y_i \neq 0$ for $i = 1,2$. Next, consider the operator $R(\cos\theta V_1 + \sin\theta V_2, \cos\theta W_1 + \sin\theta W_2)$ for $\theta \in (0, \frac{1}{2}\pi)$. It satisfies (4.6) by assumption. As the corresponding $\kappa_1^2 + \kappa_2^2$ is independent of the vector Y, putting first $Y = Y_1$ and then $Y = Y_2$, we find the equality

$$\cos^4\theta \frac{|R_1(V_1,W_1)^2Y_1|^2}{|R_1(V_1,W_1)Y_1|^2} = \sin^4\theta \frac{|R_2(V_2,W_2)^2Y_2|^2}{|R_2(V_2,W_2)Y_2|^2}.$$

Note that $R_i(V_i,W_i)^2Y_i$ is non-zero for $i = 1,2$, since otherwise we would have $0 = g_i(R_i(V_i,W_i)^2Y_i, Y_i) = -|R_i(V_i,W_i)Y_i|^2$, contrary to the hypothesis. But then we see that the above inequality cannot be valid for all $\theta \in (0, \frac{1}{2}\pi)$, which gives a contradiction. Hence one of the factors must be flat. □

We now prove a converse of theorem 4.1 within the class of locally symmetric spaces.

THEOREM 4.17. *Let (M^n, g), $n \geqslant 3$, be a locally symmetric space such that the projection $x = \pi \circ \gamma$ of any geodesic γ of $(T_1 M, g_S)$ has vanishing third curvature κ_3. Then (M^n, g) is either a space of constant curvature or a local product of a flat space and a space of constant curvature.*

Proof. The result is clearly valid in dimension three. We first exclude the four-dimensional Kähler manifolds of constant holomorphic sectional curvature c, $c \neq 0$. For such a manifold, we can calculate explicitly

$$R(x, Jx)z = -\tfrac{1}{2}c(g(x, z)Jx - g(Jx, z)x + g(x, x)Jz),$$
$$R(x, Jx)^3 z = \tfrac{1}{8}c^3 g(x, x)^2 (7g(x, z)Jx - 7g(Jx, z)x + g(x, x)Jz).$$

If we take linearly independent vectors x, Jx and z such that z is not orthogonal to both x and Jx, then these two expressions are not proportional, and condition (4.6) is not fulfilled.

Without loss of generality, we may assume that (M^n, g) is complete and simply connected. If the rank of M is one and if (M^n, g) has non-constant sectional curvature, then it contains a totally geodesic complex projective plane $\mathbb{C}P^2$ or a totally geodesic complex hyperbolic plane $\mathbb{C}H^2$ (see, for example, [16, 42]), and the above calculation shows that condition (4.6) is not fulfilled.

Next, assume that M is irreducible and of rank two. We first treat the case when M is of compact type and consider the classification of maximal totally geodesic submanifolds in rank-two symmetric spaces according to Chen and Nagano [16]. According to this classification, (M^n, g) contains a totally geodesic $\mathbb{C}P^2$ unless M is $SU(3)/SO(3)$, $G^R(2, 2)$, $G^R(2, 3)$, $G^H(2, 2)$, $SU(3)$ or $Sp(2)$. So, if M is not one of these spaces, condition (4.6) is not fulfilled by the above calculation. The real Grassmannian $G^R(2, 2)$ is isometric to $S^2 \times S^2$ and thus we may ignore it. If M is $G^R(2, 3)$, $G^H(2, 2)$, $SU(3)$ or $Sp(2)$, it contains a totally geodesic Riemannian product of two spaces of non-zero constant curvature and, from proposition 4.16, we see that condition (4.6) does not hold. Thus the only remaining space is $SU(3)/SO(3)$.

Let $M = SU(3)/SO(3)$, $G = SU(3)$ and $K = SO(3)$ the isotropy group of G at $o \in M$. Denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Then $\mathfrak{g} = \mathfrak{su}(3)$ is the Lie algebra of all skew-Hermitian traceless (3×3)-matrices with complex coefficients and $\mathfrak{k} = \mathfrak{so}(3)$ is the Lie algebra of all skew-symmetric (3×3)-matrices with real coefficients. Let $\mathfrak{m}$ be the linear subspace of $\mathfrak{g}$ consisting of all skew-Hermitian traceless (3×3)-matrices with purely imaginary coefficients, which gives us the usual Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g}$. We identify $\mathfrak{m}$ with the tangent space $T_o M$ of M at o in the usual way. Then the Riemannian curvature tensor R_o of M at o is given by $R_o(A, B)C = -[[A, B], C]$ for all $A, B, C \in \mathfrak{m}$. We now define

$$A = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By a straightforward calculation we get that $R_o(A, B)^3 A = -16 R_o(A, B) A$ on the one hand and $R_o(A, B)^3 C = -4 R_o(A, B) C$ on the other, which implies that condition (4.6) is not fulfilled.

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If M is of non-compact type and rank two and different from $SL(3,\mathbb{R})/SO(3)$, we can use duality to find a suitable totally geodesic submanifold in M showing us that condition (4.6) is not fulfilled. Using the standard Cartan decomposition of $\mathfrak{sl}(3,\mathbb{R})$, we can describe the Riemannian curvature tensor R_o of $SL(3,\mathbb{R})/SO(3)$ at a point o as the negative of the curvature tensor of $SU(3)/SO(3)$, which easily implies that also in this case condition (4.6) is not fulfilled. Altogether, it now follows that if M is an irreducible Riemannian symmetric space of rank two, then condition (4.6) is not fulfilled.

Now we consider the case that M is of rank two and reducible. From proposition 4.16, it follows that M is isometric to $\mathbb{R}\times M_1$ where M_1 is a complete simply connected Riemannian symmetric space of rank one. Using, again, the totally geodesic $\mathbb{C}P^2$- and $\mathbb{C}H^2$-argument, we see that condition (4.6) is not fulfilled if M_1 has non-constant curvature. We have thus proved the theorem for all locally symmetric spaces of rank one or two.

We now investigate the case that the rank of M is greater than two. We first assume that M is irreducible and of compact type. Let G be the identity component of the full isometry of M, $o\in M$, and K the isotropy group of G at o. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ be the corresponding Cartan decomposition of $\mathfrak{g}$ at o. We choose a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and denote by Δ the restricted root system of the semisimple symmetric pair $(\mathfrak{g},\mathfrak{k})$ with respect to $\mathfrak{a}$. This induces the root space decompositions

$$\mathfrak{k}=\mathfrak{k}(0)\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{k}(\alpha)\quad\text{and}\quad\mathfrak{m}=\mathfrak{a}\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{m}(\alpha).$$

Let Λ be a set of simple roots in Δ. We fix two simple roots in Λ and denote by Δ' the closed subsystem in Δ generated by these two simple roots. Recall that a subsystem Δ' of Δ is called closed if $\Delta'=-\Delta'$ and if $\alpha,\beta\in\Delta'$ and $\alpha+\beta\in\Delta$ implies $\alpha+\beta\in\Delta'$. We now define $\mathfrak{k}'\subset\mathfrak{k}$ and $\mathfrak{m}'\subset\mathfrak{m}$ by

$$\mathfrak{k}'=\bigoplus_{\alpha\in\Delta'}[\mathfrak{k}(\alpha),\mathfrak{k}(\alpha)]_{\mathfrak{k}(0)}\oplus\bigoplus_{\alpha\in\Delta'}\mathfrak{k}(\alpha)\quad\text{and}\quad\mathfrak{m}'=\mathfrak{a}'\oplus\bigoplus_{\alpha\in\Delta'}\mathfrak{m}(\alpha),$$

where $[\cdot,\cdot]_{\mathfrak{k}(0)}$ denotes the $\mathfrak{k}(0)$-component and $\mathfrak{a}'\subset\mathfrak{a}$ is the dual space of the linear span of Δ'. Moreover, we define

$$\mathfrak{g}'=\mathfrak{k}'\oplus\mathfrak{m}'.$$

Since Δ' is a closed subsystem of Δ, general properties of root systems imply that $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{k}'$ is a subalgebra of $\mathfrak{k}$. It is easy to see that the centre of $\mathfrak{g}'$ is trivial, which implies that $(\mathfrak{g}',\mathfrak{k}')$ is a semisimple symmetric pair (see also [40]). Denote by G' the connected Lie subgroup of G with Lie algebra $\mathfrak{g}'$ and consider the orbit $M'=G'\cdot o$ of G' through o. Since $\mathfrak{g}'$ is invariant under the Cartan involution corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$, this orbit is totally geodesic in M. Hence it is also a symmetric space and, by construction, semisimple and of rank two. Thus M' is a totally geodesic Riemannian symmetric space of compact type and with rank two. Since M' is of compact type, it has no Euclidean factor, and since we already know that condition (4.6) is not fulfilled for such symmetric spaces, this implies that condition (4.6) is not fulfilled for M as well.

In the non-compact case, we can use duality to produce a suitable totally geodesic submanifold and apply an analogous argument.

Finally, if M is reducible, the assertion follows from proposition 4.16 by taking into account the solution for the irreducible case. □

Appendix A.

In this appendix we give an alternative proof for theorem 4.12 based on a simple relation between the rank and the dimension of an irreducible Riemannian symmetric space.

LEMMA A.1. *The rank* $\mathrm{rk}\, M$ *of an irreducible Riemannian symmetric space* (M, g) *satisfies* $2\, \mathrm{rk}\, M \leqslant \dim M$.

First proof. In [4, pp. 312–317], all irreducible symmetric spaces are listed, along with their dimensions and their rank. A case-by-case check reveals that the above inequality holds. □

Second proof. As in the proof of proposition 4.10, we consider the symmetric space (M, g) as a quotient manifold G/H and take a Cartan decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G. Again, we first treat the case where M is an irreducible symmetric space of *non-compact* type. We fix a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ and consider the corresponding root space decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha,$$

and the associated decomposition of $\mathfrak{m}$,

$$\mathfrak{m} = \mathfrak{a} \oplus \sum_{\alpha \in \Lambda^+} \mathfrak{m}_\alpha, \tag{A 1}$$

where $\Lambda^+ \cap (-\Lambda^+) = \emptyset$ and $\Lambda^+ \cup (-\Lambda^+) = \Lambda$, the set of all roots. Recall also that $\dim \mathfrak{m}_\alpha = \dim \mathfrak{g}_\alpha \geqslant 1$ for any $\alpha \in \Lambda^+$.

Now define the subspace $\mathfrak{b}$ of $\mathfrak{a}$ by

$$\mathfrak{b} = \{V \in \mathfrak{a} \mid \alpha(V) = 0 \text{ for all } \alpha \in \Lambda^+\}.$$

Then $\mathfrak{b}$ centralizes $\mathfrak{g}$, since

$$\left[V, X_0 + \sum_{\alpha \in \Lambda} X_\alpha\right] = \sum_{\alpha \in \Lambda^+} \alpha(V)(X_\alpha - X_{-\alpha}) = 0$$

for any $V \in \mathfrak{b}$, $X_0 \in \mathfrak{g}_0$, $X_\alpha \in \mathfrak{g}_\alpha$. Hence we have $R_o(Y, Z)\mathfrak{b} = -[[Y, Z], \mathfrak{b}] = \{0\}$ and, by the de Rham decomposition, M is the product manifold $M = B \times M'$, where B is a Euclidean factor satisfying $T_o B = \mathfrak{b}$. As M is irreducible by hypothesis, $\mathfrak{b} = \{0\}$. Consequently, the number of linearly independent equations in the defining system of equations for $\mathfrak{b}$, $\{\alpha(V) = 0 \mid \alpha \in \Lambda^+,\ V \in \mathfrak{a}\}$, is greater than or equal to $\dim \mathfrak{a} = \mathrm{rk}\, M$. So, also using (A 1), we have

$$\dim M - \mathrm{rk}\, M = \dim \mathfrak{m} - \dim \mathfrak{a} \geqslant |\Lambda^+| \geqslant \mathrm{rk}\, M,$$

where $|\Lambda^+|$ is the cardinality of Λ^+. This proves the lemma for symmetric spaces of non-compact type.

For a symmetric space $\tilde{M}$ of compact type, consider its dual symmetric space M of non-compact type (cf. the proof of proposition 4.10). Then

$$\dim \tilde{M} = \dim M \geqslant 2 \operatorname{rk} M = 2 \operatorname{rk} \tilde{M},$$

and the lemma is proved also in this case. □

Third proof. Let $M = G/H$ be a Riemannian symmetric space of *non-compact* type with Iwasawa decomposition $G = KAN$ for G. Then AN is a solvable Lie group acting simply transitively on M. Furthermore, $\dim A = \operatorname{rk} M$ and N is the nilradical of AN. In 1966, Mubarakzianov proved (see [32]) that the dimension of the nilradical of a solvable Lie algebra is greater than or equal to one half of the sum of the dimension of the Lie algebra and of its centre. Since AN is centreless, we have

$$\operatorname{rk} M = \dim A = \dim AN - \dim N = \dim M - \dim N \leqslant \tfrac{1}{2} \dim M.$$

Using duality as before, we get the same result in the compact case. □

With this inequality, we can give an alternative proof for theorem 4.12.

Proof of theorem 4.12. Let (M, g) be a non-flat locally irreducible symmetric space satisfying the hypothesis of theorem 4.12. Then, at $o \in M$, we have $R_o(X,Y)^2 = -\lambda(X,Y)^2 \operatorname{Id}$ for all $X, Y \in T_oM$ and $\lambda(X,Y) = 0$ if and only if $R_o(X,Y) = 0$. By the skew-symmetry of $R_o(X,Y)$, we get

$$g_o(R_o(X,Y)Z, R_o(X,Y)W) = -g_o(R_o(X,Y)^2 Z, W) = \lambda(X,Y)^2 g_o(Z,W). \quad \text{(A 2)}$$

If $R_o(X,Y) \neq 0$, then $(1/|\lambda(X,Y)|)R_o(X,Y)$ is an almost complex structure on T_oM. In particular, $\dim M$ is even.

Recall that an operator T on a Euclidean vector space $(\mathfrak{m}, \langle\cdot,\cdot\rangle)$ is called a *similarity* if it satisfies $\langle TZ, TW\rangle = \sigma_T \langle Z, W\rangle$ for all $Z, W \in \mathfrak{m}$, where σ_T is a scalar factor (see [39]). From (A 2), we see that all curvature operators $R_o(X,Y)$, $X, Y \in T_oM$, are similarities of (T_oM, g_o).

Now consider the symmetric space $M = G/H$ with a decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ as before. In particular, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, $\mathfrak{m}$ can be identified with T_oM and $R_o(X,Y) = -\operatorname{ad}([X,Y])_{|\mathfrak{m}}$. In the proof of proposition 4.10, we showed that the subspace $\mathfrak{k} = \{W \in \mathfrak{h} \mid [W, \mathfrak{m}] = \{0\}\}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$ and that, consequently, $\mathfrak{k} = \{0\}$ (see [22, theorem 3.3, p. 173]). Hence, for any vector $U \in \mathfrak{m} \cong T_oM$, the subspace $\{X \in \mathfrak{m} \mid R_o(U,X) = -\operatorname{ad}([U,X])_{|\mathfrak{m}} = 0\}$ coincides with $\mathfrak{a}_U = \{X \in \mathfrak{m} \mid [U,X] = 0\}$.

Next we recall that an element $U \in \mathfrak{m}$ is called a *regular* vector if $\mathfrak{a}_U$ is an Abelian subalgebra of $\mathfrak{m}$ (and hence $\mathfrak{a}_U$ is a maximal Abelian subalgebra of $\mathfrak{m}$). According to [19, p. 79], such vectors exist in $\mathfrak{m}$. So, let U be a regular vector. But then the kernel of the linear map $\mathcal{R}_U : \mathfrak{m} \to \operatorname{End}(\mathfrak{m}) : X \mapsto R_o(U,X)$ is a maximal Abelian subalgebra $\mathfrak{a}_U$ of $\mathfrak{m}$. Hence

$$\dim(\mathcal{R}_U(\mathfrak{m})) = \dim \mathfrak{m} - \dim \mathfrak{a}_U = \dim M - \operatorname{rk} M.$$

On the other hand, the linear subspace $\mathcal{R}_U(\mathfrak{m}) \subset \operatorname{End}(\mathfrak{m})$ consists of similarities of $\mathfrak{m}$, so we have

$$\dim(\mathcal{R}_U(\mathfrak{m})) \leqslant \max\{\dim S\},$$

where S ranges over all linear subspaces of $\mathrm{End}(\mathfrak{m})$ consisting of similarities of $\mathfrak{m}$. From [39, theorem 2.12], we obtain

$$\max\{\dim S\} = \rho(n),$$

where the *Hurwitz function* $\rho : \mathbb{N}_0 \to \mathbb{N}_0$ is given by

$$\rho(2^m n_o) = \begin{cases} 2m+1 & \text{if } m \equiv 0 \pmod 4, \\ 2m & \text{if } m \equiv 1,2 \pmod 4, \\ 2m+2 & \text{if } m \equiv 3 \pmod 4, \end{cases}$$

with n_0 an odd number. Combining the above equalities and inequalities, we get

$$n - \mathrm{rk}\, M \leqslant \rho(n). \tag{A 3}$$

Finally, from lemma A.1, we obtain

$$n \leqslant 2\rho(n). \tag{A 4}$$

We will see that this inequality is only satisfied for a few values of n.

For that purpose, we fix an odd integer n_0 and investigate the function

$$\kappa_{n_0} : m \mapsto \kappa_{n_0}(m) = 2^m n_0 - 2\rho(2^m n_0).$$

We are interested in those m, n_0 for which $\kappa_{n_0}(m) \leqslant 0$. Computing

$$\kappa_{n_0}(m+1) - \kappa_{n_0}(m) = \begin{cases} 2^m n_0 - 2 & \text{if } m \equiv 0 \pmod 4, \\ 2^m n_0 - 4 & \text{if } m \equiv 1 \pmod 4, \\ 2^m n_0 - 8 & \text{if } m \equiv 2 \pmod 4, \\ 2^m n_0 - 1 & \text{if } m \equiv 3 \pmod 4, \end{cases}$$

we see at once that $\kappa_{n_0}(m)$ is monotonely increasing for $m \geqslant 4$. Furthermore, we have

m	1	2	3	4	$m \geqslant 5$
$\kappa_{n_0}(m)$	$2n_0 - 4$	$4n_0 - 8$	$8n_0 - 16$	$16n_0 - 18$	$\kappa_{n_0}(m) > 0$

So the inequality $\kappa_{n_0}(m) \leqslant 0$ is satisfied only for $n_0 = 1$ and $m = 1, 2, 3, 4$. Hence $\dim M \in \{2, 4, 8, 16\}$.

It remains to show that the dimension of M cannot be 4, 8 or 16. For $\dim M = 4$, this was done in proposition 4.11. For $\dim M = 16$, the inequality (A 3) says that $\mathrm{rk}\, M \geqslant 7$. However, from the list of irreducible symmetric spaces in [4, pp. 312–317], we see that $\mathrm{rk}\, M < 7$ when $\dim M = 16$. So we can also exclude this possibility.

At this point, we have proved theorem 4.12, provided the dimension of M is different from eight. Dealing with this last case is somewhat more involved. We proceed as in the proof of theorem 4.12, using totally geodesic submanifolds. First, we list all eight-dimensional irreducible symmetric spaces from [4, pp. 312–317] and we note that the rank of these spaces is at most two. Explicitly, we have the rank-one symmetric spaces

$$\mathbb{C}P^4,\ \mathbb{R}P^8,\ \mathbb{H}P^2 \text{ and their non-compact duals}$$

and the rank-two symmetric spaces

$$G_2(\mathbb{C}^4),\ G_2(\mathbb{R}^6),\ \frac{G_2}{SU(2)\times SU(2)},\ SU(3) \text{ and their non-compact duals.}$$

It suffices to show that all these symmetric spaces have a totally geodesic submanifold of dimension between three and seven. Indeed, these submanifolds are themselves locally symmetric and satisfy the curvature condition (4.3), hence they are flat. Consequently, the rank of the ambient space must be greater than two, which gives a contradiction.

In [16], a desired totally geodesic submanifold for each of the symmetric spaces above is given, except for $G_2(\mathbb{C}^4)$. However, this symmetric space is isometric to the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-planes in $\mathbb{R}^6$ and has $\mathbb{C}P^2$ as totally geodesic submanifold (see [1, p. 9]). So, the possibility that the dimension of M equals eight can also be excluded. □

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