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A HARMONIC SUM OVER NONTRIVIAL ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract

We consider the sum $\sum 1/\gamma$, where γ ranges over the ordinates of nontrivial zeros of the Riemann zeta-function in an interval $(0, T]$, and examine its behaviour as $T \to \infty$. We show that, after subtracting a smooth approximation $(1/4\pi)\log^2(T/2\pi)$, the sum tends to a limit $H \approx -0.0171594$, which can be expressed as an integral. We calculate H to high accuracy, using a method which has error $O((\log T)/T^2)$. Our results improve on earlier results by Hassani ['Explicit approximation of the sums over the imaginary part of the non-trivial zeros of the Riemann zeta function', *Appl. Math. E-Notes* **16** (2016), 109–116] and other authors.

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1. Introduction

Let the nontrivial zeros of the Riemann zeta-function $\zeta(s)$ be denoted by $\rho = \sigma + i\gamma$. In order of increasing height, the ordinates of these zeros in the upper half-plane are $\gamma_1 \approx 14.13 < \gamma_2 < \gamma_3 < \cdots$. Define

$$G(T) := \sum_{0<\gamma\leqslant T} 1/\gamma,$$

where multiple zeros (if they exist) are weighted according to their multiplicity. We consider the behaviour of $G(T)$ as $T \to \infty$. Answering a question of Hassani [7], we show in Theorem 2.1 of Section 2 that there exists

$$H := \lim_{T\to\infty}\left(G(T) - \frac{\log^2(T/2\pi)}{4\pi}\right). \tag{1.1}$$

There is an analogy with the harmonic series $\sum 1/n$, which appears in the usual definition of Euler's constant:

$$C := \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right) = 0.577215\ldots.$$

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It is well known that one can compute C accurately using Euler–Maclaurin summation or faster algorithms (see [1, 2, 5] and the references given there). However, it is not so easy to compute H accurately, because of the irregular spacing of the nontrivial zeros of $\zeta(s)$ (for which see [9]).

In Section 4 we consider numerical approximation of H, after giving some preliminary lemmas in Section 3. If the definition (1.1) is used directly with the zeros up to height T, then the error is $O((\log T)/T)$. In Theorem 4.1 we show how to improve this, without much extra computation, to $O((\log T)/T^2)$. In Corollary 4.2 we give an explicit bound on H with error of order 10^{-18}.

Finally, in Section 5, we comment briefly on related results in the literature.

2. Existence of the limit

Before proving Theorem 2.1, we define some notation. Let $\mathcal{F}$ denote the set of positive ordinates of zeros of $\zeta(s)$. Following Titchmarsh [11, Sections 9.2–9.3], if $0 < T \notin \mathcal{F}$, then we let $N(T)$ denote the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leqslant T$, and $S(T)$ denote the value of $\pi^{-1} \arg \zeta(\frac{1}{2} + iT)$ obtained by continuous variation along the straight lines joining 2, $2 + iT$ and $\frac{1}{2} + iT$, starting with the value 0. If $T \in \mathcal{F}$, we could take $S(T) = \lim_{\delta \to 0}[S(T - \delta) + S(T + \delta)]/2$, and similarly for $N(T)$, but we avoid this exceptional case. Note that $N(T)$ and $S(T)$ are piecewise continuous, with jumps at $T \in \mathcal{F}$.

By [11, Theorem 9.3], $N(T) = L(T) + Q(T)$, where

$$L(T) = \frac{T}{2\pi}\left(\log\left(\frac{T}{2\pi}\right) - 1\right) + \frac{7}{8} \quad \text{and} \quad Q(T) = S(T) + O(1/T).$$

An explicit bound from Trudgian [13, Corollary 1] is

$$Q(T) = S(T) + \frac{0.2\vartheta}{T}, \tag{2.1}$$

where (here and elsewhere) $\vartheta \in \mathbb{R}$ satisfies $|\vartheta| \leqslant 1$.

Let $S_1(T) := \int_0^T S(t)\,dt$. By [11, Theorems 9.4 and 9.9(A)], $S(T) = O(\log T)$ and $S_1(T) = O(\log T)$, and it follows from (2.1) that $Q(T) = O(\log T)$ also.

Explicit bounds on $S_1(T)$ are known. For certain constants c, $A_0 \geqslant 0$, $A_1 \geqslant 0$ and $T_0 > 0$, there is a bound

$$|S_1(T) - c| \leqslant A_0 + A_1 \log T \quad \text{for all } T \geqslant T_0. \tag{2.2}$$

From [12, Theorem 2.2], we could take $c = S_1(168\pi), A_0 = 2.067, A_1 = 0.059$ and $T_0 = 168\pi$. However, a small computation shows that (2.2) also holds for $T \in [2\pi, 168\pi]$. Hence, we take $T_0 = 2\pi$ in (2.2).

Our first result is the following theorem.

THEOREM 2.1. *The limit H in (1.1) exists. Also,*

$$H = \int_{2\pi}^{\infty} \frac{Q(t)}{t^2}\,dt - \frac{1}{16\pi},$$

where $Q(T) = N(T) - L(T)$ is as above.

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PROOF. Suppose that $2\pi \leqslant T \notin \mathcal{F}$. Using Stieltjes integrals, and noting that $\gamma_1 > 2\pi$ and $Q(2\pi) = \frac{1}{8}$,

$$\begin{aligned} G(T) = \sum_{0<\gamma\leqslant T} \frac{1}{\gamma} &= \int_{2\pi}^{T} \frac{dN(t)}{t} = \int_{2\pi}^{T} \frac{dL(t)}{t} + \int_{2\pi}^{T} \frac{dQ(t)}{t} \\ &= \frac{1}{2\pi} \int_{2\pi}^{T} \frac{\log(t/2\pi)}{t}\, dt + \Big[\frac{Q(t)}{t} + \int \frac{Q(t)}{t^2}\, dt\Big]_{2\pi}^{T} \\ &= \frac{\log^2(T/2\pi)}{4\pi} + \frac{Q(T)}{T} - \frac{1}{16\pi} + \int_{2\pi}^{T} \frac{Q(t)}{t^2}\, dt. \end{aligned} \tag{2.3}$$

Thus,

$$G(T) - \frac{\log^2(T/2\pi)}{4\pi} = \int_{2\pi}^{T} \frac{Q(t)}{t^2}\, dt - \frac{1}{16\pi} + O\Big(\frac{\log T}{T}\Big).$$

Letting $T \to \infty$, the last integral converges, so the limit of the left-hand side exists and

$$H = \lim_{T\to\infty} \Big(G(T) - \frac{\log^2(T/2\pi)}{4\pi}\Big) = \int_{2\pi}^{\infty} \frac{Q(t)}{t^2}\, dt - \frac{1}{16\pi}.$$

This completes the proof. □

3. Two lemmas

We now give two lemmas that are used in the proof of Theorem 4.1.

LEMMA 3.1. *If $2\pi \leqslant T \notin \mathcal{F}$, then*

$$\int_{2\pi}^{T} \frac{Q(t)}{t^2}\, dt = G(T) - \frac{Q(T)}{T} + \frac{1}{16\pi} - \frac{\log^2(T/2\pi)}{4\pi}.$$

PROOF. This is just a rearrangement of (2.3) in the proof of Theorem 2.1. □

LEMMA 3.2. *If $T \geqslant 2\pi$ and*

$$E_2(T) := \int_{T}^{\infty} \frac{Q(t)}{t^2}\, dt, \tag{3.1}$$

then

$$|E_2(T)| \leqslant \frac{4.27 + 0.12 \log T}{T^2}.$$

PROOF. To bound $E_2(T)$, we note that, from (2.1),

$$\int_{T}^{\infty} \frac{Q(t)}{t^2}\, dt = \int_{T}^{\infty} \frac{S(t)}{t^2}\, dt + \frac{0.1\vartheta}{T^2}. \tag{3.2}$$

Also, using integration by parts,

$$\int_{T}^{\infty} \frac{S(t)}{t^2}\, dt = -\frac{S_1(T) - c}{T^2} + 2 \int_{T}^{\infty} \frac{S_1(t) - c}{t^3}\, dt. \tag{3.3}$$

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Using (2.2),

$$\begin{aligned}\left|\int_T^\infty \frac{S(t)}{t^2}\,dt\right| &\leqslant \frac{|S_1(T)-c|}{T^2} + 2\int_T^\infty \frac{|S_1(t)-c|}{t^3}\,dt \\ &\leqslant \frac{A_0 + A_1\log T}{T^2} + 2\int_T^\infty \frac{A_0 + A_1\log t}{t^3}\,dt \\ &= \frac{2A_0 + 0.5A_1 + 2A_1\log T}{T^2}. \end{aligned} \tag{3.4}$$

Using (3.2),

$$|E_2(T)| \leqslant \frac{2A_0 + 0.5A_1 + 0.1 + 2A_1\log T}{T^2}.$$

Inserting the values $A_0 = 2.067$ and $A_1 = 0.059$ gives the result. □

The bound (3.4) might be improved by using a result of Fujii [6, Theorem 2] to bound the integral of $S_1(t)/t^3$ in (3.3), although we are not aware of any explicit version of Fujii's estimate. The bound would then be dominated by the term $-S_1(T)/T^2$ in (3.3). This term is $o((\log T)/T^2)$ if and only if the Lindelöf hypothesis (LH) is true; see [11, Theorem 13.6(B) and Note 13.8]. Thus, obtaining an order-of-magnitude improvement in the bound on $E_2(T)$ is equivalent to proving LH.

4. Numerical approximation of H

We consider two methods to approximate H numerically. The first method truncates the sum and integral in the definition (1.1) at height $T \geqslant 2\pi e$, giving an approximation with error $E(T) = O((\log T)/T)$. An explicit bound

$$H = G(T) - \frac{\log^2(T/2\pi)}{4\pi} + A\vartheta\left(\frac{2\log T + 1}{T}\right) \tag{4.1}$$

follows from Lehman [8, Lemma 1]. Lehman gives $A = 2$, but, from [3, Corollary 1], we may take $A = 0.28$. Thus, we can obtain about five decimal places by summing over the first 10^6 zeros of $\zeta(s)$, that is, to height $T = 600270$. In this manner we find $H \approx -0.01716$. It is difficult to obtain many more correct digits because of the slow convergence. However, the result is sufficient to show that H is negative, which is significant in the proof of [3, Lemma 8].

Convergence can be accelerated using Theorem 4.1, which improves the error bound $E(T) = O((\log T)/T)$ of (4.1) to $E_2(T) = O((\log T)/T^2)$. Note that the error term $E_2(T)$ is a continuous function of T. This is unlike $E(T)$, which has jumps for $T \in \mathcal{F}$.

THEOREM 4.1. *For all $T \geqslant 2\pi$,*

$$H = \sum_{0<\gamma\leqslant T}\left(\frac{1}{\gamma} - \frac{1}{T}\right) - \frac{\log^2(T/2\pi e) + 1}{4\pi} + \frac{7}{8T} + E_2(T), \tag{4.2}$$

where $E_2(T)$ is as in (3.1) and $|E_2(T)| \leqslant (4.27 + 0.12\log T)/T^2$.

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TABLE 1. Numerical estimation of H using Theorem 4.1.

n	H estimate
10	−0.017372393877
100	−0.017159765533
1000	−0.017159603500
10000	−0.017159404875
100000	−0.017159404244
1000000	−0.017159404307

PROOF. First assume that $T \notin \mathcal{F}$. From Theorem 2.1 and Lemma 3.1,

$$H = G(T) - \frac{Q(T)}{T} - \frac{\log^2(T/2\pi)}{4\pi} + E_2(T),$$

but $Q(T) = N(T) - L(T)$, so

$$H = \sum_{0<\gamma\leqslant T}\left(\frac{1}{\gamma} - \frac{1}{T}\right) + \frac{\log(T/2\pi) - 1}{2\pi} + \frac{7}{8T} - \frac{\log^2(T/2\pi)}{4\pi} + E_2(T).$$

Simplification gives (4.2) and a continuity argument shows that (4.2) holds if $T \in \mathcal{F}$. Finally, the bound on $E_2(T)$ follows from Lemma 3.2. □

COROLLARY 4.2. *Let H be defined by (1.1). We have*

$$H = -0.0171594043070981495 + \vartheta(10^{-18}).$$

PROOF. This follows from Theorem 4.1 by an interval-arithmetic computation using the first $n = 10^{10}$ zeros, with $T = \gamma_n \approx 3293531632.4$. □

To illustrate Theorem 4.1, we give some numerical results in Table 1. The first column (n) gives the number of zeros used and the second column is the estimate of H obtained from (4.2), using $T = \gamma_n$. The first incorrect digit of each entry is underlined.

5. Related results in the literature

Büthe [4, Lemma 3] gives the inequality

$$G(T) \leqslant \frac{\log^2(T/2\pi)}{4\pi} \quad \text{for } T \geqslant 5000. \tag{5.1}$$

In [3, Lemma 8], we give a different proof of (5.1), and show that it holds for $T \geqslant 4\pi e$.

Hassani [7] shows (in our notation) that

$$G(T) = \frac{\log^2(T/2\pi)}{4\pi} + O(1)$$

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and gives numerical bounds for the '$O(1)$' term. A similar bound is given in [10, Lemma 2.10]. Hassani does not prove existence of the limit (1.1), but asks (see [7, page 114]) whether it exists. We have answered this in our Theorem 2.1.

In fact, Hassani works with

$$\Delta_N := \sum_{n=1}^{N} \frac{1}{\gamma_n} - \left(\frac{1}{4\pi}\log^2\gamma_N - \frac{\log(2\pi)}{2\pi}\log\gamma_N\right),$$

so in our notation

$$\Delta_N = G(\gamma_N) - \frac{\log^2(\gamma_N/2\pi)}{4\pi} + \frac{\log^2(2\pi)}{4\pi}.$$

Thus, the (hypothetical) limit to which Hassani refers is, in our notation,

$$H + \frac{\log^2(2\pi)}{4\pi} = 0.25163675131270596 65 + \vartheta(10^{-18}).$$

This is consistent with the value 0.25163 that Hassani gives based on his calculations using $2 \cdot 10^6$ nontrivial zeros. Hassani also uses an averaging technique to obtain values in the range $[0.2516372, 0.2516375]$, but apparently decreasing, without an obvious limit. The acceleration technique of Theorem 4.1 is more effective and has the virtue of giving a rigorous error bound.

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