

CREDIBLE LOSS RATIO CLAIMS RESERVES: THE BENKTANDER, NEUHAUS AND MACK METHODS REVISITED

BY

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ABSTRACT

The Benktander (1976) and Neuhaus (1992) credibility claims reserving methods are reconsidered in the framework of a credible loss ratio reserving method. As a main contribution we provide a simple and practical optimal credibility weight for combining the chain-ladder or individual loss ratio reserve (grossed up latest claims experience of an origin period) with the Bornhuetter-Ferguson or collective loss ratio reserve (experience based burning cost estimate of the total ultimate claims of an origin period). The obtained simple optimal credibility weights minimize simultaneously the mean squared error and the variance of the claims reserve. We note also that the standard Chain-Ladder, Cape Cod and Bornhuetter-Ferguson methods can be reinterpreted in the credible context and extended to optimal credible standard methods. The new approach is inspired from Mack (2000). Two advantages over the Mack method are worthwhile to be mentioned. First, a pragmatic estimation of the required parameters leads to a straightforward calculation of the optimal credibility weights and mean squared errors of the credible reserves. An advantage of the collective loss ratio claims reserve over the Bornhuetter-Ferguson reserve in Mack (2000) is that different actuaries come always to the same results provided they use the same actuarial premiums.

KEYWORDS

Claims reserve, loss ratio method, credibility method, minimum mean squared error, minimum variance, chain-ladder, Cape Cod, Bornhuetter-Ferguson reserve, Benktander reserve, Neuhaus reserve

1. INTRODUCTION

The design of our new claims reserving method is inspired from and similar but different from the Benktander method reviewed in Mack (2000), Section 2. While Mack worked for simplicity in the context of a single origin period (accident year or underwriting year), our approach is based on a full loss

development triangle of paid claims (or alternatively incurred claims) with several origin periods. As in other methods we require additionally the knowledge of a measure of exposure like premiums for each origin period.

The standard claims reserving methods include the commonly used reserving methods as the Chain-Ladder, the Cape Cod and the Bornhuetter-Ferguson methods (consult the Swiss Re brochure by Boulter and Grubbs (2000) for a useful elementary description of these methods). While the standard methods apply the so-called *chain-ladder factors* defined by link ratios (average ratio of cumulative paid claims between two consecutive development periods) our approach is based on *loss ratios* (average ratio of incremental paid claims to exposure for each development period). The main contribution is Theorem 6.1, which provides an optimal credibility weight for combining the chain-ladder or individual loss ratio reserve (grossed up latest claims experience of an origin period) with the Bornhuetter-Ferguson or collective loss ratio reserve (experience based burning cost estimate of the total ultimate claims of an origin period). The obtained simple optimal credibility weights minimize simultaneously the mean squared error and the variance of the claims reserve.

The organization of the paper is as follows. Section 2 contains the necessary prerequisites and the definition of the mentioned collective and individual loss ratio claims reserves. Section 3 defines the credible loss ratio claims reserves. Formulas for the optimal credibility weights, which minimize the mean squared errors of the credible loss ratio reserves, as well as formulas for the mean squared errors are derived in Section 4. The practical evaluation of these quantities is based on a pragmatic estimation method, which is motivated in Section 5. In Section 6, the remaining unspecified parameters of the pragmatic estimation method are chosen such that they minimize additionally the variance of the optimal credible claims reserve. This is a desirable property because from a statistical point of view estimates with lower variances are usually preferred. Fortunately, the minimum variance optimal credible claims reserve is parameter-free and comparable in simplicity to the Neuhaus and Benktander loss ratio reserves. Finally, Section 7 presents numerical examples. Based on our pragmatic estimation method, we observe that the Neuhaus and Benktander loss ratio reserves are quite close to the optimal credible reserve. In our examples, the Neuhaus reserve is closer to the optimal one than the Benktander reserve for all origin periods. Through application of a credible loss ratio reserving method, the reduction in mean squared error is substantial. In absence of sufficient information to estimate the optimal credibility weights more precisely, the three simple credible methods are highly recommended for actuarial practice.

2. THE COLLECTIVE AND INDIVIDUAL LOSS RATIO CLAIMS RESERVES

Let n be the number of origin periods, say one-year periods, for which historical data on paid claims is available. Let S_{ik} , $1 \leq i, k \leq n$, be the paid claims from origin period i reported in period $i+k-1$. Under the assumption that after

n development periods all claims incurred in an origin period are known and closed, the amount $\sum_{k=1}^n S_{ik}$ is the *total ultimate claims* from origin period i . The sums $C_{ik} = \sum_{j=1}^k S_{ij}$, $1 \leq i, k \leq n$, denote the accumulated paid claims incurred in calendar year i and reported after $i+k-1$ years of development. At the end of the current calendar year only the amount $C_{i,n-i+1} = \sum_{k=1}^{n-i+1} S_{ik}$ is known. The required amount for the incurred but unpaid claims of period i , called *i -th period claims reserve*, is equal to $R_i = \sum_{k=n-i+2}^n S_{ik}$, $i = 2, \dots, n$. The total required amount of incurred but unpaid claims over all periods $R = \sum_{i=2}^n R_i$ is called total claims reserve.

To fix ideas, we suppose that the measure of exposure V_i is the premium belonging to the origin period $i = 1, \dots, n$. Our analysis is based on (expected) loss ratios representing the incremental amount of expected paid claims per unit of premium in each development period, which are defined by

$$m_k = \frac{E \left[\sum_{i=1}^{n-k+1} S_{ik} \right]}{\sum_{i=1}^{n-k+1} V_i}, \quad k = 1, \dots, n. \tag{2.1}$$

Since the sum $\sum_{k=1}^n m_k$ represents the loss ratio over all reporting periods, the quantity

$$E[U_i^{BC}] = V_i \cdot \sum_{k=1}^n m_k \tag{2.2}$$

is nothing else than the expected value of the burning cost U_i^{BC} of the total ultimate claims required for the origin period i . This quantity is similar to the expected value of the Bornhuetter-Ferguson prior estimate U_0 of the total ultimate claims in Mack (2000). By definition of the loss ratios, the quantities $V_i \cdot \sum_{k=1}^{n-i+1} m_k$, $i = 1, \dots, n$, represent the expected burning cost of the paid claims for the origin period i , which are required in the current calendar period. The *loss ratio payout factor* (also called *loss ratio lag-factor*) defined by

$$p_i = \frac{V_i \cdot \sum_{k=1}^{n-i+1} m_k}{E[U_i^{BC}]} = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^n m_k}, \quad i = 1, \dots, n, \tag{2.3}$$

represents the proportion of the total ultimate claims from origin period i , which is expected to be paid in the development period $n-i+1$. The quantity

$$q_i = 1 - p_i = \frac{\sum_{k=n-i+2}^n m_k}{\sum_{k=1}^n m_k}, \quad i = 1, \dots, n, \tag{2.4}$$

called *loss ratio reserve factor*, represents the proportion of the total ultimate claims from origin period i , which remain unpaid in the development period $n - i + 1$.

From this an estimate of the total ultimate claims is obtained by grossing up the latest accumulated paid claims amount. Since it is based solely on the individual latest claims experience of an origin period, it is called *individual total ultimate claims* amount and is given by

$$U_i^{ind} = \frac{C_{i,n-i+1}}{p_i}, \quad i = 1, \dots, n. \quad (2.5)$$

This estimate is similar to the chain-ladder estimate in Mack (2000). A corresponding estimate of the claims reserve, called *individual loss ratio claims reserve*, is defined by

$$R_i^{ind} = U_i^{ind} - C_{i,n-i+1} = q_i \cdot U_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}, \quad i = 1, \dots, n. \quad (2.6)$$

On the other side, the burning cost of the total ultimate claims leads to the alternative claims reserve

$$R_i^{coll} = q_i \cdot U_i^{BC}, \quad U_i^{BC} = V_i \cdot \sum_{k=1}^n \frac{\sum_{i=1}^{n-k+1} S_{ik}}{\sum_{i=1}^n V_i}, \quad i = 1, \dots, n. \quad (2.7)$$

It is called *collective loss ratio claims reserve* because it depends solely on the portfolio claims experience of all origin periods. It coincides with the claims reserve set according to the loss ratio reserving method as defined in Mack (1997), Section 3.2.2, p. 230-234. The associated *collective total ultimate claims* is given by

$$U_i^{coll} = R_i^{coll} + C_{i,n-i+1}, \quad i = 1, \dots, n. \quad (2.8)$$

This estimate is similar to the Bornhuetter-Ferguson posterior estimate of the total ultimate claims in Mack (2000). An advantage of the collective loss ratio claims reserve over the Bornhuetter-Ferguson reserve in Mack (2000) is that different actuaries come always to the same results provided they use the same premiums. This is also true for the individual claims reserve without restriction.

3. CREDIBLE LOSS RATIO CLAIMS RESERVES

Like the Bornhuetter-Ferguson and the chain-ladder estimates in Mack (2000), the considered collective and individual loss ratio claims reserve estimates represent extreme positions. Indeed, the individual claims reserve considers the

latest accumulated paid claims amount $C_{i,n-i+1}$ to be fully credible predictive for future claims and ignores the burning cost U_i^{BC} of the total ultimate claims, while the collective claims reserve ignores the current accumulated paid claims and relies fully on the burning cost. Therefore it is natural to apply the credibility mixture to those reserves and use the *credible loss ratio claims reserve* estimate

$$R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}, \quad i = 1, \dots, n, \tag{3.1}$$

where Z_i is the credibility weight associated to the individual loss ratio reserve. It is interesting to reconsider two popular choices of the credibility weights proposed in the literature. As the credibility weight should increase similarly as the accumulated paid claims $C_{i,n-i+1}$ develop, Gunnar Benktander (1976) proposed the credibility weight $Z_i^{GB} = p_i, i = 1, \dots, n$. This leads to the *Benktander loss ratio claims reserve*

$$R_i^{GB} = p_i \cdot R_i^{ind} + q_i \cdot R_i^{coll}, \quad i = 1, \dots, n. \tag{3.2}$$

According to Mack (1997), p. 242, the choice made by Walter Neuhaus (1992) corresponds to the credibility weight $Z_i^{WN} = \sum_{k=1}^{n-i+1} m_k = p_i \cdot \sum_{k=1}^n m_k$. It leads to the *Neuhaus loss ratio claims reserve*

$$R_i^{WN} = Z_i^{WN} \cdot R_i^{ind} + (1 - Z_i^{WN}) \cdot R_i^{coll}, \quad i = 1, \dots, n. \tag{3.3}$$

It is remarkable that in numerical examples these simple choices are both quite close to an *optimal credible loss ratio claims reserve*, whose credibility weights are derived in Section 6.

On the other side, remarks similar to those made by Mack (2000) at the end of Section 2 can be made. The functions $R_i(U_i) = q_i U_i$ and $U_i(R_i) = R_i + C_{i,n-i+1}$ are not inverse to each other except for $U_i = U_i^{ind}$. Similarly to the “iterated Bornhuetter-Ferguson method”, there is an “iterated collective loss ratio reserving method”. The successive iteration of the collective and Benktander loss ratio reserving methods for an arbitrary start point U_i^0 leads in the infinite limit to the individual loss ratio reserving method. It is worthwhile to restate this result, which paraphrases Theorem 1 in Mack (2000).

Theorem 3.1. For an arbitrary starting point $U_i^{(0)} = U_i^0$, the iteration rule

$$R_i^{(m)} = q_i \cdot U_i^{(m)}, \quad U_i^{(m+1)} = C_{i,n-i+1} + R_i^{(m)}, \quad m = 0, 1, 2, \dots, \tag{3.4}$$

gives credibility mixtures

$$\begin{aligned} U_i^{(m)} &= (1 - q_i^m) \cdot U_i^{ind} + q_i^m \cdot U_i^0, \\ R_i^{(m)} &= (1 - q_i^m) \cdot R_i^{ind} + q_i^m \cdot R_i^0, \end{aligned} \tag{3.5}$$

between the collective and individual loss ratio reserving methods, which starts at the collective method and lead via the Benktander method finally to the individual method for $m = \infty$.

4. THE OPTIMAL CREDIBILITY WEIGHTS AND THE MEAN SQUARED ERROR

In the following, we suppose that the burning cost estimate U_i^{BC} , defined in (2.7), of the total ultimate claims is independent from $C_{i,n-i+1}$, R_i and $U_i = R_i + C_{i,n-i+1}$, and has expectation $E[U_i^{BC}] = E[U_i^{ind}] = E[U_i]$ (usual unbiasedness) and variance $Var[U_i^{BC}]$. This key working assumption is similar to the assumption made to get Theorem 2 in Mack (2000).

Theorem 4.1. The optimal credibility weights Z_i^* which minimize the mean squared error $mse(R_i^c) = E[(R_i^c - R_i)^2]$ is given by

$$Z_i^* = \frac{p_i}{q_i} \cdot \frac{Cov[C_{i,n-i+1}, R_i] + p_i q_i \cdot Var[U_i^{BC}]}{Var[C_{i,n-i+1}] + p_i^2 \cdot Var[U_i^{BC}]} \tag{4.1}$$

Proof. By definition (3.1) of the credible loss reserve one has

$$\begin{aligned} E[(R_i^c - R_i)^2] &= E[(Z_i \cdot (R_i^{ind} - R_i^{coll}) + R_i^{coll} - R_i)^2] \\ &= Z_i^2 \cdot E[(R_i^{ind} - R_i^{coll})^2] - 2Z_i \cdot E[(R_i^{ind} - R_i^{coll})(R_i - R_i^{coll})] + E[(R_i^{coll} - R_i)^2] \end{aligned}$$

From the first order condition

$$\frac{\partial}{\partial Z_i} E[(R_i^c - R_i)^2] = 2Z_i \cdot E[(R_i^{ind} - R_i^{coll})^2] - 2 \cdot E[(R_i^{ind} - R_i^{coll})(R_i - R_i^{coll})] = 0,$$

one obtains that

$$Z_i^* = \frac{E[(R_i^{ind} - R_i^{coll})(R_i - R_i^{coll})]}{E[(R_i^{ind} - R_i^{coll})^2]}, \quad i = 1, \dots, n. \tag{4.2}$$

Inserting the expressions $R_i^{ind} = q_i \cdot U_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}$ and $R_i^{coll} = q_i \cdot U_i^{BC}$ one gets

$$Z_i^* = \frac{p_i}{q_i} \cdot \frac{Cov[C_{i,n-i+1} - p_i U_i^{BC}, R_i - q_i U_i^{BC}]}{Var[C_{i,n-i+1} - p_i U_i^{BC}]} \tag{4.3}$$

By definition (2.5) and the made assumption one has $E[C_{i,n-i+1}] = p_i E[U_i^{ind}] = p_i E[U_i^{BC}]$, hence $E[R_i] = E[U_i - C_{i,n-i+1}] = (1 - p_i) \cdot E[U_i^{BC}] = q_i \cdot E[U_i^{BC}]$. Using this and the assumption that U_i^{BC} is independent from $C_{i,n-i+1}$, R_i and U_i , one gets the desired formula (4.1). \square

To estimate the optimal credibility weights, one needs a model for $Var[C_{i,n-i+1}]$ and $Cov[C_{i,n-i+1}, R_i]$. Consider the following conditional model for the loss ratio payout (see Mack (2000), p. 338):

$$E\left[\frac{C_{i,n-i+1}}{U_i} \mid U_i\right] = p_i, \quad Var\left[\frac{C_{i,n-i+1}}{U_i} \mid U_i\right] = p_i q_i \beta_i^2(U_i), \quad i = 1, \dots, n. \quad (4.4)$$

The factor q_i ensures that $Var\left[\frac{C_{i,n-i+1}}{U_i} \mid U_i\right] = 0$ when $i = 1$ and that $Var\left[\frac{C_{i,n-i+1}}{U_i} \mid U_i\right] \rightarrow 0$ in case of very small values p_i . In the following the notation $\alpha_i^2(U_i) = U_i^2 \cdot \beta_i^2(U_i)$ is used.

Theorem 4.2. Under the assumption of model (4.4), the optimal credibility weights Z_i^* which minimize the mean squared error $mse(R_i^c) = E[(R_i^c - R_i)^2]$ is given by

$$Z_i^* = \frac{p_i}{p_i + t_i}, \quad \text{with} \quad (4.5)$$

$$t_i = \frac{E[\alpha_i^2(U_i)]}{Var[U_i^{BC}] + Var[U_i] - E[\alpha_i^2(U_i)]}, \quad i = 1, \dots, n. \quad (4.6)$$

Proof. From (4.4) one obtains

$$E[C_{i,n-i+1} \mid U_i] = p_i U_i, \quad Var[C_{i,n-i+1} \mid U_i] = p_i q_i \alpha_i^2(U_i), \quad i = 1, \dots, n. \quad (4.7)$$

It follows that

$$\begin{aligned} Var[C_{i,n-i+1}] &= E[Var[C_{i,n-i+1} \mid U_i]] + Var[E[C_{i,n-i+1} \mid U_i]] \\ &= p_i q_i E[\alpha_i^2(U_i)] + p_i^2 Var[U_i] \\ &= p_i E[\alpha_i^2(U_i)] + p_i^2 \cdot (Var[U_i] - E[\alpha_i^2(U_i)]), \end{aligned} \quad (4.8)$$

and

$$Cov[C_{i,n-i+1}, U_i] = E[Cov[C_{i,n-i+1}, U_i \mid U_i]] + Cov[E[C_{i,n-i+1} \mid U_i], U_i] = p_i Var[U_i]. \quad (4.9)$$

From (4.8) and (4.9) one obtains further

$$\text{Cov}[C_{i,n-i+1}, R_i] = \text{Cov}[C_{i,n-i+1}, U_i] - \text{Var}[C_{i,n-i+1}] = p_i q_i \cdot (\text{Var}[U_i] - E[\alpha_i^2(U_i)]). \tag{4.10}$$

Inserting (4.8) and (4.10) into (4.1) the desired formula follows. □

Simple formulas for the mean squared errors are derived in a similar way.

Theorem 4.3. Under the assumption of model (4.4), the following formulas for the mean squared errors hold:

$$\begin{aligned} \text{mse}(R_i^{\text{coll}}) &= E[\alpha_i^2(U_i)] \cdot q_i \left(1 + \frac{q_i}{t_i}\right), \\ \text{mse}(R_i^{\text{ind}}) &= E[\alpha_i^2(U_i)] \cdot \frac{q_i}{p_i}, \\ \text{mse}(R_i^c) &= E[\alpha_i^2(U_i)] \cdot \left(\frac{Z_i^2}{p_i} + \frac{1}{q_i} + \frac{(1 - Z_i)^2}{t_i}\right) \cdot q_i^2. \end{aligned} \tag{4.11}$$

Proof. Using (4.8) and (4.9) one obtains

$$\begin{aligned} \text{Var}[R_i] &= \text{Var}[U_i - C_{i,n-i+1}] = \text{Var}[U_i] - 2\text{Cov}[C_{i,n-i+1}, U_i] + \text{Var}[C_{i,n-i+1}] \\ &= \text{Var}[U_i] \cdot (1 - 2p_i + p_i^2) + p_i q_i E[\alpha_i^2(U_i)] = q_i^2 \cdot \text{Var}[U_i] + p_i q_i E[\alpha_i^2(U_i)] \\ &= q_i E[\alpha_i^2(U_i)] + q_i^2 \cdot (\text{Var}[U_i] - E[\alpha_i^2(U_i)]) \end{aligned} \tag{4.12}$$

Since $E[R_i^{\text{coll}}] = q_i E[U_i^{BC}] = q_i E[U_i] = E[U_i - p_i U_i^{\text{ind}}] = E[U_i - C_{i,n-i+1}] = E[R_i]$ and by assumption $\text{Cov}[R_i^{\text{coll}}, R_i] = q_i \text{Cov}[U_i^{BC}, R_i] = 0$, one has

$$\begin{aligned} \text{mse}(R_i^{\text{coll}}) &= E[(R_i^{\text{coll}} - R_i)^2] = \text{Var}[R_i^{\text{coll}} - R_i] = \text{Var}[R_i^{\text{coll}}] + \text{Var}[R_i] \\ &= q_i^2 \cdot \text{Var}[U_i^{BC}] + q_i^2 \cdot (\text{Var}[U_i] - E[\alpha_i^2(U_i)]) + q_i \cdot E[\alpha_i^2(U_i)] \\ &= E[\alpha_i^2(U_i)] \cdot \left(q_i + \frac{q_i^2}{t_i}\right), \end{aligned} \tag{4.13}$$

where the last equality follows from (4.6) of Theorem 4.2. Similarly, one has $E[R_i^{\text{ind}}] = E[R_i]$ and it follows that

$$\begin{aligned} \text{mse}(R_i^{\text{ind}}) &= E[(R_i^{\text{ind}} - R_i)^2] = \text{Var}[R_i^{\text{ind}} - R_i] = \text{Var}[R_i^{\text{coll}}] - 2\text{Cov}[R_i^{\text{ind}}, R_i] + \text{Var}[R_i] \\ &= \left(\frac{q_i}{p_i}\right)^2 \cdot \text{Var}[C_{i,n-i+1}] - 2\frac{q_i}{p_i} \text{Cov}[C_{i,n-i+1}, R_i] + \text{Var}[R_i]. \end{aligned} \tag{4.14}$$

Inserting (4.8), (4.10) and (4.12) one gets without difficulty the desired formula (4.11). The third formula follows from

$$\begin{aligned}
 mse(R_i^c) &= E[(Z_i(R_i^{ind} - R_i) + (1 - Z_i)(R_i^{coll} - R_i))^2] \\
 &= Z_i^2 mse(R_i^{ind}) + 2Z_i(1 - Z_i) E[(R_i^{ind} - R_i)(R_i^{coll} - R_i)] + (1 - Z_i)^2 mse(R_i^{coll}),
 \end{aligned}
 \tag{4.15}$$

and

$$\begin{aligned}
 E[(R_i^{ind} - R_i)(R_i^{coll} - R_i)] &= Cov[R_i^{ind} - R_i, R_i^{coll} - R_i] = Var[R_i] - Cov[R_i^{ind}, R_i] \\
 &= Var[R_i] - \frac{q_i}{p_i} Cov[C_{i,n-i+1}, R_i] = q_i E[\alpha_i^2(U_i)]
 \end{aligned}
 \tag{4.16}$$

using the formulas for $mse(R_i^{ind})$ and $mse(R_i^{coll})$. □

5. A PRAGMATIC ESTIMATION METHOD

To evaluate the optimal credibility weights and the mean squared errors, it is necessary to estimate the quantities $Var[U_i^{BC}]$, $Var[U_i]$ and $E[\alpha_i^2(U_i)]$. The proposed estimators are based on a full loss triangle of paid claims statistics S_{ik} , $1 \leq i, k \leq n$, subject to the restriction $i + k - 1 \leq n$, and the knowledge of exposures V_i for each origin period $i = 1, \dots, n$. Working in an unconditional environment we use the following standard estimate for $Var[U_i^{BC}]$, which follows from the analysis by Mack (1997), p. 231-233:

$$\widehat{Var}[U_i^{BC}] = V_i^2 \cdot \left(\sum_{k=1}^n \frac{s_k^2}{w_k} \right), \text{ with}
 \tag{5.1}$$

$$\begin{aligned}
 w_k &= \sum_{i=1}^{n-k+1} V_i, \quad s_k^2 = \frac{1}{n-k} \sum_{i=1}^{n-k+1} V_i \left(\frac{S_{ik}}{V_i} - \widehat{m}_k \right)^2, \\
 \widehat{m}_k &= \frac{\sum_{i=1}^{n-k+1} S_{ik}}{\sum_{i=1}^{n-k+1} V_i}, \quad k = 1, \dots, n-1, \quad s_n^2 = \min\{s_k^2 \mid k = 1, \dots, n-1\}
 \end{aligned}
 \tag{5.2}$$

It is intuitively appealing that U_i should be at least as volatile than the burning cost estimate U_i^{BC} (similar to the fact that $Var[U]$ should be larger than $Var[U_0]$ in Mack (2000)). As pragmatic estimates we assume that

$$\widehat{Var}[U_i] = f_i \cdot \widehat{Var}[U_i^{BC}], \quad \widehat{E}[U_i] = U_i^{BC},
 \tag{5.3}$$

with some factor $f_i \geq 1$, and that an estimate of $\beta_i^2(U_i)$ in (4.4) is a constant β_i^2 for all underwriting periods $i = 1, \dots, n$. The quantities f_i and β_i can be statistically justified as follows. Arguing that U_i^{BC} is a good unbiased point estimate of the total ultimate claims, it is reasonable to assume that U_i belongs to the confidence interval $\left[U_i^{BC} - c_i \cdot \sqrt{\widehat{Var}[U_i^{BC}]}, U_i^{BC} + c_i \cdot \sqrt{\widehat{Var}[U_i^{BC}]} \right]$ for some constant $c_i > 1$. On the other side U_i should belong to the confidence interval $\left[\widehat{E}[U_i] - d_i \cdot \sqrt{\widehat{Var}[U_i]}, \widehat{E}[U_i] + d_i \cdot \sqrt{\widehat{Var}[U_i]} \right]$ for some constant $d_i \leq c_i$ (U_i at least as volatile than U_i^{BC}). If $\widehat{E}[U_i] = U_i^{BC}$, one must have $\widehat{Var}[U_i] = f_i \cdot \widehat{Var}[U_i^{BC}]$ for some constant $f_i \geq 1$. On the other side, if the ratio $C_{i,n-i+1}/U_i$ given U_i has a Beta($a_i p_i, a_i q_i$) distribution with some constant $a_i > 0$, then one has necessarily $\beta_i^2(U_i) = (a_i + 1)^{-1}$, which is a constant independent of U_i . Recalling that $\alpha_i^2(U_i) = U_i^2 \cdot \beta_i^2(U_i)$, one obtains from (5.3) the following estimate

$$\widehat{E}[\alpha_i^2(U_i)] = \beta_i^2 \cdot \left\{ f_i \cdot \widehat{Var}[U_i^{BC}] + (U_i^{BC})^2 \right\}. \tag{5.4}$$

The above estimates are inserted in the formulas of Theorems 4.2 and 4.3 to get estimates of the optimal credibility weights and the mean squared errors. In particular, an optimal credibility estimate is obtained from

$$\hat{t}_i = \frac{\beta_i^2 \cdot (f_i + \hat{u})}{1 + f_i - \beta_i^2 \cdot (f_i + \hat{u})}, \tag{5.5}$$

where $\hat{u}_i = \frac{(U_i^{BC})^2}{\widehat{Var}[U_i^{BC}]} = \frac{\left(\sum_{k=1}^n \hat{m}_k \right)^2}{\sum_{k=1}^n \frac{s_k^2}{w_k}}$ is an estimate of the inverse of the

coefficient of variation of the burning cost estimate U_i^{BC} of the total ultimate claims, which is independent of the origin period.

6. THE OPTIMAL CREDIBLE CLAIMS RESERVE WITH MINIMUM VARIANCE

The pragmatic estimation method of Section 5 depends on the unknown parameters $(f_i, \beta_i)_{1 \leq i \leq n}$. As in the preceding Section, we work in an unconditional framework and compare the variances of the individual, collective and optimal credible claims reserves in order to determine a set of parameters $(f_i, \beta_i)_{1 \leq i \leq n}$, which minimizes the variance of the optimal credible claims reserve. This is a desirable property because from a statistical point of view estimates with lower variances are usually preferred. Fortunately, the minimum variance optimal credible claims reserve is parameter-free and attained at the parameter value

$$t_i^* = \sqrt{p_i}, \quad i = 1, \dots, n, \tag{6.1}$$

in Theorem 4.2. It compares with the Benktander estimate

$$t_i^{GB} = q_i, \quad i = 1, \dots, n, \tag{6.2}$$

and the Neuhaus estimate

$$t_i^{WN} = q_i + \frac{1 - \sum_{k=1}^n m_k}{\sum_{k=1}^n m_k}, \quad i = 1, \dots, n. \tag{6.3}$$

All three methods yield monotone decreasing credibility weights in the origin periods. Since $t_i^* = 1$ the optimal credibility weights satisfy the inequality

$$Z_i^* \leq \frac{1}{2}, \quad i = 1, \dots, n, \tag{6.4}$$

where the equality is attained for the first origin period. Note that usually the Benktander and Neuhaus methods lead to higher credibility weights. The proposed new method is the special case $f_i = 1$ of the following more general result. This choice, which implies identical volatilities for U_i and U_i^{BC} , is the most appealing one. Besides the minimum variance of the credible loss reserves among all choices $f_i \geq 1$, it yields the smallest credibility weights for the individual loss reserves putting thus more emphasis on the collective loss reserves.

Theorem 6.1. Under the assumption of Sections 4 and 5, the optimal credibility weights Z_i^* which minimize the mean squared error $mse(R_i^c) = E[(R_i^c - R_i)^2]$ and the variance $Var[R_i^c]$ are given by

$$Z_i^* = \frac{p_i}{p_i + t_i^*}, \text{ with} \tag{6.5}$$

$$t_i^* = \frac{f_i - 1 + \sqrt{(f_i + 1) \cdot (f_i - 1 + 2 \cdot p_i)}}{2}, \quad i = 1, \dots, n. \tag{6.6}$$

Proof. Recall that $R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}$, $i = 1, \dots, n$, with $R_i^{ind} = \frac{q_i}{p_i} \cdot C_{i,n-i+1}$, $R_i^{coll} = q_i \cdot U_i^{BC}$. By the assumption at the beginning of Section 4, the covariance between the individual and the collective reserve vanishes, that is $Cov[R_i^{ind}, R_i^{coll}] = \frac{q_i^2}{p_i} \cdot Cov[C_{i,n-i+1}, U_i^{BC}] = 0$. It follows that

$$Var[R_i^c] = Z_i^2 \cdot Var[R_i^{ind}] + (1 - Z_i)^2 \cdot Var[R_i^{coll}], \text{ with}$$

$$Var[R_i^{ind}] = \left(\frac{q_i}{p_i}\right)^2 \cdot Var[C_{i,n-i+1}], \quad Var[R_i^{coll}] = q_i^2 \cdot Var[U_i^{BC}].$$

Using (4.8) and the estimates (5.3), (5.4) and the definition of \hat{u} after (5.5), one gets the estimate

$$\widehat{Var}[C_{i,n-i+1}] = (f_i \cdot p_i \cdot [(1-p_i)\beta_i^2 + p_i] + p_i(1-p_i)\beta_i^2\hat{u}) \cdot Var[U_i^{BC}].$$

Inserted into the above formulas, together with the estimate $\hat{Z}_i = \frac{p_i}{p_i + \hat{t}_i}$ of the optimal credibility weight (4.5), one obtains the estimate

$$\begin{aligned} \widehat{Var}[R_i^c] &= \hat{Z}_i^2 \cdot \left(\frac{q_i}{p_i}\right)^2 [f_i \cdot p_i \cdot [(1-p_i)\beta_i^2 + p_i] + p_i(1-p_i)\beta_i^2\hat{u}] \\ &\quad \cdot Var[U_i^{BC}] + (1-\hat{Z}_i)^2 \cdot Var[R_i^{coll}] \\ &= \left(\hat{Z}_i^2 \cdot \left[f_i + \frac{1-p_i}{p_i}\beta_i^2(f_i + \hat{u})\right] + (1-\hat{Z}_i)^2\right) \cdot Var[R_i^{coll}]. \end{aligned}$$

From (5.5) one has $\beta_i^2(f_i + \hat{u}) = (1 + f_i) \frac{\hat{t}_i}{1 + \hat{t}_i}$, which yields further

$$\begin{aligned} \widehat{Var}[R_i^c] &= \left(\hat{Z}_i^2 \cdot (1 + f_i) \cdot \left[1 + \frac{1-p_i}{p_i} \frac{\hat{t}_i}{1 + \hat{t}_i}\right] - 2\hat{Z}_i + 1\right) \cdot Var[R_i^{coll}] \\ &= \left(1 - \frac{2\hat{t}_i - (f_i - 1)}{1 + \hat{t}_i} \cdot \frac{p_i}{p_i + \hat{t}_i}\right) \cdot Var[R_i^{coll}] \end{aligned}$$

An optimization with respect to \hat{t}_i shows that the minimum variance is attained at (6.6). □

Remark 6.1.

An important difference with the standard approach to claims reserving must be emphasized. Though in Mack (2000) the meaning of the payout factor p_i is general, in practice it has up to now most often been estimated using standard link ratios leading to *chain-ladder payout* or *lag-factors* p_i^{CL} instead of the loss ratio based factors (2.3). Clearly, our main result can also be reused in the traditional as well as more advanced and recent stochastic chain-ladder context. In practice, the standard methods use the so-called *chain-ladder factors* defined by the average link ratios (here by abuse of stochastic notation as statistical estimates)

$$f_k^{CL} = \sum_{i=1}^{n-k} S_{ik+1} / \sum_{i=1}^{n-k} S_{ik}, \quad k = 1, \dots, n-1. \tag{6.7}$$

From this one gets the ultimate loss development factors, so-called *LDF paid factors*,

$$F_k^{CL} = \prod_{j=k}^{n-1} f_j^{CL}, \quad k = 1, \dots, n-1, \quad F_n^{CL} = 1, \tag{6.8}$$

which represent the average ratio of the ultimate claims to the (cumulative) paid claims of each origin period after k years of development. From the LDF paid factors one gets immediately the *chain-ladder lag-factors*

$$p_i^{CL} = \frac{1}{F_{n-i+1}^{CL}}, \quad i = 1, \dots, n, \tag{6.9}$$

and the *chain-ladder reserve factors*

$$q_i^{CL} = 1 - p_i^{CL}, \quad i = 1, \dots, n. \tag{6.10}$$

The standard methods can be reinterpreted in our context and extended to optimal credible standard methods as follows.

The Chain-Ladder Method

It is similar to the individual loss ratio method with the loss ratio lag-factors (2.3) replaced by the chain-ladder lag-factors (6.9):

$$R_i^{CL} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}, \quad i = 1, \dots, n. \tag{6.11}$$

The Cape Cod Method

It is a (Benktander type) credibility mixture of the type (3.1) with

$$R_i^{ind} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}, \quad R_i^{coll} = q_i^{CL} \cdot LR \cdot V_i, \quad LR = \frac{\sum_{i=1}^n C_{i,n-i+1}}{\sum_{i=1}^n p_i^{CL} \cdot V_i}, \tag{6.12}$$

$$Z_i = p_i^{CL}, \quad i = 1, \dots, n.$$

The Optimal Cape Cod Method

It is the modified credibility mixture (6.12) with optimal credibility weights

$$Z_i = \frac{p_i^{CL}}{p_i^{CL} + \sqrt{p_i^{CL}}}, \quad i = 1, \dots, n. \tag{6.13}$$

The Bornhuetter-Ferguson Method

It is a (Benktander type) credibility mixture of the type (3.1) with

$$R_i^{ind} = \frac{q_i^{CL}}{p_i^{CL}} \cdot C_{i,n-i+1}, R_i^{coll} = q_i^{CL} \cdot LR_i \cdot V_i, Z_i = p_i^{CL}, i = 1, \dots, n, \quad (6.14)$$

where LR_i is some selected initial loss ratio for each origin period (e.g. Boulter and Grubbs (2000), p. 19-21).

The Optimal Bornhuetter-Ferguson Method

It is the modified credibility mixture (6.14) with optimal credibility weights (6.13).

7. NUMERICAL EXAMPLES

It is instructive to illustrate the obtained results at some practical examples. Let us start with the published full loss triangle of paid claims and exposures in Mack (1997), Table 3.1.5.1:

TABLE 7.1
LOSS TRIANGLE OF PAID CLAIMS

Origin period	Development period					
	1	2	3	4	5	6
1	4'370	1'923	3'999	2'168	1'200	647
2	2'701	2'590	1'871	1'783	393	–
3	4'483	2'246	3'345	1'068	–	–
4	3'254	2'550	2'547	–	–	–
5	8'010	4'108	–	–	–	–
6	5'582	–	–	–	–	–

The used exposures and relevant parameters, which are obtained through application of the described method, are summarized in Table 7.2.

TABLE 7.2
PARAMETERS OF CREDIBLE LOSS RATIO METHOD

Origin period	Parameters					
	V	m	p	q	t	Z^*
1	13'085	0.29667	1	0	1.00000	0.50000
2	14'258	0.17770	0.94496	0.05504	0.97209	0.49292
3	16'114	0.20072	0.88010	0.11990	0.93814	0.48404
4	15'142	0.11549	0.75153	0.24847	0.86691	0.46435
5	16'905	0.05826	0.52808	0.47192	0.72669	0.42086
6	20'224	0.04945	0.33026	0.66974	0.57469	0.36495

The next two Tables compare the loss ratio reserves and the total ultimate claims obtained through application of the collective, individual, Neuhaus, Benktander and optimal methods.

TABLE 7.3
CREDIBLE LOSS RATIO RESERVES

Origin period	Method				
	collective	individual	Neuhaus	Benktander	optimal
all periods	25'154	26'972	25'913	25'999	25'914
2	705	544	568	553	626
3	1'736	1'518	1'564	1'544	1'630
4	3'380	2'761	2'962	2'915	3'092
5	7'166	10'829	8'904	9'101	8'708
6	12'167	11'320	11'916	11'887	11'858

TABLE 7.4
CREDIBLE LOSS RATIO ULTIMATE CLAIMS

Origin period	Method				
	collective	individual	Neuhaus	Benktander	optimal
all periods	86'752	87'810	86'751	86'837	86'486
1	14'307	14'307	14'307	14'307	14'307
2	9'964	9'882	9'906	9'891	9'966
3	12'772	12'660	12'706	12'686	12'779
4	11'443	11'112	11'313	11'266	11'484
5	20'826	22'947	21'022	21'219	20'364
6	17'440	16'902	17'498	17'469	17'586

The Table 7.5 displays mean squared errors of the different methods expressed as ratios to the minimal mean squared error of the optimal credible reserve. For this the minimum variance estimator of Section 6 is applied with $f_i = 1$ and $t_i = \sqrt{p_i}$.

TABLE 7.5
MEAN SQUARED STANDARD ERRORS (RATIO TO MINIMAL ERROR)

Origin period	Method				
	collective	individual	Neuhaus	Benktander	optimal
2	1.027133	1.028713	1.014146	1.022818	1
3	1.058036	1.065943	1.003002	1.038856	1
4	1.115378	1.153525	1.002692	1.044128	1
5	1.198612	1.376096	1.120972	1.012892	1
6	1.244417	1.740080	1.409648	1.002206	1

The Neuhaus and Benktander loss ratio reserves are quite close to the optimal credible reserve. In the present situation, the Neuhaus reserve is closer to the optimal one than the Benktander reserve for all origin periods. Through application of a credible loss ratio reserving method, the reduction in mean squared error is substantial. In absence of sufficient information to estimate the optimal credibility weights, the three simple credible methods are highly recommended for actuarial practice.

The next practical example stems from a slightly modified real life project. The same conclusions as before are made. Again, the minimum variance estimator of Section 6 is applied with $f_i = 1$ and $t_i = \sqrt{p_i}$.

TABLE 7.6
LOSS TRIANGLE OF PAID CLAIMS

Origin period	Development period					
	1	2	3	4	5	6
1	3'789'045	2'860'826	506'651	151'996	65'141	24'203
2	3'582'774	2'687'080	1'250'163	535'784	880'143	–
3	4'221'853	3'166'390	2'249'388	207'853	–	–
4	4'074'429	2'949'557	1'162'885	–	–	–
5	1'227'618	3'906'617	–	–	–	–
6	6'839'930	–	–	–	–	–

TABLE 7.7
PARAMETERS OF CREDIBLE LOSS RATIO METHOD

Origin period	Parameters					
	V	m	p	q	t	Z^*
1	8'000'000	0.40230	1	0	1.00000	0.50000
2	9'000'000	0.33129	0.99687	0.00314	0.99843	0.49961
3	10'000'000	0.13971	0.93925	0.06075	0.96915	0.49217
4	10'000'000	0.03317	0.90488	0.09512	0.95125	0.48751
5	10'000'000	0.05561	0.76012	0.23988	0.87185	0.46577
6	12'000'000	0.00303	0.41685	0.58315	0.64564	0.39233

TABLE 7.8
CREDIBLE LOSS RATIO RESERVES

Origin period	Method				
	collective	individual	Neuhaus	Benktander	optimal
all periods	10'600'143	12'714'477	11'219'478	11'241'879	11'378'317

Origin period	Method				
	collective	individual	Neuhaus	Benktander	optimal
2	27'228	28'101	28'067	28'098	27'664
3	586'303	636'809	632'085	633'741	611'161
4	918'019	860'619	867'892	866'079	890'036
5	2'315'070	1'620'276	1'805'379	1'786'943	1'991'456
6	6'753'523	9'568'672	7'886'055	7'927'018	7'858'000

TABLE 7.9

CREDIBLE LOSS RATIO ULTIMATE CLAIMS

Origin period	Method				
	collective	individual	Neuhaus	Benktander	optimal
all periods	56'940'469	59'054'803	57'559'804	57'582'205	57'718'643
1	7'397'862	7'397'862	7'397'862	7'397'862	7'397'862
2	8'963'172	8'964'045	8'964'011	8'964'042	8'963'608
3	10'431'787	10'482'293	10'477'569	10'479'225	10'456'645
4	9'104'890	9'047'490	9'054'763	9'052'950	9'076'907
5	7'449'305	6'754'511	6'939'614	6'921'178	7'125'691
6	13'593'453	16'408'602	14'725'985	14'766'948	14'697'930

TABLE 7.10

MEAN SQUARED STANDARD ERRORS (RATIO TO MINIMAL ERROR)

Origin period	Method				
	collective	individual	Neuhaus	Benktander	optimal
2	1.001629	1.001634	1.001405	1.001615	1
3	1.029900	1.031834	1.021187	1.024673	1
4	1.046368	1.051243	1.029035	1.033985	1
5	1.111731	1.146991	1.036943	1.044625	1
6	1.228790	1.548854	1.000149	1.000894	1

As a third example, let us analyze whether the published and practically used A.M. Best loss development factors are “best” in the sense of the proposed credible loss ratio method. We compare the inverse of the loss ratio payout factors obtained from the ratio $\frac{U_i}{C_{i,n-i+1}}$ for the various methods. As a single illustration, we just look at the 2004 A.M. Best Table of paid claims for General

Liability claims made policies, but note that similar results hold for other insurance categories. Table 7.11 lists the used triangle of paid claims and Table 7.12 displays the calculated factors. The optimal credibility weights are calculated using the minimum variance estimator of Section 6 with $f_i = 1$ and $t_i = \sqrt{p_i}$.

TABLE 7.11
LOSS TRIANGLE OF PAID CLAIMS FOR GENERAL LIABILITY CLAIMS MADE POLICIES

Origin period	Development period									
	1	2	3	4	5	6	7	8	9	10
1994	394'645	991'758	1'601'166	2'048'315	2'286'923	2'464'913	2'578'083	2'632'786	2'678'889	2'691'701
1995	353'192	969'542	1'592'009	1'974'155	2'272'665	2'438'559	2'566'985	2'671'750	2'741'844	
1996	388'218	1'078'557	1'646'205	2'171'862	2'443'991	2'709'179	2'827'232	2'934'048		
1997	389'758	1'082'588	1'874'868	2'427'603	2'867'146	3'125'564	3'277'475			
1998	449'549	1'312'383	2'190'150	2'989'522	3'671'202	4'103'170				
1999	391'147	1'398'570	2'506'347	3'448'315	4'134'317					
2000	683'853	1'720'774	3'136'015	4'112'141						
2001	586'225	1'962'547	3'283'757							
2002	810'359	2'149'450								
2003	648'230									

TABLE 7.12
INVERSE OF LOSS RATIO PAYOUT FACTORS

	A.M. Best	optimal	Benktander	Neuhaus	collective	individual
1994	1.000	1.000	1.000	1.000	1.000	1.000
1995	1.010	1.005	1.005	1.005	1.005	1.005
1996	1.030	1.027	1.027	1.027	1.027	1.027
1997	1.066	1.062	1.062	1.062	1.061	1.062
1998	1.114	1.112	1.113	1.113	1.112	1.113
1999	1.226	1.220	1.221	1.221	1.219	1.222
2000	1.471	1.439	1.439	1.439	1.441	1.438
2001	1.986	1.917	1.914	1.915	1.927	1.903
2002	3.475	3.322	3.328	3.331	3.366	3.245
2003	9.903	9.595	9.652	9.655	9.696	9.285

One notes that the A.M. Best factors slightly but systematically overestimate the optimal and nearly optimal Benktander and Neuhaus factors.

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REFERENCES

- BENKTANDER, G. (1976) An approach to credibility in calculating IBNR for casualty excess reinsurance. In: *The Actuarial Review*, p. 7.
- BOULTER, A. and GRUBBS, D. (2000) *Late claims reserves in reinsurance*. Swiss Re publications, available at www.swissre.com.
- MACK, Th. (1997) *Schadenversicherungsmathematik*. Verlag Versicherungswirtschaft, Karlsruhe.
- MACK, Th. (2000) Credible claims reserves: the Benktander method. *ASTIN Bulletin* **30(2)**, 333-347.
- NEUHAUS, W. (1992) Another pragmatic loss reserving method or Bornhuetter/Ferguson revisited. *Scandinavian Actuarial Journal*, 151-162.

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