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THE COMMON POINTS OF FAMILIES OF NORMAL FUNCTIONS

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In this paper we show that the common points of any nonempty "family" of normal functions form a normal function and from this we derive various significant consequences such as Theorem 2 below.

As usual, by a *normal function* [**1**, p. 31] we mean an ordinal valued function F defined on the class of all ordinals such that for every ordinal u and v and any set S of ordinals

$$(1) \qquad u < v \text{ implies } F(u) < F(v) \quad \text{and} \quad F(\sup S) = \sup F[S].$$

Thus, a normal function is strictly increasing and continuous.

Although there is no class having a normal function as an element, we shall use the illicit convenience of speaking of a *family* $(F_i)_{i\in A}$ *of normal functions* F_i *indexed by a set* A. The fact that A is a set will prevent the occurrence of any of the known paradoxes and will allow us to consider families of normal functions which are not restricted to finitely many normal functions.

In what follows, by a *nonempty family* of normal functions we shall mean a family $(F_i)_{i\in A}$ of normal functions where A is a *nonempty set.*

An ordinal v is called a *common point* of a family $(F_i)_{i\in A}$ of normal functions if and only if there exists an ordinal x such that

$$(2) \qquad v = F_i(x) \text{ for every } i \in A.$$

Next, we prove:

LEMMA 1. *Let* $(F_i)_{i\in A}$ *be a nonempty family of normal functions and* u *an ordinal. Then there exists a common point* v *of* $(F_i)_{i\in A}$ *such that* $v > u$.

Proof. Let us consider the family

$$(3) \qquad \{F_j,\ F_iF_jF_j,\ F_jF_k,\ F_iF_kF_j,\ F_kF_k,\ F_iF_i, \ldots\}$$

of all finite iterations (with repetitions) of F_i's where $i, j, k, \ldots$ are elements of A. Now, let

$$(4) \qquad v = \sup\{F_j(u+1),\ F_iF_jF_j(u+1),\ F_jF_k(u+1),\ F_iF_kF_j(u+1),\ F_kF_k(u+1),\ F_iF_i(u+1), \ldots\}.$$

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The above supremum exists since A is a set. From (1) it follows that for every $i \in A$ we have:

$$(5) \qquad F_i(v) = \sup\{F_iF_j(u+1), F_iF_iF_jF_j(u+1), F_iF_jF_k(u+1), F_iF_iF_kF_j(u+1), F_iF_kF_k(u+1), F_iF_iF_i(u+1), \ldots\}.$$

By [1, p. 69] we have $F_i(x) \geqq x$ and since (3) is the family of all finite iterations (with repetitions) of F_i's, from (4) and (5) it follows that the supremum in (4) is equal to the supremum in (5). Therefore,

$$(6) \qquad v = F_i(v) \text{ for every } i \in A.$$

By (6) we see that v is a common point (in fact a common *fixed point*) of $(F_i)_{i\in A}$. However, since $F_i(x) \geqq x$ it follows that $v \geqq u + 1 > u$, as desired.

Lemma 1 can be rephrased as follows:

COROLLARY 1. *Every nonempty family of normal functions has arbitrary large common points.*

Let $(F_i)_{i\in A}$ be a nonempty family of normal functions. By virtue of Corollary 1 we may consider the function C defined on the class of all ordinals as follows:

$$(7) \qquad C(v) = \text{the smallest common point of } (F_i)_{i\in A} \text{ which is larger than } C(u) \text{ for every } u < v.$$

Accordingly, $C(0)$ is the smallest common point of $(F_i)_{i\in A}$; next is $C(1)$, then $C(2)$, . . . and the smallest common point of $(F_i)_{i\in A}$ which is larger than $C(0)$, $C(1)$, $C(2)$, . . . is $C(\omega)$, and so on.

In view of (7), it is natural to refer to C as *the function which enumerates the common points of* $(F_i)_{i\in A}$.

Next we prove our main theorem.

THEOREM 1. *Let C be the function which enumerates the common points of a nonempty family $(F_i)_{i\in A}$ of normal functions. Then C is a normal function.*

Proof. From (7) it follows that $u < v$ implies $C(u) < C(v)$. Thus, in view of (1), it remains to show that for every set P of ordinals $C(\sup P) = \sup C[P]$. However, since C enumerates the common points of $(F_i)_{i\in A}$ we see that for every set P of ordinals there exists a set S of ordinals such that

$$\sup C[P] = \sup F_i[S] \text{ for every } i \in A$$

which by (1) implies $\sup C[P] = F_i(\sup S)$ for every $i \in A$. Consequently, by (6) we see that $\sup C[P]$ is a common point of $(F_i)_{i\in A}$. Hence, by (7) we have $\sup C[P] = C(\sup P)$, as desired.

Now, let us denote by I the *identity* normal function, i.e., $I(x) = x$ for every ordinal x.

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By adjoining I to any nonempty family of normal functions and by observing that every ordinal is a fixed point of I, in view of Corollary 1 (in fact, in view of the proof of Lemma 1), we obtain:

COROLLARY 2. *Every nonempty family of normal functions has arbitrary large common fixed points.*

For every nonempty family $(F_i)_{i \in A}$ of normal functions, the above Corollary ensures the existence of a function B defined on the class of all ordinals as follows:

(8) $B(v)$ = *the smallest common fixed point of* $(F_i)_{i \in A}$ *which is larger than* $B(u)$ *for every* $u < v$.

Again, in view of (8), it is natural to refer to B as *the function which enumerates the common fixed points of* $(F_i)_{i \in A}$.

Next, based on Theorem 1 we have:

COROLLARY 3. *Let B be the function which enumerates the common fixed points of a nonempty family $(F_i)_{i \in A}$ of normal functions. Then B is a normal function.*

Proof. Enlarge the family $(F_i)_{i \in A}$ by adjoining I to it. Clearly, B is the function which enumerates the common points of the enlarged family. But then, from Theorem 1 it follows that B is a normal function.

As in [**2**, p. 101] a cardinal c is called *e-inaccessible* (inaccessible with respect to cardinal exponentiation) if and only if $a < c$ implies $2^a < c$. Clearly, 0, and $\aleph_0$ are the first two e-inaccessible cardinals. It is easy to show that the smallest e-inaccessible cardinal larger than a cardinal $k \geqq \aleph_0$ is

$$\sup\{k_0, k_1, k_2, \ldots\} \qquad \text{where} \qquad k_0 = k$$

and for every natural number n we let $k_{n+1} = 2^{k_n}$. Thus, there exist arbitrary large e-inaccessible cardinals which ensure the existence of the function E defined on the class of all ordinals as follows:

(9) $E(v)$ = *the smallest e-inaccessible cardinal which is larger than* $E(u)$ *for every* $u < v$.

Clearly, E is a normal function. Moreover, in view of (9), it is natural to refer to E as *the function which enumerates all the e-inaccessible cardinals.*

By adjoining E and I to any nonempty family of normal functions, just as in the case of Corollary 2, we obtain:

COROLLARY 4. *Every nonempty family of normal functions has arbitrary large common e-inaccessible fixed points.*

As in the previous case, for every nonempty family $(F_i)_{i \in A}$ of normal functions, Corollary 4 ensures the existence of a function defined (in an

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obvious way) on the class of all ordinals which *enumerates the common e-inaccessible points of* $(F_i)_{i\in A}$. Also, Corollary 4 ensures the existence of a function defined (in an obvious way) on the class of all ordinals which *enumerates the common e-inaccessible fixed points of* $(F_i)_{i\in A}$. Again, by enlarging $(F_i)_{i\in A}$ by E or by E and I, based on Theorem 1 and just as in the case of Corollary 3, we have:

COROLLARY 5. *Let* $(F_i)_{i\in A}$ *be a nonempty family of normal functions. Then the function which enumerates the common e-inaccessible points of* $(F_i)_{i\in A}$ *as well as the function which enumerates the common e-inaccessible fixed points of* $(F_i)_{i\in A}$ *is a normal function.*

Since there are arbitrary many ordinal numbers, from Corollary 5 it follows that any nonempty family of normal functions has arbitrary many common e-inaccessible fixed points. In particular, *every normal function has arbitrary many e-inaccessible fixed points.*

Let us recall that a cardinal c is called *regular* if and only if c is not a sum of fewer than c cardinals each less than c. As shown in [**3**, p. 311], a cardinal is *strongly inaccessible* if and only if it is regular and e-inaccessible.

Next we prove our main consequence of Theorem 1.

THEOREM 2. *If every normal function has at least one regular cardinal in its range then every nonempty family of normal functions has at least one common strongly inaccessible fixed point.*

Proof. By Corollary 5 the function G which enumerates the common e-inaccessible fixed points of a nonempty family $(F_i)_{i\in A}$ of normal functions is a normal function. Now assume the hypothesis of the theorem. But then G has a regular cardinal s in its range. However, every cardinal in the range of G is a common e-inaccessible fixed point of $(F_i)_{i\in A}$. Thus, s is a common strongly inaccessible fixed point of $(F_i)_{i\in A}$.

Finally, let us consider the following three schemas:

(i) *Every normal function has at least one regular cardinal in its range.*

(ii) *Every normal function has at least one strongly inaccessible cardinal in its range.*

(iii) *Every nonempty family of normal functions has at least one common strongly inaccessible fixed point.*

Schema (ii) is Levy's axiom schema of strong infinity [**4**, p. 227]. In [**5**, p. 653] as well as in [**6**] it is shown that (i) and (ii) are equivalent. Theorem 2 above shows that (i) and (iii) are equivalent. Clearly, (ii) is a special case of (iii). Thus, we have:

PROPOSITION. *Schemas* (i), (ii), (iii) *are pairwise equivalent.*

We conclude by observing that in Theorem 2 as well as in (i), (ii), (iii) the words "*at least one*" obviously imply "*arbitrary many*" or "*arbitrary large*".

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