Consider the square $ABA'B'$ constructed on the same side of AB as C, as in Figure 2. Then, since they are copies of triangle ABC under rotations of 90° (in opposite directions), the triangles $A'BF$ and $AB'E$ are congruent to one another, with $A'F$ parallel to EA. Hence $AFA'E$ is a parallelogram and its diagonals bisect one another at M. As this is the midpoint of AA' as well as that of EF, it is independent of C.

It is clear that this argument works equally for squares which are constructed internally on the sides of the triangle ABC.

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Reference

1. A. Bogomolny, (1996). https://www.cut-the-knot.org.

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106.12 A new proof of the *n*-dimensional Pythagorean theorem

We shall use a useful tool of functional analysis, Parseval's identity, to give a new proof for the n-dimensional Pythagorean theorem in [1]. We recall Parseval's identity in n-dimensional Euclidean space $\mathbb{R}^n$ as follows.

Theorem 1: (Parseval's identity [2]). Let $\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}$ be an orthonormal basis of n-dimensional Euclidean space $\mathbb{R}^n$. Then for every vector $\vec{u} \in \mathbb{R}^n$, we have

$$|\vec{u}|^2 = (\vec{u} \cdot \vec{e}_1)^2 + (\vec{u} \cdot \vec{e}_2)^2 + \dots + (\vec{u} \cdot \vec{e}_n)^2. \quad (1)$$

Throughout this Note, in n-dimensional Euclidean space $\mathbb{R}^n$, we denote by XY the Euclidean distance between two points X and Y, and $\overrightarrow{XY}$ the Euclidean vector connecting an initial point X with a terminal point Y. We recall the n-dimensional Pythagorean theorem in [1].

Theorem 2: (n-dimensional Pythagorean theorem). In n-dimensional Euclidean space $\mathbb{R}^n$, if the edges $OP_1, OP_2, \dots, OP_n$ of a simplex $OP_1P_2 \dots P_n$ are all perpendicular, and if the bounding simplexes opposite to the vertices $O, P_1, P_2, \dots, P_n$ have $(n-1)$-dimensional contents $A, A_1, A_2, \dots, A_n$ respectively, then

$$A^2 = A_1^2 + A_2^2 + \dots + A_n^2. \quad (2)$$

Proof: Let H be the orthogonal projection of O on hyperplane $(P_1P_2 \dots P_n)$, and let V be volume of the simplex $OP_1P_2 \dots P_n$. Since the edges

$OP_1, OP_2, \dots, OP_n$ are mutually orthogonal, using formula of volume of simplex, (2) is equivalent to

$$\frac{(nV)^2}{OH^2} = \frac{(nV)^2}{OP_1^2} + \frac{(nV)^2}{OP_2^2} + \dots + \frac{(nV)^2}{OP_n^2} \tag{3}$$

which is equivalent to

$$\frac{1}{OH^2} = \frac{1}{OP_1^2} + \frac{1}{OP_2^2} + \dots + \frac{1}{OP_n^2}. \tag{4}$$

Note that in [1], Donchian and Coxeter also showed that (4) is an equivalent form of n-dimensional Pythagorean theorem by using Cartesian coordinates. We now prove (4) by using Parseval's identity (Theorem 1).

Since the edges $OP_1, OP_2, \dots, OP_n$ are mutually orthogonal, we have the vectors

$$\vec{e}_1 = \frac{\overrightarrow{OP_1}}{|\overrightarrow{OP_1}|}, \vec{e}_2 = \frac{\overrightarrow{OP_2}}{|\overrightarrow{OP_2}|}, \dots, \vec{e}_n = \frac{\overrightarrow{OP_n}}{|\overrightarrow{OP_n}|}$$

are an orthonormal basis of $\mathbb{R}^n$. Since H is orthogonal projection of O on hyperplane $(P_1P_2\dots P_n)$, the triangle OHP_i is right-angled at H $(i = 1, \dots, n)$ so we have the identity of dot product

$$\overrightarrow{OH} \cdot \overrightarrow{OP_i} = OH^2, \qquad 1 \leqslant i \leqslant n.$$

From this and using Parseval's identity (Theorem 1), we have

$$OH^2 = \left|\overrightarrow{OH}\right|^2 = \sum_{i=1}^{n} \left(\overrightarrow{OH} \cdot \vec{e}_i\right)^2 = \sum_{i=1}^{n} \left(\overrightarrow{OH} \cdot \frac{\overrightarrow{OP_i}}{\left|\overrightarrow{OP_i}\right|}\right)^2$$

$$= \sum_{i=1}^{n} \frac{\left(\overrightarrow{OH} \cdot \overrightarrow{OP_i}\right)^2}{\left|\overrightarrow{OP_i}\right|^2} = \sum_{i=1}^{n} \frac{\left(OH^2\right)^2}{OP_i^2}. \tag{5}$$

By dividing both sides of (5) by OH^4, we obtain (4). This completes the proof.

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References

1. P. S. Donchian and H. S. M. Coxeter, An n-dimensional extension of Pythagoras' Theorem, *Math. Gaz.* **19** (July 1935) p. 206.
2. M. Hazewinkel, Parseval equality, *Encyclopedia of Mathematics*, Springer.

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