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RANK TWO TOPOLOGICAL AND INFINITESIMAL EMBEDDED JUMP LOCI OF QUASI-PROJECTIVE MANIFOLDS

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Abstract We study the germs at the origin of G-representation varieties and the degree 1 cohomology jump loci of fundamental groups of quasi-projective manifolds. Using the Morgan–Dupont model associated to a convenient compactification of such a manifold, we relate these germs to those of their infinitesimal counterparts, defined in terms of flat connections on those models. When the linear algebraic group G is either $\mathrm{SL}_2(\mathbb{C})$ or its standard Borel subgroup and the depth of the jump locus is 1, this dictionary works perfectly, allowing us to describe in this way explicit irreducible decompositions for the germs of these embedded jump loci. On the other hand, if either $G = \mathrm{SL}_n(\mathbb{C})$ for some $n \geqslant 3$, or the depth is greater than 1, then certain natural inclusions of germs are strict.

Keywords: representation variety; variety of flat connections; cohomology jump loci; analytic germ; differential graded algebra model; quasi-projective manifold; admissible map; Deligne weight filtration; holonomy Lie algebra; semisimple Lie algebra

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Contents

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1. Introduction and statement of results

1.1. Representation varieties and cohomology jump loci

Let X be a path-connected space having the homotopy type of a finite CW-complex, and let G be a complex, linear algebraic group. The representation variety $\mathrm{Hom}(\pi_1(X), G)$ is the parameter space for locally constant sheaves on X whose monodromies factor through G. The *characteristic varieties* of X with respect to a rational representation $\iota\colon G \to \mathrm{GL}(V)$ are the jump loci

$$\mathscr{V}^i_r(X, \iota) = \{\rho \in \mathrm{Hom}(\pi_1(X), G) \mid \dim H^i(X, V_{\iota\circ\rho}) \geqslant r\}, \tag{1.1}$$

defined for each degree $i \geqslant 0$ and depth $r \geqslant 0$. These loci, which record the variation of twisted cohomology inside the parameter space, encode subtle information about the topology of X and its covering spaces. We focus here mainly on the degree 1 jump loci, which depend only on the fundamental group $\pi_1(X)$ and the representation ι.

We work throughout over $\mathbb{C}$, unless otherwise mentioned. For an affine variety $\mathscr{X}$, we denote by $\mathscr{X}_{(x)}$ the analytic germ of $\mathscr{X}$ at a point $x \in \mathscr{X}$. Affine varieties and analytic germs are always reduced.

The study of analytic germs of embedded cohomology jump loci is a basic problem in deformation theory with homological constraints. Building on the foundational work of Goldman and Millson [13], it was shown in [8] that the germs at the origin $1 \in \mathrm{Hom}(\pi_1(X), G)$ of those loci are isomorphic to the germs at the origin of embedded infinitesimal jump loci of a commutative differential graded algebra (for short, CDGA) that is a finite model for the space X. In [3], Budur and Wang extended this result away from the origin, by developing a theory of differential graded Lie algebra modules which control the corresponding deformation problem.

The case when $G = \mathrm{SL}_2(\mathbb{C})$ has received a lot of attention in the literature. For instance, results from [21] reveal an unexpected connection between $\mathrm{SL}_2(\mathbb{C})$ representation varieties and the monodromy action on the homology of Milnor fibers of central hyperplane arrangements: for a line arrangement in $\mathbb{CP}^2$, combinatorial information (namely the nonexistence of points of intersection with multiplicity properly divisible by 3) implies the fact that all roots of the characteristic polynomial of the monodromy action on the first homology of the Milnor fiber of order a power of 3 have multiplicity at most 2 (a delicate topological property). The proof from [21, § 7.3] uses a construction related to the irreducible decomposition of the analytic germ $\mathrm{Hom}(\pi_1(X), \mathrm{SL}_2(\mathbb{C}))_{(1)}$, where X is the arrangement complement. On the other hand, the universality theorem of Kapovich and Millson [16] shows that rank 2 representation varieties may have arbitrarily bad

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singularities away from 1. This leads us to focus on germs at the origin of such varieties, and look for explicit descriptions via infinitesimal CDGA methods.

1.2. Flat connections and resonance varieties

The infinitesimal analog of the G-representation variety around the origin is the set $\mathscr{F}(A, \mathfrak{g})$ of $\mathfrak{g}$-valued flat connections on a commutative, differential, positively graded $\mathbb{C}$-algebra A, where $\mathfrak{g}$ is the Lie algebra of the Lie group G. This set consists of all elements $\omega \in A^1 \otimes \mathfrak{g}$ which satisfy the Maurer–Cartan equation, $d\omega + \frac{1}{2}[\omega, \omega] = 0$. If A^1 is finite-dimensional, then the set of flat connections is a Zariski-closed subset of the affine space $A^1 \otimes \mathfrak{g}$. Furthermore, $\mathscr{F}(A, \mathfrak{g})$ contains the closed subvariety $\mathscr{F}^1(A, \mathfrak{g})$ consisting of all tensors of the form $\eta \otimes g$ with $d\eta = 0$.

To define the infinitesimal counterpart of the jump loci (1.1), let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation. For each $\mathfrak{g}$-valued flat connection ω, there is an associated covariant derivative, $d_\omega\colon A^\bullet \otimes V \to A^{\bullet+1} \otimes V$, given by $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$, and satisfying $d_\omega^2 = 0$. The *resonance varieties* of A with respect to θ are the sets

$$\mathscr{R}^i_r(A, \theta) = \{\omega \in \mathscr{F}(A, \mathfrak{g}) \mid \dim H^i(A \otimes V, d_\omega) \geqslant r\}. \tag{1.2}$$

If A is finite-dimensional, then these sets are Zariski-closed in $\mathscr{F}(A, \mathfrak{g})$. Furthermore, if $H^i(A) \neq 0$, then $\mathscr{R}^i_1(A, \theta)$ contains the closed subvariety $\Pi(A, \theta)$ consisting of all elements $\eta \otimes g \in \mathscr{F}^1(A, \mathfrak{g})$ with $\det \theta(g) = 0$.

1.3. Quasi-Kähler manifolds and admissible maps

We now turn our attention to a class of spaces for which the characteristic varieties are constrained by some powerful structural results. Let M be a quasi-Kähler manifold, that is, the complement of a normal crossing divisor D in a compact, connected Kähler manifold $\overline{M}$. A map $f\colon M \to C$ from such a manifold to a smooth complex curve C is said to be *admissible* if f is holomorphic and surjective, and f admits a holomorphic, surjective extension between suitable compactifications, $\bar{f}\colon \overline{M} \to \overline{C}$, such that all the fibers of $\bar{f}$ are connected.

As shown by Arapura in [1], there exists a finite set $\mathscr{E}(M)$ of equivalence classes of 'admissible' maps from M to smooth curves of negative Euler characteristic, up to reparametrization in the target. For each such map $f\colon M \to M_f$, we denote by $f_\sharp\colon \pi \to \pi_f$ the induced homomorphism on fundamental groups; the admissibility condition insures that $f_\sharp$ is surjective. Let $\mathrm{abf}\colon \pi \twoheadrightarrow \pi_{\mathrm{abf}}$ be the projection of the group π onto its maximal torsion-free abelian quotient. We will denote by $f_0\colon M \to K(\pi_{\mathrm{abf}}, 1)$ the corresponding classifying map, which is determined up to homotopy by the property that $(f_0)_\sharp = \mathrm{abf}$. Furthermore, we will write $E(M) = \mathscr{E}(M) \cup \{f_0\}$.

Let G be a complex linear algebraic group, let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation, and let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be its tangential representation. For all $r \geqslant 0$, we have inclusions

$$\mathscr{V}^1_r(\pi, \iota) \supseteq \bigcup_{f \in E(M)} f_\sharp^* \mathscr{V}^1_r(\pi_f, \iota), \tag{1.3}$$

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where $f_\sharp^*\colon \operatorname{Hom}(\pi_f, G) \to \operatorname{Hom}(\pi, G)$ denotes the induced morphism on representation varieties. For $r = 0$ and 1, the inclusions from (1.3) are equivalent to the two inclusions

$$\operatorname{Hom}(\pi, G) \supseteq \operatorname{abf}^* \operatorname{Hom}(\pi_{\text{abf}}, G) \cup \bigcup_{f \in \mathscr{E}(M)} f_\sharp^* \operatorname{Hom}(\pi_f, G), \tag{1.4}$$

$$\mathscr{V}_1^1(\pi, \iota) \supseteq \operatorname{abf}^* \mathscr{V}_1^1(\pi_{\text{abf}}, \iota) \cup \bigcup_{f \in \mathscr{E}(M)} f_\sharp^* \operatorname{Hom}(\pi_f, G). \tag{1.5}$$

The case when $b_1(M) = 0$ is trivial, since $\operatorname{Hom}(\pi, G)_{(1)} = \{1\}$, while both $\mathscr{V}_1^1(\pi, \iota)_{(1)}$ and $\mathscr{E}(M)$ are empty in that situation. So, it is harmless to assume that $b_1(M) > 0$.

In the rank 1 case, i.e., the case when $G = \mathbb{C}^\times$ and ι identifies $\mathbb{C}^\times$ with $\mathrm{GL}_1(\mathbb{C})$, equality near 1 in (1.5) holds, by a deep result of Arapura [1] on the structure of $\mathscr{V}_1^1(\pi, \iota)_{(1)}$. In particular, every nontrivial character $\rho \in \operatorname{Hom}(\pi, \mathbb{C}^\times)$ sufficiently close to the trivial character and such that $H^1(\pi, \mathbb{C}_\rho) \neq 0$ must belong to $f_\sharp^* \operatorname{Hom}(\pi_f, \mathbb{C}^\times)$, for some $f \in \mathscr{E}(M)$. For a more general treatment of factorization results of this nature we refer to the book by Zuo [27] and to the recent work of Campana *et al.* [4, 5].

1.4. Quasi-projective manifolds and transversality

We specialize now to the case when M is a connected, smooth quasi-projective variety (for short, a quasi-projective manifold). Let $(\overline{M}, D)$ be a convenient compactification of M, where $\overline{M}$ is now a projective manifold, and D is a union of smooth hypersurfaces, intersecting locally like hyperplanes. Work of Morgan [20], as recently sharpened by Dupont in [11], associates to these data a bigraded, rationally defined CDGA, $A = \mathrm{OS}(\overline{M}, D)$, called the Orlik–Solomon model of M. This CDGA is a finite model of M, i.e., it is connected ($A^0 = \mathbb{C}$), finite-dimensional as a $\mathbb{C}$-vector space, and weakly equivalent to the de Rham algebra of M. Furthermore, A is functorial with respect to regular morphisms of pairs $(\overline{M}, D)$ as above.

For an admissible map $f\colon M \to M_f$, we will denote by $\Phi_f\colon A_f \to A$ the induced CDGA map between the respective Orlik–Solomon models. Let $\Phi_f^*\colon \mathscr{F}(A_f, \mathfrak{g}) \to \mathscr{F}(A, \mathfrak{g})$ be the morphism induced by Φ_f between the respective varieties of flat connections. Assuming as before that $b_1(M) > 0$, we obtain the following infinitesimal counterparts of inclusions (1.4)–(1.5):

$$\mathscr{F}(A, \mathfrak{g}) \supseteq \mathscr{F}^1(A, \mathfrak{g}) \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g}), \tag{1.6}$$

$$\mathscr{R}_1^1(A, \theta) \supseteq \Pi(A, \theta) \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g}). \tag{1.7}$$

This brings us to our first result, which can be viewed as a 'transversality' theorem for the subvarieties which appear on the right-hand side of inclusions (1.4)–(1.7). This result summarizes Theorems 7.2 and 7.10, and will be proved in §§ 7.1 and 7.5.

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Theorem 1.1. *Let M be a quasi-Kähler manifold, and let $f, g \in \mathscr{E}(M)$ be two distinct admissible maps.*

(1) *If M is a quasi-projective manifold, then*

$$\Phi_f^* \mathscr{F}(A_f, \mathfrak{g}) \cap \Phi_g^* \mathscr{F}(A_g, \mathfrak{g}) = \{0\}.$$

(2) *If M is either a compact, connected Kähler manifold or the complement of a central complex hyperplane arrangement, then*

$$f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \cap g_\sharp^* \operatorname{Hom}(\pi_g, G)_{(1)} = \{1\}.$$

In the rank 1 case, part (2) of the theorem also follows from results in [10]. Moreover, if M is an arrangement complement, an equivalent statement can be found in [17], again only in the rank 1 case. The novelty here is that a completely analogous statement holds for arbitrary complex linear algebraic groups G.

1.5. Topological versus infinitesimal factorizations

Our main goal in this paper is to analyze the decomposition into irreducible components of the germs at 1 of the embedded jump loci $(\operatorname{Hom}(\pi, G), \mathscr{V}_1^1(\pi, \iota))$ and the germs at 0 of their infinitesimal analogs, $(\mathscr{F}(A, \mathfrak{g}), \mathscr{R}_1^1(A, \theta))$, in the case when π is the fundamental group of a quasi-projective manifold.

A key step in this direction is the next theorem, which establishes a very strong connection between equalities in (1.4)–(1.7), and opens the way for using infinitesimal computations to derive factorization results near 1.

Theorem 1.2. *Let M be quasi-projective manifold with $b_1(M) > 0$. For an arbitrary rational representation of $G = \mathrm{SL}_2(\mathbb{C})$ or its standard Borel subgroup $\mathrm{Sol}_2(\mathbb{C})$, the following statements are equivalent.*

(1) *The inclusion (1.4) becomes an equality near* 1*.*
(2) *Both (1.4) and (1.5) become equalities near* 1*.*
(3) *The inclusion (1.6) is an equality, for some convenient compactification of M.*
(4) *Both (1.6) and (1.7) are equalities, for any convenient compactification of M.*

This theorem, which will be proved in § 7.2, provides a topological interpretation for [22, Question 8.4], which asks whether statement (4) from above always holds.

1.6. Irreducible decompositions for germs of embedded jump loci

The next theorem, which will be proved in § 7.3, is our main result regarding the irreducible decomposition around the origin of the rank 2 topological and infinitesimal embedded jump loci of quasi-projective manifolds.

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Theorem 1.3. *With notation as above, suppose the equivalent properties from Theorem 1.2 are satisfied.*

(1) *If $b_1(M_f) \neq b_1(M)$ for all $f \in \mathscr{E}(M)$, then we have the following decompositions into irreducible components of analytic germs:*

$$\operatorname{Hom}(\pi, G)_{(1)} = \mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cup \bigcup_{f \in \mathscr{E}(M)} f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)}, \tag{1.8}$$

$$\mathscr{V}_1^1(\pi, \iota)_{(1)} = \mathrm{abf}^* \mathscr{V}_1^1(\pi_{\mathrm{abf}}, \iota)_{(1)} \cup \bigcup_{f \in \mathscr{E}(M)} f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)}, \tag{1.9}$$

$$\mathscr{F}(A, \mathfrak{g})_{(0)} = \mathscr{F}^1(A, \mathfrak{g})_{(0)} \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}, \tag{1.10}$$

$$\mathscr{R}_1^1(A, \theta)_{(0)} = \Pi(A, \theta)_{(0)} \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}. \tag{1.11}$$

(2) *If $b_1(M_f) = b_1(M)$ for some $f \in \mathscr{E}(M)$, then we have the following equalities of irreducible germs:*

$$\operatorname{Hom}(\pi, G)_{(1)} = f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)}, \tag{1.12}$$

$$\mathscr{V}_1^1(\pi, \iota)_{(1)} = f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)}, \tag{1.13}$$

$$\mathscr{F}(A, \mathfrak{g})_{(0)} = \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}, \tag{1.14}$$

$$\mathscr{R}_1^1(A, \theta)_{(0)} = \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}. \tag{1.15}$$

(3) *For any two distinct admissible maps $f, g \in \mathscr{E}(M)$,*

$$f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \cap g_\sharp^* \operatorname{Hom}(\pi_g, G)_{(1)} = \{1\}.$$

Under our assumptions, this theorem gives a local, more precise and simple, classification for representations of π into G, when compared to the similar global, more sophisticated classification obtained by Corlette–Simpson [6] and Loray *et al.* [18] in the case when $G = \mathrm{SL}_2(\mathbb{C})$, see Remark 7.8. Furthermore, as explained in Remark 7.6, all irreducible components appearing in Theorem 1.3 are known, for an arbitrary quasi-projective manifold with $b_1(M) > 0$, and for an arbitrary rational representation of $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{Sol}_2(\mathbb{C})$,

1.7. Applying the decomposition results

By now, the reader may wonder whether our structural results on the irreducible decompositions of germs of embedded jump loci apply in any meaningful way. The next theorem, which will be proved in § 7.4, seeks to dispel such possible doubts, by providing a rich supply of quasi-projective manifolds for which both (1.4) and (1.5) hold as equalities near 1, in the rank 2 case.

Theorem 1.4. *Suppose M is quasi-projective manifold satisfying one of the following hypotheses.*

(1) *M is projective.*

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(2) *The Deligne weight filtration has the property that* $W_1 H^1(M) = 0$.

(3) M *is the partial configuration space of a projective curve associated to an arbitrary finite simple graph.*

(4) $\mathscr{R}_1^1(H^\bullet(M), d = 0) = \{0\}$.

(5) $M = S \setminus \{0\}$, *where* S *is a quasi-homogeneous affine surface having a normal, isolated singularity at* 0.

Then, for $G = \mathrm{SL}_2(\mathbb{C})$ *or* $\mathrm{Sol}_2(\mathbb{C})$, *the equivalent properties from Theorem 1.2 are satisfied, and thus, the conclusions of Theorem 1.3 hold.*

It would be interesting to decide whether the conclusions of Theorem 1.3 hold for arbitrary quasi-projective manifolds, not just the ones from the above list.

1.8. Beyond depth 1 or rank 2

As shown in the next result (which will be proved in Theorem 8.1), the higher-rank analog of the local equality (1.8) fails, even for some very simple, weighted-homogeneous quasi-projective surfaces.

Theorem 1.5. *Let* $M = S \setminus \{0\}$, *where* S *is a quasi-homogeneous affine surface having a normal, isolated singularity at* 0. *If* $b_1(M) > 0$ *and* $G = \mathrm{SL}_n(\mathbb{C})$ *with* $n \geqslant 3$, *then inclusion (1.4) is strict near* 1.

Finally, as shown in the next result (which will be proved in Theorem 9.3), the higher-depth analog of Theorem 1.3 also may fail, even in rank 2.

Theorem 1.6. *Let* M *be a connected, compact Kähler manifold, or the complement of a central complex hyperplane arrangement, and suppose there exists an admissible map* $f\colon M \to M_f$ *such that* $b_1(M_f) < b_1(M)$. *If* $\iota\colon G \to \mathrm{GL}(V)$ *is a rational representation of* $G = \mathrm{SL}_2(\mathbb{C})$ *or* $\mathrm{Sol}_2(\mathbb{C})$, *having a non-zero fixed vector* $v \in V^G$, *there is then an integer* $r > 1$ *such that inclusion (1.3) is strict near* 1.

Concrete instances where this theorem applies are given in Examples 9.4 and 9.5. On the other hand, when M is a connected, compact Kähler manifold or the complement of a central complex hyperplane arrangement and $\iota\colon G \to \mathrm{GL}(V)$ is an arbitrary rational representation of $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{Sol}_2(\mathbb{C})$, then, as shown in [23], the local equalities (1.8) and (1.9) hold.

2. Local analytic germs

2.1. Irreducible decompositions

This section contains the necessary preparatory material pertaining to irreducible decompositions of complex affine varieties and analytic germs. We start with a lemma which will be used repeatedly in the sequel.

Lemma 2.1. *Let X be an analytic germ, and assume that*

$$\bigcup_{i\in I} Y_i \subseteq X = \bigcup_{i\in I} X_i,$$

where the indexing set I is finite, all germs X_i, Y_i are irreducible, $\dim X_i = \dim Y_i$ *for all i, and $Y_i \not\subseteq Y_j$ for $i \neq j$. Then:*

(1) *There is a bijection $b\colon I \to I$ such that $Y_i = X_{b(i)}$, for all i;*

(2) $X = \bigcup_{i\in I} Y_i$.

Proof. Let $J \subseteq I$ be a subset. To prove part (1), we will construct by induction on the cardinality of J an injection $b\colon J \hookrightarrow I$ with the property that $Y_i = X_{b(i)}$, for all $i \in J$, starting with $J = \emptyset$. It will be useful to consider the dimension partition, $I = \coprod I_d$, where

$$I_d = \{i \in I \mid \dim X_i = \dim Y_i = d\}. \tag{2.1}$$

Clearly, the injection b must respect the partition blocks.

For the induction step, assume $J \neq I$ and pick $i_0 \in I \setminus J$ such that $d_0 = \dim Y_{i_0}$ maximizes $\dim Y_i$ for $i \in I \setminus J$. Plainly, $I_d \subseteq J$ and $b(I_d) = I_d$, for $d > d_0$.

Since Y_{i_0} is irreducible, $Y_{i_0} \subseteq X_i$, for some $i \in I$. Set $d = \dim X_i \geqslant d_0$. If $d > d_0$, then $i = b(j)$ for some $j \in I_d \subseteq J$, by the previous remark. This implies that $X_i = Y_j$, by the induction assumption. We infer that $Y_{i_0} \subseteq Y_j$, a contradiction. Hence, $d = d_0$, and therefore $Y_{i_0} = X_i$, by irreducibility.

Set $J' = J \coprod \{i_0\}$ and extend b to J' by defining $b(i_0) = i$. Then clearly $Y_j = X_{b(j)}$, for all $j \in J'$. To finish the induction, we have to check that i cannot be of the form $b(j)$ with $j \in J$. Otherwise, $X_i = Y_j$, by the induction hypothesis. This implies that $Y_{i_0} = Y_j$, a contradiction.

Finally, the equality from part (2) is a direct consequence of part (1). □

2.2. From inclusions to equalities

Next, we delineate conditions under which inclusions of affine varieties or analytic germs become equalities.

Lemma 2.2. *Let $\mathscr{X}$ be an affine variety with the property that all irreducible components pass through $x \in \mathscr{X}$. If $\mathscr{X}'$ is another affine variety such that the inclusion $\mathscr{X} \subseteq \mathscr{X}'$ holds near x, then the inclusion holds globally.*

Proof. The argument from the first paragraph of [8, § 9.23] establishes the claim. □

Lemma 2.3. *If X and Y are isomorphic germs, and $X \subseteq Y$, then $X = Y$.*

Proof. The inclusion $X \subseteq Y$ induces an epimorphism on coordinate rings, which must be an isomorphism, by the Hopfian property of Noetherian rings, see e.g., [25, p. 65]. □

2.3. Local versus global irreducibility

Finally, we describe a setting in which global and local irreducibility are equivalent. We say that an affine subvariety $\mathscr{X} \subseteq \mathbb{C}^n$ has *positive weights* if $\mathscr{X}$ is invariant with respect to a $\mathbb{C}^{\times}$-action on $\mathbb{C}^n$ with positive weights.

Lemma 2.4. *If $\mathscr{X}$ has positive weights, then all its irreducible components pass through 0. Moreover, the global irreducibility of $\mathscr{X}$ is equivalent to the local irreducibility of the germ $\mathscr{X}_{(0)}$.*

Proof. Since the algebraic group $\mathbb{C}^\times$ is connected, the action by $\mathbb{C}^\times$ on $\mathbb{C}^n$ leaves the irreducible components of $\mathscr{X}$ invariant. Fix such a component $\mathcal{Y}$, and let $x \in \mathcal{Y}$. Then $tx \in \mathcal{Y}$ for all $t \in \mathbb{C}^\times$, and thus $0 = \lim_{t\to 0} tx$ must also belong to $\mathcal{Y}$, since the action has positive weights.

To prove the second claim, let us consider the canonical algebra morphisms, $R \to S \to T$, relating polynomials, convergent series and formal series in n variables. Given an ideal $I \subset R$, denote by J and K the ideals generated by I in S and T, respectively. When I is the defining ideal of an affine subvariety $\mathscr{X} \subseteq \mathbb{C}^n$ having positive weights, we know that I is generated by finitely many polynomials which are homogeneous with respect to the positive weights of the variables. We infer that an element $f \in R$ (respectively $f \in T$) belongs to I (respectively K) if and only if all its weighted-homogeneous components f_i are in I. Hence, R/I canonically embeds into T/K. Consequently, if T/K is a reduced ring (or a domain), the ring R/I has the same property. We claim that both implications above are actually equivalences.

Granting this claim, we may finish our proof, as follows. Given an arbitrary ideal $J \subset S$, it is well known that the canonical algebra morphism, $S/J \to T/K$ (where K is as above), is injective, cf. [25, p. 36]. It follows from [25, Corollary II.4.2 and Theorem II.4.5] that T/K is a reduced ring (a domain) if and only if the ring S/J has the same property. Together with the above claim, this shows that R/I is a reduced ring (a domain) if and only if the ring S/J has the same property. Since R/I and S/J are the coordinate rings of $\mathscr{X}$ and $\mathscr{X}_{(0)}$, respectively, we infer that $\mathscr{X}$ is irreducible if and only if $\mathscr{X}_{(0)}$ is irreducible, as asserted.

Going back to the above claim, let us show that if R/I a domain then T/K is a domain as well. (The reduced property can be verified by a similar argument.) Otherwise, we may find two formal series with the property that $f \not\equiv 0$ (modulo K) and $g \not\equiv 0$ (modulo K), for which $fg \equiv 0$ (modulo K). Plainly, we may assume that their weighted initial terms, $f_p \in R$ (respectively, $g_q \in R$), do not belong to I. But then the initial term of the product, $f_p g_q$, must belong to I. This contradiction completes our proof. □

3. Embedded jump loci

3.1. Representation varieties and characteristic varieties

Let π be a discrete group, and let G be a $\mathbb{C}$-linear algebraic group. The set $\mathrm{Hom}(\pi, G)$ of group homomorphisms from π to G, called the *G-representation variety* of π, depends bi-functorially on π and G. Furthermore, this set comes equipped with a natural base point, namely, the trivial representation, 1.

Assuming now that π is a finitely generated group, the set $\mathrm{Hom}(\pi, G)$ acquires a natural structure of affine variety. Furthermore, every homomorphism $\varphi\colon \pi \to \pi'$ induces an algebraic morphism between the corresponding representation varieties, $\varphi^*\colon \mathrm{Hom}(\pi', G) \to \mathrm{Hom}(\pi, G)$.

Now let (X, x_0) be a pointed, path-connected space, and let $\pi = \pi_1(X, x_0)$ be its fundamental group. Then the representation variety $\mathrm{Hom}(\pi, G)$ is the parameter space for finite-dimensional local systems on X of type G, see e.g., [26, Ch. VI]. Given a representation $\tau\colon \pi \to \mathrm{GL}(V)$, we let V_τ denote the local system on X associated to τ, that is, the left π-module V defined by $g \cdot v = \tau(g)v$. Furthermore, we let $H^\bullet(X, V_\tau)$ be the twisted cohomology of X with coefficients in this local system, as in [26, Ch. VI]. (If X is semilocally 1-connected, a classical result of S. Eilenberg identifies twisted homology on X with equivariant homology on the universal cover of X.)

Definition 3.1. The *characteristic varieties* of the space X in degree $i \geqslant 0$ and depth $r \geqslant 0$ with respect to a representation $\iota\colon G \to \mathrm{GL}(V)$ are the sets

$$\mathcal{V}^i_r(X, \iota) = \{\rho \in \mathrm{Hom}(\pi, G) \mid \dim H^i(X, V_{\iota\circ\rho}) \geqslant r\}.$$

For instance, $\mathcal{V}^0_1(X, \iota)$ consists of those representations $\rho\colon \pi \to G$ for which there exists a non-zero vector $v \in V$ such that $\iota(\rho(g))(v) = v$, for all $g \in \pi$. In the rank 1 case, i.e., when ι is the canonical identification $\mathbb{C}^\times \to \mathrm{GL}_1(\mathbb{C})$, we will drop the map ι from the notation, and simply write $\mathcal{V}^i_r(X)$. For each $i \geqslant 0$, the sequence $\{\mathcal{V}^i_r(X, \iota)\}_{r\geqslant 0}$ is a descending filtration of $\mathrm{Hom}(\pi, G) = \mathcal{V}^i_0(X, \iota)$. We will refer to the pairs

$$(\mathrm{Hom}(\pi, G), \mathcal{V}^i_r(X, \iota)) \tag{3.1}$$

as the (global) *embedded jump loci* of X with respect to ι. Clearly, such pairs depend only on the homotopy type of X and on the representation ι.

Assume now that the space X has the homotopy type of a finite, connected CW-complex (in particular, X is path-connected and locally simply connected), and that the map $\iota\colon G \to \mathrm{GL}(V)$ is a rational representation. Then the sets $\mathcal{V}^i_r(X, \iota)$ are closed subvarieties of the representation variety $\mathrm{Hom}(\pi, G)$. The following simple example will be useful later on.

Example 3.2. Let X be a connected, 2-dimensional CW-complex, and assume $\chi(X) < 0$. Then $\mathcal{V}^1_1(X, \iota) = \mathrm{Hom}(\pi_1(X), G)$, for any rational representation $\iota\colon G \to \mathrm{GL}(V)$. Indeed, let $\rho\colon \pi_1(X) \to G$ be a homomorphism. Writing $b_i(X, \rho) = \dim H^i(X, V_{\iota\circ\rho})$, we have that

$$b_0(X, \rho) - b_1(X, \rho) + b_2(X, \rho) = \chi(X)\dim(V) < 0.$$

This forces $b_1(X, \rho) > 0$, thereby showing that $\rho \in \mathcal{V}^1_1(X, \iota)$.

The embedded jump loci enjoy a useful naturality property, which we record in the next lemma (see [23, Corollary 5.8] for a proof, in a more general setting).

Lemma 3.3. *Let $f\colon X \to X'$ be a pointed map between two spaces as above. Assume that the induced homomorphism on fundamental groups, $f_\sharp\colon \pi_1(X) \to \pi_1(X')$, is surjective. Then the morphism induced by $f_\sharp$ on representation varieties,*

$$f^*_\sharp\colon \mathrm{Hom}(\pi_1(X'), G) \longrightarrow \mathrm{Hom}(\pi_1(X), G), \tag{3.2}$$

is a closed embedding, which restricts to embeddings $\mathcal{V}^i_r(X', \iota) \to \mathcal{V}^i_r(X, \iota)$ *for all* $i \leqslant 1$ *and* $r \geqslant 0$*, and induces isomorphisms between* $\mathcal{V}^0_r(X', \iota)$ *and* $\mathcal{V}^0_r(X, \iota) \cap f^*_\sharp \operatorname{Hom}(\pi_1(X'), G)$*, for all* $r \geqslant 0$*.*

Finally, suppose $K(\pi, 1)$ is a classifying space for the group π. In this case, we will simply denote the corresponding characteristic varieties by $\mathcal{V}^i_r(\pi, \iota)$. If X is a pointed space, and $f: X \to K(\pi, 1)$ is a classifying map for its fundamental group, then the induced isomorphism $f^*_\sharp: \operatorname{Hom}(\pi, G) \to \operatorname{Hom}(\pi, G)$ restricts to isomorphisms $\mathcal{V}^1_r(\pi, \iota) \cong \mathcal{V}^1_r(X, \iota)$ for all $r \geqslant 0$, see [23, Corollary 5.11].

3.2. Flat connections

We now turn to the infinitesimal counterparts of the above constructions, following closely the exposition from [19, §§ 2–3]. Let (A, d) be commutative, differential graded algebra (for short, a CDGA) over $\mathbb{C}$, and let $\mathfrak{g}$ be a Lie algebra, also over $\mathbb{C}$. The tensor product $A \otimes \mathfrak{g}$ has the structure of a graded, differential Lie algebra, with Lie bracket given by $[\alpha \otimes x, \beta \otimes y] = \alpha\beta \otimes [x, y]$, and differential given by $\partial(\alpha \otimes x) = d\alpha \otimes x$. Clearly, this construction is functorial in both A and $\mathfrak{g}$.

The algebraic analog of the G-representation variety $\operatorname{Hom}(\pi, G)$ is the (bi-functorial) pointed set $(\mathcal{F}(A, \mathfrak{g}), 0)$ of $\mathfrak{g}$-*valued flat connections* on A, consisting of degree 1 elements in $A \otimes \mathfrak{g}$ that satisfy the Maurer–Cartan equation,

$$\partial\omega + \tfrac{1}{2}[\omega, \omega] = 0. \tag{3.3}$$

The CDGA A is said to be connected if A^0 is the $\mathbb{C}$-span of 1. For such an algebra, the bilinear map $P: A^1 \times \mathfrak{g} \to A^1 \otimes \mathfrak{g}$, $(\eta, g) \mapsto \eta \otimes g$ induces a map $P: H^1(A) \times \mathfrak{g} \to \mathcal{F}(A, \mathfrak{g})$. The *essentially rank 1* part of $\mathcal{F}(A, \mathfrak{g})$ is the set

$$\mathcal{F}^1(A, \mathfrak{g}) := P(H^1(A) \times \mathfrak{g}). \tag{3.4}$$

Suppose now that both A^1 and $\mathfrak{g}$ are finite-dimensional. Then the set $\mathcal{F}(A, \mathfrak{g})$ has a natural structure of affine variety, which we shall call the $\mathfrak{g}$-*variety of flat connections* on A. Moreover, $\mathcal{F}^1(A, \mathfrak{g})$ is an irreducible, Zariski-closed subset of $\mathcal{F}(A, \mathfrak{g})$. More precisely, $\mathcal{F}^1(A, \mathfrak{g})$ is either $\{0\}$, or the cone on $\mathbb{P}(H^1(A)) \times \mathbb{P}(\mathfrak{g})$.

An alternate interpretation of these varieties is given in [19, § 4]. Set $A_i = (A^i)^*$, and let $\mathbb{L}(A_1)$ be the free Lie algebra on the dual vector space A_1. We then define the *holonomy Lie algebra* of A as

$$\mathfrak{h}(A) := \mathbb{L}(A_1)/\operatorname{ideal}(\operatorname{im}(d^* + \cup^*)), \tag{3.5}$$

where $d^*: A_2 \to A_1 = \mathbb{L}^1(A_1)$ and $\cup^*: A_2 \to A_1 \wedge A_1 = \mathbb{L}^2(A_1)$ are the maps dual to the differential and the multiplication map in A, respectively. Clearly, this construction is functorial. Moreover, as shown in [19, Proposition 4.5], the canonical isomorphism $A^1 \otimes \mathfrak{g} \cong \operatorname{Hom}(A_1, \mathfrak{g})$ restricts to a natural isomorphism

$$\mathcal{F}(A, \mathfrak{g}) \cong \operatorname{Hom}_{\texttt{Lie}}(\mathfrak{h}(A), \mathfrak{g}) \tag{3.6}$$

which identifies $\mathcal{F}^1(A, \mathfrak{g})$ with the set of Lie algebra morphisms from $\mathfrak{h}(A)$ to $\mathfrak{g}$ whose image is a vector subspace of dimension at most 1.

Finally, let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation, and consider the set

$$\Pi(A,\theta) = P(H^1(A) \times \mathbf{V}(\det \circ \theta)), \tag{3.7}$$

where $\det\colon \mathfrak{gl}(V) \to \mathbb{C}$ is the determinant, and $\mathbf{V}(f) := \{x \mid f(x) = 0\}$ is the zero set of a polynomial function f. Then $\Pi(A,\theta)$ is a Zariski-closed subset of $\mathscr{F}^1(A,\mathfrak{g})$ containing 0. Both $\mathscr{F}^1(A,\mathfrak{g})$ and $\Pi(A,\theta)$ behave functorially with respect to CDGA maps. Moreover, CDGA maps inducing an H^1-isomorphism also induce $\mathscr{F}^1$ and Π-isomorphisms, since the variety $\mathscr{F}^1(A,\mathfrak{g})$ depends only on $H^1(A)$ and $\mathfrak{g}$, and similarly $\Pi(A,\theta)$ depends only on $H^1(A)$ and θ.

3.3. Resonance varieties

Once again, consider a representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$. For each flat connection $\omega \in \mathscr{F}(A,\mathfrak{g})$, we turn the tensor product $A \otimes V$ into a cochain complex,

$$(A \otimes V, d_\omega)\colon\ A^0 \otimes V \xrightarrow{d_\omega} A^1 \otimes V \xrightarrow{d_\omega} A^2 \otimes V \xrightarrow{d_\omega} \cdots, \tag{3.8}$$

using as differential the covariant derivative $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$. Here, if we write $\omega = \sum_k a_k \otimes g_k$, for some $a_k \in A^1$ and $g_k \in \mathfrak{g}$, then $\mathrm{ad}_\omega(a \otimes v) = \sum_k a_k a \otimes \theta(g_k)(v)$, for all $a \in A$ and $v \in V$. It is readily checked that the flatness condition on ω insures that $d_\omega^2 = 0$.

Definition 3.4. The *resonance varieties* of the CDGA $A^\bullet$ in degree $i \geqslant 0$ and depth $r \geqslant 0$ with respect to a representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ are the sets

$$\mathscr{R}^i_r(A,\theta) = \{\omega \in \mathscr{F}(A,\mathfrak{g}) \mid \dim H^i(A \otimes V, d_\omega) \geqslant r\}. \tag{3.9}$$

For instance, $\mathscr{R}^0_1(A,\theta)$ consists of those flat connections $\omega = \sum_k a_k \otimes g_k$ for which there exists a non-zero vector $v \in V$ such that $\theta(g_k)(v) = 0$, for all k, see [19, Lemma 2.3], provided A is connected. For each $i \geqslant 0$, the sequence $\{\mathscr{R}^i_r(A,\theta)\}_{r\geqslant 0}$ is a descending filtration of the set $\mathscr{F}(A,\mathfrak{g}) = \mathscr{R}^i_0(A,\theta)$. We will refer to the pairs

$$(\mathscr{F}(A,\mathfrak{g}), \mathscr{R}^i_r(A,\theta)) \tag{3.10}$$

as the (global) *infinitesimal embedded jump loci* of A with respect to θ. In the rank 1 case, i.e., the case when θ is the canonical identification $\mathbb{C} \to \mathfrak{gl}_1(\mathbb{C})$, we will simply write $\mathscr{R}^i_r(A)$ for the corresponding sets. If $H^\bullet(A)$ is the cohomology algebra of A, we will view it as a CDGA with differential $d = 0$, and will denote the corresponding jump loci as $\mathscr{R}^i_r(H^\bullet(A),\theta)$, or simply $\mathscr{R}^i_r(H^\bullet(A))$ in the rank 1 case.

Now suppose A is a connected, finite-dimensional CDGA, and the map $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite-dimensional representation of a finite-dimensional Lie algebra. Then the sets $\mathscr{R}^i_r(A,\theta)$ are closed subvarieties of $\mathscr{F}(A,\mathfrak{g})$, often referred to as the *resonance varieties* of A with respect to θ.

They enjoy the following useful naturality property, proved in greater generality in [23, Corollary 5.10].

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Lemma 3.5. *In the above setup, suppose $\psi\colon A'\to A$ is a morphism between two such* CDGA*s, which is injective in degree* 1*. Then the natural morphism*

$$\psi^* = \psi\otimes \mathrm{id}\colon \mathscr{F}(A',\mathfrak{g}) \longrightarrow \mathscr{F}(A,\mathfrak{g}) \tag{3.11}$$

is a closed embedding, which restricts to embeddings $\mathscr{R}^i_r(A',\theta)\to\mathscr{R}^i_r(A,\theta)$ for all $i\leqslant 1$ and $r\geqslant 0$, and induces isomorphisms between $\mathscr{R}^0_r(A',\theta)$ and $\mathscr{R}^0_r(A,\theta)\cap\psi^*\mathscr{F}(A',\mathfrak{g})$, for all $r\geqslant 0$.

3.4. Algebraic models and germs of jump loci

Given a topological space X, we let $\Omega^\bullet(X)$ be the Sullivan algebra [24] of piecewise polynomial $\mathbb{C}$-forms on X. This CDGA has the property that $H^\bullet(\Omega(X))\cong H^\bullet(X,\mathbb{C})$, as graded rings. A CDGA A is said to be a *model* for X if A may be connected by a zig-zag of quasi-isomorphisms to $\Omega(X)$. For instance, if M is a smooth manifold, then the de Rham algebra $\Omega_{\mathrm{dR}}(M)$ of smooth $\mathbb{C}$-forms on M is a model for this manifold. We say that A is a *finite* model for X if the dimension of A (viewed as a $\mathbb{C}$-vector space) is finite, and A is connected.

Assume now that X is a path-connected space having the homotopy type of a finite CW-complex. Let $\pi=\pi_1(X)$, let $\iota\colon G\to\mathrm{GL}(V)$ be a rational representation, and let $\theta\colon\mathfrak{g}\to\mathfrak{gl}(V)$ be its tangential representation. We will use frequently the following result, proved in [8, Theorem B(1)].

Theorem 3.6. *Suppose that X admits a finite* CDGA *model A. There is then an analytic isomorphism of germs, $\mathscr{F}(A,\mathfrak{g})_{(0)}\xrightarrow{\simeq}\mathrm{Hom}(\pi,G)_{(1)}$, restricting to isomorphisms $\mathscr{R}^i_r(A,\theta)_{(0)}\xrightarrow{\simeq}\mathscr{V}^i_r(X,\iota)_{(1)}$, for all i,r.*

For later use, we will need the following lemmas.

Lemma 3.7. *Let $\pi=\mathbb{Z}^n$, with $n\geqslant 1$. Denote by A_0 the* CDGA *$(\bigwedge^\bullet H^1(\pi), d=0)$. Then A_0 is a finite model for the torus $T^n=K(\pi,1)$, and*

$$(\mathscr{F}(A_0,\mathfrak{g}),\mathscr{R}^1_1(A_0,\theta)) = (\mathscr{F}^1(A_0,\mathfrak{g}),\Pi(A_0,\theta)),$$

for every finite-dimensional Lie representation $\theta\colon\mathfrak{g}\to\mathfrak{gl}(V)$ of $\mathfrak{g}=\mathfrak{sl}_2$ or $\mathfrak{sol}_2$.

Proof. Since T^n is a formal space in the sense of Sullivan [24], the CDGA A_0 is a finite model of T^n. On the other hand, A_0 is the Chevalley–Eilenberg cochain CDGA of the (abelian) Malcev Lie algebra of $\pi=\mathbb{Z}^n$. Lemma 4.14 and Theorem 4.15 from [19] together imply our second claim. □

Lemma 3.8. *Let $\iota\colon G\to\mathrm{GL}(V)$ be a rational representation of a unipotent group. If $\theta = d_1(\iota)\colon\mathfrak{g}\to\mathfrak{gl}(V)$, then $\mathscr{F}^1(A,\mathfrak{g})=\Pi(A,\theta)$, for any connected* CDGA *A.*

Proof. Since the group G is unipotent, the homomorphism ι takes values in the upper-triangular unipotent subgroup of GL_m, where $m=\dim V$, by a classical result in representation theory [15]. Hence, the function $\det\circ\theta\colon\mathfrak{g}\to\mathbb{C}$ is identically 0. The claim then follows from the construction of $\mathscr{F}^1$ and Π. □

In the case when $G = \mathbb{C}$, more can be said.

Lemma 3.9. $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{C})_{(1)} = \mathscr{V}^1_1(\mathbb{Z}^n, \iota)_{(1)}$*, for all* $n \geqslant 1$.

Proof. Consider the CDGA A_0 from Lemma 3.7. We infer from Theorem 3.6 that

$$(\mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}), \mathscr{V}^1_1(\mathbb{Z}^n, \iota))_{(1)} \cong (\mathscr{F}(A_0, \mathbb{C}), \mathscr{R}^1_1(A_0, \theta))_{(0)}. \tag{3.12}$$

Clearly, $\mathscr{F}(A_0, \mathbb{C}) = \mathscr{F}^1(A_0, \mathbb{C})$. On the other hand, $\mathscr{F}^1(A_0, \mathbb{C}) = \Pi(A_0, \theta)$, by the preceding lemma. Finally, $\Pi(A_0, \theta) \subseteq \mathscr{R}^1_1(A_0, \theta)$, by [19, Theorem 1.2]. Therefore, $\mathscr{F}(A_0, \mathbb{C}) = \mathscr{R}^1_1(A_0, \theta)$. Our claim then follows from (3.12). □

4. Quasi-Kähler manifolds and admissible maps

4.1. Admissible maps to curves

Let M be a connected, complex manifold. We say that M is a *quasi-Kähler manifold* if $M = \overline{M} \setminus D$, where $\overline{M}$ is a connected, compact Kähler manifold and D is a normal crossing divisor. A map $f : M \to C$ from such a manifold to a smooth complex curve C is said to be *admissible* if f is holomorphic and surjective, and f admits a holomorphic, surjective extension between suitable compactifications, $\bar{f} : \overline{M} \to \overline{C}$, such that all the fibers of $\bar{f}$ are connected. It is readily checked that the homomorphism on fundamental groups induced by such a map, $f_\sharp : \pi_1(M) \to \pi_1(C)$, is surjective.

We denote by $\mathscr{E}(M)$ the family of admissible maps $f : M \to M_f$ to curves with negative Euler characteristic, modulo automorphisms of the target, and we denote by $f_\sharp : \pi \twoheadrightarrow \pi_f$ the corresponding induced homomorphisms. Deep work of Arapura [1] characterizes those irreducible components of the rank 1 characteristic variety $\mathscr{V}^1_1(M)$ which contain the origin of the character group $\mathrm{Hom}(\pi, \mathbb{C}^\times)$: all such components are connected, affine subtori, which can be described in terms of admissible maps, as follows.

Theorem 4.1 [1]. *For a quasi-Kähler manifold M, the set $\mathscr{E}(M)$ is finite. Moreover, the correspondence $f \rightsquigarrow f_\sharp^* \mathrm{Hom}(\pi_f, \mathbb{C}^\times)$ establishes a bijection between $\mathscr{E}(M)$ and the set of positive-dimensional, irreducible components of $\mathscr{V}^1_1(M)$ passing through* 1.

Let $\mathrm{abf} : \pi \twoheadrightarrow \pi_{\mathrm{abf}}$ be the projection of the group π onto its maximal torsion-free abelian quotient. We will denote by $f_0 : M \to K(\pi_{\mathrm{abf}}, 1)$ the corresponding classifying map, which is determined up to homotopy by the property that $(f_0)_\sharp = \mathrm{abf}$. Furthermore, we will write

$$E(M) = \mathscr{E}(M) \cup \{f_0\}. \tag{4.1}$$

4.2. Cohomology jump loci of quasi-Kähler manifolds

Now let G be a complex linear algebraic group, and let $\iota : G \to \mathrm{GL}(V)$ be a rational representation. By Lemma 3.3, the natural inclusion

$$\mathrm{Hom}(\pi, G) \supseteq \bigcup_{f \in E(M)} f_\sharp^* \mathrm{Hom}(\pi_f, G) \tag{4.2}$$

induces an inclusion

$$\mathscr{V}^i_r(\pi, \iota) \supseteq \bigcup_{f \in E(M)} f_\sharp^* \mathscr{V}^i_r(\pi_f, \iota) \tag{4.3}$$

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for $i = 0$ and 1, and for all $r \geqslant 0$. In order to prove Theorem 1.2 from the Introduction, we want to establish some criteria under which the inclusions (4.3) become equalities near 1, for $i = 1$ and $r = 0, 1$. In the case when $G = \mathbb{C}^\times$, $\iota = \mathrm{id}_{\mathbb{C}^\times}$, and $i = r = 1$, equality near 1 always holds in (4.3), and in fact is equivalent to Arapura's Theorem 4.1. To attack the general case, we start with some preliminary observations.

Suppose first that $b_1(M) = 0$. Plainly, $1 \notin \mathscr{V}^1_1(\pi, \iota)$, and therefore $\mathscr{V}^1_1(\pi, \iota)_{(1)} = \emptyset$. Hence, equality (4.3) follows trivially. Moreover, the natural map $\Omega(K(1, 1)) \to \Omega(K(\pi, 1))$ is a 1-equivalence; hence, π has the same 1-minimal model as the trivial group, cf. [7, 24]. It then follows from [8, Theorem A] that

$$\mathrm{Hom}(\pi, G)_{(1)} = \{1\}. \tag{4.4}$$

Therefore, equality (4.2) also holds trivially in this case. Thus, we may assume from now on that $b_1(M) > 0$.

In view of the discussion at the end of § 3.1, we may replace in (4.3) the group π by the manifold M, and likewise π_f by M_f. Moreover, for $i = r = 1$, the characteristic variety $\mathscr{V}^1_1(M_f, \iota)$ may be replaced in (4.3) by the representation variety $\mathrm{Hom}(\pi_f, G)$, when $f \in \mathscr{E}(M)$. Indeed, for each $f \in \mathscr{E}(M)$, the manifold M_f is a connected, 2-dimensional CW-complex with $\chi(M_f) < 0$. Thus, by the computation from Example 3.2, we have that

$$\mathscr{V}^1_1(M_f, \iota) = \mathrm{Hom}(\pi_f, G). \tag{4.5}$$

Finally, let $f \in E(M)$. By Lemma 3.3, the set $f^*_\sharp \mathrm{Hom}(\pi_f, G)$ is Zariski-closed in $\mathrm{Hom}(\pi, G)$, and the set $f^*_\sharp \mathscr{V}^1_1(\pi_f, \iota)$ is Zariski-closed in $\mathscr{V}^1_1(\pi, \iota)$. Furthermore, the analytic germ $f^*_\sharp \mathrm{Hom}(\pi_f, G)_{(1)}$ is isomorphic to $\mathrm{Hom}(\pi_f, G)_{(1)}$, and similarly $f^*_\sharp \mathscr{V}^1_1(\pi_f, \iota)_{(1)} \cong \mathscr{V}^1_1(\pi_f, \iota)_{(1)}$.

Remark 4.2. We also deduce from Lemma 3.3 that equality near 1 in (4.2) implies equality near 1 in (4.3), for $i = 0$ and all $r \geqslant 1$.

5. Quasi-projective manifolds and Orlik–Solomon models

5.1. Orlik–Solomon models

We now restrict our attention to a class of quasi-Kähler manifolds of great importance in complex algebraic geometry. Recall that a space M is said to be a quasi-projective variety if M is a Zariski open subset of a projective variety. By resolution of singularities, a connected, smooth, complex quasi-projective variety M can realized as $M = \overline{M} \setminus D$, where $\overline{M}$ is a connected, smooth, complex projective variety, and D is a normal crossing divisor. For short, we will say that M is a *quasi-projective manifold*.

Let $\overline{M}$ and $\overline{M'}$ be two projective manifolds, and let $D \subset \overline{M}$ and $D' \subset \overline{M'}$ be two divisors. A *regular morphism* of pairs, $\bar{f}\colon (\overline{M}, D) \to (\overline{M}', D')$, is a regular map $\bar{f}\colon \overline{M} \to \overline{M}'$ with the property that $\bar{f}^{-1}(D') \subseteq D$. Clearly, the restriction $f\colon \overline{M} \setminus D \to \overline{M}' \setminus D'$ is also a regular map. Conversely, any regular map between quasi-projective manifolds is induced by a regular morphism between suitable compactifications with normal crossing divisors, see [20]. Consequently, a map between two quasi-projective manifolds, $f\colon M \to M'$, is admissible (in the sense of § 4.1) if and only if M' is a smooth curve and f is a regular surjection with connected generic fiber.

We will consider a class of divisors broader than the normal crossing type, namely the *hypersurface arrangements* investigated in [11]. Extending Morgan's Gysin models from [20], Dupont constructs in [11] a bigraded $\mathbb{Q}$-CDGA, $\mathrm{OS}^{\bullet}_{\bullet}(\overline{M}, D)$, associated to a hypersurface arrangement D in $\overline{M}$, functorial with respect to regular morphisms of such pairs. He proves that $\mathrm{OS}^{\bullet}(\overline{M}, D)$ is a finite model of the quasi-projective manifold $M = \overline{M} \setminus D$. It is straightforward to extract from the results in [11] that $\mathrm{OS}^{\bullet}_{\bullet}(\overline{M}, D)$ is a model with positive weights for M, in the sense from [22]. Moreover, there is an identification, $H^{\bullet}(\mathrm{OS}(\overline{M}, D)) \equiv H^{\bullet}(\overline{M} \setminus D)$, natural with respect to regular morphisms of pairs.

Given a quasi-projective manifold M, a compactification $\overline{M}$ obtained by adding a hypersurface arrangement D is called a *convenient compactification* if every element of $\mathcal{E}(M)$ is represented by an admissible map $f: M \to M_f$ which is induced by a regular morphism of pairs, $\bar{f}: (\overline{M}, D) \to (\overline{M}_f, D_f)$, where $\overline{M}_f$ is the canonical compactification of the curve M_f, obtained by adding a finite set of points D_f. It is known that convenient compactifications always exist, see [20]. Fixing such an object, we will use the following simplified notation: for each $f \in \mathcal{E}(M)$, we denote the weight-preserving CDGA map $\mathrm{OS}(\bar{f}): \mathrm{OS}(\overline{M}_f, D_f) \to \mathrm{OS}(\overline{M}, D)$ by $\Phi_f: A_f \to A$.

Remark 5.1. Let M be a quasi-projective manifold with fundamental group π, and let $A = \mathrm{OS}(\overline{M}, D)$ be an Orlik–Solomon model for M. If G is a linear algebraic group whose Lie algebra $\mathfrak{g}$ is abelian, then, as shown in [8, Theorem B(2)], there is an analytic isomorphism of germs,

$$\mathscr{F}(A, \mathfrak{g})_{(0)} \xrightarrow{\cong} \mathrm{Hom}(\pi, G)_{(1)}\,, \tag{5.1}$$

which is *natural* with respect to the action on flat connections of CDGA maps induced by regular morphisms of pairs, and the action on representation varieties of induced homomorphisms on fundamental groups. Furthermore, this isomorphism restricts to isomorphisms $\mathscr{R}^i_r(A, \theta)_{(0)} \xrightarrow{\simeq} \mathscr{V}^i_r(M, \iota)_{(1)}$, for all i, r.

The naturality of the isomorphism (5.1) for $G = \mathrm{SL}_2(\mathbb{C})$ and $\mathrm{Sol}_2(\mathbb{C})$ would simplify the proof of Theorem 1.2. As explained in [23, § 7.5], though, the argument from [8, Theorem B(2)] that establishes the naturality of the isomorphism (5.1) breaks down in the non-abelian case. This is the reason why we chose to prove Theorem 1.2 with the aid of Lemma 2.1, instead.

5.2. Flat connections and infinitesimal jump loci

We now proceed to describe infinitesimal analogs of the inclusions (4.2) and (4.3) for $i = r = 1$. Let $\theta: \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of a finite-dimensional Lie algebra. By naturality of the set of flat connections, we have an inclusion,

$$\mathscr{F}(A, \mathfrak{g}) \supseteq \mathscr{F}^1(A, \mathfrak{g}) \cup \bigcup_{f \in \mathcal{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})\,, \tag{5.2}$$

where Φ_f^* denotes the map $\Phi_f \otimes \mathrm{id}_{\mathfrak{g}}: A^1_f \otimes \mathfrak{g} \to A^1 \otimes \mathfrak{g}$. This inclusion is then the analog of (4.2). For the analog of (4.3), we need some preparation.

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Lemma 5.2. *For every map $f \in \mathscr{E}(M)$, the following hold:*

(1) $A_f^\bullet = A_f^{\leqslant 2}$;

(2) $\chi(H^\bullet(A_f)) < 0$;

(3) Φ_f *is injective.*

Proof. The first claim follows easily from the construction of the Orlik–Solomon model A_f of M_f, while the second claim simply translates the fact that $\chi(M_f) < 0$.

For the last assertion, we recall from [11] that there is a regular morphism of pairs, $\bar{b}\colon (\widetilde{M}, \widetilde{D}) \to (\overline{M}, D)$, constructed by iterated blow-up, where $\widetilde{D}$ is a normal crossing divisor. We deduce that $\mathrm{OS}(\bar{f} \circ \bar{b})$ coincides with the map between Gysin models constructed by Morgan. This latter map is injective, as shown in [8, Example 5.3], and so we are done. □

Recall now that $b_1(M) > 0$, and thus $H^1(A) \neq 0$. The infinitesimal analog of (4.3) is the following inclusion,

$$\mathscr{R}_1^1(A, \theta) \supseteq \Pi(A, \theta) \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g}). \tag{5.3}$$

Let us verify that (5.3) holds. To start with, the inclusion $\Pi(A, \theta) \subseteq \mathscr{R}_1^1(A, \theta)$ is given by [19, Corollary 3.8]. Next, for every $f \in \mathscr{E}(M)$, we have that $\mathscr{F}(A_f, \mathfrak{g}) = \mathscr{R}_1^1(A_f, \theta)$, by [19, Proposition 2.4] and Lemma 5.2. Finally, $\Phi_f^* \mathscr{R}_1^1(A_f, \theta) \subseteq \mathscr{R}_1^1(A, \theta)$, by Lemmas 3.5 and 5.2.

5.3. Properties of the infinitesimal inclusions

Before proceeding, let us make a couple of simple remarks about the inclusions in displays (5.2) and (5.3). All terms appearing on the right-hand side are Zariski-closed subsets of $\mathscr{F}(A, \mathfrak{g})$, respectively $\mathscr{R}_1^1(A, \theta)$. Indeed, for $\mathscr{F}^1(A, \mathfrak{g})$ and $\Pi(A, \theta)$, this follows by construction, cf. [19, § 1.5]; moreover, these two varieties depend only on $H^1(A)$ and θ. On the other hand, for $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})$ the claim follows from Lemmas 3.5 and 5.2. Furthermore, the analytic germs $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}$ and $\mathscr{F}(A_f, \mathfrak{g})_{(0)}$ are isomorphic.

Lemma 5.3. *In both inclusions (5.2) and (5.3), equality is equivalent to equality near* 0.

Proof. The positive-weight decomposition of A^1 gives rise to a positive-weight $\mathbb{C}^\times$-action on $A^1 \otimes \mathfrak{g}$, leaving both $\mathscr{F}(A, \mathfrak{g})$ and $\mathscr{R}_1^1(A, \theta)$ invariant, as explained in [8, § 9.17]. It follows from Lemmas 2.2 and 2.4 that equality near 0 implies global equality. □

For later use, let us note that the above $\mathbb{C}^\times$-action on $A^1 \otimes \mathfrak{g}$ also endows with positive weights the subvarieties $\mathscr{F}^1(A, \mathfrak{g})$, $\Pi(A, \theta)$, and $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})$, for all $f \in \mathscr{E}(M)$.

In the rank 1 case, i.e., when $\theta = \mathrm{id}_{\mathbb{C}}$, we have that $\mathscr{F}(A, \mathbb{C}) = H^1(A)$, for every connected CDGA A. Thus, inclusion (5.3) becomes

$$\mathscr{R}_1^1(A) \supseteq \{0\} \cup \bigcup_{f \in \mathscr{E}(M)} \operatorname{im} H^1(\Phi_f) \tag{5.4}$$

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in this case. Actually, more can be said about this. [8, Theorem C] implies the following infinitesimal analog of the bijection from Theorem 4.1:

$$\mathscr{R}^1_1(A) = \{0\} \cup \bigcup_{f \in \mathscr{E}(M)} \operatorname{im} H^1(\Phi_f). \tag{5.5}$$

Moreover, this is the irreducible decomposition of $\mathscr{R}^1_1(A)$, where $\{0\}$ is omitted when $\mathscr{E}(M) \neq \emptyset$, as in [19, (50)].

Corollary 5.4. *Equality in (5.2) implies equality in (5.3).*

Proof. By Lemma 5.2 and formula (5.5), we may apply [2, Proposition 4.1] to the family $\{\Phi_f\}_{f \in \mathscr{E}(M)}$ to obtain the desired conclusion. □

6. Irreducibility, dimension, redundancies

In this section, M will be a quasi-projective manifold with fundamental group $\pi = \pi_1(M)$, and $A = \mathrm{OS}^{\bullet}(\overline{M}, D)$ will be an OS-model for M associated to a fixed convenient compactification of M. Let G be a complex linear algebraic group, let $\iota \colon G \to \mathrm{GL}(V)$ be a rational representation, and let $\theta \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be its tangential representation. Unless otherwise mentioned, we suppose in this section that G is either SL_2 or its standard Borel subgroup Sol_2, consisting of upper-triangular matrices with determinant 1. To avoid trivialities, we will assume throughout that $b_1(M) > 0$.

6.1. Dimension and irreducibility

Our strategy is to compare the union of germs at 1 from (1.4) with the union of germs at 0 from (1.6), and similarly for (1.5) and (1.7), using Theorem 3.6 and Lemma 2.1. We start this approach by verifying the dimension and irreducibility assumptions from that lemma.

Lemma 6.1. *Let $f \colon M \to M_f$ be an admissible map, and let $\Phi_f \colon A_f \to A$ be the corresponding morphism of* CDGA *models. For any complex linear algebraic group G, the germs $f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)}$ and $\Phi^*_f \mathscr{F}(A_f, \mathfrak{g})_{(0)}$ are isomorphic to $\mathscr{F}(H^{\bullet}(M_f), \mathfrak{g})_{(0)}$. Moreover, if $G = \mathrm{SL}_2$ or Sol_2, then*

1. *These germs are irreducible;*
2. *$\mathscr{F}(A_f, \mathfrak{g})_{(0)} \neq \mathscr{F}^1(A_f, \mathfrak{g})_{(0)}$.*

Proof. As noted before, there are isomorphisms of analytic germs,

$$f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)} \cong \operatorname{Hom}(\pi_f, G)_{(1)} \quad \text{and} \quad \Phi^*_f \mathscr{F}(A_f, \mathfrak{g})_{(0)} \cong \mathscr{F}(A_f, \mathfrak{g})_{(0)}. \tag{6.1}$$

On the other hand, by Theorem 3.6, we also have an isomorphism

$$\mathscr{F}(A_f, \mathfrak{g})_{(0)} \cong \operatorname{Hom}(\pi_f, G)_{(1)}. \tag{6.2}$$

Observe that the curve M_f is a formal space in the sense of Sullivan [24]. Hence, the CDGA $H^{\bullet}(M_f)$, endowed with zero differential, is another finite model of M_f. Again by

Theorem 3.6, we have an isomorphism $\mathscr{F}(A_f, \mathfrak{g})_{(0)} \cong \mathscr{F}(H^\bullet(M_f), \mathfrak{g})_{(0)}$, and this proves the first claim.

For Parts (1) and (2), we will use [19, Lemma 7.3], which says that, for a smooth complex curve C with $\chi(C) < 0$, the variety $\mathscr{F}(H^\bullet(C), \mathfrak{g})$ is irreducible and strictly contains $\mathscr{F}^1(H^\bullet(C), \mathfrak{g})$.

Next, we prove (1). By the aforementioned result, the variety $\mathscr{F}(H^\bullet(M_f), \mathfrak{g})$ is irreducible. Since this variety is homogeneous, and thus has positive weights, Lemma 2.4 implies that the germ $\mathscr{F}(H^\bullet(M_f), \mathfrak{g})_{(0)}$ is also irreducible.

Finally, we prove (2). Since $H^1(A_f) \cong H^1(M_f)$, there is an isomorphism of germs, $\mathscr{F}^1(A_f, \mathfrak{g})_{(0)} \cong \mathscr{F}^1(H^\bullet(M_f), \mathfrak{g})_{(0)}$. Suppose that $\mathscr{F}(A_f, \mathfrak{g})_{(0)} = \mathscr{F}^1(A_f, \mathfrak{g})_{(0)}$. Then the germ at 0 of the variety $\mathscr{F}(H^\bullet(M_f), \mathfrak{g})$ would be isomorphic to the germ at 0 of the closed subvariety $\mathscr{F}^1(H^\bullet(M_f), \mathfrak{g})$. Hence, Lemma 2.3 would imply that $\mathscr{F}(H^\bullet(M_f), \mathfrak{g}) = \mathscr{F}^1(H^\bullet(M_f), \mathfrak{g})$, in contradiction with [19, Lemma 7.3]. The proof is thus complete. $\square$

The next lemma completes the verification of the dimension and irreducibility assumptions from Lemma 2.1.

Lemma 6.2. *With notation as above, the following hold for $G = \mathrm{SL}_2$ or Sol_2.*

(1) *The germ $\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$ is isomorphic to $\mathscr{F}^1(A, \mathfrak{g})_{(0)}$.*

(2) *The germ $\mathrm{abf}^* \mathscr{V}^1_1(\pi_{\mathrm{abf}}, \iota)_{(1)}$ is isomorphic to $\Pi(A, \theta)_{(0)}$.*

(3) *All the above germs are irreducible.*

Proof. As mentioned previously, we have an isomorphism

$$(\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G), \mathrm{abf}^* \mathscr{V}^1_1(\pi_{\mathrm{abf}}, \iota))_{(1)} \cong (\operatorname{Hom}(\pi_{\mathrm{abf}}, G), \mathscr{V}^1_1(\pi_{\mathrm{abf}}, \iota))_{(1)}. \tag{6.3}$$

Denote by A_0 the CDGA $(\bigwedge^\bullet H^1(M), d = 0)$. We deduce from Theorem 3.6 and Lemma 3.7 that

$$(\operatorname{Hom}(\pi_{\mathrm{abf}}, G), \mathscr{V}^1_1(\pi_{\mathrm{abf}}, \iota))_{(1)} \cong (\mathscr{F}(A_0, \mathfrak{g}), \mathscr{R}^1_1(A_0, \theta))_{(0)}. \tag{6.4}$$

Again by Lemma 3.7,

$$(\mathscr{F}(A_0, \mathfrak{g}), \mathscr{R}^1_1(A_0, \theta)) = (\mathscr{F}^1(A_0, \mathfrak{g}), \Pi(A_0, \theta)). \tag{6.5}$$

Since plainly $H^1(A_0) \cong H^1(A)$, we have that

$$(\mathscr{F}^1(A_0, \mathfrak{g}), \Pi(A_0, \theta)) \cong (\mathscr{F}^1(A, \mathfrak{g}), \Pi(A, \theta)). \tag{6.6}$$

Putting things together verifies claims (1) and (2).

We now prove claim (3). The irreducibility of $\mathscr{F}^1(A, \mathfrak{g})_{(0)}$ follows from [19, Lemma 3.3] and Lemma 2.4. By the construction of $\Pi(A, \theta)$ (see [19, (18)]), the irreducibility of the zero set $\mathbf{V}(\det \circ \theta)$ implies the irreducibility of $\Pi(A, \theta)_{(0)}$. When $\mathfrak{g} = \mathfrak{sl}_2$, the variety $\mathbf{V}(\det \circ \theta)$ is irreducible, by [19, Lemma 3.9].

Finally, we let $\mathfrak{g}$ be the 2-dimensional solvable Lie algebra $\mathfrak{sol}_2$, with 1-dimensional abelianization. Set $m = \dim V$. Since $\mathfrak{sol}_2$ is solvable, θ takes values in the upper-triangular Lie subalgebra of $\mathfrak{gl}_m(\mathbb{C})$, by a classical result in Lie theory [14]. Composing θ with the projection onto the diagonal matrices, we obtain a Lie algebra map $\theta' \colon \mathfrak{sol}_2 \to \mathbb{C}^m$, with components $\lambda_1, \ldots, \lambda_m$, having the property that $\det \circ \theta = \prod_i \lambda_i$.

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Since θ' factors through the abelianization, we infer that $\det \circ \theta = c \cdot \lambda^m$, for some constant c and some linear map λ on $\mathfrak{sol}_2$, which clearly implies the irreducibility of $\mathbf{V}(\det \circ \theta)$. This completes our proof. $\square$

6.2. Non-redundant cases

Next, we have to analyze the unions of germs at the origin from (1.4)–(1.7) from the viewpoint of their redundancies.

Lemma 6.3. *For any two distinct maps $f, g \in \mathscr{E}(M)$, the following hold for $G = \mathrm{SL}_2$ or Sol_2:*

(1) $f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \not\subseteq g_\sharp^* \operatorname{Hom}(\pi_g, G)_{(1)}$;

(2) $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)} \not\subseteq \Phi_g^* \mathscr{F}(A_g, \mathfrak{g})_{(0)}$.

Proof. We assume first that $f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \subseteq g_\sharp^* \operatorname{Hom}(\pi_g, G)_{(1)}$. Intersecting this inclusion with $\operatorname{Hom}(\pi, \mathbb{C}^\times)$ where $\mathbb{C}^\times$ is the subtorus of diagonal matrices from G, we infer that $f_\sharp^* \operatorname{Hom}(\pi_f, \mathbb{C}^\times)_{(1)} \subseteq g_\sharp^* \operatorname{Hom}(\pi_g, \mathbb{C}^\times)_{(1)}$. Since both $f_\sharp^* \operatorname{Hom}(\pi_f, \mathbb{C}^\times)$ and $g_\sharp^* \operatorname{Hom}(\pi_g, \mathbb{C}^\times)$ are connected tori, it follows from Lemma 2.2 that $f_\sharp^* \operatorname{Hom}(\pi_f, \mathbb{C}^\times) \subseteq g_\sharp^* \operatorname{Hom}(\pi_g, \mathbb{C}^\times)$, in contradiction with the bijection from Theorem 4.1.

Finally, suppose that $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)} \subseteq \Phi_g^* \mathscr{F}(A_g, \mathfrak{g})_{(0)}$. Let $\mathbb{C} \subseteq \mathfrak{g}$ be the Lie algebra of the torus considered above. Intersecting this inclusion with $\mathscr{F}(A, \mathbb{C})$, we infer as before that $\Phi_f^* \mathscr{F}(A_f, \mathbb{C})_{(0)} \subseteq \Phi_g^* \mathscr{F}(A_g, \mathbb{C})_{(0)}$, which implies that $\operatorname{im} H^1(\Phi_f) \subseteq \operatorname{im} H^1(\Phi_g)$, in contradiction with (5.5). $\square$

Lemma 6.4. *For all $f \in \mathscr{E}(M)$ and for all complex linear algebraic groups G, the following equality of germs holds:*

$$f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \cap \mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} = f_\sharp^* \mathrm{abf}^* \operatorname{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)}. \tag{6.7}$$

Furthermore, if $G = \mathrm{SL}_2$ or Sol_2, then

(1) $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)} \not\subseteq \mathscr{F}^1(A, \mathfrak{g})_{(0)}$;

(2) $f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \not\subseteq \mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$.

Proof. We first establish equality (6.7). As explained in [23, Remark 7.8], the analytic germ $\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$ coincides with the abelian part near 1 of the representation variety $\operatorname{Hom}(\pi, G)$, and similarly for π_f. It follows that we may replace the map abf by the abelianization map ab in (6.7).

The inclusion '$\supseteq$' is an immediate consequence of naturality of abelianization. Hence, it is enough to prove that any homomorphism $\rho \colon \pi_f \to G$ for which $\rho \circ f_\sharp$ factors through abelianization has the same property. This in turn follows from the fact that the epimorphism $f_\sharp$ induces a surjection on derived subgroups, plus naturality. Our first claim is proved.

To prove (1), suppose that $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)} \subseteq \mathscr{F}^1(A, \mathfrak{g})_{(0)}$. Then, by the injectivity of Φ_f, we would have that $\mathscr{F}(A_f, \mathfrak{g})_{(0)} \subseteq \mathscr{F}^1(A_f, \mathfrak{g})_{(0)}$, in contradiction with Lemma 6.1(2).

To prove (2), suppose that $f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \subseteq \mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$. From (6.7), we deduce that $\operatorname{Hom}(\pi_f, G)_{(1)} = \mathrm{abf}^* \operatorname{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)}$, by the surjectivity of $f_\sharp$. We now consider the inclusion $\mathscr{F}^1(A_f, \mathfrak{g})_{(0)} \subseteq \mathscr{F}(A_f, \mathfrak{g})_{(0)}$. We claim that our assumption leads to the equality $\mathscr{F}^1(A_f, \mathfrak{g})_{(0)} = \mathscr{F}(A_f, \mathfrak{g})_{(0)}$, the same contradiction as before.

In view of Lemma 2.3, our claim follows from the existence of an isomorphism of germs, $\mathscr{F}^1(A_f, \mathfrak{g})_{(0)} \cong \mathscr{F}(A_f, \mathfrak{g})_{(0)}$. To construct such an isomorphism, we consider the CDGA $A_0 = (\bigwedge^\bullet H^1(M_f), d = 0)$ from Lemma 3.7. Since A_f is a finite model of M_f, we have an isomorphism

$$\mathscr{F}(A_f, \mathfrak{g})_{(0)} \cong \mathrm{Hom}(\pi_f, G)_{(1)}, \tag{6.8}$$

by Theorem 3.6. Next,

$$\mathrm{Hom}(\pi_f, G)_{(1)} = \mathrm{abf}^* \,\mathrm{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)}, \tag{6.9}$$

by our assumption, and clearly

$$\mathrm{abf}^* \,\mathrm{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)} \cong \mathrm{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)}. \tag{6.10}$$

Again by Theorem 3.6,

$$\mathrm{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)} \cong \mathscr{F}(A_0, \mathfrak{g})_{(0)}. \tag{6.11}$$

By Lemma 3.7,

$$\mathscr{F}(A_0, \mathfrak{g})_{(0)} = \mathscr{F}^1(A_0, \mathfrak{g})_{(0)}. \tag{6.12}$$

Finally,

$$\mathscr{F}^1(A_0, \mathfrak{g})_{(0)} \cong \mathscr{F}^1(A_f, \mathfrak{g})_{(0)}, \tag{6.13}$$

since $H^1(A_0) \cong H^1(A_f)$. Putting things together completes our proof. □

6.3. Redundant cases

We now complete our analysis of the redundancies in the unions

$$\mathrm{abf}^* \,\mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cup \bigcup_{f \in \mathscr{E}(M)} f_\sharp^* \,\mathrm{Hom}(\pi_f, G)_{(1)}, \tag{6.14}$$

$$\mathrm{abf}^* \,\mathscr{V}_1^1(\pi_{\mathrm{abf}}, \iota)_{(1)} \cup \bigcup_{f \in \mathscr{E}(M)} f_\sharp^* \,\mathrm{Hom}(\pi_f, G)_{(1)}, \tag{6.15}$$

$$\mathscr{F}^1(A, \mathfrak{g})_{(0)} \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}, \tag{6.16}$$

$$\Pi(A, \theta)_{(0)} \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}. \tag{6.17}$$

By the results from § 6.2, we need to consider the following two cases. If either (6.14) or (6.15) is redundant, then

$$\mathrm{abf}^* \,\mathscr{V}_1^1(\pi_{\mathrm{abf}}, \iota)_{(1)} \subseteq f_\sharp^* \,\mathrm{Hom}(\pi_f, G)_{(1)}, \quad \text{for some } f \in \mathscr{E}(M). \tag{6.18}$$

If either (6.16) or (6.17) is redundant, then

$$\Pi(A, \theta)_{(0)} \subseteq \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}, \quad \text{for some } f \in \mathscr{E}(M). \tag{6.19}$$

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Lemma 6.5. *If condition (6.19) holds, then* $b_1(M_f) = b_1(M)$.

Proof. Let $\mathbb{C} \subseteq \mathfrak{g}$ be the Lie algebra of the unipotent group $\mathbb{C} \subseteq G$ consisting of the matrices of Sol_2 with 1's on the diagonal. Denote by $\iota'\colon \mathbb{C} \to \mathrm{GL}(V)$ the restriction of ι. Clearly, $d_1(\iota') = \theta'$, the restriction of θ to the Lie algebra $\mathbb{C}$. Our assumption implies that $\Pi(A, \theta')_{(0)} \subseteq \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}$. We infer from Lemma 3.8 that

$$\mathscr{F}(A, \mathbb{C})_{(0)} = \mathscr{F}^1(A, \mathbb{C})_{(0)} \subseteq \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}. \tag{6.20}$$

Therefore, $\mathscr{F}(A, \mathbb{C})_{(0)} \subseteq \Phi_f^* \mathscr{F}(A_f, \mathbb{C})_{(0)}$. In other words, $H^1(A)_{(0)} \subseteq \operatorname{im} H^1(\Phi_f)_{(0)}$. Hence,

$$b_1(M) = \dim H^1(A)_{(0)} \leqslant \dim \operatorname{im} H^1(\Phi_f)_{(0)}. \tag{6.21}$$

On the other hand, $H^1(\Phi_f)$ is identified with $H^1(f)$, as recalled in § 5.1, and $H^1(f)$ is injective, since $f_\sharp$ is surjective. In conclusion, $b_1(M) \leqslant b_1(M_f)$. Since clearly $b_1(M) \geqslant b_1(M_f)$, we are done. □

Lemma 6.6. *If condition (6.18) holds, then* $b_1(M_f) = b_1(M)$.

Proof. Define ι' and θ' as before. Note that $\mathscr{V}_1^1(\pi_{\mathrm{abf}}, \iota') \subseteq \mathscr{V}_1^1(\pi_{\mathrm{abf}}, \iota)$, by construction. By Lemma 3.9, $\mathscr{V}_1^1(\pi_{\mathrm{abf}}, \iota')_{(1)} = \mathrm{Hom}(\pi_{\mathrm{abf}}, \mathbb{C})_{(1)}$. Thus, we infer from our assumption that

$$\mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, \mathbb{C})_{(1)} \subseteq f_\sharp^* \, \mathrm{Hom}(\pi_f, G)_{(1)}. \tag{6.22}$$

Hence,

$$\mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, \mathbb{C})_{(1)} \subseteq f_\sharp^* \, \mathrm{Hom}(\pi_f, \mathbb{C})_{(1)}. \tag{6.23}$$

Consequently, $\dim \mathrm{Hom}(\pi_{\mathrm{abf}}, \mathbb{C})_{(1)} \leqslant \dim \mathrm{Hom}(\pi_f, \mathbb{C})_{(1)}$, that is, $b_1(M) \leqslant b_1(M_f)$. Proceeding now as in the proof of Lemma 6.5 completes the proof of this lemma. □

Lemma 6.7. *Suppose* $b_1(M) = b_1(M_f)$ *for some* $f \in \mathscr{E}(M)$. *Then:*

(1) $H^1(f)$ *is an isomorphism.*

(2) $\mathscr{E}(M) = \{f\}$.

(3) $\mathscr{F}^1(A, \mathfrak{g}) \subseteq \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})$, *for any finite-dimensional Lie algebra* $\mathfrak{g}$.

(4) $\mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \subseteq f_\sharp^* \, \mathrm{Hom}(\pi_f, G)_{(1)}$, *for any linear algebraic group* G.

Proof. (1) This claim is clear, since $H^1(f)$ is injective.

(2) Fix $g \in \mathscr{E}(M)$. We start by noting that $g_\sharp^* \, \mathrm{Hom}(\pi_g, \mathbb{C}^\times)$ is a connected affine subtorus of dimension $b_1(M_g)$ of the connected affine torus $\mathrm{Hom}(\pi_{\mathrm{abf}}, \mathbb{C}^\times)$ of dimension $b_1(M)$, since $(\pi_g)_{\mathrm{ab}} = H_1(M_g, \mathbb{Z})$ has no torsion. Thus, our assumption implies that $f_\sharp^* \, \mathrm{Hom}(\pi_f, \mathbb{C}^\times) = \mathrm{Hom}(\pi_{\mathrm{abf}}, \mathbb{C}^\times)$. We infer that $g_\sharp^* \, \mathrm{Hom}(\pi_g, \mathbb{C}^\times) \subseteq f_\sharp^* \, \mathrm{Hom}(\pi_f, \mathbb{C}^\times)$. By Theorem 4.1, we must have $g = f$, and we are done.

(3) It is enough to note that $\Phi_f^* \mathscr{F}^1(A_f, \mathfrak{g}) = \mathscr{F}^1(A, \mathfrak{g})$, since $H^1(\Phi_f) \equiv H^1(f)$ is an isomorphism.

(4) Let $f_{\mathrm{abf}}\colon \pi_{\mathrm{abf}} \twoheadrightarrow (\pi_f)_{\mathrm{abf}}$ be the (surjective) homomorphism induced by $f_\sharp$. Since $b_1(M) = b_1(M_f)$, we have that $\pi_{\mathrm{abf}} \cong (\pi_f)_{\mathrm{abf}}$. Hence, f_{abf} is an isomorphism, and consequently

$$f_{\mathrm{abf}}^* \, \mathrm{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)} = \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}. \tag{6.24}$$

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This equality of germs implies that

$$\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} = f_\sharp^* \circ \mathrm{abf}^* \operatorname{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)} \subseteq f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)}, \tag{6.25}$$

as asserted. This completes the proof. □

7. Proofs of the main results

In this section, we provide proofs to Theorems 1.1–1.4 from the Introduction.

7.1. Transversality in the quasi-projective setting

Let M be a quasi-projective manifold, and fix a compactification $\overline{M} = M \cup D$, where $\overline{M}$ is a connected, smooth projective variety, and D is a hypersurface arrangement in $\overline{M}$.

As before, let $(A^{\bullet}_{\bullet}, d) = \mathrm{OS}^{\bullet}_{\bullet}(\overline{M}, D)$ be the corresponding Orlik–Solomon model for M. By construction, the lower degree (called weight) is concentrated in the interval $[i, 2i]$, in (upper) degree i. The terminology comes from the fact that the induced lower grading in cohomology splits Deligne's weight filtration [11]. Thus, an element $\Omega \in A^1 \otimes \mathfrak{g}$ has weight decomposition $\Omega = \Omega_1 + \Omega_2$, where $\Omega_j \in A^1_j \otimes \mathfrak{g}$, for $j = 1, 2$. Note also that $H^1(A) = A^1 \cap \ker(d)$, since A is connected. When we write $\Omega \in H^1(A) \otimes \mathfrak{g}$, we mean that $\partial\Omega = 0$, where ∂ denotes $d \otimes \mathrm{id}_{\mathfrak{g}}$. By construction, $\Omega_1 \in H^1(A) \otimes \mathfrak{g}$.

Lemma 7.1. *Let $(A, d) = \mathrm{OS}(\overline{M}, D)$ be any Orlik–Solomon model. If $\Omega \in \mathscr{F}(A, \mathfrak{g})$ and $\Omega_1 = 0$, then $\Omega \in H^1(A) \otimes \mathfrak{g}$, for any finite-dimensional Lie algebra $\mathfrak{g}$.*

Proof. For an element $\Omega \in A^1 \otimes \mathfrak{g}$, let us examine the flatness equation, $\partial\Omega + \frac{1}{2}[\Omega, \Omega] = 0$. Let $\Omega = \Omega_1 + \Omega_2$ be the weight decomposition. We recall from § 5.1 that both the differential and the product of A have degree zero with respect to the weights of A. Using this fact, the weight 2 component of the flatness equation translates to the equality $\partial\Omega_2 + \frac{1}{2}[\Omega_1, \Omega_1] = 0$, which proves our claim. □

Fix now a convenient compactification $\overline{M} = M \cup D$. Then any admissible map $f : M \to M_f$ induces a morphism $\Phi_f : A_f \to A$ between the corresponding OS-models, and this, in turn, induces a morphism $\Phi_f^* : \mathscr{F}(A_f, \mathfrak{g}) \to \mathscr{F}(A, \mathfrak{g})$ between the corresponding varieties of $\mathfrak{g}$-flat connections. The next result proves Theorem 1.1, Part (1) from the Introduction.

Theorem 7.2. *Let M be a quasi-projective manifold, and let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For any distinct $f, g \in \mathscr{E}(M)$,*

$$\Phi_f^* \mathscr{F}(A_f, \mathfrak{g}) \cap \Phi_g^* \mathscr{F}(A_g, \mathfrak{g}) = \{0\}.$$

Proof. We start by noting that

$$\operatorname{im} H^1(\Phi_f) \cap \operatorname{im} H^1(\Phi_g) = \{0\}. \tag{7.1}$$

Indeed, $\operatorname{im} H^1(\Phi_f)$ is naturally identified with $\operatorname{im} H^1(f)$, and similarly for g. On the other hand, Theorem 4.1 yields a natural identification of $\operatorname{im} H^1(f)$ with the tangent space $T_1(f_\sharp^* \operatorname{Hom}(\pi_f, \mathbb{C}^\times))$ to the corresponding irreducible component through 1 of $\mathscr{V}^1_1(M)$, and similarly for g. Finally, as shown in [10, Theorem C(2)],

$$T_1(f_\sharp^* \operatorname{Hom}(\pi_f, \mathbb{C}^\times)) \cap T_1(g_\sharp^* \operatorname{Hom}(\pi_g, \mathbb{C}^\times)) = \{0\}. \tag{7.2}$$

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Suppose now that $\Phi_f^*(\Omega) = \Phi_g^*(\Omega')$, for some $\Omega \in \mathscr{F}(A_f, \mathfrak{g})$ and $\Omega' \in \mathscr{F}(A_g, \mathfrak{g})$. Consider the weight decompositions, $\Omega = \Omega_1 + \Omega_2$ and $\Omega' = \Omega'_1 + \Omega'_2$. We infer that $\Phi_f^*(\Omega_1) = \Phi_g^*(\Omega'_1)$, since Orlik–Solomon CDGA maps preserve weight. As mentioned before, $\Omega_1 \in H^1(A_f) \otimes \mathfrak{g}$ and $\Omega'_1 \in H^1(A_g) \otimes \mathfrak{g}$. We infer then from (7.1) that $\Phi_f^*(\Omega_1) = \Phi_g^*(\Omega'_1) = 0$. Hence, $\Omega_1 = \Omega'_1 = 0$, since Φ_f and Φ_g are injective, by Lemma 5.2. Our assumption becomes then $\Phi_f^*(\Omega_2) = \Phi_g^*(\Omega'_2)$. On the other hand, we know from Lemma 7.1 that $\Omega_2 \in H^1(A_f) \otimes \mathfrak{g}$ and $\Omega'_2 \in H^1(A_g) \otimes \mathfrak{g}$. Therefore, $\Phi_f^*(\Omega_2) = \Phi_g^*(\Omega'_2) = 0$, by the same argument as before. Our proof is complete. □

Let us point out that the transversality property from Theorem 7.2 is a non-abelian generalization of the aforementioned rank 1 result from [10, Theorem C(2)].

7.2. Topological and infinitesimal jump loci

We now turn to the proof of Theorem 1.2 from the Introduction. As before, we shall work with a fixed convenient compactification $\overline{M}$ of a quasi-projective manifold M (which we will assume satisfies $b_1(M) > 0$), and we shall let A denote the corresponding Orlik–Solomon model for M. The bulk of the proof is contained in the next three lemmas.

Lemma 7.3. *If the inclusion (1.6) is an equality, then the inclusion (1.4) becomes an equality near* 1.

Proof. Recall from Theorem 3.6 that $\mathscr{F}(A, \mathfrak{g})_{(0)} \cong \mathrm{Hom}(\pi, G)_{(1)}$, as analytic germs. By assumption, the inclusion (1.6) is an equality near 0. Suppose first that the union (6.14) has no redundancies. By Lemma 2.1 and the results from § 6.1, the inclusion (1.4) is then an equality near 1, thereby verifying our claim.

Now suppose that the union (6.14) is redundant. Then (6.18) also holds. Hence, by Lemmas 6.6 and 6.7, we have that $\mathscr{E}(M) = \{f\}$ and $H^1(f)$ is an isomorphism. Furthermore, by Lemma 6.7 again, our claim in this case reduces to proving the equality

$$\mathrm{Hom}(\pi, G)_{(1)} = f_\sharp^* \,\mathrm{Hom}(\pi_f, G)_{(1)}. \tag{7.3}$$

Our hypothesis regarding (1.6) gives the equality

$$\mathscr{F}(A, \mathfrak{g})_{(0)} = \mathscr{F}^1(A, \mathfrak{g})_{(0)} \cup \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}. \tag{7.4}$$

Again by Lemmas 6.6 and 6.7, equation (7.4) becomes

$$\mathscr{F}(A, \mathfrak{g})_{(0)} = \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}. \tag{7.5}$$

As seen before, $\mathrm{Hom}(\pi, G)_{(1)} \cong \mathscr{F}(A, \mathfrak{g})_{(0)}$. Plainly, $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)} \cong \mathscr{F}(A_f, \mathfrak{g})_{(0)}$. Furthermore, $\mathscr{F}(A_f, \mathfrak{g})_{(0)} \cong \mathrm{Hom}(\pi_f, G)_{(1)}$, again by Theorem 3.6. Finally, $\mathrm{Hom}(\pi_f, G)_{(1)} \cong f_\sharp^* \,\mathrm{Hom}(\pi_f, G)_{(1)}$.

In conclusion, the equality from (7.5) implies that $\mathrm{Hom}(\pi, G)_{(1)} \cong f_\sharp^* \,\mathrm{Hom}(\pi_f, G)_{(1)}$. Therefore, by Lemma 2.3, equality (7.3) holds, and we are done. □

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Lemma 7.4. *If the inclusion (1.6) is an equality, then the inclusion (1.5) becomes an equality near* 1*.*

Proof. We infer from our assumption that the inclusion (1.7) is an equality, by Corollary 5.4. Set $X = \mathscr{R}^1_1(A,\theta)_{(0)} \cong \mathscr{V}^1_1(\pi,\iota)_{(1)}$, cf. Theorem 3.6. If the inclusion (1.7) is an equality near 0 and the union (6.15) has no redundancies, then the inclusion (1.5) is an equality near 1, as claimed, by Lemma 2.1 and the results from § 6.1.

If the union (6.15) is redundant, we may assume also that (6.18) holds. Hence, $\mathscr{E}(M) = \{f\}$ and $H^1(f)$ is an isomorphism, by Lemmas 6.6 and 6.7. By Lemma 6.7, we are left with proving the equality

$$\mathscr{V}^1_1(\pi,\iota)_{(1)} = f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)}. \tag{7.6}$$

Since the inclusion (1.7) is a global equality, we deduce the local equality

$$\mathscr{R}^1_1(A,\theta)_{(0)} = \Pi(A,\theta)_{(0)} \cup \Phi^*_f \mathscr{F}(A_f,\mathfrak{g})_{(0)}. \tag{7.7}$$

Again by Lemmas 6.6 and 6.7, equality (7.7) becomes

$$\mathscr{R}^1_1(A,\theta)_{(0)} = \Phi^*_f \mathscr{F}(A_f,\mathfrak{g})_{(0)}. \tag{7.8}$$

As we mentioned before, $\mathscr{V}^1_1(\pi,\iota)_{(1)} \cong \mathscr{R}^1_1(A,\theta)_{(0)}$. Next, we have that

$$\Phi^*_f \mathscr{F}(A_f,\mathfrak{g})_{(0)} \cong \mathscr{F}(A_f,\mathfrak{g})_{(0)} \cong \operatorname{Hom}(\pi_f, G)_{(1)} \cong f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)},$$

as in the proof of Lemma 7.3. Therefore, (7.8) implies that $\mathscr{V}^1_1(\pi,\iota)_{(1)} \cong f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)}$. Hence, equality (7.6) holds, by Lemma 2.3, and this completes our proof. □

Lemma 7.5. *If the inclusion (1.4) is an equality near* 1*, then the inclusion (1.6) is also an equality.*

Proof. By Lemma 5.3, it is enough to show that the inclusion (1.6) becomes an equality near 0. Set $X = \mathscr{F}(A,\mathfrak{g})_{(0)} \cong \operatorname{Hom}(\pi, G)_{(1)}$, cf. Theorem 3.6. If the inclusion (1.4) is an equality near 1 and the union (6.16) has no redundancies, then the inclusion (1.6) is an equality near 0, by Lemma 2.1 and the results from § 6.1.

If the union (6.16) is redundant, we may assume also that (6.19) holds. Hence, $\mathscr{E}(M) = \{f\}$ and $H^1(f)$ is an isomorphism, by Lemmas 6.5 and 6.7. By Lemma 6.7, our claim reduces to verifying the equality

$$\mathscr{F}(A,\mathfrak{g})_{(0)} = \Phi^*_f \mathscr{F}(A_f,\mathfrak{g})_{(0)}. \tag{7.9}$$

Our assumption related to (1.4) gives the equality

$$\operatorname{Hom}(\pi, G)_{(1)} = \mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cup f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)}. \tag{7.10}$$

Again by Lemmas 6.5 and 6.7, formula (7.10) reduces to

$$\operatorname{Hom}(\pi, G)_{(1)} = f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)}. \tag{7.11}$$

As seen before, $\mathscr{F}(A,\mathfrak{g})_{(0)} \cong \operatorname{Hom}(\pi, G)_{(1)}$. Clearly, $f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)} \cong \operatorname{Hom}(\pi_f, G)_{(1)}$. Next, $\operatorname{Hom}(\pi_f, G)_{(1)} \cong \mathscr{F}(A_f,\mathfrak{g})_{(0)}$, again by Theorem 3.6. Finally, $\mathscr{F}(A_f,\mathfrak{g})_{(0)} \cong \Phi^*_f \mathscr{F}(A_f,\mathfrak{g})_{(0)}$.

In conclusion, (7.11) implies that $\mathscr{F}(A,\mathfrak{g})_{(0)} \cong \Phi^*_f \mathscr{F}(A_f,\mathfrak{g})_{(0)}$. Hence, by Lemma 2.3, equality (7.9) holds, and we are done. □

Proof of Theorem 1.2. The implication (1) $\Rightarrow$ (4) follows from Lemma 7.5 and Corollary 5.4. Implication (4) $\Rightarrow$ (3) is clear. The implication (3) $\Rightarrow$ (2) follows from Lemmas 7.3 and 7.4. Finally, the implication (2) $\Rightarrow$ (1) is obvious. $\square$

7.3. Irreducible decompositions

We are now in a position to prove Theorem 1.3 from the Introduction, regarding the decomposition into irreducible components of germs of embedded jump loci of a quasi-projective manifold M satisfying one of the equivalent properties from Theorem 1.2.

Proof of Theorem 1.3. We start with Parts (1) and (2). The equalities (1.8)–(1.11) follow from Theorem 1.2. We also know from Lemmas 6.1 and 6.2 that all subgerms appearing in these unions are irreducible. If any one of these unions has redundancies, then, in view of the results from § 6.3, either (6.18) or (6.19) holds. In Part (1), this violates our assumption on first Betti numbers, by Lemmas 6.5 and 6.6. In Part (2), we have to verify equalities (1.12)–(1.15): these follow at once from (1.8)–(1.11) and Lemma 6.7.

To prove Part (3), we start by examining the irreducible decomposition (1.10). By Theorem 7.2, all components different from $\mathscr{F}^1(A, \mathfrak{g})_{(0)}$ intersect pairwise in a single point. Next, we claim that the irreducible decomposition (1.8) has the following property: all components different from $\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$ have positive-dimensional intersection with $\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$. Indeed, such an intersection is isomorphic to $\operatorname{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)}$, by Lemma 6.4. Since $(\pi_f)_{\mathrm{abf}} \cong \mathbb{Z}^n$ with $n \geqslant 1$, Lemma 3.7 gives the isomorphism $\operatorname{Hom}((\pi_f)_{\mathrm{abf}}, G)_{(1)} \cong \mathscr{F}^1(A_0, \mathfrak{g})_{(0)}$. On the other hand, the homogeneous variety $\mathscr{F}^1(A_0, \mathfrak{g})$ is isomorphic to the cone on the product of projective spaces $\mathbb{P}^{n-1} \times \mathbb{P}(\mathfrak{g})$, which implies that its germ at 0 is positive-dimensional.

By Theorem 3.6, the germs $\operatorname{Hom}(\pi, G)_{(1)}$ and $\mathscr{F}(A, \mathfrak{g})_{(0)}$ are isomorphic. Clearly, in Part (3) we may suppose that $\mathscr{E}(M)$ has at least two elements. With this assumption, we infer from Part (1) and the above discussion that the isomorphism identifies the components $\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$ and $\mathscr{F}^1(A, \mathfrak{g})_{(0)}$. Indeed, the isomorphism identifies the components of $\operatorname{Hom}(\pi, G)_{(1)}$ with those of $\mathscr{F}(A, \mathfrak{g})_{(0)}$, modulo a permutation τ of the index set $E(M)$. Assume that $\mathscr{F}^1(A, \mathfrak{g})_{(0)}$ is identified with $\operatorname{Hom}(\pi_{f'}, G)_{(1)}$, for some $f' \in \mathscr{E}(M)$, and pick an element $g' \in \mathscr{E}(M)$ different from f'. By the above property of the irreducible decomposition (1.10), $\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$ must intersect $\operatorname{Hom}(\pi_{g'}, G)_{(1)}$ in a single point. On the other hand, the above property of the irreducible decomposition (1.8) implies that this intersection is positive-dimensional. This contradiction proves that $\tau(f_0) = f_0$, as claimed. Our assertion in Part (3) follows then from Theorem 7.2. $\square$

Remark 7.6. We point out that all irreducible components appearing in Theorem 1.3 are known, for any quasi-projective manifold M with $b_1(M) > 0$ and any rational representation of $G = \mathrm{SL}_2$ or Sol_2. Indeed, Lemmas 6.1 and 6.2 provide isomorphisms of germs, $f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)} \cong \Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)} \cong \mathscr{F}(H^\bullet(M_f), \mathfrak{g})_{(0)}$, for any $f \in \mathscr{E}(M)$, as well as isomorphisms $\mathrm{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cong \mathscr{F}^1(A, \mathfrak{g})_{(0)}$ and $\mathrm{abf}^* \mathscr{V}_1^1(\pi_{\mathrm{abf}}, \iota)_{(1)} \cong \Pi(A, \theta)_{(0)}$. Finally, the affine varieties $\mathscr{F}(H^\bullet(M_f), \mathfrak{g})$ and $\mathscr{F}^1(A, \mathfrak{g})$, $\Pi(A, \theta)$ are described in [19, Lemmas 7.3 and 3.3].

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Remark 7.7. The equivalent properties from Theorem 1.2 also imply global equalities in (1.10) and (1.11), by Lemmas 2.2 and 2.4. When all Betti numbers $b_1(M_f)$ are different from $b_1(M)$, these equalities are in fact global irreducible decompositions, by Theorem 1.3(1) and Lemma 2.4. When $b_1(M_f) = b_1(M)$ for some $f \in \mathscr{E}(M)$, the local equalities (1.14) and (1.15) are actually global equalities of irreducible varieties, by a similar argument.

Remark 7.8. Building on the seminal work of Corlette and Simpson [6], Loray *et al.* establish in [18, Corollary B] the following striking result. Let M be a quasi-projective manifold, and let $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a representation which is not virtually abelian. There is then an orbifold morphism, $f\colon M \to N$, such that the associated representation, $\tilde{\rho}\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$, factors through the induced homomorphism $f_\sharp\colon \pi_1(M) \to \pi_1^{\mathrm{orb}}(N)$, where N is either a 1-dimensional complex orbifold, or a polydisk Shimura modular orbifold.

The equality from Theorem 1.3, display (1.8) provides a simpler, more precise local classification: If the representation ρ is sufficiently close to the origin, then either ρ is abelian, or there is an admissible map $f\colon M \to C$ such that ρ factors through the homomorphism $f_\sharp\colon \pi_1(M) \to \pi_1(C)$, where C is a smooth curve with $\chi(C) < 0$.

Example 7.9. Let M be the product $\Sigma_g \times N$, where Σ_g is a projective curve of genus $g > 1$ and N is a projective manifold with $b_1(N) = 0$. This simple example shows that the case from Theorem 1.3(2) really does occur. Indeed, it is clear that the canonical projection, $f\colon M \to \Sigma_g = M_f$, gives an element $f \in \mathscr{E}(M)$ with $b_1(M_f) = b_1(M)$.

7.4. On the structure of rank 2 jump loci

The results we have obtained so far enable us to derive structural decompositions near 1 of the non-abelian rank 2 topological embedded jump loci in low degree, for several large classes of quasi-projective manifolds. These structural decompositions are summarized in Theorem 1.4 from the Introduction, which we now proceed to prove.

Proof of Theorem 1.4. Let M be a quasi-projective manifold; as explained in § 4.2, we may suppose that $b_1(M) > 0$. Fix a convenient compactification $\overline{M} = M \cup D$, and let $(A^\bullet, d) = \mathrm{OS}^\bullet(\overline{M}, D)$ be the corresponding model for M.

We need to verify that equalities (1.8) and (1.9) hold in the five cases from our list. By Theorem 1.2, it is enough to check that the infinitesimal inclusion (1.6) is an equality in each case, that is, we need to verify that

$$\mathscr{F}(A, \mathfrak{g}) = \mathscr{F}^1(A, \mathfrak{g}) \cup \bigcup_{f \in \mathscr{E}(M)} \Phi_f^* \mathscr{F}(A_f, \mathfrak{g}), \tag{7.12}$$

for each of the corresponding Orlik–Solomon models.

(1) First suppose that M is projective. In this case, $\mathrm{OS}^\bullet(M, \emptyset) = (H^\bullet(M), d = 0)$. In particular, M is a formal space and π is a 1-formal group, and similarly for each curve M_f. It follows from [19, Corollary 7.2] that (7.12) holds.

(2) Next, suppose that the Deligne weight filtration has the property that $W_1 H^1(M) = 0$. In this case, equality (7.12) holds by [2, Theorem 4.2].

(3) Now suppose that M is the partial configuration space of a projective curve associated to a finite simple graph. Then the needed equality is established in [2, Theorem 1.3].

(4) Next, suppose that $\mathscr{R}^1_1(H^\bullet(M), d = 0) = \{0\}$. Then (7.12) holds by [19, Corollary 7.7].

(5) Finally, suppose that $M = S \setminus \{0\}$, where S is a quasi-homogeneous affine surface having a normal, isolated singularity at 0. Equality (7.12) is then proved in displays (35) and (36) from [22, Theorem 9.6]. □

7.5. Transversality for Kähler manifolds and hyperplane arrangements

When M is either a compact, connected Kähler manifold or the complement of a central complex hyperplane arrangement, the local analytic equalities (1.8) and (1.9) were obtained in [23, Theorem 1.3], by a completely different approach. (In the compact Kähler case, a map $f : M \to M_f$ is admissible if it is a holomorphic surjection with connected fibers onto a compact Riemann surface; the finite set $E(M)$ is defined as before.)

The method used in [23] is based on the fact that the family of maps $E(M)$ has the uniform formality property (in the sense of [23, Definitions 3.2 and 6.3]), in the above two cases: this is proved in [23, Proposition 7.4] for compact Kähler manifolds, respectively in [23, Proposition 9.3] for the arrangement case. This means that, for all $f \in E(M)$, there are zig-zags of augmentation-preserving quasi-isomorphisms connecting the Sullivan algebras of M and M_f to the respective cohomology algebras, as well as augmented CDGA maps Φ_i making the following ladder commute, up to augmented homotopy of CDGA maps,

$$\begin{array}{ccccccccc}
\Omega(M) & \xleftarrow{\psi_0} & A_1 & \xrightarrow{\psi_1} & \cdots & \longleftarrow & A_{\ell-1} & \xrightarrow{\psi_{\ell-1}} & H^*(M) \\
{\scriptstyle \Omega(f)}\uparrow & & \uparrow{\scriptstyle \Phi_1} & & & & \uparrow{\scriptstyle \Phi_{\ell-1}} & & \uparrow{\scriptstyle f^*} \\
\Omega(M_f) & \xleftarrow{\psi'_0} & A'_1 & \xrightarrow{\psi'_1} & \cdots & \longleftarrow & A'_{\ell-1} & \xrightarrow{\psi'_{\ell-1}} & H^*(M_f)\,,
\end{array} \tag{7.13}$$

with the property that the isomorphism induced by the top zig-zag on deformation functors (i.e., the appropriate moduli spaces of flat connections) is independent of f.

Using this uniform formality property, we obtain the following topological analog of the transversality property from Theorem 7.2, which proves Theorem 1.1, Part (2) from the Introduction.

Theorem 7.10. *Let M be either a compact, connected Kähler manifold or the complement of a central complex hyperplane arrangement. Let G be a linear algebraic group. Then*

$$f^*_\sharp \operatorname{Hom}(\pi_f, G)_{(1)} \cap g^*_\sharp \operatorname{Hom}(\pi_g, G)_{(1)} = \{1\},$$

for any two distinct maps $f, g \in \mathscr{E}(M)$.

Proof. In the arrangement case, we may suppose by a standard slicing argument that the hyperplanes lie in $\mathbb{C}^3$, since our claim depends only on the fundamental group $\pi = \pi_1(M)$. In both cases, we may choose a basepoint in M, and assume that all elements of $E(M)$ are represented by pointed maps.

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For a map $f \in E(M)$, consider the CDGA map $H^{\bullet}(f)\colon (H^{\bullet}(M_f), d=0) \to (H^{\bullet}(M), d = 0)$, denoted $\Phi_f\colon A_f \to A$. We want to apply [23, Theorem 6.4] to the finite families $\{f\}$ and $\{\Phi_f\}$, for $q=1$. Clearly, all spaces and all CDGAs appearing in these families are finite objects. Since each $f_\sharp$ is an epimorphism, both f and Φ_f are 0-connected maps. Denote by $\Omega(f)$ the CDGA map induced by f between Sullivan de Rham algebras. As mentioned before, $\Omega(f) \simeq \Phi_f$ in the category of augmented CDGAs, uniformly with respect to $f \in E(M)$.

Theorem 6.4 from [23] provides then a local analytic isomorphism, $\operatorname{Hom}(\pi, G)_{(1)} \cong \mathscr{F}(A, \mathfrak{g})_{(0)}$, that identifies $f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)}$ with $\Phi_f^* \mathscr{F}(A_f, \mathfrak{g})_{(0)}$, for all $f \in \mathscr{E}(M)$. Thus, our assertion will follow from the global transversality property

$$\Phi_f^* \mathscr{F}(A_f, \mathfrak{g}) \cap \Phi_g^* \mathscr{F}(A_g, \mathfrak{g}) = \{0\}. \tag{7.14}$$

In our situation, a stronger transversality holds, namely

$$\Phi_f^* A_f^1 \otimes \mathfrak{g} \cap \Phi_g^* A_g^1 \otimes \mathfrak{g} = \{0\}. \tag{7.15}$$

Indeed, property (7.15) becomes

$$\operatorname{im} H^1(f) \otimes \mathfrak{g} \cap \operatorname{im} H^1(g) \otimes \mathfrak{g} = \{0\}, \tag{7.16}$$

by the construction of Φ. On the other hand,

$$\operatorname{im} H^1(f) \cap \operatorname{im} H^1(g) = \{0\}, \tag{7.17}$$

by the argument from the proof of Theorem 7.2, which also works for quasi-Kähler manifolds. Thus, equality (7.16) holds, and this completes our proof. □

It is proved in [9, Theorem 4.2] that all pairs of distinct irreducible components of $\mathscr{V}_1^1(M)$ intersect in a finite set, for any quasi-projective manifold M. In light of the bijection from Theorem 4.1, Theorem 7.10 may be viewed as a non-abelian analog of this rank 1 result.

8. Rank greater than 2

In this section, we consider in more detail the case when M is a punctured quasi-homogeneous, isolated surface singularity, as in Theorem 1.4, Part (5). For the group $\pi = \pi_1(M)$, we will examine the natural inclusion

$$\operatorname{Hom}(\pi, G)_{(1)} \supseteq \operatorname{abf}^* \operatorname{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cup \bigcup_{f \in \mathscr{E}(M)} f_\sharp^* \operatorname{Hom}(\pi_f, G)_{(1)}, \tag{8.1}$$

where $G = \mathrm{SL}_n(\mathbb{C})$ with $n \geqslant 3$. We begin by recalling from [22, § 9] several relevant facts.

Since M is a quasi-homogeneous variety, there is a positive-weight $\mathbb{C}^\times$-action on M with finite isotropy groups. The orbit space $M/\mathbb{C}^\times$ is a smooth projective curve Σ_g, where $g = b_1(M)/2$. Thus, our standard assumption that $b_1(M) > 0$ translates to $g > 0$.

It is readily seen that the canonical projection, $f\colon M \to M/\mathbb{C}^\times = M_f$, is an admissible map. Furthermore,

$$\mathscr{E}(M) = \begin{cases} \emptyset & \text{if } g = 1, \\ \{f\} & \text{if } g > 1. \end{cases} \tag{8.2}$$

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Set $H^\bullet = (H^\bullet(\Sigma_g), d = 0)$. Define a CDGA (A, d) by $A^\bullet = H^\bullet \otimes \bigwedge(t)$, with t of degree 1, where $d = 0$ on $H^\bullet$ and $dt = \omega$, where $\omega \in H^2$ is the orientation class. Then A (respectively H) is a finite model of M (respectively M_f).

Theorem 8.1. *Let $M = S \setminus \{0\}$, where S is a quasi-homogeneous affine surface having a normal, isolated singularity at 0. If $b_1(M) > 0$ and $G = \mathrm{SL}_n(\mathbb{C})$ with $n \geqslant 3$, then inclusion (8.1) is strict.*

Proof. Assuming the contrary, we infer for $g > 1$ that

$$\mathrm{Hom}(\pi, G)_{(1)} = f_\sharp^* \, \mathrm{Hom}(\pi_f, G)_{(1)}. \tag{8.3}$$

Indeed, in this case equality in (8.1) becomes

$$\mathrm{Hom}(\pi, G)_{(1)} = \mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cup f_\sharp^* \, \mathrm{Hom}(\pi_f, G)_{(1)}, \tag{8.4}$$

since $\mathscr{E}(M) = \{f\}$. On the other hand,

$$\mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \subseteq f_\sharp^* \, \mathrm{Hom}(\pi_f, G)_{(1)}, \tag{8.5}$$

by Lemma 6.7.

If $g = 1$, equality in (8.1) becomes

$$\mathrm{Hom}(\pi, G)_{(1)} = \mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}, \tag{8.6}$$

since $\mathscr{E}(M) = \emptyset$.

We will show that both (8.3) and (8.6) lead to a contradiction. We denote by $\varphi \colon H \hookrightarrow A$ the canonical CDGA inclusion. Note that both H and A are CDGAs with positive weights, preserved by the map φ; see [22, Proposition 9.1].

First, we claim that equality (8.3) implies that

$$\mathscr{F}(A, \mathfrak{g}) = \varphi^* \mathscr{F}(H, \mathfrak{g}). \tag{8.7}$$

To verify this claim, let us note that, by Lemmas 2.2–2.4, it is enough to construct a local analytic isomorphism,

$$\mathscr{F}(A, \mathfrak{g})_{(0)} \cong \varphi^* \mathscr{F}(H, \mathfrak{g})_{(0)}. \tag{8.8}$$

In turn, such an isomorphism is obtained as follows. First, $\mathscr{F}(A, \mathfrak{g})_{(0)} \cong \mathrm{Hom}(\pi, G)_{(1)}$, by Theorem 3.6. Next, $\mathrm{Hom}(\pi, G)_{(1)} = f_\sharp^* \, \mathrm{Hom}(\pi_f, G)_{(1)}$, by (8.3). On the other hand, we clearly have that $f_\sharp^* \, \mathrm{Hom}(\pi_f, G)_{(1)} \cong \mathrm{Hom}(\pi_f, G)_{(1)}$ and $\varphi^* \mathscr{F}(H, \mathfrak{g})_{(0)} \cong \mathscr{F}(H, \mathfrak{g})_{(0)}$. Finally, $\mathrm{Hom}(\pi_f, G)_{(1)} \cong \mathscr{F}(H, \mathfrak{g})_{(0)}$, again by Theorem 3.6. Thus, our claim is established.

Now, we claim that (8.6) also implies equality (8.7). By the previous argument, it is enough to construct the isomorphism (8.8). As before, $\mathscr{F}(A, \mathfrak{g})_{(0)} \cong \mathrm{Hom}(\pi, G)_{(1)}$. Next, $\mathrm{Hom}(\pi, G)_{(1)} = \mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$, by (8.6). Plainly, $\mathrm{abf}^* \, \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cong \mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)}$ and $\varphi^* \mathscr{F}(H, \mathfrak{g})_{(0)} \cong \mathscr{F}(H, \mathfrak{g})_{(0)}$. Set $A_0 = (\bigwedge^\bullet H^1(M), d = 0)$. We infer from Lemma 3.7 and Theorem 3.6 that $\mathrm{Hom}(\pi_{\mathrm{abf}}, G)_{(1)} \cong \mathscr{F}(A_0, \mathfrak{g})_{(0)}$. Finally, the CDGAs A_0 and H are isomorphic, since $b_1(M) = b_1(M_f)$ and $g = 1$.

In conclusion, equality in (8.1) implies (8.7), in all cases.

It will be convenient to rephrase equality (8.7) in terms of the holonomy Lie algebra $\mathfrak{h}(A)$ described in § 3.2. In view of the isomorphism (3.6), the equality (8.7) holds if and only if the natural morphism,

$$\mathfrak{h}(\varphi)^*\colon \operatorname{Hom}_{\texttt{Lie}}(\mathfrak{h}(H), \mathfrak{g}) \longrightarrow \operatorname{Hom}_{\texttt{Lie}}(\mathfrak{h}(A), \mathfrak{g}), \tag{8.9}$$

is surjective.

Let $a_1, b_1, \ldots, a_g, b_g$ be the dual of a symplectic basis of H^1, and let $\mathbb{L}$ be the free Lie algebra with this generating set. Write $r = \sum_{i=1}^{g}[a_i, b_i]$. It is straightforward to check that $\mathfrak{h}(H)$ is the quotient of $\mathbb{L}$ by the ideal generated by r, while $\mathfrak{h}(A)$ is the quotient of $\mathbb{L}$ by the ideal generated by $[a_i, r]$ and $[b_i, r]$, for $i = 1, \ldots, g$. Moreover, the Lie morphism $\mathfrak{h}(\varphi)\colon \mathfrak{h}(A) \to \mathfrak{h}(H)$ is the identity on free generators. To disprove surjectivity in (8.9), we have to construct a Lie algebra map $\rho\colon \mathfrak{h}(A) \to \mathfrak{g}$ which does not factor through $\mathfrak{h}(H)$.

To achieve this goal, we first need to recall from [14] a couple of classical facts from the structure theory of semisimple Lie algebras. The elements of the root system R of $\mathfrak{g} = \mathfrak{sl}_n$ are $t_{ij} := t_i - t_j$, $1 \leqslant i \neq j \leqslant n$, where t_i denotes the ith projection of the Cartan subalgebra consisting of the diagonal matrices in $\mathfrak{sl}_n$. For each such root, the corresponding 1-dimensional root space $\mathfrak{g}_{ij}$ is of the form $\mathbb{C} \cdot X_{ij}$, for some $X_{ij} \in \mathfrak{g}$. It is known that $[X_{ij}, X_{kl}] = 0$ if $0 \neq t_{ij} + t_{kl} \notin R$, and $[X_{ij}, X_{kl}] = c \cdot X_{i'j'}$ for some $c \in \mathbb{C}^\times$ if $t_{ij} + t_{kl} = t_{i'j'} \in R$.

Assuming that $n \geqslant 3$, we may now define the morphism $\rho\colon \mathfrak{h}(A) \to \mathfrak{g}$ by sending the free Lie generators to $\rho(a_1) = X_{12}$, $\rho(b_1) = X_{23}$, and $\rho(a_i) = \rho(b_i) = 0$ for $1 < i \leqslant g$. By the above discussion, $\rho(r) = c \cdot X_{13}$, for some $c \in \mathbb{C}^\times$. Since clearly $0 \neq t_{12} + t_{13} \notin R$ and $0 \neq t_{23} + t_{13} \notin R$, we have that $\rho([a_i, r]) = \rho([b_i, r]) = 0$, for $i = 1, \ldots, g$. Hence, $\rho \in \operatorname{Hom}_{\texttt{Lie}}(\mathfrak{h}(A), \mathfrak{g})$. Plainly, the map ρ does not factor through $\mathfrak{h}(H)$, since $\rho(r) \neq 0$. This completes the proof. □

9. Depth greater than 1

Let M be a quasi-Kähler manifold, and let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation of a $\mathbb{C}$-linear algebraic group G. By Lemma 3.3, we have for each $r \geqslant 0$ an inclusion of affine varieties,

$$\mathscr{V}^1_r(\pi, \iota) \supseteq \bigcup_{f \in E(M)} f^*_\sharp \mathscr{V}^1_r(\pi_f, \iota). \tag{9.1}$$

For each $f \in E(M)$, we may view the induced homomorphism in cohomology, $\Phi_f := H^\bullet(f)\colon H^\bullet(M_f) \to H^\bullet(M)$, as a map of CDGAs with zero differentials. Let $\theta := d_1(\iota)\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be the tangential Lie algebra representation. By Lemma 3.5, for each $r \geqslant 0$ we have an inclusion of affine varieties,

$$\mathscr{R}^1_r(H^\bullet(M), \theta) \supseteq \bigcup_{f \in E(M)} \Phi^*_f \mathscr{R}^1_r(H^\bullet(M_f), \theta). \tag{9.2}$$

Lemma 9.1. *If M is either a connected, compact Kähler manifold or the complement of a central complex hyperplane arrangement, then the inclusion (9.1) becomes an equality near 1 if and only if the inclusion (9.2) is an equality.*

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Proof. By the argument from the proof of Theorem 7.10, we may apply [23, Theorem 6.4] to the families $\{f \in E(M)\}$ and $\{\Phi_f \mid f \in E(M)\}$, for $q = 1$. We obtain in this manner a local analytic identification, $\mathscr{V}^1_r(\pi, \iota)_{(1)} \cong \mathscr{R}^1_r(H^{\bullet}(M), \theta)_{(0)}$, which induces similar identifications, $f^*_{\sharp}\mathscr{V}^1_r(\pi_f, \iota)_{(1)} \cong \Phi^*_f\mathscr{R}^1_r(H^{\bullet}(M_f), \theta)_{(0)}$, for all $f \in E(M)$. Hence, (9.1) becomes an equality near 1 if and only if (9.2) becomes an equality near 0.

On the other hand, the CDGA $(H^{\bullet}(M), d = 0)$ has positive weights, equal to the degrees. As explained in [8, § 9.17], this endows the variety $\mathscr{R}^1_r(H^{\bullet}(M), \theta)$ with positive weights. Our claim follows then from Lemmas 2.2 and 2.4. □

We continue with a more detailed analysis of inclusion (9.1) near 1, in the rank 2 case, i.e., when $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{Sol}_2(\mathbb{C})$. In the context of Lemma 9.1, we know from [23, Theorem 1.3] that in this case (9.1) holds as an equality near 1 for any ι, when $r = 0$ or 1. What about depth greater than 1?

Lemma 9.2. *Let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional Lie algebra representation of $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sol}_2$ having a non-zero vector $v \in V$ annihilated by $\mathfrak{g}$. If M is a quasi-Kähler manifold with the property that there is an $f \in \mathscr{E}(M)$ with $b_1(M_f) < b_1(M)$, then there is an integer $r > 1$ such that inclusion (9.2) is strict.*

Proof. By [19, Lemma 7.3], there is a flat connection $\omega \in \mathscr{F}(H^{\bullet}(M_f), \mathfrak{g})$ which is not in $\mathscr{F}^1(H^{\bullet}(M_f), \mathfrak{g})$. Set $\Omega = \Phi^*_f(\omega)$. Clearly, $\Omega \in \mathscr{F}(H^{\bullet}(M), \mathfrak{g}) \setminus \mathscr{F}^1(H^{\bullet}(M), \mathfrak{g})$, since $H^1(f)$ is injective.

Next, recall from § 3.3 that, given a finite CDGA A, a finite-dimensional Lie algebra representation $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$, and a flat connection $\omega \in \mathscr{F}(A, \mathfrak{g})$, there is an associated Aomoto cochain complex, $(A^{\bullet} \otimes V, d_\omega)$, with differential $d^i_\omega\colon A^i \otimes V \to A^{i+1} \otimes V$. This gives rise to Aomoto–Betti numbers, $b_i(\omega, \theta) := \dim H^i(A \otimes V, d_\omega)$. By definition, $\omega \in \mathscr{R}^i_k(A, \theta)$ if and only if $b_i(\omega, \theta) \geqslant k$. In the above setup, write $s = b_1(\omega, \theta)$ and $r = b_1(\Omega, \theta)$. By [19, Proposition 2.4], we have that $s \geqslant 1$. We claim that $r > s$.

To verify the claim, let us consider the natural cochain map from [8],

$$H^{\bullet}(f) \otimes \mathrm{id}_V\colon (H^{\bullet}(M_f) \otimes V, d_\omega) \to (H^{\bullet}(M) \otimes V, d_\Omega). \tag{9.3}$$

Since $H^0(M_f) = H^0(M) = \mathbb{C}$ and $H^1(f)$ is injective, we infer that $\dim \mathrm{im}(d^0_\omega) = \dim \mathrm{im}(d^0_\Omega)$. Thus, we need to show that $\dim \ker(d^1_\omega) < \dim \ker(d^1_\Omega)$. To this end, we use our assumption that $b_1(M_f) < b_1(M)$ and pick a class $\eta \in H^1(M) \setminus \mathrm{im}\, H^1(f)$. Our hypothesis on θ yields the subspace

$$H^1(f) \otimes \mathrm{id}_V(\ker(d^1_\omega)) \oplus \mathbb{C} \cdot \eta \otimes v \subseteq H^1(M) \otimes V. \tag{9.4}$$

By (9.3), we have an inclusion $H^1(f) \otimes \mathrm{id}_V(\ker(d^1_\omega)) \subseteq \ker(d^1_\Omega)$. If $\eta \otimes v \in \ker(d^1_\Omega)$, our claim follows. On the other hand, the property that $d^1_\Omega(\eta \otimes v) = 0$ is an immediate consequence of the fact that $\mathfrak{g}$ annihilates v, by the construction of d_Ω recalled in § 3.3.

Finally, we will show that

$$\Omega \in \mathscr{R}^1_r(H^{\bullet}(M), \theta) \setminus \bigcup_{g \in E(M)} \Phi^*_g\mathscr{R}^1_r(H^{\bullet}(M_g), \theta). \tag{9.5}$$

First, let us suppose that $\Omega \in \Phi^*_0\mathscr{R}^1_r(H^{\bullet}(M_0), \theta)$. We know from Lemma 3.7 that $\mathscr{F}(H^{\bullet}(M_0), \mathfrak{g}) = \mathscr{F}^1(H^{\bullet}(M_0), \mathfrak{g})$. This leads to $\Omega \in \mathscr{F}^1(H^{\bullet}(M), \mathfrak{g})$, a contradiction.

Next, assume that $\Omega \in \Phi_g^* \mathscr{R}_r^1(H^{\bullet}(M_g), \theta)$, for some $g \in \mathscr{E}(M)$. Consequently,

$$\Omega \in \operatorname{im} H^1(g) \otimes \mathfrak{g} \cap \operatorname{im} H^1(f) \otimes \mathfrak{g}, \tag{9.6}$$

by the construction of Ω. If $g \neq f$, then $\operatorname{im} H^1(g) \cap \operatorname{im} H^1(f) = 0$, by [10, Theorem C(2)]. This leads to $\Omega = 0$, again a contradiction.

Hence, $\Omega = \Phi_f^*(\omega')$, for some $\omega' \in \mathscr{R}_r^1(H^{\bullet}(M_f), \theta)$. Since $H^1(f)$ is injective, we infer that $\omega = \omega'$. Therefore, $s = b_1(\omega, \theta) = b_1(\omega', \theta) \geqslant r$. This last contradiction completes our proof. □

Putting together Lemmas 9.1 and 9.2, we obtain the following theorem.

Theorem 9.3. *Let M be either a connected, compact Kähler manifold, or the complement of a central complex hyperplane arrangement. Let $\iota \colon G \to \mathrm{GL}(V)$ be a rational representation of $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{Sol}_2(\mathbb{C})$, having a non-zero fixed vector $v \in V^G$. If there exists an admissible map $f \colon M \to M_f$ with $b_1(M_f) < b_1(M)$, then there is an integer $r > 1$ such that inclusion (9.1) is strict near 1.*

Example 9.4. Suppose M is the complement of a central arrangement in $\mathbb{C}^3$. There are then two cases to consider. If the lines of the associated projective arrangement in $\mathbb{CP}^2$ intersect only in double points, it is well known that the group $\pi = \pi_1(M)$ is free abelian. Hence, $\mathrm{abf}^* \colon \mathscr{V}_r^1(\pi_{\mathrm{abf}}, \iota) \to \mathscr{V}_r^1(\pi, \iota)$ is an isomorphism, and the inclusion (9.1) becomes a global equality, for any ι and all $r \geqslant 0$.

On the other hand, if the lines in $\mathbb{CP}^2$ have an intersection point of multiplicity $m \geqslant 3$, we claim that Theorem 9.3 applies to M. Indeed, [12, Lemma 3.14] implies that $\mathscr{R}_1^1(H^{\bullet}(M))$ has an irreducible component of dimension $m - 1$. By [10, Theorem C(3)], this component is of the form $\operatorname{im}(H^1(f))$, for some $f \in \mathscr{E}(M)$. Finally, $b_1(M_f) < b_1(M)$, since otherwise clearly $\mathscr{R}_1^1(H^{\bullet}(M)) = H^1(M)$, in contradiction with [12, Theorem 2.8].

Compact examples for Theorem 9.3 are also easy to construct.

Example 9.5. Let M be the product $\Sigma_g \times N$, where Σ_g is a projective curve of genus $g > 1$ and N is a projective manifold with $b_1(N) > 0$. Plainly, the canonical projection $f \colon M \to \Sigma_g = M_f$ gives an element $f \in \mathscr{E}(M)$ with $b_1(M_f) < b_1(M)$.

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