

The Evolution of the Cover Time

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The cover time of a graph is a celebrated example of a parameter that is easy to approximate using a randomized algorithm, but for which no constant factor deterministic polynomial time approximation is known. A breakthrough due to Kahn, Kim, Lovász and Vu [25] yielded a $(\log \log n)^2$ polynomial time approximation. We refine the upper bound of [25], and show that the resulting bound is sharp and explicitly computable in random graphs. Cooper and Frieze showed that the cover time of the largest component of the Erdős–Rényi random graph $G(n, c/n)$ in the supercritical regime with $c > 1$ fixed, is asymptotic to $\varphi(c)n \log^2 n$, where $\varphi(c) \rightarrow 1$ as $c \downarrow 1$. However, our new bound implies that the cover time for the critical Erdős–Rényi random graph $G(n, 1/n)$ has order n , and shows how the cover time evolves from the critical window to the supercritical phase. Our general estimate also yields the order of the cover time for a variety of other concrete graphs, including critical percolation clusters on the Hamming hypercube $\{0, 1\}^n$, on high-girth expanders, and on tori \mathbb{Z}_n^d for fixed large d . This approach also gives a simpler proof of a result of Aldous [2] that the cover time of a uniform labelled tree on k vertices is of order $k^{3/2}$. For the graphs we consider, our results show that the *blanket* time, introduced by Winkler and Zuckerman [45], is within a constant factor of the cover time. Finally, we prove that for any connected graph, adding an edge can increase the cover time by at most a factor of 4.

1. Introduction

The cover time $t_{\text{cov}}(G)$ of a graph G is the expected number of steps a simple random walk takes to visit every vertex of the graph G , starting from the worst-possible vertex. It has been studied

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extensively by computer scientists, due to its intrinsic appeal and its applications to designing universal traversal sequences [4, 8, 9], testing graph connectivity [4, 26], and protocol testing [36]; see [1] for an introduction to cover times.

Sophisticated methods to estimate the cover time have been developed [18, 19, 35, 25]. One of the most precise bounds was obtained by Kahn, Kim, Lovász and Vu [25]. They gave polynomially computable upper and lower bounds that differ by a factor of order $(\log \log n)^2$. This breakthrough left several questions open.

- (i) Can the bounds in [25] be represented by an explicit formula for concrete graphs of interest?
- (ii) For such graphs, can the $(\log \log n)^2$ factor be removed?

In this work we improve the upper bound from [25] and show that the resulting estimate is sharp up to a constant factor, and explicitly computable, for a large variety of graphs, in particular random graphs.

Let $G = (V, E)$ be a graph (not necessarily simple) and write $R_{\text{eff}}(x, y)$ and $d(x, y)$ for the *effective resistance* and graph distance between two vertices $x, y \in V$, respectively. See, e.g., [34] or [31] for definitions and properties of effective resistance. It is known that $R_{\text{eff}}(x, y) \leq d(x, y)$ and that $R_{\text{eff}}(\cdot, \cdot)$ forms a metric on G . For $x \in V$ and a real number $R > 0$ we write $B_{\text{eff}}(x, R)$ for the ball of radius R in the resistance metric, that is,

$$B_{\text{eff}}(x, R) = \{v \in G : R_{\text{eff}}(x, v) \leq R\}.$$

Theorem 1.1. *Let $G = (V, E)$ be a finite graph with diameter R in the resistance metric. For $i \in \mathbb{N}$, let $A_i = A_i(G)$ be a set of minimal size such that*

$$G \subset \bigcup_{v \in A_i} B_{\text{eff}}\left(v, \frac{R}{2^i}\right), \tag{1.1}$$

and write $\alpha_i = 2^{-i} \log |A_i|$. Then there exists a universal constant $C > 0$ such that

$$t_{\text{cov}} \leq C \left(\sum_{i=1}^{\log_2 \log n} \sqrt{\alpha_i}\right)^2 R |E|. \tag{1.2}$$

The right-hand side is approximable up to constant factors in polynomial time: see Remark 2.2. Theorem 1.1 is a refinement of Theorem 1.3 of [25], stating that

$$\max_i \alpha_i R |E| \leq t_{\text{cov}}(G) \leq C (\log \log n)^2 \cdot \max_i \alpha_i R |E|. \tag{1.3}$$

The lower bound is a variant of Matthews’ estimate for cover times [35], and the upper bound is the main contribution of [25]. We refine the methods of [25] to deduce the stronger statement of Theorem 1.1 (clearly, $(\sum_{i=1}^{\log_2 \log n} \sqrt{\alpha_i})^2 \leq (\log_2 \log n)^2 \max_i \alpha_i$). This new bound turns out to be sharp in many concrete examples, where we can show that $\max_i \alpha_i$ and $(\sum_{i=1}^{\log_2 \log n} \sqrt{\alpha_i})^2$ are of the same order. Such examples are presented in Theorems 1.2 and 1.3 below.

Cooper and Frieze [11] studied the cover time of the largest component of the Erdős–Rényi [16] random graph model $G(n, p)$, that is, the random graph obtained from the complete graph K_n by retaining each edge with probability p independently. It is well known that if $p = \frac{c}{n}$ for some $c > 1$, then the largest connected component, \mathcal{C}_1 , is of size about xn with probability tending to 1, where $x = x(c)$ is the unique solution in $(0, 1)$ of $x = 1 - e^{-cx}$. Cooper and Frieze [11]

established the asymptotics for the cover time in this regime,

$$t_{\text{cov}}(\mathcal{C}_1) \sim \varphi(c)n \log^2 n \quad \text{with} \quad \varphi(c) = \frac{cx(2-x)}{4(cx - \log c)}$$

with probability tending to 1 as $n \rightarrow \infty$.

Since $\varphi(c)$ tends to 1 as $c \rightarrow 1$, one might be tempted to guess that $t_{\text{cov}}(\mathcal{C}_1)$ for $G(n, 1/n)$ is of order $n \log^2 n$. However, it is known [38] that the maximal hitting time between two vertices in \mathcal{C}_1 is typically of order n , so Matthews' bound [35] shows that $t_{\text{cov}}(\mathcal{C}_1)$ is at most $O(n \log n)$. In fact, in $G(n, 1/n)$ the largest component \mathcal{C}_1 is roughly of size $n^{2/3}$ [17, 6, 32], and with probability uniformly bounded away from 0 it is a tree. Aldous [2] proved that a random tree on k vertices has cover time of order $k^{3/2}$ (see Theorem 3.2 for a precise statement and an alternative proof). Combining these facts yields that $t_{\text{cov}}(\mathcal{C}_1)$ in $G(n, \frac{1}{n})$ is of order n with probability uniformly bounded away from 0. In the following theorem we show that this probability tends to 1, and moreover, we show how the order of the cover time continuously evolves from the critical regime $c = 1$ to the supercritical regime $c > 1$.

Theorem 1.2. *Let $t_{\text{cov}}(\mathcal{C}_1)$ denote the cover time of largest component of $G(n, p)$ and let $\lambda \in \mathbb{R}$ be fixed and $\varepsilon(n) > 0$ be a sequence such that $\varepsilon(n) \rightarrow 0$ but $n^{1/3}\varepsilon(n) \rightarrow \infty$.*

(a) *If $p = \frac{1-\varepsilon(n)}{n}$, then for any $\delta > 0$ there exists $B > 0$ such that*

$$\mathbb{P}(B^{-1}\varepsilon^{-3} \log^{3/2}(\varepsilon^3 n) \leq t_{\text{cov}}(\mathcal{C}_1) \leq B\varepsilon^{-3} \log^{3/2}(\varepsilon^3 n)) \geq 1 - \delta.$$

(b) *If $p = \frac{1+\lambda n^{-1/3}}{n}$, then then for any $\delta > 0$ there exists $B > 0$ such that*

$$\mathbb{P}(B^{-1}n \leq t_{\text{cov}}(\mathcal{C}_1) \leq Bn) \geq 1 - \delta.$$

(c) *There exists a constant $C > 0$ such that if $p = \frac{1+\varepsilon(n)}{n}$, then*

$$\mathbb{P}(C^{-1}n \log^2(\varepsilon^3 n) \leq t_{\text{cov}}(\mathcal{C}_1) \leq Cn \log^2(\varepsilon^3 n)) \rightarrow 1.$$

Theorem 1.1 also allows us to prove sharp bounds on cover time for critical percolation clusters, even when the underlying graph is *not* the complete graph. Given a graph G on n vertices and $p \in [0, 1]$, the random graph G_p is obtained from G by retaining each edge with probability p independently. In the special case of $G = K_n$, this yields the Erdős–Rényi graph $G(n, p)$. For a vertex $v \in G$ we write $\mathcal{C}(v)$ for the connected component in G_p containing v , and denote by \mathcal{C}_1 the largest connected component of G_p . We are interested in critical percolation in which $|\mathcal{C}_1| \approx n^{2/3}$. This occurs in numerous underlying graphs G . A partial list of examples is:

- (1) the complete graph on n vertices [17, 6, 32] with $p = \frac{1+\Theta(n^{-1/3})}{n}$,
- (2) a random d -regular graph [39, 41] with $p = \frac{1+\Theta(n^{-1/3})}{d-1}$,
- (3) expanders of high girth and degree d [37] with $p = \frac{1+\Theta(n^{-1/3})}{d-1}$,
- (4) the Hamming hypercube $\{0, 1\}^m$ [7] with p satisfying $\mathbb{E}_p|\mathcal{C}(v)| = \Theta(n^{1/3})$,
- (5) discrete tori \mathbb{Z}_m^d for large but fixed dimension d with $p = p_c(\mathbb{Z}^d)$ or p satisfying $\mathbb{E}_p|\mathcal{C}(v)| = \Theta(n^{1/3})$ [7, 21, 22].

In all the examples above it is known that for any $\delta > 0$ there exists $B = B(\delta) > 0$ such that

$$\mathbb{P}_p(B^{-1}n^{2/3} \leq |\mathcal{C}_1| \leq Bn^{2/3}) \geq 1 - \delta.$$

The following theorem is a generalization of Theorem 1.2(b), and states that in these cases $t_{\text{cov}}(\mathcal{C}_1)$ has order n . This means that the cover time of the largest component has the same order as the cover time of a random tree on the same number of vertices. We note that unlike the $G(n, p)$ case, in examples (4) and (5), the probability that the largest component is a tree tends to zero as the volume grows, so the Aldous estimate [2] does not apply.

Theorem 1.3. *In examples (1)–(5) above, we have that for any $\delta > 0$ there exists $B = B(\delta) > 0$ such that*

$$\mathbb{P}_p(B^{-1}n \leq t_{\text{cov}}(\mathcal{C}_1) \leq Bn) \geq 1 - \delta.$$

In fact, in Section 3 we provide a general criterion for the conclusion of Theorem 1.3 to hold, which applies to examples (1)–(5): see Theorem 3.1.

Remark 1.4. The *blanket* time t_B is the expected first time when the local times at all vertices are within a factor of 2 from each other (the local time at a vertex v is the number of visits to v divided by the degree of v). This quantity was introduced by Winkler and Zuckerman [45] (we use the definition of [25]), who conjectured that $t_B = O(t_{\text{cov}})$ for any graph. The bounds in Theorems 1.1–1.3 also apply to t_B in place of t_{cov} . This will be clear from the proofs.

Finally, it is natural to guess that adding edges to a graph can only decrease the cover time. However, this is not the case, as is shown by the following example. Let K_n^* be the graph obtained from K_n (the complete graph on n vertices) by adding a new vertex v and connecting it to one vertex of K_n . The cover time of K_n^* is easily seen to be n^2 . On the other hand, if we replace K_n by H_n , a bounded degree expander on n vertices, and construct H_n^* by adding a new vertex v and connecting it to one vertex of H_n , then the cover time of H_n^* is of order $n \log n$. Since H_n^* is a subgraph of K_n^* on the same $n + 1$ vertices, we conclude that adding an edge to a graph may increase the cover time. The increase is at most by a constant factor.

Proposition 1.5. *Let G be a connected graph and let $u, v \in G$ be two vertices. Let G^+ be the graph obtained from G by adding the edge $\{u, v\}$ (if an edge connecting these two vertices already exists, then we add a multiple edge, and if $u = v$, then we add a loop). Then we have*

$$t_{\text{cov}}(G^+) \leq 4t_{\text{cov}}(G).$$

2. Proof of Theorem 1.1

Let S_t be a simple random walk on G , and for an integer $t \geq 0$, define the *local time* L_t^v of a vertex $v \in V$ by

$$L_t^v \triangleq \frac{1}{d_v} \sum_{k=0}^t 1_{\{S_k=v\}}, \quad \text{for all } v \in V \text{ and } t \in \mathbb{N}, \tag{2.1}$$

where d_v is the degree of vertex v . Furthermore, let $\tau_k^v = \min\{t \in \mathbb{N} : L_t^v = k/d_v\}$ be the time of the k th visit of the random walk to v . The following lemma of Kahn, Kim, Lovász and Vu [25]

implies that if the local time at a vertex u is large, then with high probability, the local time is also large at vertices v that are close to u in the resistance metric.

Lemma 2.1 ([25], Lemma 5.2). *For all $u, v \in V$, numbers $\lambda \geq 0$ and $t \in \mathbb{N}$ we have*

$$\mathbb{P}_u(L_{\tau_t^u}^u - L_{\tau_t^v}^v \geq \lambda) \leq \exp\left(-\frac{\lambda^2}{4tR_{\text{eff}}(u, v)}\right).$$

We use an idea of Kolmogorov [42, page 91]. For all $i \geq 1$ and for each $u \in A_i$, we can always select $v \in A_{i-1}$ such that $R_{\text{eff}}(u, v) \leq 2^{-(i-1)}R$ (see (1.1)). Write $v = h(u)$. Set $\alpha'_i = \alpha_i \vee 2^{-i/2}$ and define

$$\Psi = 128 \left(\sum_{i=1}^{\infty} \sqrt{\alpha'_i} \right)^2, \quad \text{and} \quad \beta_i = \frac{\sqrt{\alpha'_i}}{2 \sum_{i=1}^{\infty} \sqrt{\alpha'_i}} \quad \text{for all } i \in \mathbb{N}.$$

For $i \in \mathbb{N}$, let $t_i = (1 - \sum_{j=1}^i \beta_j)\Psi$, and for $u \in A_i$ define $t_i^*(u) = \lceil t_i R d_u \rceil / d_u$. Furthermore, define $M_i(u)$ to be the difference of the local times of vertices $h(u)$ and u at time $\tau_{t_{i-1}^*(h(u))}^{h(u)}$, by

$$M_i(u) = t_{i-1}^*(h(u)) - L_{\tau_{t_{i-1}^*(h(u))}^{h(u)}}^u.$$

Note that $t_i^*(u) \leq t_i R + 1/d_u \leq (t_i + 1)R \leq 2t_i R$. Lemma 2.1 then gives that

$$\mathbb{P}(M_i(u) \geq \beta_i \Psi R) \leq \exp\left(-\frac{(\beta_i \Psi R)^2}{4R2^{-(i-1)} \cdot 2t_{i-1} R}\right) \leq e^{-2^{i+1}\alpha'_i}.$$

Define $M_i = \max_{u \in A_i} M_i(u)$. Recalling the definition of α_i and α'_i , we apply a union bound and get

$$\mathbb{P}(M_i \geq \beta_i \Psi R) \leq |A_i| e^{-2^{i+1}\alpha'_i} \leq e^{-2^i \alpha'_i}.$$

It follows that

$$\mathbb{P}\left(\bigcup_{i \geq 1} \{M_i \geq \beta_i \Psi R\}\right) \leq \sum_{i \geq 1} e^{-2^i \alpha'_i} \leq \sum_{i \geq 1} e^{-2^{i/2}} \leq \frac{2}{3}. \tag{2.2}$$

Now, take $v \in V$ and write τ_{cov} for the cover time of the random walk. Provided that the event $\mathcal{M} \triangleq \bigcap_{i \geq 1} \{M_i \leq \beta_i \Psi R\}$ occurs, we can inductively deduce that $L_{\tau_{\Psi R}^u}^u \geq t_i^*(u)$ for all $u \in A_i$ and hence $L_{\tau_{\Psi R}^v}^v \geq \Psi R/2$ by the definition of β_i and $t_i^*(u)$. In particular, on the event \mathcal{M} every vertex in the graph should have been visited at least once. Combined with (2.2), it follows that

$$\mathbb{P}_v(\tau_{\text{cov}} \geq \tau_{\Psi R}^v) \leq \frac{2}{3}.$$

Now, let N be the minimal $i \in \mathbb{N}$ such that the random walk between time $\tau_{(i-1)\Psi R}^v$ and $\tau_{i\Psi R}^v$ covers the graph. The preceding inequality gives that $\mathbb{E}N \leq 3$. Note that $\tau_{i\Psi R}^v - \tau_{(i-1)\Psi R}^v$ is an i.i.d. sequence and the event $\{N \leq i\}$ is independent of $\tau_{(i+1)\Psi R}^v - \tau_{i\Psi R}^v$ for all i . Therefore, Wald's equation implies that $\mathbb{E}_v \tau_{\text{cov}} \leq 3\mathbb{E}_v \tau_{\Psi R}^v$. The expected return time to v satisfies $\mathbb{E}_v T_v^+ = 2|E|/d_v$, whence

$$\mathbb{E}_v \tau_{\text{cov}} \leq 3\Psi R d_v \mathbb{E}_v T_v^+ = 6\Psi R |E|. \tag{2.3}$$

Since the above holds for all $v \in V$, we have $t_{\text{cov}} \leq 6\Psi R|E|$. Note that $|A_i| \leq n$ for all $i \in \mathbb{N}$ and hence $\sum_{i \geq \log_2 \log n} \sqrt{\alpha_i} = O(1)$. Observing also that $\alpha'_i \leq \alpha_i + 2^{-i/2}$, we get

$$\Psi \leq 256 \left(\left(\sum_{i=1}^{\log_2 \log n} \sqrt{\alpha_i} \right)^2 + 16 \right).$$

It completes the proof of the theorem, together with the fact that $\alpha_1 \geq \frac{1}{2} \log 2$ (since $|A_1|$ has to be at least 2).

Remark 2.2. Note that the sum $\sum_i \sqrt{\alpha_i}$ can be easily approximated up to a constant. To see this, one can use a greedy algorithm to find a maximal collection of centres \tilde{A}_i such that

$$\{B_{\text{eff}}(v, 2^{-(i+1)}R) : v \in \tilde{A}_i\}$$

forms a collection of disjoint balls. Thus, $|\tilde{A}_{i-1}| \leq |A_i| \leq |\tilde{A}_i|$ and

$$\frac{1}{\sqrt{2}} \sum_i \sqrt{2^{-i} \log |\tilde{A}_i|} \leq \sum_i \sqrt{\alpha_i} \leq \sum_i \sqrt{2^{-i} \log |\tilde{A}_i|}.$$

3. Cover time of critical percolation clusters

We are interested in critical percolation in which $|C_1| \approx n^{2/3}$. This occurs in numerous underlying graphs G as listed in the Introduction (examples (1)–(5)). Recall the definition of G_p , and write $d_{G_p}(x, y)$ for the length of the shortest path between x and y in G_p , or ∞ if there is no such path. We call d the *intrinsic metric* on G_p . Define the random sets

$$B_p(x, r; G) = \{u : d_{G_p}(x, u) \leq r\}, \quad \partial B_p(x, r; G) = \{u : d_{G_p}(x, u) = r\},$$

and the event $H_p(x, r; G) = \{\partial B_p(x, r; G) \neq \emptyset\}$. Finally, define

$$\Gamma_p(x, r; G) = \sup_{G' \subset G} \mathbb{P}_{G'}(H_p(x, r; G')),$$

where $\mathbb{P}_{G'}$ is the percolation probability measure over subgraphs of G' . The reason for taking a supremum in the definition of Γ_p is that the event $H_p(x, r; G)$ is *not* monotone with respect to edge addition (indeed, adding an edge can potentially shorten a shortest path and make $\partial B_p(x, r; G)$ empty even if it were not empty before). The exponent describing the polynomial decay of Γ_p is called the *intrinsic metric arm exponent* and was introduced in [38]: see Theorem 2.1 of that paper or [29, Section 1.3] for further details.

Theorem 3.1. *Let $G = (V, E)$ be a graph and let $p \in [0, 1]$. Suppose that for some constants $c_1, c_2 > 0$ and all vertices $x \in V$ the following two conditions are satisfied:*

- (i) $\mathbb{E}|B_p(x, r; G)| \leq c_1 r$,
- (ii) $\Gamma_p(x, r; G) \leq c_2/r$.

Then, for any $\beta, \delta > 0$ there exists $B > 0$ such that

$$\mathbb{P}(\exists C \text{ with } |C| \geq \beta n^{2/3} \text{ and } t_{\text{cov}}(C) \notin [B^{-1}n, Bn]) \leq \delta.$$

Proof. The fact that there exists $B > 0$ such that

$$\mathbb{P}(\exists \mathcal{C} \text{ with } |\mathcal{C}| \geq \beta n^{2/3} \text{ and } t_{\text{cov}}(\mathcal{C}) \leq B^{-1}n) \leq \delta/2$$

follows immediately from the corresponding lower bound on the maximal hitting time: see part (c.2) of Theorem 2.1 of [38] and Lemma 4.1 in that paper. Also from [38] we have that for any $\beta, \delta' > 0$ there exists $D = D(\beta, \delta') > 0$ such that

$$\mathbb{P}(|\mathcal{C}(v)| \geq \beta n^{2/3} \text{ and } \text{diam}(\mathcal{C}(v)) \notin [D^{-1}n^{1/3}, Dn^{1/3}]) \leq \delta'n^{-1/3}. \tag{3.1}$$

To see this, combine (3.1) and (3.3) of [38]. Denote $\text{diam}_{\text{eff}}(\mathcal{C}(v))$ for the diameter of $\mathcal{C}(v)$ according to the resistance metric. We first show that with high probability components of size $n^{2/3}$ have diam_{eff} of order $n^{1/3}$. Indeed, the upper bound follows immediately from [38, Theorem 2.1] and the fact that $R_{\text{eff}}(x, y) \leq d(x, y)$. For the lower bound, we use [38, Proposition 5.6], the Nash–Williams inequality and (3.1) to deduce that, for large enough $D = D(\beta, \delta') > 0$, we have

$$\mathbb{P}(|\mathcal{C}(v)| \geq \beta n^{2/3} \text{ and } \text{diam}_{\text{eff}}(\mathcal{C}(v)) \leq D^{-1}n^{1/3}) \leq \delta'n^{-1/3}. \tag{3.2}$$

We now proceed to construct covering sets of G on different scales. Fix an integer $i \geq 0$, and we define a sequence of radii $\{r_j\}_{j \leq 2D^{2^i}}$ which have the following properties:

$$(a) \ r_0 = 0, \quad (b) \ \frac{(j - 1/2)n^{1/3}}{2D^{2^i}} \leq r_j \leq \frac{jn^{1/3}}{2D^{2^i}}, \quad (c) \ \mathbb{E} \partial B_p(v, r_j; G) \leq 4D^2 c_1 2^i.$$

This is possible by condition (i) of the theorem, which implies that for each $j \leq 2D^{2^i}$

$$\sum_{\ell=(j-1/2)n^{1/3}/(2D^{2^i})}^{jn^{1/3}/(2D^{2^i})} \mathbb{E} \partial B_p(v, \ell; G) \leq c_1 D n^{1/3},$$

and so there must exist $\ell \in [(j - 1/2)n^{1/3}/(2D^{2^i}), jn^{1/3}/(2D^{2^i})]$ such that $r_j = \ell$ satisfies condition (c). Given such radii $\{r_j\}$ we say that a vertex $u \in \partial B_p(v, r_j; G)$ is *i-good* if there exists a path between u and $\partial B_p(v, r_{j+1}; G)$ which does not go through $B_p(v, r_j; G)$. We now construct a sequence of sets $\{A'_i\}$ which will serve as a covering. Define

$$A'_i = \bigcup_{j \leq 2D^{2^i}} \{u \in \partial B_p(v, r_j; G) : u \text{ is } i\text{-good}\}.$$

Observe that if $\text{diam}(\mathcal{C}(v)) \leq Dn^{1/3}$ then we have that

$$\mathcal{C}(v) \subset \bigcup_{u \in A'_i} B_p(u, 2^{-i}D^{-1}n^{1/3}; G).$$

Further, if in addition $R = \text{diam}_{\text{eff}}(\mathcal{C}(v)) \geq D^{-1}n^{1/3}$, then $\mathcal{C}(v) \subset \bigcup_{u \in A'_i} B_p(u, 2^{-i}R; G)$. Given these two events and the fact that $B_p(u, r; G) \subset B_{\text{eff}}(u, r; \mathcal{C}(v))$, we deduce that

$$\mathcal{C}(v) \subset \bigcup_{u \in A'_i} B_{\text{eff}}(u, 2^{-i}R; \mathcal{C}(v)),$$

and therefore $|A_i| = |A_i(\mathcal{C}(v))| \leq |A'_i|$ for all $i \in \mathbb{N}$ (see (1.1)). By (3.1) and (3.2), we get that

$$\mathbb{P}(|\mathcal{C}(v)| \geq \beta n^{2/3}, \exists i \in \mathbb{N} : |A'_i| < |A_i|) \leq 2\delta'n^{-1/3}. \tag{3.3}$$

Now, by condition (ii) of our theorem and our construction of $\{r_j\}$, we get that

$$\mathbb{E}|A'_i| \leq \sum_{j \leq 2D^{2^i}} \mathbb{E} \delta B_p(v, r_j; G) \cdot \frac{4Dc_2 2^i}{n^{1/3}} \leq 16D^3 c_1 c_2 2^{2i} n^{-1/3}.$$

So we can choose a large integer $m = m(c_1, c_2, D, \delta')$ such that

$$\sum_{i=1}^{\infty} \mathbb{P}(|A'_i| \geq e^{m \cdot 2^{i/2}}) \leq \sum_{i=1}^{\infty} \frac{16D^3 c_1 c_2 2^{2i} n^{-1/3}}{e^{m \cdot 2^{i/2}}} \leq \delta' n^{-1/3}. \tag{3.4}$$

Recalling that $\alpha_i = 2^{-i} \log |A_i|$ (see Theorem 1.1) and combining the above estimate with (3.3), we obtain that

$$\begin{aligned} \mathbb{P}\left(|\mathcal{C}(v)| \geq \beta n^{2/3}, \sum_{i=1}^{\infty} \sqrt{\alpha_i} \geq 4m\right) &\leq \mathbb{P}(|\mathcal{C}(v)| \geq \beta n^{2/3}, \exists i \in \mathbb{N} : |A'_i| < |A_i|) \\ &\quad + \sum_{i=1}^{\infty} \mathbb{P}(|A'_i| \geq e^{m \cdot 2^{i/2}}) \leq 3\delta' n^{-1/3}. \end{aligned} \tag{3.5}$$

We say that $\mathcal{C}(v)$ is *bad* if $|\mathcal{C}(v)| \geq \beta n^{2/3}$ and one of the following holds:

- $\sum_{i=1}^{\infty} \sqrt{\alpha_i} \geq 4m,$
- $\text{diam}_{\text{eff}}(\mathcal{C}(v)) \geq Dn^{1/3},$
- $|E(\mathcal{C}(v))| \geq Dn^{2/3}.$

By (3.5) and [38, Theorem 2.1], we learn that we can choose D large enough so that the probability that $\mathcal{C}(v)$ is bad is at most $5\delta' n^{-1/3}$, whence $\mathbb{E}X \leq 5\delta' n^{2/3}$. Note that if there exists v such that $\mathcal{C}(v)$ is bad, then $X \geq \beta n^{2/3}$. By Theorem 1.1 we learn that there exists some large constant $B = B(D, m)$ such that if $|\mathcal{C}(v)| \geq \beta n^{2/3}$ and $t_{\text{cov}}(\mathcal{C}(v)) \geq Bn$, then $\mathcal{C}(v)$ is bad (taking $B = 16Cm^2D^2$, where C is the constant of Theorem 1.1, suffices). Hence, by Markov’s inequality

$$\mathbb{P}(\exists \mathcal{C} \text{ with } |\mathcal{C}| \geq \beta n^{2/3} \text{ and } t_{\text{cov}}(\mathcal{C}) \geq Bn) \leq \mathbb{P}(X \geq \beta n^{2/3}) \leq 5\delta' / \beta,$$

which concludes the proof of the theorem by setting $\delta' = \delta / (10\beta)$. □

Proof of Theorem 1.3. We only need to show that the conditions of Theorem 3.1 hold in examples (1)–(5). Indeed, it is shown in [38] that the conditions hold for examples (1)–(3), and in [29] and [30] it is shown for examples (4)–(5). In [22, 21] it is shown for example (5) that at $p = p_c(\mathbb{Z}^d)$ the largest cluster size is of order $n^{2/3}$. □

We will require the following result of Aldous [2]. For the reader’s convenience we provide a simpler proof of this theorem based on Theorem 1.1.

Theorem 3.2. *Let T be a Galton–Watson tree with progeny mean 1 and variance $\sigma^2 < \infty$. Then, for any $\delta > 0$ there exists $A = A(\delta, \sigma^2) > 0$ such that*

$$\mathbb{P}(t_{\text{cov}}(T) \notin [A^{-1}k^{3/2}, Ak^{3/2}] \mid |T| \in [k, 2k]) \leq \delta.$$

Proof. This is very similar to the proof of Theorem 3.1. Firstly, we claim that there exists $D > 0$ such that

$$\mathbb{P}(\text{diam}(T) \notin [D^{-1}k^{1/2}, Dk^{1/2}], |T| \in [k, 2k]) \leq k^{-1/2}\delta/2.$$

Indeed, it is a classical fact [27] that $\mathbb{P}(\text{diam}(T) \geq Dk^{1/2}) = O(D^{-1}k^{-1/2})$. Furthermore, the expected number of particles in T up to level $D^{-1}k^{1/2}$ is precisely $D^{-1}k^{1/2}$, and the event $\{\text{diam}(T) \leq D^{-1}k^{1/2}, |T| \geq k\}$ implies that this quantity is at least k . Hence, by Markov’s inequality we have that $\mathbb{P}(\text{diam}(T) \leq D^{-1}k^{1/2}, |T| \geq k) \leq D^{-1}k^{-1/2}$.

Now, for each i we define $r_j = j2^{-i-1}D^{-1}\sqrt{k}$ for $j = 0, \dots, 2^{i+1}D^2$ and define A'_i to be the set of particles at level r_j which survive up to level r_{j+1} . As in the proof of Theorem 3.1, if $\text{diam}(T) \in [D^{-1}k^{1/2}, Dk^{1/2}]$, then

$$T \subset \bigcup_{u \in A'_i} B_{\text{eff}}(u, 2^{-i}R; T),$$

where R is the diameter of T with respect to the resistance metric. Now, for each j the expected number of particles in level r_j is precisely 1 and the probability of surviving up to level r_{j+1} is of order $(r_{j+1} - r_j)^{-1}$ (see [27] again), hence $\mathbb{E}|A'_i| \leq C2^{2i+2}D^3k^{-1/2}$ and the proof continues as in (3.4), to show using Theorem 1.1 that there exists A such that

$$\mathbb{P}(t_{\text{cov}}(T) \geq Ak^{3/2}, |T| \in [k, 2k]) \leq k^{-1/2}\delta/2.$$

Let L be the offspring random variable of T . We have that $|T|$ is distributed as the first hitting time of 0 of a random walk starting at 1 with increments distributed as $L - 1$ (see [34, exercise 5.26]). We use this and Theorem 1a of Chapter XII.7 in [20] to deduce that

$$\mathbb{P}(|T| \in [k, 2k]) = (1 + o(1))Ck^{1/2},$$

for some constant $C > 0$. This gives the required upper bound on the cover time. The corresponding lower bound follows immediately from the lower bound on the maximal hitting time, which we obtain via the \sqrt{k} lower bound on the diameter of T together with commute time identity. □

Proof of Theorem 1.2(a,b). Part (b) of the theorem follows immediately from Theorem 3.1, so we are only left with proving part (a). In this case it is known that the largest cluster is a uniform random tree of order $\varepsilon^{-2} \log(\varepsilon^3 n)$ (see [23]). It is a classical fact (see [28, Chapter 2.2]) that a uniform random tree of size k is distributed as a Poisson(1) Galton–Watson tree T conditioned on $|T| = k$. Hence the following statement concludes the proof: let T be a Poisson(1) Galton–Watson tree; then for any $\delta > 0$ there exists $A > 0$ such that

$$\mathbb{P}(t_{\text{cov}}(T) \notin [A^{-1}k^{3/2}, Ak^{3/2}] \mid |T| = k) \leq \delta. \tag{3.6}$$

Note that this assertion does not immediately follow from Theorem 3.2. To fill in the gap, we will infer from a result of Luczak and Winkler [33] that there exists a coupling between a random tree T_k of size k and a random tree T_{k+1} of size $k + 1$ such that $T_k \subset T_{k+1}$. This, together with Theorem 3.2, shows the upper bound on the cover time of (3.6) and concludes the proof (the lower bound on the cover time is easier, and follows, as in the remark above, by the easy lower bound on the maximal hitting time).

To see that such a coupling exists, write $T_k^{(d)}$ for a $\text{Bin}(d, 1/d)$ Galton–Watson tree conditioned on being of size k . Theorem 4.1 in [33] shows that there exists a coupling between $T_k^{(d)}$ and $T_{k+1}^{(d)}$ such that $T_k^{(d)} \subset T_{k+1}^{(d)}$. Now, for any fixed k we may take $d \rightarrow \infty$ and we get the required coupling between $\text{Poisson}(1)$ Galton–Watson trees. This concludes our coupling since the latter trees are uniform random trees. \square

4. Cover time for mildly supercritical Erdős–Rényi graph

In this section, we prove Theorem 1.2(c), which incorporates the order of the cover time for the largest component of Erdős–Rényi graph $G(n, p)$ with $p = \frac{1+\varepsilon}{n}$, where $\varepsilon = o(1)$ and $\varepsilon^3 n \rightarrow \infty$. Our proof makes use of the following structure result of Ding, Kim, Lubetzky and Peres [12].

Theorem 4.1 ([12]). *Let \mathcal{C}_1 be the largest component of $G(n, p)$ for $p = \frac{1+\varepsilon}{n}$, where $\varepsilon^3 n \rightarrow \infty$ and $\varepsilon \rightarrow 0$. Let $\mu < 1$ denote the conjugate of $1 + \varepsilon$, that is, $\mu\varepsilon^{-\mu} = (1 + \varepsilon)e^{-(1+\varepsilon)}$. Then \mathcal{C}_1 is contiguous to the model $\tilde{\mathcal{C}}_1$ constructed in the following 3 steps.*

- (a) *Let $\Lambda \sim \mathcal{N}(1 + \varepsilon - \mu, \frac{1}{\varepsilon n})$ and assign i.i.d. variables $D_u \sim \text{Poisson}(\Lambda)$ ($u \in [n]$) to the vertices, conditioned that $\sum D_u \mathbf{1}_{D_u \geq 3}$ is even. Let $N_k = \#\{u : D_u = k\}$ and $N = \sum_{k \geq 3} N_k$. Select a random graph \mathcal{K} on N vertices, uniformly among all graphs with N_k vertices of degree k for $k \geq 3$.*
- (b) *Replace the edges of \mathcal{K} by paths of lengths i.i.d. $\text{Geom}(1 - \mu)$.*
- (c) *Attach an independent $\text{Poisson}(\mu)$ Galton–Watson tree (PGW tree from now on) to each vertex.*

That is, $\mathbb{P}(\tilde{\mathcal{C}}_1 \in \mathcal{A}) \rightarrow 0$ implies $\mathbb{P}(\mathcal{C}_1 \in \mathcal{A}) \rightarrow 0$ for any set of graphs \mathcal{A} .

By the above theorem, it suffices to analyse the cover time of $\tilde{\mathcal{C}}_1$. In what follows, we will repeatedly use some known facts about $\tilde{\mathcal{C}}_1$, and one can see [12, 13] for references.

4.1. Lower bound

We first show that w.h.p. there are $(\varepsilon^3 n)^{1/4}$ attached trees, as in part (c) of the construction of $\tilde{\mathcal{C}}_1$, of height at least $\frac{1}{2}\varepsilon^{-1} \log(\varepsilon^3 n)$. To this end, note that the height H of a $\text{PGW}(\mu)$ tree satisfies the following for some constant $c > 0$ (see, e.g., [13, Lemma 4.2]):

$$\mathbb{P}\left(H \geq \frac{1}{2}\varepsilon^{-1} \log(\varepsilon^3 n)\right) \geq c\varepsilon(\varepsilon^3 n)^{-1/2+o(1)}, \tag{4.1}$$

where we used the fact that $\mu = (1 - (1 + o(1))\varepsilon)$. It is an immediate consequence of parts (a) and (b) of the construction of $\tilde{\mathcal{C}}_1$ that w.h.p. there are $(2 + o(1))\varepsilon^2 n$ i.i.d. attached $\text{PGW}(\mu)$ trees. Hence, by (4.1), we learn that with high probability there are at least $(\varepsilon^3 n)^{1/4}$ PGW trees of height at least $\frac{1}{2}\varepsilon^{-1} \log(\varepsilon^3 n)$. Now, take exactly one leaf in the bottom level from each of these trees and denote by B the set of these leaves. We will use the following lemma (see, e.g., [44], and also see [34, Proposition 2.19]) to bound the hitting time between vertices in B .

Lemma 4.2. *Given a finite network with a vertex v and a subset of vertices Z such that $v \notin Z$, let $\text{vol}(\cdot)$ be the voltage when a unit current flows from v to Z and $\text{vol}(Z) = 0$. Then we have that*

$\mathbb{E}_v[\tau_Z] = \sum_{x \in V} c(x) \text{vol}(x)$, where $c(x) = \sum_{x \sim y} c(x, y)$ and $c(x, y)$ is the conductance between (x, y) .

In our setting, $c(x, y) = 1$ if (x, y) is an edge of $\tilde{\mathcal{C}}_1$, and otherwise $c(x, y) = 0$. Let $u, v \in B$, and let $T(v)$ be the attached PGW tree that contains v . It is clear that for all $w \notin T(v)$ the effective resistance between w and v satisfies $R_{\text{eff}}(w, v) \geq (2\varepsilon)^{-1} \log(\varepsilon^3 n)$. Now, if a unit current flows from u to v and the voltage at v is set to be 0, we can then deduce that the voltage at vertex w is at least $(2\varepsilon)^{-1} \log(\varepsilon^3 n)$, for all $w \notin T(v)$. Note that w.h.p. simultaneously for all $v \in B$ we have $|\tilde{\mathcal{C}}_1 \setminus T(v)| = (2 + o(1))\varepsilon n$ (see [12]) and we then assume this. Lemma 4.2 then yields that, for all $u, v \in B$,

$$\mathbb{E}_u \tau_v \geq (2\varepsilon)^{-1} \log(\varepsilon^3 n)(2 + o(1))\varepsilon n = (1 + o(1))n \log(\varepsilon^3 n).$$

At this point, an application of Matthews' lower bound [35] (see also, e.g., [31]), stating that for any subset $A \subset G$ we have $t_{\text{cov}}(G) \geq \log |A| \min_{u, v \in A} \mathbb{E}_u \tau_v$, completes the proof of the lower bound. □

4.2. Upper bound

In this section we establish the upper bound on the cover time. In light of Theorem 1.1, it suffices to show that w.h.p. for $\tilde{\mathcal{C}}_1$ we have that $|A_i| \leq (\varepsilon^3 n)^{2i}$ simultaneously for all $i \geq 1$. Let R be the diameter of $\tilde{\mathcal{C}}_1$ in the resistance metric. As is shown in [13], with high probability the diameter in the graph metric is $(3 + o(1))\varepsilon^{-1} \log(\varepsilon^3 n)$ and also the two highest attached trees have height $(1 + o(1))\varepsilon^{-1} \log(\varepsilon^3 n)$ each. It implies that $(2 + o(1))\varepsilon^{-1} \log(\varepsilon^3 n) \leq R \leq (3 + o(1))\varepsilon^{-1} \log(\varepsilon^3 n)$ w.h.p., and we assume this in what follows.

Fix $i \in \mathbb{N}$. We now construct A'_i such that balls of radius $2^{-i}R$ around vertices in A'_i form a covering of $\tilde{\mathcal{C}}_1$. We first cover the 2-core \mathcal{H} of $\tilde{\mathcal{C}}_1$ by balls of radius $2^{-(i+1)}R$. To this end, consider the disjoint balls of radius $2^{-(i+2)}R$ that can be packed in \mathcal{H} . Take such a maximal packing and denote by $A'_{i,1}$ the set of these centres. Since the packing is maximal, we have that

$$\mathcal{H} \subseteq \bigcup_{v \in A'_{i,1}} B_{\text{eff}}(v, 2^{-(i+1)}R).$$

Since $R_{\text{eff}}(x, y) \leq d(x, y)$, it follows that $|B_{\text{eff}}(v, 2^{-(i+2)}R) \cap \mathcal{H}| \geq 2^{-(i+2)}R$ for all $v \in A'_{i,1}$. Therefore, since the balls $B_{\text{eff}}(v, 2^{-(i+2)}R)$ for $v \in A'_{i,1}$ are disjoint, we conclude that $|A'_{i,1}| \leq 4 \cdot 2^i |\mathcal{H}| / R$.

We now turn to covering the attached trees. For a rooted tree T , let $H(T)$ be the height of T . For $v \in T$, denote by T_v the subtree of T rooted at v that contains all the descendants of v . Also, denote by L_k the vertices in level $k2^{-(i+1)}R$ of T . Define

$$F_T \triangleq \bigcup_{k=1}^{\infty} \{v \in L_k : H(T_v) \geq 2^{-(i+1)}R\}.$$

Let \mathcal{T} be the collection of attached PGW trees in $\tilde{\mathcal{C}}_1$ and let $A'_{i,2} = \bigcup_{T \in \mathcal{T}} F_T$. Defining $A'_i = A'_{i,1} \cup A'_{i,2}$, we deduce from the definition that $\tilde{\mathcal{C}}_1 \subseteq \bigcup_{v \in A'_i} B_{\text{eff}}(v, 2^{-i}R)$. It remains to bound $|A'_{i,2}|$. Using [13, Lemma 4.2] again, we obtain that for a PGW(μ) tree T and some absolute

constant C ,

$$\mathbb{P}(H(T) \geq 2^{-(i+1)}R) \leq \begin{cases} C\varepsilon & \text{if } 2^i \leq \log(\varepsilon^3 n), \\ \frac{C}{2^{-(i+1)}R} & \text{if } 2^i \geq \log(\varepsilon^3 n). \end{cases} \tag{4.2}$$

Also, it is immediate that $\mathbb{E}[|L_k|] = \mu^{k2^{-(i+1)}R}$. Furthermore, by the Markov property, given $|L_k|$ the set $\{T_v : v \in L_k\}$ is distributed as $|L_k|$ independent copies of T . By this and (4.2) we get that, for some absolute constant $C > 0$,

$$\begin{aligned} \mathbb{E}[F_T] &= \sum_{k \geq 1} \mathbb{E}|\{v \in L_k : H(T_v) \geq 2^{-(i+1)}R\}| = \sum_{k \geq 1} \mathbb{E}[|L_k|] \mathbb{P}(H(T_v) \geq 2^{-(i+1)}R) \\ &\leq \begin{cases} \sum_{k \geq 1} \mu^{k2^{-i}R/2} \cdot C\varepsilon \leq C^2\varepsilon & \text{if } 2^i \leq \log(\varepsilon^3 n), \\ \sum_{k \geq 1} \mu^{k2^{-(i+1)}R} \cdot \frac{C}{2^{-(i+1)}R} \leq C^2 2^{2i}/R & \text{if } 2^i \geq \log(\varepsilon^3 n). \end{cases} \end{aligned}$$

Hence, we can always get $\mathbb{E}[F_T] \leq C^2\varepsilon 2^{2i}$. Furthermore, it is known that $|\mathcal{H}| = (2 + o(1))\varepsilon^2 n$ with high probability, so we may assume this. By Markov’s inequality and the fact that $|A'_{i,1}| \leq 4 \cdot 2^i |\mathcal{H}|/R = o((\varepsilon^3 n)^{2i})$, we have that

$$\begin{aligned} \mathbb{P}(|A'_i| \geq (\varepsilon^3 n)^{2i}) &= \mathbb{P}(|A'_{i,2}| \geq (\varepsilon^3 n)^{2i} - |A'_{i,1}|) \leq \frac{\mathbb{E}[|A'_{i,2}|]}{(\varepsilon^3 n)^{2i} - |A'_{i,1}|} = \frac{|\mathcal{H}| \mathbb{E}[F_T]}{(1 + o(1))(\varepsilon^3 n)^{2i}} \\ &\leq (2 + o(1))C^2\varepsilon^3 n 2^{2i} (\varepsilon^3 n)^{-2i} \leq o(1)C^2(\varepsilon^3 n/8)^{-2(i-1)}. \end{aligned}$$

A simple union bound gives that with high probability $|A'_i| \leq (\varepsilon^3 n)^{2i}$ simultaneously for all $i \geq 1$. Recalling the facts that $|E(\tilde{C}_1)| = (2 + o(1))\varepsilon n$ and $R \leq 3 + o(1)\varepsilon^{-1} \log(\varepsilon^3 n)$, we conclude the proof of the upper bound by an application of Theorem 1.1. \square

5. Proof of Proposition 1.5

We may assume that $|E(G)| \geq 2$. Let π be the stationary distribution of G and let $\{S_t^+\}_{t \geq 0}$ be a random walk on G^+ starting from the initial distribution π (note that π is *not* the stationary distribution for G^+). Let $\tau_0 = \tau'_0 = 0$, and for all $i \geq 1$ define

$$\tau_i \triangleq \min\{t \geq \tau'_{i-1} : \{S_t^+, S_{t+1}^+\} = \{u, v\}\}, \quad X_i \triangleq S_{\tau_i}^+, \quad \text{and} \quad \tau'_i \triangleq \min\{t > \tau_i : S_t^+ = X_i\}.$$

Write $T_i = \{t : \tau_i < t \leq \tau'_i\}$, and for all $t \in \mathbb{N}$ define

$$\Phi(t) = \min\{k : |[0, k] \setminus \cup_{i=1}^\infty T_i| = t\}.$$

Now let $S_t = S_{\Phi(t)}^+$. We first claim that S_t is a simple random walk on the graph G . In order to see that, one just needs to note that S_t is obtained from S_t^+ by omitting all the excursions started with travelling through the edge (u, v) . Let τ_{cov} be the first time S_t visits every vertex of G , and it then remains to bound $\mathbb{E}[\Phi(\tau_{\text{cov}})]$.

To this end, it is more convenient to consider the first time τ_{cov}^* when S_t visits every vertex of G and returns to the starting point. We wish to bound the number of steps spent on the above defined excursions before τ_{cov}^* . Define

$$\begin{aligned} L_u(\tau_{\text{cov}}^*) &= |\{t \leq \tau_{\text{cov}}^* : S_t = u\}|, & L_v(\tau_{\text{cov}}^*) &= |\{t \leq \tau_{\text{cov}}^* : S_t = v\}|, \\ N_u(\tau_{\text{cov}}^*) &= |\{i : T_i \subseteq [0, \Phi(\tau_{\text{cov}}^*)], X_i = u\}|, & N_v(\tau_{\text{cov}}^*) &= |\{i : T_i \subseteq [0, \Phi(\tau_{\text{cov}}^*)], X_i = v\}|. \end{aligned}$$

Note that every time $S_t = u$, the corresponding random walk $S_{\Phi(t)}^+$ is also at u and has chance $\frac{1}{d_u+1}$ to travel to v , and thus starts an excursion, and moreover, once started, the number of excursions has law $\text{Geom}(1/(d_u + 1))$ independent of $\{S_i\}$. Therefore, we have

$$N_u(\tau_{\text{cov}}^*) = \sum_{i=1}^{L_u(\tau_{\text{cov}}^*)} Y_i Z_i,$$

where $\{(Y_i, Z_i)\}$ are independent and $Y_i \sim \text{Ber}(1/(d_u + 1))$ and $Z_i \sim \text{Geom}(1/(d_u + 1))$. Thus,

$$\mathbb{E}[N_u(\tau_{\text{cov}}^*)] = \frac{1}{d_u} \mathbb{E}[L_u(\tau_{\text{cov}}^*)].$$

By [3, Chapter 2, Proposition 3], we know that

$$\mathbb{E}[L_u(\tau_{\text{cov}}^*)] = \frac{d_u}{2|E(G)|} \mathbb{E}[\tau_{\text{cov}}^*]$$

and therefore

$$\mathbb{E}[N_u(\tau_{\text{cov}}^*)] = \frac{1}{2|E(G)|} \mathbb{E}[\tau_{\text{cov}}^*].$$

Suppose $X_i = u$; each T_i is distributed as $1 + \tau_u^+$, where τ_u^+ is the hitting time of S_t^+ to u started at v . Observing that $\{T_i\}$ are independent of $N_u(\tau_{\text{cov}}^*)$, we can then obtain that

$$\text{Exc}(u) \triangleq \mathbb{E}[\cup_i \{T_i \in [0, \Phi(\tau_{\text{cov}}^*)] : X_i = u\}] = \frac{1}{2|E(G)|} \mathbb{E}[\tau_{\text{cov}}^*](1 + \mathbb{E}_v[\tau_u^+]).$$

In the same manner, we derive that

$$\text{Exc}(v) \triangleq \mathbb{E}[\cup_i \{T_i \in [0, \Phi(\tau_{\text{cov}}^*)] : X_i = v\}] = \frac{1}{2|E(G)|} \mathbb{E}[\tau_{\text{cov}}^*](1 + \mathbb{E}_u[\tau_v^+]).$$

Note that $\mathbb{E}_v[\tau_u^+] + \mathbb{E}_u[\tau_v^+]$ is the expected commute time between u and v , and hence by commute identity [10] we have $\mathbb{E}_v[\tau_u^+] + \mathbb{E}_u[\tau_v^+] = 2|E(G^+)|R^+(u, v)$, where $R^+(u, v)$ is the resistance between u and v in G^+ . Since G is connected, we get

$$R^+(u, v) \leq \frac{|E(G)|}{|E(G)| + 1}.$$

Altogether,

$$\begin{aligned} t_{\text{cov}}(G^+) &= \mathbb{E}[\Phi(\tau_{\text{cov}})] \leq t_{\text{cov}}(G) + \text{Exc}(u) + \text{Exc}(v) \\ &\leq 3t_{\text{cov}}(G) + \frac{2}{|E(G)|} t_{\text{cov}}(G) \leq 4t_{\text{cov}}(G), \end{aligned}$$

where we used the inequality $\mathbb{E}[\tau_{\text{cov}}^*] \leq 2t_{\text{cov}}$ and the assumption that $|E(G)| \geq 2$. □

Remark 5.1. If G^+ is obtained from a connected graph G by adding k extra edges, a similar argument gives that

$$t_{\text{cov}}(G^+) \leq \left(2k + 1 + \frac{2k^2}{|E|}\right) t_{\text{cov}}(G).$$

6. A concluding remark

The bound (1.2) is reminiscent of Dudley's entropy bound for Gaussian process [15]. Motivated by this, Ding, Lee and Peres [14] show that the link to Gaussian processes is much tighter. In particular, Talagrand's majorizing measures bound for Gaussian processes (see [43]) can be used to estimate the cover time up to a multiplicative constant.

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