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ON ω_1-STRONGLY COMPACT CARDINALS

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Abstract. An uncountable cardinal κ is called ω_1-*strongly compact* if every κ-complete ultrafilter on any set I can be extended to an ω_1-complete ultrafilter on I. We show that the first ω_1-strongly compact cardinal, κ_0, cannot be a successor cardinal, and that its cofinality is at least the first measurable cardinal. We prove that the Singular Cardinal Hypothesis holds above κ_0. We show that the product of Lindelöf spaces is κ-Lindelöf if and only if $\kappa \geq \kappa_0$. Finally, we characterize κ_0 in terms of second order reflection for relational structures and we give some applications. For instance, we show that every first-countable nonmetrizable space has a nonmetrizable subspace of size less than κ_0.

§1. Preliminaries. Recall that an uncountable cardinal κ is *strongly compact* if for every set I, every κ-complete filter on I can be extended to a κ-complete ultrafilter on I ([12]; see also [7]). In [1], we studied the notion of a δ-*strongly compact* cardinal in the context of infinite abelian group theory. Let us recall the definition.

DEFINITION 1.1. *If $\delta < \kappa$ are uncountable cardinals, which may be singular, we say that κ is δ-*strongly compact *if for every set I, every κ-complete filter on I can be extended to a δ-complete ultrafilter on I.*

An uncountable limit cardinal κ is almost strongly compact *if κ is δ-strongly compact for every uncountable cardinal $\delta < \kappa$.*

To avoid confusion, let us note that in recent literature "κ is δ-strongly compact" is sometimes defined as: κ is regular, $\delta \geq \kappa$, and there is a κ-complete fine ultrafilter on $\mathcal{P}_\kappa(\delta)$ (see below).

Notice that if κ is δ-strongly compact and λ is a cardinal greater than κ, then λ is also δ-strongly compact. Note also that if κ is regular and ω_1-strongly compact, then the filter $\{X \subseteq \kappa : |\kappa - X| < \kappa\}$ is κ-complete, and therefore can be extended to an ω_1-complete ultrafilter. Hence, there exists a measurable cardinal less than or equal to κ.

Suppose κ is δ-strongly compact. Let I be any nonempty set, and for every $a \in I$, let $X_a := \{x \in \mathcal{P}_\kappa(I) : a \in x\}$, where $\mathcal{P}_\kappa(I) := \{x \subseteq I : |x| < \kappa\}$. If κ is regular, then the set $\{X_a : a \in I\}$ generates a κ-complete filter on $\mathcal{P}_\kappa(I)$, which can be extended to a δ-complete ultrafilter on $\mathcal{P}_\kappa(I)$. A δ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(I)$ that contains the sets X_a, for $a \in I$, is called a δ-*complete fine measure* on $\mathcal{P}_\kappa(I)$.

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The following characterizations of δ-strong compactness from [1] will be useful.

THEOREM 1.2. *The following are equivalent for any uncountable cardinals $\delta < \kappa$:*

(1) *κ is δ-strongly compact.*
(2) *For every α greater than or equal to κ, there exists an elementary embedding $j : V \to M$, with M transitive, and critical point greater than or equal to δ, such that j is definable in V, and there exists $D \in M$ such that $j''\alpha := \{j(\beta) : \beta < \alpha\} \subseteq D$ and $M \models |D| < j(\kappa)$.*
(3) *For every set I, there exists a δ-complete fine measure on $\mathcal{P}_\kappa(I)$.*

Observe that an equivalent formulation of (2) above is the following:

(2)' *For every set I of cardinality greater than or equal to κ, there exists an elementary embedding $j : V \to M$, with M transitive, and critical point greater than or equal to δ, such that j is definable in V, and there exists $D \in M$ such that $j''I := \{j(x) : x \in I\} \subseteq D$ and $M \models |D| < j(\kappa)$.*

The point is that, given I as in (2)', we can fix a bijection $\pi : \alpha \to I$, for some ordinal α, so that given j and D for α, as in (2), and assuming without loss of generality that $D \subseteq j(\alpha)$, we have that the set $D' := j(\pi)''D \in M$ satisfies $j''I \subseteq D'$ and $M \models |D'| < j(\kappa)$.

If λ is the least measurable cardinal and κ is ω_1-strongly compact, κ not necessarily regular, then κ is λ-strongly compact. For, if $\mathcal{U}$ is an ω_1-complete ultrafilter on a set I that is not λ-complete, then there is a partition $\{X_\alpha : \alpha < \beta\}$ of I, some $\beta < \lambda$, such that none of the X_α belongs to $\mathcal{U}$. But then the set $\{X \subseteq \beta : \bigcup\{X_\alpha : \alpha \in X\} \in \mathcal{U}\}$ is a nonprincipal ω_1-complete ultrafilter on β, contradicting the minimality of λ.

1.1. Group radicals. For X an abelian group, let $R_X : Ab \to Ab$ be the functor given by:

$$R_X(A) = \bigcap\{Ker(f) : f \in Hom(A, X)\}.$$

R_X is called *the radical singly generated by X* (see [5] and [1]).

For κ a cardinal, let

$$R_X^\kappa(A) = \sum\{R_X(B) : B \subseteq A,\ |B| < \kappa\}.$$

R_X is called *the κ-radical singly generated by X* (see [2] and [1]).

Notice that, since $B \subseteq A$ implies $R_X(B) \subseteq R_X(A)$, we have $R_X^\kappa(A) \subseteq R_X(A)$, for all A. And trivially, $R_X^\kappa(A) = R_X(A)$ for all A of cardinality $< \kappa$. Note also that if $\kappa < \lambda$ and $R_X = R_X^\kappa$, then $R_X = R_X^\lambda$ as well.

We will make use of the following theorem of Eda and Abe (see also [1]).

THEOREM 1.3 ([3]). *$R_{\mathbb{Z}} = R_{\mathbb{Z}}^\kappa$ if and only if κ is ω_1-strongly compact.*

§2. On the first ω_1-strongly compact cardinal. Magidor [8] showed that it is consistent for the first ω_1-strongly compact cardinal κ to be also the first measurable cardinal, in which case κ is strongly compact. However, in [1] it is shown that the first ω_1-strongly compact cardinal may also be singular of cofinality the first measurable cardinal. In the model given in [1], the first measurable cardinal is thus smaller than the first ω_1-strongly compact cardinal, which in turn is smaller than the first strongly compact cardinal. We will show (Theorem 2.3 below) that the first

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measurable cardinal is indeed the least possible cofinality for the least ω_1-strongly compact cardinal.

THEOREM 2.1. *The least ω_1-strongly compact cardinal is not a successor cardinal.*

PROOF. Suppose, aiming for a contradiction, that κ is the least ω_1-strongly compact cardinal and $\kappa = \lambda^+$ for some cardinal λ. Hence λ is not ω_1-strongly compact. Thus, there exists an ordinal $\alpha \geq \lambda$ such that for every definable elementary embedding $j : V \to M$, with M transitive, and every $D \in M$, if $j''\alpha \subseteq D$, then $M \models |D| \geq j(\lambda)$. Without loss of generality, α is a regular cardinal.

Since κ is ω_1-strongly compact, there exists a definable elementary embedding $j_0 : V \to M_0$, with M_0 transitive, and there exists $D_0 \in M$ such that $j''\alpha^+ \subseteq D_0$ and $M_0 \models |D_0| < j_0(\kappa) = j_0(\lambda)^+$.

Notice that, since $M_0 \not\models |D_0| < j_0(\lambda)$, we must have $M_0 \models |D_0| = j_0(\lambda)$.

Now, let $\beta := sup(j_0''\alpha)$ and $\beta^* := sup(j_0''\alpha^+)$.

CLAIM 2.2. $M_0 \models cof(\beta) = cof(\beta^*) = j_0(\lambda)$.

PROOF OF CLAIM. Let us first show that $M_0 \models cof(\beta) = j_0(\lambda)$. Clearly, $M_0 \models cof(\beta) \leq j_0(\lambda)$, because $M_0 \models |D_0| = j_0(\lambda)$. So, towards a contradiction, suppose $E \subseteq \beta$ is a club in M_0 with $ot(E) < j_0(\lambda)$. Let A be the set $\{\gamma : \gamma < \alpha \text{ and } j_0(\gamma) \in E\}$. Then A is unbounded in α: given $\gamma_0 < \alpha$, let $\gamma_1 \in E$ be such that $j_0(\gamma_0) < \gamma_1$, then let $\gamma_2 < \alpha$ be such that $\gamma_1 < j_0(\gamma_2)$, and so on. Then $sup(\gamma_i)_i < \alpha$, because α is regular and uncountable, and $sup(\gamma_i)_i = sup(j_0(\gamma_i))_i = j_0(sup(\gamma_i)_i) \in E$. It follows that $|A| = \alpha$, because α is regular. Moreover, we have that $E \in M_0$, $j_0''A \subseteq E$, and $M_0 \models |E| < j_0(\lambda)$. Fixing a bijection $\pi : A \to \alpha$, we have that $D := j_0(\pi)[E \cap j_0(A)] \in M_0$, $j_0''\alpha \subseteq D$, and $M_0 \models |D| < j_0(\lambda)$, which is impossible.

The same argument, taking β^* instead of β and α^+ instead of α, shows that $cof(\beta^*) = j_0(\lambda)$. ⊣

The claim yields a contradiction because, in V, β has cofinality α and β^* has cofinality α^+, and we can easily express that β and β^* have different cofinalities by a Π_1 sentence. So, since Π_1 sentences are downwards absolute for transitive models, β and β^* have different cofinalities in M_0. ⊣

THEOREM 2.3. *The least ω_1-strongly compact cardinal has cofinality greater than or equal to the first measurable cardinal.*

PROOF. Let κ be the first ω_1-strongly compact cardinal, and let λ be the first measurable cardinal. Thus, $\lambda \leq \kappa$. Suppose, towards a contradiction, that $\gamma := cof(\kappa) < \lambda$. Since κ is a limit cardinal (Theorem 2.1), we can fix an increasing sequence $\langle \kappa_\delta : \delta < \gamma \rangle$ of cardinals converging to κ. So, none of the κ_δ is ω_1-strongly compact. Hence, for each $\delta < \gamma$ there exists an ordinal $\alpha_\delta \geq \kappa_\delta$ such that for every definable elementary embedding $j : V \to M$, with M transitive, and for every $D \in M$, if $j''\alpha_\delta \subseteq D$, then $M \models |D| \geq j(\kappa_\delta)$. Let $\alpha = sup\{\alpha_\delta : \delta < \gamma\}$.

Since κ is ω_1-strongly compact, and hence λ-strongly compact, there exists a definable elementary embedding $j_0 : V \to M_0$, with M_0 transitive and with $crit(j_0) \geq \lambda$, and there exists $D_0 \in M$ such that $j_0''\alpha \subseteq D_0$ and $M_0 \models |D_0| < j_0(\kappa)$.

Since $j_0(\kappa) = j_0(sup\{\kappa_\delta : \delta < \gamma\}) = sup\{j_0(\kappa_\delta) : \delta < \gamma\}$, there is $\delta_0 < \gamma$ such that $M_0 \models |D_0| < j_0(\kappa_{\delta_0})$, which is impossible, because $j_0''\alpha_{\delta_0} \subseteq D_0$. ⊣

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§3. **Reflection of stationary sets.** It is a well-known fact (Solovay [11]) that if κ is a supercompact cardinal and $\lambda \geq \kappa$, then Jensen's square principle $\square_\lambda$ fails. It is also well known that if $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$ there exists a stationary $T \subseteq S$ that does not reflect, i.e., $T \cap \alpha$ is nonstationary, for all $\alpha < \lambda^+$.

The following theorem extends a similar result for strongly compact cardinals due to Solovay [11].

THEOREM 3.1. *Suppose κ is ω_1-strongly compact and $\lambda \geq \kappa$ is regular. Then every stationary subset of λ consisting of ordinals of countable cofinality reflects. Hence, $\square_\lambda$ fails for all $\lambda \geq \kappa$.*

PROOF. Let S be a stationary subset of λ consisting of ordinals of countable cofinality. Let $j : V \to M$ be a definable elementary embedding, with M transitive, and let $D \in M$ be such that $j''\lambda \subseteq D$ and $M \models |D| < j(\kappa)$. Let $\beta := sup(j''\lambda)$.

Suppose $C \subseteq \beta$ is a club in M. Then, using the fact that λ has uncountable cofinality and arguing as in the Proof of Claim in Theorem 2.1 above, the set $A := \{\alpha < \lambda : j(\alpha) \in C\}$ is unbounded in λ, and in fact it is an ω_1-club. Hence, $A \cap S \neq \emptyset$, and therefore $M \models$ "$j(S) \cap C \neq \emptyset$". Thus, we have shown that $M \models$ "$j(S) \cap \beta$ is stationary in β". Moreover,

$$M \models \text{“}cof(\beta) \leq |D| < j(\kappa) \leq j(\lambda)\text{”}$$

and also $M \models$ "$\beta \leq j(\lambda)$". Hence, since $M \models$ "$j(\lambda)$ is regular", we have that $M \models$ "$\beta < j(\lambda)$". Thus,

$$M \models \text{“}\exists \beta < j(\lambda)(j(S) \cap \beta \text{ is stationary in } \beta)\text{”}.$$

Hence, by elementarity,

$$V \models \text{“}\exists \beta < \lambda(S \cap \beta \text{ is stationary in } \beta)\text{”}$$

as wanted. ⊣

§4. **Good scales and the SCH.** Recall that the Singular Cardinal Hypothesis (SCH) asserts that if λ is a singular cardinal and $2^{cof(\lambda)} < \lambda$, then $\lambda^{cof(\lambda)} = \lambda^+$. We will show that above the first ω_1-strongly compact cardinal the SCH holds.

Suppose λ is a cardinal of cofinality ω, and $\langle \lambda_n : n < \omega \rangle$ is an increasing sequence of regular cardinals with limit λ.

Let us call a sequence $\langle f_\alpha : \alpha < \lambda^+ \rangle$ a λ^+-*scale*, relative to $\langle \lambda_n : n < \omega \rangle$, if $f_\alpha \in \prod_{n<\omega} \lambda_n$, and if $\alpha < \beta$ then $f_\alpha <^* f_\beta$, i.e., there exists $n < \omega$ such that $f_\alpha(m) < f_\beta(m)$, for all $m > n$. (Notice that we do not require, as in the usual standard definition of λ^+-scale, that $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is cofinal, in the $<^*$ order, in $\prod_{n<\omega} \lambda_n$.)

A λ^+-scale $\langle f_\alpha : \alpha < \lambda^+ \rangle$ for λ is *good* if for every limit $\alpha < \lambda^+$ of uncountable cofinality, there exists $D \subseteq \alpha$ cofinal in α, and there exists n such that $f_\beta(m) < f_\gamma(m)$ for all $\beta < \gamma$ in D and $m > n$.

THEOREM 4.1. *Suppose κ is an ω_1-strongly compact cardinal. Then for every $\lambda > \kappa$ with $cof(\lambda) = \omega$ there is no good λ^+-scale for λ.*

PROOF. Towards a contradiction, suppose $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good λ^+-scale, relative to an increasing sequence of regular cardinals $\langle \lambda_n : n < \omega \rangle$ with limit λ.

Let $\mathcal{U}$ be an ω_1-complete fine measure on $\mathcal{P}_\kappa(\lambda^+)$, and let

$$j : V \to Ult(V^{\lambda^+}, \mathcal{U}) \cong M$$

be the corresponding ultrapower embedding, with M transitive. The well-foundedness of the ultrapower $Ult(V^{\lambda^+}, \mathcal{U})$ follows from the ω_1-completeness of $\mathcal{U}$.

Since $\mathcal{U}$ is fine, $j''\lambda^+ \subseteq [id]_\mathcal{U}$, where id is the identity function on $\mathcal{P}_\kappa(\lambda^+)$. So, since $[id]_\mathcal{U}$ represents in the ultrapower a subset of $j(\lambda^+)$ of cardinality less than $j(\kappa)$, hence less than $j(\lambda^+)$, and since $j(\lambda^+)$ is a regular cardinal in M, we have that $\lambda^+ < j(\lambda^+)$.

Let $\beta := sup(j''\lambda^+)$. In M, β is the supremum of a subset of $j(\lambda^+)$ of cardinality less than $j(\kappa)$, hence less than $j(\lambda^+)$, and therefore bounded in $j(\lambda^+)$ since $j(\lambda^+)$ is regular. Thus, $\beta < j(\lambda^+)$.

We have:

$$M \models \text{“}j(\langle f_\alpha : \alpha < \lambda^+\rangle) \text{ is a good } j(\lambda^+)\text{-scale, relative to } \langle j(\lambda_n) : n < \omega\rangle\text{”}.$$

Say $j(\langle f_\alpha : \alpha < \lambda^+\rangle) := \langle f^*_\alpha : \alpha < j(\lambda^+)\rangle$.

Since $\beta < j(\lambda^+)$, and since β has uncountable cofinality in M, there exists $D \subseteq \beta$ in M, cofinal in β, and there exists n such that for every $\gamma < \gamma'$ in D and every $m > n$,

$$f^*_\gamma(m) < f^*_{\gamma'}(m).$$

We will define by induction on $\delta < \lambda^+$ an increasing sequence of ordinals $D^* = \{\gamma_\delta : \delta < \lambda^+\}$ contained in D.

Let γ_0 be the first ordinal in D. Let α_0 be the least ordinal such that $\gamma_0 < j(\alpha_0)$. Then let $\gamma_1 \in D$ be such that $j(\alpha_0) < \gamma_1$. Then, let α_1 be the least ordinal such that $\gamma_1 < j(\alpha_1)$. And so on. At limit stages, take the least $\gamma \in D$ greater than all the ordinals γ_δ picked so far. Notice that, $\alpha_\delta < \lambda^+$, for all $\delta < \lambda^+$. We have

$$f^*_{\gamma_0} <^* f^*_{j(\alpha_0)} <^* f^*_{\gamma_1} <^* f^*_{j(\alpha_1)} <^* \dots$$

For each $\delta < \lambda^+$, let $n_\delta > n$ be such that for every $m > n_\delta$,

$$f^*_{\gamma_\delta}(m) < f^*_{j(\alpha_\delta)}(m) < f^*_{\gamma_{\delta+1}}(m).$$

Let $E \subseteq D^*$ of cardinality λ^+ be such that for all $\delta \in E$, the n_δ is the same, say k. Then, for every limit $\delta < \delta' \in E$ we have

$$f^*_{\gamma_\delta}(k) < f^*_{j(\alpha_\delta)}(k) < f^*_{\gamma_{\delta+1}}(k) < f^*_{\gamma_{\delta'}}(k) < f^*_{j(\alpha_{\delta'})}(k) < f^*_{\gamma_{\delta'+1}}(k).$$

Note that for every $\delta < \lambda^+$,

$$f^*_{j(\alpha_\delta)}(k) = j(f_{\alpha_\delta}(k)) \in j''\lambda_k.$$

But this is impossible, since the sequence $\langle f^*_{\gamma_\delta}(k) : \delta \in E\rangle$ has order-type λ^+, and $j''\lambda_k$ has order type $\lambda_k < \lambda^+$. ⊣

The following theorem is due to Shelah [10] (see also [4], Section 4.7).

THEOREM 4.2. *If λ is a singular cardinal and the Singular Cardinal Hypothesis fails at λ, then there is a good λ^+-scale for λ.*

COROLLARY 4.3. *If κ is a ω_1-strongly compact cardinal, then the SCH holds above κ.*

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PROOF. By Silver's Theorem (see [6], Theorem 8.13) it is enough to show that the SCH holds at singular cardinals $\lambda > \kappa$ of countable cofinality. And by Shelah's theorem above it suffices to show that there is no good λ^+-scale for such λ. The conclusion now follows from Theorem 4.1. ⊣

§5. On the product of Lindelöf spaces. Recall that a topological space X is κ-*Lindelöf* (also known in the literature as κ-*compact*) if every open covering has a subcovering of cardinality less than κ. The space is called *Lindelöf* if it is ω_1-Lindelöf.

The product of Lindelöf spaces need not be Lindelöf. The Sorgenfrey plane is a well-known counterexample.

THEOREM 5.1. *κ is an ω_1-strongly compact cardinal if and only if every product of Lindelöf spaces is κ-Lindelöf.*

PROOF. Suppose κ is an ω_1-strongly compact cardinal. Let $X = \prod_{i<\alpha} X_i$, where X_i is Lindelöf, for all $i < \alpha$. Let $\mathcal{C}$ be a basic open cover of X and suppose, towards a contradiction, that it has no subcover of cardinality less than κ.

We shall find an ω_1-complete ultrafilter over X that contains the set $\{\mathcal{U}^c : \mathcal{U} \in \mathcal{C}\}$. By Theorem 1.2 and the remarks that follow it, there exists a definable elementary embedding $j : V \to M$ such that the set $\{j(\mathcal{U}) : \mathcal{U} \in \mathcal{C}\}$ is contained in a set $D \in M$ and such that $M \models |D| < j(\kappa)$. Without loss of generality, $D \subseteq j(\mathcal{C})$. By the elementarity of j, $\bigcup D$ does not cover $j(X)$. So let $y \in j(X)$ be a point that is not in $\bigcup D$. Define now an ultrafilter $\mathcal{F}$ over X by:

$$A \in \mathcal{F} \quad \text{if and only if} \quad y \in j(A).$$

One can easily check that $\mathcal{F}$ is an ω_1-complete ultrafilter over X such that $\mathcal{U} \notin \mathcal{F}$, for every $\mathcal{U} \in \mathcal{C}$.

For each $i < \alpha$, consider the set

$$Y_i := \{\mathcal{U} \subseteq X_i : \mathcal{U} \text{ is open and } \prod_{j<i} X_j \times \mathcal{U} \times \prod_{i<j<\alpha} X_j \notin \mathcal{F}\}.$$

We claim that $\bigcup Y_i \neq X_i$. Otherwise, $\bigcup Y_i$ is an open cover of X_i. So, it contains a countable subcover, say $\{\mathcal{V}_n : n < \omega\}$. Since $\mathcal{F}$ is an ultrafilter,

$$(\prod_{j<i} X_j \times \mathcal{V}_n \times \prod_{i<j<\alpha} X_j)^c = \prod_{j<i} X_j \times \mathcal{V}_n^c \times \prod_{i<j<\alpha} X_j \in \mathcal{F}$$

for all $n < \omega$. Hence, since $\mathcal{F}$ is ω_1-complete,

$$\bigcap_{n<\omega} (\prod_{j<i} X_j \times \mathcal{V}_n^c \times \prod_{i<j<\alpha} X_j) = \prod_{j<i} X_j \times \bigcap_{n<\omega} \mathcal{V}_n^c \times \prod_{i<j<\alpha} X_j \in \mathcal{F}$$

and therefore

$$(\prod_{j<i} X_j \times \bigcap_{n<\omega} \mathcal{V}_n^c \times \prod_{i<j<\alpha} X_j)^c = \prod_{j<i} X_j \times \bigcup_{n<\omega} \mathcal{V}_n \times \prod_{i<j<\alpha} X_j = X \notin \mathcal{F}$$

contradicting the fact that $\mathcal{F}$ is a filter on X.

For each $i < \alpha$, let $a_i \in X_i \setminus \bigcup Y_i$, and let $\bar{a} := \langle a_i : i < \alpha \rangle$. So $\bar{a}$ is covered by some $\mathcal{U} \in \mathcal{C}$.

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Let $i_0, \dots, i_n$ be an increasing enumeration of the support of $\mathcal{U}$, so that $\mathcal{U} = \prod_{i<\alpha} \mathcal{U}_i$, where $\mathcal{U}_i = X_i$ for all $i \neq i_0, \dots, i_n$. Then since $a_{i_k} \notin \bigcup Y_{i_k}$,

$$\prod_{j<i_k} X_j \times \mathcal{U}_{i_k} \times \prod_{i_k<j<\alpha} X_j \in \mathcal{F}$$

for all $0 \leq k \leq n$. Hence, since $\mathcal{F}$ is a filter,

$$\bigcap_{0\leq k\leq n} (\prod_{j<i_k} X_j \times \mathcal{U}_{i_k} \times \prod_{i_k<j<\alpha} X_j) = \mathcal{U} \in \mathcal{F}.$$

But this contradicts the fact that $\mathcal{U}^c \in \mathcal{F}$.

For the converse, suppose that every product of Lindelöf spaces is κ-Lindelöf. To show that κ is ω_1-strongly compact we will use the characterization from Theorem 1.3, namely, κ is ω_1-strongly compact if and only if $R_{\mathbb{Z}} = R^{\kappa}_{\mathbb{Z}}$.

Since $R^{\kappa}_{\mathbb{Z}}(G) \subseteq R_{\mathbb{Z}}(G)$ for all G, we only need to show that $R_{\mathbb{Z}}(G) \subseteq R^{\kappa}_{\mathbb{Z}}(G)$, for every abelian group G. So fix G and $x \in R_{\mathbb{Z}}(G)$, and suppose, towards a contradiction, that $x \notin R^{\kappa}_{\mathbb{Z}}(G)$. So for every subgroup $H \subseteq G$ of cardinality less than κ such that $x \in H$ there exists a homomorphism $h_H : H \to \mathbb{Z}$ such that $h_H(x) \neq 0$. Now consider the space $\mathbb{Z}^G$, i.e., the space of all functions $f : G \to \mathbb{Z}$ with the pointwise discrete topology on $\mathbb{Z}$. For each $a, b \in G$, let

$$\mathcal{U}_{a,b} := \{f \in \mathbb{Z}^G : \text{ either } f(a+b) \neq f(a) + f(b) \text{ or } f(x) = 0\}.$$

Since $x \in R_{\mathbb{Z}}(G)$, we have that $\bigcup\{\mathcal{U}_{a,b} : a, b \in G\} = \mathbb{Z}^G$.

Notice that, $\mathcal{U}_{a,b}$ is an open subset of $\mathbb{Z}^G$. For given $k, l, m \in \mathbb{Z}$ with $k \neq l + m$, let $\mathcal{U}_{k,l,m}$ be the open subset of $\mathbb{Z}^G$ consisting of all functions f such that $f(a+b) = k$, $f(a) = l$, and $f(b) = m$. Let $\mathcal{U}_x$ be the open set of all functions $f : \mathbb{Z} \to G$ such that $f(x) = 0$. Then

$$\mathcal{U}_{a,b} = \bigcup\{\mathcal{U}_{k,l,m} : k, l, m \in \mathbb{Z} \text{ and } k \neq l + m\} \cup \mathcal{U}_x.$$

So $\mathcal{C} := \{\mathcal{U}_{a,b} : a, b \in G\}$ is an open cover of $\mathbb{Z}^G$. Let $\bar{\mathcal{C}}$ be a subcover of $\mathcal{C}$ of cardinality λ, for some $\lambda < \kappa$.

Let H be the subgroup of G generated by x and all a, b's such that $\mathcal{U}_{a,b} \in \bar{\mathcal{C}}$. So, $|H| < \kappa$. By assumption, pick a homomorphism $h : H \to \mathbb{Z}$ such that $h(x) \neq 0$. Extend h to a function $h^* : G \to \mathbb{Z}$ (h^* need not be a homomorphism). Since $\bar{\mathcal{C}}$ covers $\mathbb{Z}^G$ there exist a, b such that $\mathcal{U}_{a,b} \in \bar{\mathcal{C}}$ and $h^* \in \mathcal{U}_{a,b}$. Since $a, b \in H$ and h^* extends h we get: $h^*(a + b) = h^*(a) + h^*(b)$ and $h^*(x) \neq 0$, so $h^* \notin \mathcal{U}_{a,b}$. This is a contradiction. $\dashv$

§6. Reflection. The existence of a supercompact cardinal is equivalent to a reflection property for second-order formulas: There is a supercompact cardinal iff there is a cardinal κ that reflects every second-order formula φ, i.e., given a second order formula $\varphi(\vec{Y}, \vec{y})$, where $\vec{Y} = \langle Y_i^{k_i} \rangle_i$ is a finite sequence of second order relational variables, each of arity k_i, for every structure $\mathcal{A}$ (with universe A), every sequence $\vec{R} = \langle R_i^{k_i} \rangle_i$, with $R_i^{k_i} \subseteq A^{k_i}$, and every sequence $\vec{a}$ of the same length as $\vec{y}$ of elements of A, if $\mathcal{A} \models \varphi(\vec{R}, \vec{a})$, then there exists a substructure $\mathcal{B} \subseteq \mathcal{A}$ of cardinality less than κ such that $\mathcal{B} \models \varphi(\langle R_i^{k_i} \cap B^{k_i} \rangle_i, \vec{a})$, where B is the universe of $\mathcal{B}$. See [9]. When one analyzes the proof in that paper it can be shown that the existence of a supercompact cardinal is equivalent to having a cardinal that reflects every second order formula of the form $\forall \vec{X} \exists \vec{x} \forall \vec{y} \psi$, where $\vec{X}$ is a finite sequence of

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second-order relational variables, $\vec{x}, \vec{y}$ are finite sequences of first-order variables, and ψ is quantifier free. In this section, we show (Theorem 6.1) that a cardinal is ω_1-strongly compact if and only if it reflects every formula in a somewhat similar class of second-order formulas where we only have the existential first-order quantifiers but we allow countable conjunctions and disjunctions. Namely, we work in the domain of the $\mathcal{L}_{\omega_1,\omega_1}$ version of second order logic. Note that we are talking about a language with no function symbols.

So, let Φ be the class of formulas of the form $\forall\vec{X}\exists\vec{x}\,\psi$, where $\vec{X}$ is a countable sequence $\langle X_i^{k_i}\rangle_i$ of second-order relational variables, each of them of arity k_i, for some $k_i \in \omega$, $\vec{x}$ is a countable sequence of first-order variables, and ψ is a formula of $\mathcal{L}_{\omega_1,\omega}$ in a countable language with only relation and constant symbols, without quantifiers, and which may have countably many free first-order variables and second order relational variables of any arity.

Given a formula $\varphi(\vec{Y}, \vec{y})$ in Φ, where $\vec{Y} = \langle Y_i^{k_i}\rangle_i$, we say that a cardinal κ *reflects* $\varphi(\vec{Y}, \vec{y})$ if for every structure $\mathcal{A}$ (with universe A), every sequence $\vec{R} = \langle R_i^{k_i}\rangle_i$, with $R_i^{k_i} \subseteq A^{k_i}$, and every sequence $\vec{a} = \langle a_j\rangle_j$ of the same length as $\vec{y}$ of elements of A, if $\mathcal{A} \models \varphi(\vec{R}, \vec{a})$, then there exists a substructure $\mathcal{B} \subseteq \mathcal{A}$ of cardinality less than κ and with $\{a_j\}_j \subseteq B$ such that $\mathcal{B} \models \varphi(\langle R_i^{k_i} \cap B^{k_i}\rangle_i, \vec{a})$, where B is the universe of $\mathcal{B}$.

THEOREM 6.1. *A cardinal κ reflects every formula in Φ if and only if κ is ω_1-strongly compact.*

PROOF. Consider the following second-order formula, with an abelian group G and $\mathbb{Z}$ as second-order parameters, 0 as a first-order parameter, and z the only free first-order variable:

$$\forall F(F : G \to \mathbb{Z} \text{ is a homomorphism} \to F(z) = 0).$$

Thus, the formula expresses that $z \in R_{\mathbb{Z}}(G)$. Let us see how this can be expressed by a Φ formula in the language

$$\{G, \mathbb{Z}, +_G, 0_G, +_{\mathbb{Z}}, \langle n\rangle_{n\in\mathbb{Z}}\}$$

where G and $\mathbb{Z}$ are unary relation symbols, $+_G$ and $+_{\mathbb{Z}}$ are ternary relation symbols, and 0_G and n, for $n \in \mathbb{Z}$, are constant symbols. We will first express in this language the following facts, where F is taken as a binary relation variable:

(1) F is a function on G with range contained in $\mathbb{Z}$.
(2) For every $a, b \in G$, $F(a +_G b) = F(a) +_{\mathbb{Z}} F(b)$.
(3) $F(0_G) = 0_{\mathbb{Z}}$.
(4) $F(z) = 0_{\mathbb{Z}}$.

(1) can be written in the form $\forall a(G(a) \to \psi(F, a))$, where the formula $\psi(F, a)$ is the following $\mathcal{L}_{\omega_1,\omega}$ formula, expressing that $F(a)$ belongs to $\mathbb{Z}$ and is unique:

$$\bigvee_{n\in\mathbb{Z}} F(a, n) \wedge \bigwedge_{m,n\in\mathbb{Z};\ m\neq n} \neg(F(a, m) \wedge F(a, n)).$$

(2) can be written as:

$$\forall a, b, c, m, n, k(G(a) \wedge G(b) \wedge G(c) \wedge \mathbb{Z}(m) \wedge \mathbb{Z}(n) \wedge \mathbb{Z}(k)\wedge$$
$$\wedge +_G (a, b, c) \wedge F(a, m) \wedge F(b, n) \wedge +_{\mathbb{Z}}(m, n, k) \to F(c, k)).$$

Call this formula $\theta(F)$.

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Finally, (3) and (4) can be written as $F(0_G, 0_{\mathbb{Z}})$ and $F(z, 0_{\mathbb{Z}})$, respectively.

Let now $\varphi(z)$ be the following formula, where F is a binary relation variable:

$$\forall F(\forall a(G(a) \rightarrow \psi(F, a)) \wedge \theta(F) \wedge F(0_G, 0_{\mathbb{Z}}) \rightarrow F(z, 0_{\mathbb{Z}}))$$

which is clearly equivalent to a Φ formula.

Suppose $\langle G, +_G, 0_G\rangle$ is an abelian group. Let $A := G \cup \mathbb{Z}$, and consider the structure $\mathcal{A} := \langle A, G, \mathbb{Z}, +_G, 0_G, +_{\mathbb{Z}}, \langle n\rangle_{n\in\mathbb{Z}}\rangle$, with $+_G$ and $+_{\mathbb{Z}}$ taken as ternary relations. Then $\mathcal{A}$ satisfies $\varphi(z)$, with z interpreted as some $b \in G$, if and only if $b \in R_{\mathbb{Z}}(G)$.

So suppose $\mathcal{A} \models \varphi(b)$, for some $b \in G$. Let

$$\mathcal{B} := \langle B, G \cap B, \mathbb{Z}, +_G \cap B^3, 0_G, +_{\mathbb{Z}}, \langle n\rangle_{n\in\mathbb{Z}}\rangle \subseteq \mathcal{A}$$

with B of cardinality less than κ be such that $\mathcal{B} \models \varphi(b)$. Let $H := \langle G \cap B\rangle$ be the subgroup of G generated by $G \cap B$. Note that, H has cardinality less than κ. If $f : H \rightarrow \mathbb{Z}$ is a homomorphism, then letting $F := f \restriction G \cap B$, we have that

$$\mathcal{B} \models \forall a(G(a) \rightarrow \psi(F, a)) \wedge \theta(F) \wedge F(0_G, 0_{\mathbb{Z}})$$

where F is a second order parameter. Since $\mathcal{B} \models \varphi(b)$, it follows that $\mathcal{B} \models F(b, 0_{\mathbb{Z}})$. Hence, $f(b) = 0_{\mathbb{Z}}$. And this shows that $b \in R^{\kappa}_{\mathbb{Z}}(G)$.

Thus, we have shown that $R_{\mathbb{Z}} = R^{\kappa}_{\mathbb{Z}}$. Hence, by Theorem 1.3, κ is ω_1-strongly compact.

For the converse, assume κ is ω_1-strongly compact, and suppose that some structure $\mathcal{A}$, in a countable language with only relational and constant symbols $\{R^{k_n}_n\}_n \cup \{r_m\}_m$, satisfies a formula $\varphi(\vec{Y}, \vec{y}) = \forall \vec{X} \exists \vec{x} \psi(\vec{X}, \vec{x}, \vec{Y}, \vec{y}) \in \Phi$, with $\vec{Y}$ interpreted as $\vec{C} = \langle C^{k_i}_i\rangle_i$ and $\vec{y}$ interpreted as $\vec{c} = \langle c_\ell\rangle_\ell$. Let A be the universe of $\mathcal{A}$.

Let $j : V \rightarrow M$ be a definable elementary embedding such that $j''A \subseteq D$, for some $D \in M$ such that $M \models |D| < j(\kappa)$. Notice that, by elementarity, $j''(A^k) \subseteq D^k$, for every $k < \omega$. We may assume that $D \subseteq j(A)$ (and hence $D^k \subseteq j(A^k)$, for all $k < \omega$), for otherwise we may work with $D \cap j(A)$ instead of D. Thus, we may turn D into a substructure $\mathcal{D}$ of $j(\mathcal{A})$ by restricting all the relations $(R^{k_n}_n)^{j(\mathcal{A})}$ to D. Notice that, $j(\vec{C}) = \langle j(C^{k_i}_i)\rangle_i$ and $j(\vec{c}) = \langle j(c_\ell)\rangle_\ell$.

Claim 6.2. *In M, $\mathcal{D} \models \varphi(\langle j(C^{k_i}_i) \cap D^{k_i}\rangle_i, \langle j(c_\ell)\rangle_\ell)$.*

Proof of the Claim. Otherwise, for some $\vec{X} = \langle X^{k_m}_m\rangle_m$ with $X^{k_m}_m \subseteq D^{k_m}$, and $X^{k_m}_m \in M$, all m, we have that

$$\mathcal{D} \models \forall \vec{x} \neg \psi(\vec{X}, \vec{x}, \langle j(C^{k_i}_i) \cap D^{k_i}\rangle_i, \langle j(c_\ell)\rangle_\ell).$$

Then, since $j''A \subseteq D$, again we may turn $j''A$ into a substructure $\mathcal{E}$ of $\mathcal{D}$, with universe $j''A$. Since ψ has no quantifiers,

$$\mathcal{E} \models \forall \vec{x} \neg \psi(\langle X^{k_m}_m \cap j''A^{k_m}\rangle_m, \vec{x}, \langle j(C^{k_i}_i) \cap j''A^{k_i}\rangle_i, \langle j(c_\ell)\rangle_\ell).$$

Hence, by elementarity,

$$\mathcal{A} \models \forall \vec{x} \neg \psi(\langle j^{-1}[(X^{k_m}_m \cap j''A^{k_m})]\rangle_m, \vec{x}, \vec{C}, \vec{c}).$$

But this contradicts the fact that $\mathcal{A} \models \varphi(\vec{C}, \vec{c})$. ⊣

Thus,

$$M \models \exists \mathcal{D} \subseteq j(\mathcal{A})(|\mathcal{D}| < j(\kappa) \wedge \mathcal{D} \models \varphi(\langle j(C_i^{k_i}) \cap D^{k_i}\rangle_i, \langle j(c_\ell)\rangle_\ell))$$

where D is the universe of $\mathcal{D}$. So, by elementarity,

$$V \models \exists \mathcal{D} \subseteq \mathcal{A}(|\mathcal{D}| < \kappa \wedge \mathcal{D} \models \varphi(\langle C_i^{k_i} \cap D^{k_i}\rangle_i, \langle c_\ell\rangle_\ell))$$

as wanted. ⊣

Let us see next a few applications of Theorem 6.1.

6.1. On noncountably chromatic graphs. We say that a graph $G = \langle G, R\rangle$ is *countably chromatic* if its chromatic number is $\leq \aleph_0$. This means that there exists a function $f : G \to \omega$ such that for every $x \neq y$, if $R(x, y)$ then $f(x) \neq f(y)$.

THEOREM 6.3. *If κ is ω_1-strongly compact, then for every graph G that is not countably chromatic there is a subgraph H of G of cardinality less than κ that is not countably-chromatic.*

PROOF. Suppose $G = \langle G, R\rangle$ is a graph. Let $\psi(F)$ be the formula of $\mathcal{L}_{\omega_1,\omega}$, with F as a free second order binary relation variable, consisting of the conjunction of the following three formulas in the language $\{G, \omega\}$, where G and ω are unary relation symbols:

(1) $\forall x, n(F(x, n) \to G(x) \wedge \omega(n))$;
(2) $\forall x(G(x) \to \bigvee_{n<\omega} F(x, n))$;
(3) $\forall x, m, n(F(x, m) \wedge F(x, n) \to m = n)$.

Thus, $\psi(F)$ expresses that F is a function, with G contained in its domain, and into ω. Let now φ be the following formula:

$$\forall F(\psi(F) \to \exists x, y(x \neq y \wedge G(x) \wedge G(y) \wedge R(x, y) \wedge \bigwedge_{m,n}(F(x, m) \wedge F(y, n) \to m = n))).$$

Clearly, φ is equivalent to a formula in Φ and expresses that G is not countably chromatic.

So, if G is not countably chromatic, then

$$\mathcal{G} := \langle G \cup \omega, G, \omega, \langle n\rangle_{n\in\omega}, R\rangle \models \varphi.$$

Since κ is ω_1-strongly compact, by Theorem 6.1 there is a substructure $\mathcal{H} = \langle H \cup \omega, H, \omega, \langle n\rangle_{n\in\omega}, R \cap H^2\rangle$ of $\mathcal{G}$ of size $< \kappa$ such that $\mathcal{H} \models \varphi$. And this implies that the graph $H = \langle H, R \cap H^2\rangle$ is not countably chromatic. ⊣

6.2. On first-countable nonmetrizable spaces.

THEOREM 6.4. *If κ is ω_1-strongly compact, then for every first-countable nonmetrizable topological space X there is a subspace $Y \subseteq X$ of cardinality less than κ that is nonmetrizable.*

PROOF. Suppose κ is ω_1-strongly compact and X is a first-countable topological space. For each $x \in X$, fix a countable neighborhood base $\langle U_n^x\rangle_n$ of x. Now, for each $n \in \omega$, let R_n be the binary relation on X given by:

$$R_n(x, y) \text{ if and only if } y \in U_n^x.$$

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Let $\theta(D)$ be the formula of $\mathcal{L}_{\omega_1,\omega}$, with D a ternary second-order variable, that is the conjunction of the following seven formulas in the language $\{X, \mathbb{Q}, +_{\mathbb{Q}}, \leq_{\mathbb{Q}}, \langle r\rangle_{r\in\mathbb{Q}}\}$, where X and $\mathbb{Q}$ are unary relation symbols, $\leq_{\mathbb{Q}}$ is a binary relation symbol, $+_{\mathbb{Q}}$ is a ternary relation symbol, and r, for $r \in \mathbb{Q}$, are constant symbols. We intend $D(x, y, r)$ to mean "the distance between x and y is $\leq r$". So the set $\{r \in \mathbb{Q} | D(x, y, r)\}$ is the lower part of the Dedekind cut representing the distance between x and y.

(1) $\forall x, y, r(D(x, y, r) \to X(x) \wedge X(y) \wedge \mathbb{Q}(r))$;
(2) $\forall x, y, r, s(\leq_{\mathbb{Q}} (s, r) \wedge D(x, y, r) \to D(x, y, s))$;
(3) $\forall x, y(X(x) \wedge X(y) \to D(x, y, 0))$;
(4) $\forall x, y(X(x) \wedge X(y) \to \exists r(\mathbb{Q}(r) \wedge \neg D(x, y, r)))$;
(5) $\forall x, y(X(x) \wedge X(y) \to (\forall r(D(x, y, r) \to \leq_{\mathbb{Q}} (r, 0)) \leftrightarrow x = y))$;
(6) $\forall x, y(\bigwedge_{r\in\mathbb{Q}}(D(x, y, r) \leftrightarrow D(y, x, r)))$;
(7) (The triangle inequality) $\forall x, y, z, r(X(x) \wedge X(y) \wedge X(z) \wedge D(x, z, r) \to \exists s, t(D(x, y, s) \wedge D(y, z, t) \wedge +_{\mathbb{Q}}(s, t, r)))$.

Thus, the intended meaning of $\theta(D)$ is that the function d on X^2 given by: $d(x, y) = s$ if and only if $s = sup\{r \in \mathbb{Q} | D(x, y, r)\}$, is a distance function.

Now let φ be the following second-order sentence in the relational language

$$\{X, \mathbb{Q}, \langle R_n\rangle_{n<\omega}, +_{\mathbb{Q}}, \leq_{\mathbb{Q}}, \langle r\rangle_{r\in\mathbb{Q}}\}$$

with D a second-order ternary relation variable:

$$\forall D(\theta(D) \to \exists x(X(x) \wedge ($$
$$\exists\langle y_n\rangle_n(\bigwedge_{n\in\omega}(\bigwedge_{r\in\mathbb{Q}}(D(x, y_n, r) \to \leq_{\mathbb{Q}} (r, \frac{1}{2^n}))) \wedge \bigvee_{m\in\omega}\bigwedge_{n\in\omega} \neg R_m(x, y_n)) \vee$$
$$\exists\langle y_m\rangle_m(\bigwedge_{m\in\omega} R_m(x, y_m) \wedge \bigvee_{n\in\omega}\bigwedge_{m\in\omega}\bigwedge_{r\in\mathbb{Q}}(\leq_{\mathbb{Q}} (r, \frac{1}{2^n}) \to \neg D(x, y_m, r)))))).$$

Thus, φ is saying that for every distance function d on X there is a point $x \in X$ such that either some basic neighborhood U_m^x of x does not contain any open d-ball centered at x, or some open d-ball centered at x does not contain any of the basic neighborhoods U_m^x, $m \in \omega$.

Consider now the relational structure

$$\mathcal{X} := \langle X \cup \mathbb{Q}, X, \mathbb{Q}, \langle R_n\rangle_{n<\omega}, +_{\mathbb{Q}}, \leq_{\mathbb{Q}}, \langle r\rangle_{r\in\mathbb{Q}}\rangle.$$

Claim 6.5. *X is nonmetrizable if and only if $\mathcal{X} \models \varphi$.*

Proof of Claim. If X is metrizable, say via a metric d, then letting

$$D := \{\langle x, y, r\rangle : x, y \in X \wedge r \in \mathbb{Q} \wedge r \leq d(x, y)\}$$

we have that $\theta(D)$ holds in $\mathcal{X}$. But if φ holds in $\mathcal{X}$, then there is $x \in X$ such that either some U_m^x does not contain any open d-ball centered at x, or some open d-ball centered at x does not contain any of the U_m^x, $m \in \omega$, which is impossible.

For the converse, suppose $\mathcal{X}$ is nonmetrizable. Given any $D \subseteq X \times X \times \mathbb{Q}$ such that $\mathcal{X} \models \theta(D)$, let d be the distance function on X given by: $d(x, y) = sup\{r \in \mathbb{Q} | D(x, y, r)\}$. Then there must exist some $x \in X$ such that either some U_m^x does not contain any open d-ball centered at x, or some open d-ball centered

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at x does not contain any of the U^x_m, $m \in \omega$ (for otherwise d would witness the metrizability of X). Thus, φ holds in $\mathcal{X}$. ⊣

Observe that φ is equivalent to a formula in Φ. So, since κ is ω_1-strongly compact, by Theorem 6.1 there is substructure $\mathcal{Y}$ of $\mathcal{X}$ of size $< \kappa$ that reflects φ. That is,

$$\mathcal{Y} = \langle Y \cup \mathbb{Q}, Y, \mathbb{Q}, \langle R_n \cap Y^2\rangle_{n<\omega}, +_{\mathbb{Q}}, \leq_{\mathbb{Q}}, \langle r\rangle_{r\in\mathbb{Q}}\rangle \models \varphi.$$

Hence, as in the claim above, $\mathcal{Y}$ is not metrizable. ⊣

6.3. On noncompletely regular spaces. Recall that a topological space X is *completely regular* if for every closed subset C of X and every point p not in C, there exists a continuous function $f : X \to \mathbb{R}$ such that f is 0 on p and 1 on C.

THEOREM 6.6. *If κ is ω_1-strongly compact, then every first-countable topological space that is not completely regular contains a subspace of cardinality less than κ that is not completely regular.*

PROOF. Suppose X is a first-countable topological space. For each $x \in X$, fix a countable neighborhood base $\langle U^x_n\rangle_n$ for x, and for each $n \in \omega$, let R_n be the binary relation on X given by: $R_n(x, y)$ if and only if $y \in U^x_n$.

Let $\theta(F)$ be the formula of $\mathcal{L}_{\omega_1,\omega}$, with F a binary relation variable, that is the conjunction of the following formulas in the language

$$\{X, \mathbb{Q}, \leq_{\mathbb{Q}}, +_{\mathbb{Q}}, \langle r\rangle_{r\in\mathbb{Q}}, \langle R_n\rangle_{n\in\omega}\}$$

where X and $\mathbb{Q}$ are unary relation symbols, $\leq_{\mathbb{Q}}$ and R_n, $n \in \omega$, are binary relation symbols, $+_{\mathbb{Q}}$ is a ternary relation symbol, and $\langle r\rangle_{r\in\mathbb{Q}}$ are constant symbols. We intend $F(x, r)$ to mean that r is in the Dedekind cut of $f(x)$, where $f : X \to \mathbb{R}$ is the function given by $f(x) = sup\{r \in \mathbb{Q} : F(x, r)\}$.

(1) $\forall x, r(F(x, r) \to X(x) \wedge \mathbb{Q}(r))$
(2) $\forall x(X(x) \to \bigvee_{r\in\mathbb{Q}} F(x, r))$
(3) $\forall x(X(x) \to \bigvee_{r\in\mathbb{Q}} \neg F(x, r))$
(4) $\forall x, r, s(\leq_{\mathbb{Q}} (s, r) \wedge F(x, r) \to F(x, s))$
(5) (Continuity)
$\forall x, v, w\ \forall\langle y_n\rangle_{n<\omega}(\bigwedge_{n<\omega} R_n(x, y_n) \to \bigwedge_{m>0} \bigvee_{r,s\in\mathbb{Q}} \bigvee_{n<\omega}(F(x, r) \wedge F(y_n, s) \wedge \bigwedge_{t,u\in\mathbb{Q}} \bigwedge_{k<\omega}(\leq_{\mathbb{Q}} (r, t)\wedge \leq_{\mathbb{Q}} (s, u)\wedge \leq_{\mathbb{Q}} (n, k) \wedge F(x, t) \wedge F(y_k, u) \to (+_{\mathbb{Q}}(r, \frac{1}{m}, v) \wedge +_{\mathbb{Q}}(s, \frac{1}{m}, w) \to\leq_{\mathbb{Q}} (t, v)\wedge \leq_{\mathbb{Q}} (u, w)))))$

Thus, the intended meaning of $\theta(F)$ is that the function $f : X \to \mathbb{R}$ given by $f(x) = sup\{r \in \mathbb{Q} : F(x, r)\}$ is continuous.

If X is not completely regular, then there exists $p \in X$ and a closed $C \subseteq X$ such that for no continuous function $f : X \to \mathbb{R}$ we have $f(p) = 0$ and $f[C] = \{1\}$. Then letting,

$$\mathcal{X} := \langle X \cup \mathbb{Q}, X, \mathbb{Q}, C, \langle R_n\rangle_{n<\omega}, +_{\mathbb{Q}}, \leq_{\mathbb{Q}}, \langle r\rangle_{r\in\mathbb{Q}}, p\rangle$$

we have that $\mathcal{X}$ satisfies the following formula φ in the relational language

$$\{X, \mathbb{Q}, \leq_{\mathbb{Q}}, +_{\mathbb{Q}}, \langle r\rangle_{r\in\mathbb{Q}}, \langle R_n\rangle_{n\in\omega}, C, p\}$$

where C is a unary relation symbol and p is a constant symbol:

$$\forall F(\theta(F)\to(\bigvee_{r>0} F(p, r) \vee \bigvee_{r<0}\neg F(p, r)) \vee \exists x(C(x) \wedge (\bigvee_{r>1} F(x, r) \vee \bigvee_{r<1} \neg F(p, r)))).$$

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Note that φ is equivalent to a formula in Φ. So, since κ is ω_1-strongly compact, by Theorem 6.1, there exists a substructure

$$\mathcal{Y} = \langle Y \cup \mathbb{Q}, Y, \mathbb{Q}, C \cap Y, \langle R_n \cap Y^2 \rangle_{n<\omega}, +_{\mathbb{Q}}, \leq_{\mathbb{Q}}, \langle r \rangle_{r \in \mathbb{Q}}, p \rangle$$

of $\mathcal{X}$ of size less than κ such that $\mathcal{Y} \models \varphi$. And this implies that Y is not completely regular. ⊣

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