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Existence of positive solutions for a class of p-Laplacian superlinear semipositone problems

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We consider a quasilinear elliptic problem of the form

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\lambda > 0$ is a parameter, $1 < p < 2$ and Ω is a strictly convex bounded domain in $\mathbb{R}^N$, $N > p$, with C^2 boundary $\partial\Omega$. The nonlinearity $f\colon [0,\infty) \to \mathbb{R}$ is a continuous function that is semipositone ($f(0) < 0$) and p-superlinear at infinity. Using degree theory, combined with a rescaling argument and uniform L^∞ *a priori* bound, we establish the existence of a positive solution for λ small. Moreover, we show that there exists a connected component of positive solutions bifurcating from infinity at $\lambda = 0$. We also extend our study to systems.

Keywords: p-Laplacian; systems; semipositone; superlinear; positive solutions

2010 *Mathematics subject classification:* Primary 35J60; 35J25; 35J65

1. Introduction

We consider a quasilinear elliptic problem of the form

$$\left.\begin{aligned} -\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < 2$, and $\lambda > 0$ is a parameter. We assume Ω to be a strictly convex bounded domain in $\mathbb{R}^N$, $N > p$, with C^2 boundary $\partial\Omega$.

The nonlinearity $f\colon [0,\infty) \to \mathbb{R}$ is a continuous function that satisfies $f(0) < 0$, $f(s) > 0$ for $s \gg 1$ and additional growth conditions. Such problems are referred in the literature as *semipositone problems*, and it is well documented (see [6, 17]) that the study of positive solutions is mathematically challenging.

System (1.1) has been studied in [9,11,14,18,20] for the case when f is p-sublinear at infinity, namely, $\lim_{s\to\infty} f(s)/s^{p-1} = 0$. Here our focus will be to study (1.1) when f satisfies a p-superlinear condition at infinity (to be made precise in the statement of the theorem). The existence of solutions to such nonlinear eigenvalue problems is generally studied via degree theory or the variational method. However, the semipositone structure poses an additional challenge in establishing the positivity of solutions. The existence of a positive solution when $p = 2$ and $\lim_{s\to\infty} f(s)/s^q = b$ for some $b > 0$ and $1 < q < (N + 2)/(N - 2)$ was established for λ small using degree theory in [1, 2]. To the best of our knowledge this has not been achieved for $p \neq 2$ in non-radial domains. This paper is, we believe, the first to establish such an existence result in non-radial domains.

The existence of a positive radial solution was established in a ball for $p > 1$ in [3, theorem 4.6], [16, theorem 2.28] and [10, theorem 1.2(i)].

By a solution of (1.1), we mean a pair (λ, u) that solves (1.1) in the weak sense, that is, $u \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \lambda \int_\Omega f(u)\varphi \,\mathrm{d}x$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Moreover, if $u > 0$ in Ω, then (λ, u) is called a positive solution.

We say that λ_∞ is a bifurcation point from infinity if the solution set $\mathfrak{S} := \{(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)\colon \lambda \text{ and } u \text{ solves (1.1)}\}$ contains a sequence $\{(\lambda_n, u_n)\}$ such that

$$\lambda_n \to \lambda_\infty \quad \text{and} \quad \|u_n\|_\infty \to \infty.$$

We prove the following result.

Theorem 1.1. *Assume $1 < p < 2$ and $N > p$. Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous function satisfying the following conditions:*

(1) *$f(0) < 0$; and*

(2) *there exist $b > 0$ and $q \in (p - 1, N(p - 1)/(N - p)]$ such that*

$$\lim_{s\to\infty} \frac{f(s)}{s^q} = b.$$

Then (1.1) has a positive solution (λ, u) for $\lambda > 0$ small.

Moreover, there exists a connected component $\mathfrak{C}(\subset \mathfrak{S})$, consisting of positive solutions, bifurcating from infinity at $\lambda_\infty = 0$.

An example satisfying the hypotheses of theorem 1.1 is given by $f(s) = bs^q - \varepsilon$, where b, ε are positive and $p - 1 < q \leqslant N(p - 1)/(N - p)$.

We now quote the following observations from [4] that are relevant to our result.

Remark 1.2 (Azizieh and Clement [4, remark 0.1]). When Ω is an open ball in $\mathbb{R}^N$, $N > p$, the *a priori* bound of positive solutions (for the limiting problem) used in our analysis holds for $p - 1 < q < (N(p-1) + p)/(N - p)$ (see proposition 2.3), and so does our result in theorem 1.1.

Remark 1.3 (Azizieh and Clement [4, remark 0.2]). The restrictions on $p \in (1, 2)$, the growth range for q and the strict convexity requirement on the domain all arise from the *a priori* bound result (proposition 2.3) used in our analysis.

In § 2 we recall some useful results for our functional framework, a crucial uniform *a priori* bound result for a limiting problem related to (1.1) and a continuation theorem via degree theory. In § 3, we prove theorem 1.1. In § 4, we state the existence result for a system and provide an outline of the proof.

2. Preliminaries

The following result (using slightly different notation but without modification to the content) provides the functional framework for our approach.

Proposition 2.1 (Azizieh and Clement [4, lemma 1.1]). *Consider the problem*

$$\left.\begin{aligned} -\Delta_p u &= g(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.1}$$

Then

(1) *for all $g \in L^\infty(\Omega)$, there exists a unique weak solution u of (2.1) in $C_0^1(\bar{\Omega})$,*

(2) *the solution operator $K\colon L^\infty(\Omega) \to C_0^1(\bar{\Omega})$ defined by $Kg = u$ is continuous, compact and homogeneous of order $1/(p-1)$.*

The estimate below turns out to be helpful in computing the degree near the origin.

Proposition 2.2 (Daners and Drábek [12, theorem 2.5]). *Let $g \in L^\infty(\Omega)$. Then there is a constant $C > 0$ such that the corresponding solution of (2.1) satisfies*

$$\|u\|_\infty^{p-1} \leqslant C\|g\|_\infty. \tag{2.2}$$

The following uniform *a priori* bound result will be crucial in computing the degree in a large ball.

Proposition 2.3 (Azizieh and Clement [4, theorem 0.1, remark 0.1]). *Consider*

$$\left.\begin{aligned} -\Delta_p w &= h(w(x) + t) \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \\ t &\geqslant 0, \end{aligned}\right\} \tag{2.3}$$

where $1 < p \leqslant 2$, and h satisfies the following:

(1) *$h\colon \mathbb{R} \to [0, \infty)$ is continuous on $\mathbb{R}$ and is locally Lipschitz continuous on $[0, +\infty)$;*

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(2) *there exist some constants* $C_0, C_1 > 0$ *such that*

$$C_0 s^q \leqslant h(s) \leqslant C_1 s^q \quad \forall s \in [0, \infty).$$

If $p-1 < q \leqslant N(p-1)/(N-p)$*, then there exists a constant* $R > 0$ *such that*

$$\|w\|_{C^1} + t \leqslant R$$

for all solutions (t, w) *of (2.3) with* $t \geqslant 0$*.*

Moreover, if $h(s) = \tilde{C}|s|^q$*, for some constant* $\tilde{C} > 0$ *and* Ω *is an open ball in* $\mathbb{R}^N$*, then the above result holds true for* $p-1 < q < (N(p-1)+p)/(N-p)$*.*

To prove the second part of theorem 1.1, we shall use the following special version of the result from [13, proposition 2.3] originally stated in [19, lemma 3.4].

Proposition 2.4 (de Figueiredo *et al*. [13, proposition 2.3]). *Let* X *be a Banach space, let* U *be a bounded open subset of* X *and let* $M\colon [a,b] \times \bar{U} \to X$ *be a compact map such that* $M(t,x) \neq x$ *for* $(t,x) \in [a,b] \times \partial U$*. Assume that* $\deg(I - M(t,\cdot), U, 0) \neq 0$ *for all* $t \in [a,b]$*. Then if* $\Sigma := \{(t,x) \in [a,b] \times U\colon M(t,x) = x\}$*, there exists a connected component* D *of* Σ *such that* $D \cap (\{a\} \times U)$ *and* $D \cap (\{b\} \times U)$ *are non-empty.*

3. Proof of theorem 1.1

First we extend f as an even function on $\mathbb{R}$ by setting $f(s) = f(-s)$ for $s \in \mathbb{R}$. Let $F(s) := f(s) - b|s|^q$ for all $s \in \mathbb{R}$. Then, for $\gamma > 0$, we set $\lambda = \gamma^{q-p+1}$ and rescale the solution variable using $w = \gamma u$. We see that w formally satisfies

$$\begin{aligned} -\Delta_p w &= \gamma^{p-1}\gamma^{q-p+1} f(w/\gamma) \\ &= \gamma^q f(w/\gamma) \\ &= \gamma^q [f(w/\gamma) - b|w/\gamma|^q] + b|w|^q \\ &= \gamma^q F(w/\gamma) + b|w|^q. \end{aligned}$$

Next, let $\tilde{F}(\gamma, s) := \gamma^q F(s/\gamma) + b|s|^q$ for $\gamma > 0$ and $s \in \mathbb{R}$. Using theorem 1.1(2), we see that $\lim_{\gamma\to 0} \tilde{F}(\gamma, s) = b|s|^q$, and hence we can continuously extend $\tilde{F}(\gamma, s)$ to $\gamma = 0$ by setting $\tilde{F}(0, s) = b|s|^q$ for all $s \in \mathbb{R}$. Then $\tilde{F}(\cdot,\cdot)\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.

Let X denote the Banach space $C_0(\bar{\Omega})$ equipped with the supremum norm $\|\cdot\|_\infty$. Then, for fixed $\gamma \geqslant 0$, we define the map $S(\gamma, \cdot)\colon X \to X$ by

$$S(\gamma, w) := w - K\tilde{F}(\gamma, w). \tag{3.1}$$

For $\gamma \geqslant 0$ fixed, $K \circ \tilde{F}(\gamma, \cdot)\colon X \to X$ is compact, since the Nemytskii operator $\tilde{F}(\gamma, \cdot)\colon X \to L^\infty(\Omega)$ is continuous and the solution operator $K\colon L^\infty(\Omega) \to C_0^1(\bar{\Omega})$ is compact (by proposition 2.1). Thus, $S(\gamma, \cdot)$ is the compact perturbation of the identity. Note that $S(\gamma, w) = 0$ if and only if w is a solution of

$$\left.\begin{aligned} -\Delta_p w &= \tilde{F}(\gamma, w) \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

In particular, for $\gamma = 0$, we have that $S(0, w) = 0$ if and only if w is a solution of

$$\left.\begin{aligned} -\Delta_p w &= b|w|^q && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.3}$$

To prove theorem 1.1, we shall first compute the degree of $S(0, \cdot)$ as follows.

LEMMA 3.1. *There exist* $0 < r < R$ *such that* $S(0, w) \neq 0$ *for all* $w \in X$ *with* $\|w\|_\infty \in \{r, R\}$ *and* $\deg(S(0, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = -1$.

Proof. First, for $\eta \in [0, 1]$, consider

$$\left.\begin{aligned} -\Delta_p w &= \eta^{p-1} b|w|^q && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{3.4}$$

and assume that (3.4) has a solution with $\|w\|_\infty = r$. Then, using (2.2), we obtain

$$r^{p-1} = \|w\|_\infty^{p-1} \leqslant C\|\eta^{p-1} b|w|^q\|_\infty \leqslant Cb\|w\|_\infty^q = Cbr^q.$$

Since $q > p - 1$, we arrive at a contradiction for $r \ll 1$. Thus, there exists $r > 0$ small enough that (3.4) does not have a solution w with $\|w\|_\infty = r$. Now it is easy to see, using the operator equation $w - \eta K\tilde{F}(0, w) = 0$ and homotopy invariance of degree with respect to $\eta \in [0, 1]$, that $\deg(S(0, \cdot), B_r(0), 0) = \deg(I, B_r(0), 0) = 1$.

Next, letting $h(s) = b|s|^q$ in proposition 2.3, there exists $R > 0$ such that all solutions of (t, w) of $S^t(0, w) = 0$, i.e. of

$$\left.\begin{aligned} -\Delta_p w &= b|w + t|^q && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \\ t &\geqslant 0, \end{aligned}\right\} \tag{3.5}$$

satisfy $\|w\|_\infty + t < R$. Hence, $S^t(0, w) \neq 0$ for any $w \in \partial B_R(0)$ and for any $t \geqslant 0$. In particular, we can conclude that there are no solutions to (3.5) in $B_R(0)$ for any $t \geqslant R$. Therefore, $\deg(S^t(0, \cdot), B_R(0), 0) = 0$ for all $t \geqslant R$. Then, using the homotopy invariance of degree with respect to $t \in [0, R]$, we have

$$\deg(S(0, \cdot), B_R(0), 0) = \deg(S^0(0, \cdot), B_R(0), 0) = \deg(S^R(0, \cdot), B_R(0), 0) = 0.$$

Then the excision property of the degree yields

$$\deg(S(0, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = -1.$$

□

Now we compute the degree of $S(\gamma, \cdot)$ by connecting $S(\gamma, \cdot)$ and $S(0, \cdot)$ using the homotopy invariance of degree with respect to γ. In particular, this will imply that $S(\gamma, w) = 0$ has a solution w satisfying $r < \|w\|_\infty < R$. Then we show that this solution, that is, the solution of the rescaled problem, (3.2), is positive in Ω for γ small.

LEMMA 3.2. *There exists* $\gamma_0 > 0$ *such that*

(i) $\deg(S(\gamma, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = -1$ *for all* $\gamma \in [0, \gamma_0]$,

(ii) *if* $S(\gamma, w) = 0$ *for* $\gamma \in [0, \gamma_0]$ *with* $r < \|w\|_\infty < R$, *then* $w > 0$ *in* Ω.

Proof. We shall show that there exists $\gamma_0 > 0$ such that $S(\gamma, w) \neq 0$ for all $\|w\|_\infty \in \{r, R\}$ and all $\gamma \in [0, \gamma_0]$. Suppose to the contrary that there exist $\gamma_n \to 0$ with $\|w_n\|_\infty \in \{r, R\}$ and $S(\gamma_n, w_n) = 0$. Since K is compact and $\{\tilde{F}((\gamma_n, w_n))\}$ are bounded in $L^\infty(\Omega)$, by proposition 2.1, $w_n \to w$ in $C_0^1(\bar{\Omega})$ (up to a subsequence) where $\|w\|_\infty = r$ or R and $S(0, w) = 0$. This is a contradiction to lemma 3.1. Hence, due to the homotopy invariance of degree with respect to $\gamma \in [0, \gamma_0]$, we have that $\deg(S(\gamma, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = \deg(S(0, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = -1$. This completes the proof of part (i).

We proceed to prove part (ii) by contradiction. Suppose there exist $\gamma_n \to 0$ and corresponding solution w_n of (3.2) such that, for all n, $r < \|w_n\|_\infty < R$, $S(\gamma_n, w_n) = 0$ and $\Omega_n := \{x \in \Omega\colon w_n(x) \leqslant 0\} \neq \emptyset$. Using the same argument as above, $w_n \to w$ in $C_0^1(\bar{\Omega})$ (up to a subsequence), where w satisfies (3.3). But one has $w > 0$ in Ω and $\partial w/\partial\eta < 0$ on $\partial\Omega$ by [21, theorem 5]. Now let $\{x_n\} \in \Omega_n$. Then there exists a subsequence of $\{x_n\}$ (without loss of generality we can call this subsequence $\{x_n\}$) that converges to some $z \in \bar{\Omega}$. However, $w > 0$ in Ω, and hence $z \in \partial\Omega$. Now let $\tilde{x}_n$ be the point on $\partial\Omega$ closest to x_n. Then $\mu_n = (\tilde{x}_n - x_n)/|\tilde{x}_n - x_n|$ will be the outward unit normal to $\partial\Omega$ at $\tilde{x}_n$. Since $w_n(x_n) \leqslant 0$ and $w_n(\tilde{x}_n) = 0$, there exist $\{y_n\}$, with y_n belonging to the line segment joining x_n, and $\tilde{x}_n$ such that $(\nabla w_n \cdot \mu_n)|_{y_n} \geqslant 0$. Letting $n \to \infty$, we obtain $\partial w/\partial\eta(z) \geqslant 0$, a contradiction. Hence, for large n, $w_n(x) > 0$ for all $x \in \Omega$. This completes the proof of (ii). □

Now we complete the proof of the existence part of theorem 1.1. By lemma 3.2, we have that (3.2) has a positive solution $w := w(\gamma) \in B_R(0) \setminus \bar{B}_r(0)$ for all $\gamma \in [0, \gamma_0]$. But the rescaling $\lambda = \gamma^{q-p+1}$ implies that (1.1) has a positive solution $u := \gamma^{-1} w = \lambda^{1/(p-q-1)} w$ for $0 < \lambda \leqslant \lambda_0 := \gamma_0^{q-p+1}$. Finally, since $\|w\|_\infty > r > 0$ for all $\gamma \in [0, \gamma_0]$, we have $\|u\|_\infty = \|w\|_\infty/\gamma \to +\infty$ as $\gamma \to 0$. But $\lambda \to 0$ if and only if $\gamma \to 0$, and consequently $\|u\|_\infty \to +\infty$ as $\lambda \to 0$.

Moreover, using (i) and (ii) of lemma 3.2, it follows from proposition 2.4 that there exists a connected component $\mathfrak{C}$ (continuum) of positive solutions to $S(\gamma, w) = 0$ for $\gamma \in [0, \gamma_0]$ such that $\mathfrak{C}$ connects the set $\mathfrak{C} \cap (\{0\} \times (B_R(0) \setminus \bar{B}_r(0)))$ with the set $\mathfrak{C} \cap (\{\gamma_0\} \times (B_R(0) \setminus \bar{B}_r(0)))$. This in turn implies that there exists a connected component of positive solutions of (1.1) bifurcating from infinity at $\lambda_\infty = 0$. This completes the proof of the theorem.

4. Systems case

Here we consider a quasilinear system of the form

$$\left.\begin{aligned} -\Delta_{p_1} u &= \lambda f(v) \quad \text{in } \Omega, \\ -\Delta_{p_2} v &= \lambda g(u) \quad \text{in } \Omega, \\ u = 0 &= v \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$, $N > \max\{p_1, p_2\}$, is as before a strictly convex bounded domain. The study of this special system is motivated by earlier work in [7, 8], where the case $p_1 = p_2 = 2$ was considered.

By a solution of (4.1), we mean a $(\lambda,(u,v))$ that solves (4.1) in the weak sense, that is, $(u,v)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^{p_1-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x=\lambda\int_\Omega f(v)\varphi\,\mathrm{d}x,$$
$$\int_\Omega |\nabla v|^{p_2-2}\nabla v\cdot\nabla\psi\,\mathrm{d}x=\lambda\int_\Omega g(u)\psi\,\mathrm{d}x$$

for all $(\varphi,\psi)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$. Furthermore, if $u>0$ and $v>0$ in Ω, then $(\lambda,(u,v))$ is called a positive solution.

We say that λ_∞ is a bifurcation point from infinity if the solution set

$$\mathfrak{S}:=\{(\lambda,(u,v))\in\mathbb{R}\times W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)\colon \lambda \text{ and } (u,v) \text{ solves } (4.1)\}$$

contains a sequence $(\lambda_n,(u_n,v_n))$ such that

$$\lambda_n\to\lambda_\infty \quad\text{and}\quad \max\{\|u_n\|_\infty,\|v_n\|_\infty\}\to\infty.$$

First, we state the following *a priori* bound result, analogous to proposition 2.3 for the scalar case, which allows us to state our theorem precisely.

Proposition 4.1 (Azizieh *et al.* [5, theorem 1.1]). *Consider*

$$\left.\begin{aligned} -\Delta_{p_1}w_1&=h_1(|w_2(x)|+t) && \text{in } \Omega,\\ -\Delta_{p_2}w_2&=h_2(|w_1(x)|+t) && \text{in } \Omega,\\ w_1&=0=w_2 && \text{on } \partial\Omega,\\ t&\geqslant 0, \end{aligned}\right\} \tag{4.2}$$

where $h_1,h_2\colon[0,\infty)\to[0,\infty)$ are continuous functions. Assume that one of the following conditions holds:

(a) *$p_1,p_2\in(1,2)$ and $h_1,h_2\colon[0,\infty)\to[0,\infty)$ are strictly increasing;*

(b) *$p_1\in(1,\infty)$, $p_2=2$ and $h_1,h_2\colon\mathbb{R}\to[0,\infty)$ are increasing on $[0,\infty)$.*

Moreover, suppose that h_1, h_2 are continuous on $[0,\infty)$, locally Lipschitz continuous on $[0,\infty)$ and satisfy

(c) *$C_1s^{q_1}\leqslant h_1(s)\leqslant C_2s^{q_1}$, $D_1s^{q_2}\leqslant h_2(s)\leqslant D_2s^{q_2}$ for all $s\in[0,\infty)$ for some constants $C_1,C_2,D_1,D_2>0$.*

Then if $q_1q_2>(p_1-1)(p_2-1)$ and

$$\max\left\{\frac{p_2q_1+p_1(p_2-1)}{q_2q_1-(p_1-1)(p_2-1)}-\frac{N-p_1}{p_1-1},\frac{p_1q_2+p_2(p_1-1)}{q_2q_1-(p_1-1)(p_2-1)}-\frac{N-p_2}{p_2-1}\right\}\geqslant 0, \tag{4.3}$$

then there exists a constant $R>0$ such that

$$\|w_1\|_{C^1}+\|w_2\|_{C^1}+t\leqslant R$$

for any solution (t,w_1,w_2) with $t\geqslant 0$.

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Remark 4.2. Note that condition (4.3) is equivalent to either one of the following inequalities:

$$\left.\begin{aligned} q_1 &\leqslant \frac{p_1(p_2-1)}{N-p_2} + \frac{N(p_2-1)(p_1-1)}{N-p_2}\frac{1}{q_2}, \\ q_2 &\leqslant \frac{p_2(p_1-1)}{N-p_1} + \frac{N(p_1-1)(p_2-1)}{N-p_1}\frac{1}{q_1}. \end{aligned}\right\} \tag{4.4}$$

The range of q_1 and q_2 given by (4.3) is the region bounded by the (q_1, q_2)-axes and the two hyperbolas obtained by setting equalities in (4.4).

Now we state our result.

Theorem 4.3. *Let $f, g\colon [0,\infty) \to \mathbb{R}$ be continuous functions satisfying $f(0) < 0$ and $g(0) < 0$. Assume that either (a) or (b) of proposition 4.1 holds. Then if $q_1 q_2 > (p_1-1)(p_2-1)$ and there exist positive numbers b_1, b_2 satisfying*

$$\lim_{s\to\infty} \frac{f(s)}{s^{q_1}} = b_1 > 0 \quad \textit{and} \quad \lim_{s\to\infty} \frac{g(s)}{s^{q_2}} = b_2 > 0, \tag{4.5}$$

where p_1, p_2, q_1 and q_2 satisfy (4.4), then (4.1) has a positive solution for $\lambda > 0$ small.

Moreover, there exists a connected component $\mathfrak{C}(\subset \mathfrak{S})$, consisting of positive solutions, bifurcating from infinity at $\lambda_\infty = 0$.

Remark 4.4. In the case of superlinear systems, in particular with semipositone structure, the only known result is in an annulus (see [15]). See also [5], where Azizieh *et al*. prove existence of positive solutions under the assumptions of proposition 4.1 but without semipositone structure on the nonlinearities.

Proof. The proof of this theorem uses the same functional setting and abstract theorem as the scalar case, except for the uniform *a priori* bound result (proposition 4.1). For completeness, we give an outline where the details of the extension to systems from the scalar case are trivial, and provide more detail where necessary.

First, extend f and g as even functions on $\mathbb{R}$ by setting $f(s) := f(-s)$ and $g(s) := g(-s)$, and define $F(s) := f(s) - b_1|s|^{q_1}$, $G(s) := g(s) - b_2|s|^{q_2}$ for all $s \in \mathbb{R}$.

The rescaling for the systems case is technically complicated and thus we provide the details below.

Let

$$\lambda = \gamma^{\delta}; \quad w_1 = \gamma^{\theta_1} u, \quad w_2 = \gamma^{\theta_2} v, \quad \gamma > 0,$$

where δ, θ_1 and θ_2 are to be determined. We wish to determine these parameters so that the rescaled problem corresponding to (4.1) approaches

$$\left.\begin{aligned} -\Delta_{p_1} w_1 &= b_1 |w_2|^{q_1} \quad \text{in } \Omega, \\ -\Delta_{p_2} w_2 &= b_2 |w_1|^{q_2} \quad \text{in } \Omega, \\ w_1 = 0 &= w_2 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{4.6}$$

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as $\gamma \to 0$. With this in mind, we see that w_1 formally satisfies

$$\begin{aligned}
-\Delta_{p_1} w_1 &= -\Delta_{p_1}(\gamma^{\theta_1} u) \\
&= \gamma^{\theta_1(p_1-1)}(-\Delta_{p_1} u) \\
&= \gamma^{\theta_1(p_1-1)}\lambda f(v) \\
&= \gamma^{\theta_1(p_1-1)}\gamma^{\delta} f(w_2/\gamma^{\theta_2}) \\
&= \gamma^{\theta_1(p_1-1)+\delta} f(w_2/\gamma^{\theta_2}) \\
&= \gamma^{\theta_1(p_1-1)+\delta}[f(w_2/\gamma^{\theta_2}) - b_1|w_2/\gamma^{\theta_2}|^{q_1}\gamma^{\theta_2 q_1-(\theta_1(p_1-1)+\delta)}] + b_1|w_2|^{q_1} \\
&= \gamma^{\theta_1(p_1-1)+\delta} F(w_2/\gamma^{\theta_2}) + b_1|w_2|^{q_1}
\end{aligned}$$

if

$$\theta_2 q_1 - \theta_1(p_1-1) - \delta = 0. \tag{4.7}$$

Similarly, using the other equation, w_2 formally satisfies

$$-\Delta_{p_2} w_2 = \gamma^{\theta_2(p_2-1)+\delta} G(w_1/\gamma^{\theta_1}) + b_2|w_1|^{q_2}$$

if

$$\theta_1 q_2 - \theta_2(p_2-1) - \delta = 0. \tag{4.8}$$

From (4.7) and (4.8), we get

$$\theta_2 = \theta_1 \frac{q_2+p_1-1}{q_1+p_2-1}. \tag{4.9}$$

Letting $\theta_1 = 1$, we have

$$\delta = \frac{q_1 q_2 - (p_1-1)(p_2-1)}{q_1+p_2-1}. \tag{4.10}$$

Define $\tilde{F}(\gamma, s_2) := \gamma^{\theta q_1} F(s_2/\gamma^{\theta}) + b_1|s_2|^{q_1}$ and $\tilde{G}(\gamma, s_1) := \gamma^{q_2} G(s_1/\gamma) + b_2|s_1|^{q_2}$ for $\gamma > 0$ and $(s_1, s_2) \in \mathbb{R} \times \mathbb{R}$, where $\theta := \theta_2$ and δ is given by (4.10). Then, using (4.5), we see that

$$\lim_{\gamma\to 0} \tilde{F}(\gamma, s_2) = b_1|s_2|^{q_1} \quad \text{and} \quad \lim_{\gamma\to 0} \tilde{G}(\gamma, s_1) = b_2|s_1|^{q_2}.$$

Therefore, $\tilde{F}(\gamma, s_2)$ and $\tilde{G}(\gamma, s_1)$ are continuous on $[0,\infty)\times\mathbb{R}$. It is then easy to see that (w_1, w_2) satisfy

$$\left.\begin{aligned}
-\Delta_{p_1} w_1 &= \tilde{F}(\gamma, w_2) \quad \text{in } \Omega, \\
-\Delta_{p_2} w_2 &= \tilde{G}(\gamma, w_1) \quad \text{in } \Omega, \\
w_1 &= 0 = w_2 \qquad \text{on } \partial\Omega.
\end{aligned}\right\} \tag{4.11}$$

Let $X := C_0(\bar{\Omega})\times C_0(\bar{\Omega})$ with norm $\|(u,v)\|_X := \max\{\|u\|_\infty, \|v\|_\infty\}$, where $\|\cdot\|_\infty$ denotes the usual supremum norm in $C_0(\bar{\Omega})$. For fixed $\gamma \geqslant 0$ and $w := (w_1, w_2)$, define the map $S(\gamma, \cdot)\colon X \to X$ by

$$S(\gamma, w) := w - (K_{p_1}\tilde{F}(\gamma, w_2), K_{p_2}\tilde{G}(\gamma, w_1)).$$

Since $K_{p_i}\colon L^\infty(\Omega) \to C^1_0(\bar{\Omega})$ for $i = 1, 2$ are compact and the Nemytskii operators $\tilde{F}(\gamma,\cdot), \tilde{G}(\gamma,\cdot)\colon C_0(\bar{\Omega}) \to L^\infty(\Omega)$ are continuous, $S(\gamma,\cdot)$ is a compact perturbation of the identity. We note that, for $\gamma \geqslant 0$, $S(\gamma, w) = 0$ if and only if w is a solution of (4.11). In particular, $S(0, w) = 0$ if and only if w is a solution of (4.6).

Due to the abstract functional setting of our problem, we see that the degree computations of $S(\gamma, \cdot)$ for $\gamma = 0$ and $\gamma \geqslant 0$ give same lemmas as in the scalar case. Proofs of these lemmas require some modifications, which we indicate below.

LEMMA 4.5. *There exist $0 < r < R$ such that $S(0, w) \neq 0$ for all $w \in X$ with $\|w\|_X \in \{r, R\}$ and* $\deg(S(0, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = -1$.

Proof. First, for $\eta \in [0, 1]$, consider

$$\left.\begin{aligned} -\Delta_{p_1} w_1 &= \eta^{p_1-1} b_1 |w_2|^{q_1} \quad \text{in } \Omega, \\ -\Delta_{p_2} w_2 &= \eta^{p_2-1} b_2 |w_1|^{q_2} \quad \text{in } \Omega, \\ w_1 = 0 &= w_2 \qquad\qquad\quad \text{on } \partial\Omega \end{aligned}\right\} \tag{4.12}$$

and assume that (4.12) has a solution with $\|w\|_X = r$. Without loss of generality, assume $\|w\|_X = \|w_1\|_\infty = r$. Observe that, by proposition 2.2, for each $i = 1, 2$, there exists a constant $C_i > 0$ such that (2.2) holds true with $p = p_i$ and $C = C_i$. Then, using both equations of (4.12), we obtain

$$\begin{aligned} r^{p_1-1} &= \|w_1\|_\infty^{p_1-1} \\ &\leqslant C_1 \|\eta^{p_1-1} b_1 |w_2|^{q_1}\|_\infty \\ &\leqslant C_1 b_1 \|w_2\|_\infty^{q_1} \\ &\leqslant C_1 b_1 (C_2 b_2)^{1/(p_2-1)} (\|w_1\|_\infty)^{q_1 q_2/(p_2-1)} \\ &= C_1 b_1 (C_2 b_2)^{1/(p_2-1)} r^{q_1 q_2/(p_2-1)}, \end{aligned}$$

which is a contradiction for $r \ll 1$ since $q_1 q_2 > (p_1 - 1)(p_2 - 1)$. Thus, there exists $r > 0$ small enough that (4.12) does not have a solution w with $\|w\|_X = r$. The rest of the argument is identical except that we take $h_1(s_2) = b_1 |s_2|^{q_1}$ and $h_2(s_1) = b_2 |s_1|^{q_2}$ in proposition 4.1 to prove the existence of large $R > 0$ such that $\deg(S(0, \cdot), B_R(0), 0) = 0$. □

LEMMA 4.6. *There exists $\gamma_0 > 0$ such that*

(i) $\deg(S(\gamma, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = -1$ *for all* $\gamma \in [0, \gamma_0]$,

(ii) *if $S(\gamma, w) = 0$ for $\gamma \in [0, \gamma_0]$ with $r < \|w\|_X < R$, then $w > 0$ in Ω.*

Proof. Suppose to the contrary that there exists a sequence $(\gamma_n, (w_{1_n}, w_{2_n}))$ with $\gamma_n \to 0$ without loss of generality, with

$$\|(w_{1_n}, w_{2_n})\|_X \in \{r, R\} \quad \text{and} \quad S(\gamma_n, (w_{1_n}, w_{2_n})) = 0.$$

By arguments similar to those in lemma 3.2, $w_{i_n} \to w_i$ in $C_0^1(\bar{\Omega})$ (up to a subsequence), where $\|(w_1, w_2)\|_X = r$ or R and $S(0, (w_1, w_2)) = 0$, which is a contradiction to lemma 4.5.

Hence, due to the homotopy invariance of degree with respect to the parameter $\gamma \in [0, \gamma_0]$, we have that

$$\deg(S(\gamma, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = \deg(S(0, \cdot), B_R(0) \setminus \bar{B}_r(0), 0) = -1.$$

This completes the proof of part (i).

Part (ii) follows as in the scalar case by noting that $w_i > 0$ in Ω and $\partial w_i/\partial\eta < 0$ on $\partial\Omega$ by [21, theorem 5] for $i = 1, 2$, where $w = (w_1, w_2) \in B_R(0) \setminus \bar{B}_r(0)$ is a solution of the limiting system (4.6). □

By lemma 4.6, (4.11) has a positive solution $(w_1, w_2) := (w_1(\gamma), w_2(\gamma)) \in B_R(0) \setminus \bar{B}_r(0)$ for $\gamma \in [0, \gamma_0]$. This in turn implies that (4.1) has a positive solution (u, v) where $u := \gamma^{-1} w_1$ and $v := \gamma^{-\delta\theta} w_2$ for $0 < \lambda \leqslant \lambda_0 := \gamma_0^\delta$. Since $\|(w_1, w_2)\|_X > r > 0$, it is easy to see that $\|(u, v)\|_X \to \infty$ as $\lambda \to 0$.

The last part of the theorem follows, as in the scalar case, from (i) and (ii) of lemma 4.6 and proposition 2.4. This completes the proof. □

Example 4.7. An example satisfying the hypotheses of theorem 4.3 is

$$\begin{aligned} -\Delta_{p_1} u &= \lambda(b_1 v^{q_1} - \varepsilon_1) && \text{in } \Omega, \\ -\Delta_{p_2} v &= \lambda(b_2 u^{q_2} - \varepsilon_2) && \text{in } \Omega, \\ u &= 0 = v && \text{on } \partial\Omega, \end{aligned}$$

where b_i, ε_i are positive, p_i satisfy part (a) or (b) of proposition 4.1 and the q_i satisfy (4.4) for each $i = 1, 2$.

Remark 4.8. When $p_1 = p_2 = 2$, (4.4) becomes one of the following:

$$\left.\begin{aligned} q_1 &\leqslant \frac{2}{N-2} + \frac{N}{N-2}\frac{1}{q_2}, \\ q_2 &\leqslant \frac{2}{N-2} + \frac{N}{N-2}\frac{1}{q_1}. \end{aligned}\right\} \tag{4.13}$$

The $(q_1 q_2)$-region given by these inequalities is smaller than that given by the critical hyperbola condition,

$$\frac{1}{q_1+1} + \frac{1}{q_2+1} > \frac{N-2}{N}, \quad N \geqslant 3, \tag{4.14}$$

for which the existence of a positive solution for λ small was established in [7].

Furthermore, if $q_1 = q_2 = q$, the critical hyperbola condition gives a wider growth range $(1 < q < (N+2)/(N-2))$ than the one given in (4.4) $(1 < q \leqslant N/(N-2))$; see remark 1.2.

Remark 4.9. For the case $p_1 = p_2 = 2$, the rescaling given by (4.10) agrees with that obtained in [7] with $\delta = (q_1 q_2 - 1)/(q_1 + 1)$, $\theta_1 = 1$ and $\theta_2 = (q_2 + 1)/(q_1 + 1)$.

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