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Deviations in the Franks–Misiurewicz conjecture

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Abstract. We show that if there exists a counter example for the rational case of the Franks–Misiurewicz conjecture, then it must exhibit unbounded deviations in the complementary direction of its rotation set.

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1. *Introduction*

After the seminal result due to Miziurewicz and Ziemian [**MZ89**] proving that the rotation set of a lift $F:\mathbb{R}^2\to\mathbb{R}^2$ of a homeomorphism homotopic to the identity $f:\mathbb{T}^2:=\mathbb{R}^2/_{\mathbb{Z}^2}\to\mathbb{T}^2$ given by

$$\rho(F)=\left\{\lim_i \frac{F^{n_i}(x_i)-x_i}{n_i} : x_i\in\mathbb{R}^2,\ n_i\nearrow+\infty\right\}$$

is a compact convex set, a theory has been developed with in general aims to determine a classification of the dynamics of torus homeomorphisms homotopic to the identity under this invariant. Many authors have contributed to these developments, where most results can be classified into two different categories: (i) assuming shapes (point, segments, non-empty interior) for the rotation set, derive dynamical properties (see, for instance, [**Fra89**, **Fra88**, **MZ91**, **LM91**]), (ii) try to find which convex sets can occur as rotation sets (see, for instance, [**Kwa95**, **BdCH16**]).

Concerning point (ii) there is a long-standing conjecture due to Franks and Misiurewicz [**FM90**], which claims the following: if a non-trivial interval I is attained as a rotation set then:

- if I has irrational slope, one end-point is rational;
- if I has rational slope, it contains a rational point.

For the irrational case, Avila presented a smooth counter example in 2014 (still not published), which is minimal. For the second case there has been important progress in recent years. In [**KPS16**] it is shown that there cannot be a minimal counter example. In fact it is proven there that a counter example for this case cannot be an extension of an irrational rotation, and then using the results of Kocsard [**Koc16**] and Jäger and Tal [**JT17**], one concludes that a minimal example should be an extension of an irrational rotation, so it cannot exist.

In this article, improving [**JT17**], we show that a possible counter example must exhibit *unbounded deviations* in the complementary direction of the supporting line of the interval $\rho(F)$. In fact, this suggests that such counter examples may not exist, as it was shown in several similar situations that having two different rotation vectors is an obstruction for deviations [**Dáv18**, **CT18**].

1.1. *Precise result.* We denote by $\mathrm{Homeo}_0(\mathbb{T}^2)$ the family of homotopic to the identity toral homeomorphisms. The rotation set is defined above. Let $\rho(F)$ be a non-trivial segment contained in a supporting line $\{p + \lambda v\}_{\lambda\in\mathbb{R}}$. Then the *perpendicular deviation* of f is given by the (possibly infinite) value

$$\mathrm{dev}_\perp(f) = \sup_{x\in\mathbb{R}^2} \{\mathrm{d}(\mathrm{pr}_\perp(n \cdot \rho(F)), \mathrm{pr}_\perp(F^n(x) - x))\},$$

where $\mathrm{pr}_\perp : \mathbb{R}^2 \to v^\perp$ is the projection on a unitary element of $v^\perp$, and $\mathrm{d}(\cdot, \cdot)$ is Euclidean distance in $v^\perp$.

In this article we prove that if f is a counter example for the rational case of the Franks–Misiurewicz conjecture, then it has infinite perpendicular deviation. We state the result as follows.

THEOREM. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ so that $\rho(F)$ is contained in a line of rational slope containing no rational points, and f has perpendicular bounded deviation. Then, $\rho(F)$ is a singleton.*

1.2. *Strategy.* As explained in [**KPS16**], in order to obtain the result above, we can just work with vertical rotation sets. So we must show the following.

THEOREM 1.1. *Assume that for a lift F of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ we have $\rho(F) = \{\alpha\} \times [\rho^-, \rho^+]$, $\alpha \in \mathbb{Q}^c$, and f has bounded horizontal deviation. Then $\rho^+ = \rho^-$.*

In this last sentence *unbounded horizontal deviation* stands for the value $\mathrm{dev}_\perp(f)$ being infinite when $\rho(F)$ is vertical.

For proving Theorem 1.1 we suppose a counter example with bounded horizontal deviations is possible, and then by improving [**JT17**] we get that this counter example would be an extension of an irrational rotation. This is absurd due to the result in [**KPS16**].

2. *Topological results*

We consider the torus given by $\mathbb{T}^2 = \mathbb{R}^2/_{\mathbb{Z}^2}$ and $\pi_{\mathbb{T}^2} : \mathbb{R}^2 \to \mathbb{T}^2$ the covering map. The annulus is given by $\mathbb{A} = \mathbb{R}^2/_\sim$, where $(u, v) \sim (r, s)$ if and only if $u = r$ and $v - s \in \mathbb{Z}$.

We have the natural covering maps $\pi : \mathbb{R}^2 \to \mathbb{A}$ and $\mathrm{p} : \mathbb{A} \to \mathbb{T}^2$. In $\mathbb{R}^2$ we name the projection to the first coordinate by pr_1, and to the second coordinate by pr_2.

An *annular continuum* in $\mathbb{A}$ is a continuum so that its complements is given by exactly two connected components, which are unbounded. A *circloid* in $\mathbb{A}$ is an annular continuum, which is minimal with respect to the inclusion. In this article, we call $\mathcal{A} \subset \mathbb{T}^2$ *annular continuum* if $\mathcal{A} = \mathrm{p}(A)$, where $A \subset \mathbb{A}$ is an annular continuum and $p|_A$ is a homeomorphism. A *circloid* in $\mathbb{T}^2$ is an annular continuum, which is minimal with respect to the inclusion amongst all annular continua. For any continuum X in $\mathbb{A}$ we denote by $\mathrm{Fill}(X)$ to the union of X and all the bounded connected components of $\mathbb{A} \backslash X$.

Back in $\mathbb{A}$ we can define a partial order on annular continua. Given an annular continua $A \subset \mathbb{A}$ we have two unbounded components on its complement. We write $\mathcal{U}^+(A)$ for the component which is unbounded to the right and by $\mathcal{U}^-(A)$ to the complementary one. For two annular continua A, B in $\mathbb{A}$ we say A *precedes* B if and only if $B \subset \mathrm{cl}[\mathcal{U}^+(A)]$, or equivalently $A \subset \mathrm{cl}[\mathcal{U}^-(B)]$. We denote this by $A \preceq B$.

Consider the following situation, which we refer by **(S)** in all of this article: $\mathcal{C}_1$, $\mathcal{C}_2$ are circloids in $\mathbb{A}$, and $A \subset \mathbb{A}$ is an annular continuum so that:

- $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$;
- $\mathcal{C}_1 \preceq A \preceq \mathcal{C}_2$;
- $\mathcal{C}_1 \not\subset A$, $\mathcal{C}_2 \not\subset A$.

The second and third item implies that $\mathcal{C}_1 \neq \mathcal{C}_2$. Moreover, for this setting we have that any connected component of $\mathcal{C}_1 \cap \mathcal{C}_2$ must be inessential and contained in A, and the same holds for $\mathcal{C}_1 \cap A$ and $A \cap \mathcal{C}_2$. Furthermore, if we consider lifts $\tilde{\mathcal{C}}_1$, $\tilde{\mathcal{C}}_2$ and $\tilde{A}$ of $\mathcal{C}_1$, $\mathcal{C}_2$ and A respectively, and the family

$$\mathrm{CC} = \mathrm{c.c.}(\tilde{\mathcal{C}}_1 \cap \tilde{\mathcal{C}}_2) \cup \mathrm{c.c.}(\tilde{\mathcal{C}}_1 \cap \tilde{A}) \cup \mathrm{c.c.}(\tilde{A} \cap \tilde{\mathcal{C}}_2),$$

where we denote by $\mathrm{c.c.}(X)$ the set of connected components for any space X, the following claim holds.

LEMMA 2.1. *There exists $K_0 > 0$ so that*

$$\sup_{X \in CC} \{\mathrm{diam}(\mathrm{pr}_2(X))\} < K_0.$$

Proof. Assume otherwise. Then we can consider a sequence of continua $(C_n)_{n\in\mathbb{N}}$ with $C_n \subset X \cap Y$, $X \neq Y \in \{\tilde{\mathcal{C}}_1, \tilde{A}, \tilde{\mathcal{C}}_2\}$ and $\mathrm{diam}(C_n) \to_n \infty$. Assume $X = \tilde{\mathcal{C}}_1$ and $Y = \tilde{A}$, for the complementary cases the same proof works.

Due to the basic properties of the Hausdorff metric for sub-sets in a metric space, we can assume that there exists $\mathcal{L} = \lim_{\mathrm{H}} \pi(C_n)$ given by a continuum in $\mathbb{A}$ contained in $\mathcal{C}_1 \cap \mathcal{A}$. Moreover, as $\mathrm{diam}(C_n) \to_n \infty$ then $\mathbb{A} \backslash \mathcal{L}$ contains two unbounded connected components. Furthermore, every bounded connected component of $\mathbb{A} \backslash \mathcal{L}$ is contained in $\mathcal{C}_1 \cap \mathcal{A}$, otherwise either $\mathcal{C}_1$ or A fails to be annular continuum. Thus, $\mathrm{Fill}(\mathcal{L})$ is an annular continumm contained in $\mathcal{C}_1$ and $\mathcal{A}$, which implies $\mathcal{C}_1 = \mathcal{L} \subset \mathcal{A}$ due to the definition of circloid, an absurd. □

We now introduce a definition. Given a sub-continuum $Z \subset \tilde{A}$ and $X \in \mathrm{c.c.}(\tilde{\mathcal{C}}_1 \cap \tilde{\mathcal{C}}_2)$ the *vertical homotopical intersection number* of Z and X is defined by

$$\nu(X, Z) = \#\{v \in \{0\} \times \mathbb{Z} : \ X + v \subset Z\}.$$

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Our goal is to prove the following proposition.

PROPOSITION 2.2. *Let $X \in c.c.(\tilde{\mathcal{C}}_1 \cap \tilde{\mathcal{C}}_2)$ and $(Z_n)_{n\in\mathbb{N}}$ be a sequence of sub-continua contained in $\tilde{A}$ with $\operatorname{diam}(Z_n) \to +\infty$. Then, $\nu(X, Z_n) \to +\infty$.*

Let us introduce some notation and results before providing the proof. Given two continua X and Z in $\mathbb{R}^2$ we say that X is *K-centered* with respect to Z if $\mathrm{pr}_2(Z)\setminus \mathrm{pr}_2(X)$ consists of the union of two disjoint intervals both having length larger than K.

Given a continuum Z in $\mathbb{R}^2$ we say that a continuum Y is *K-virtually to the right of Z* if it is K-centered with respect to Z and there exists a pair of disjoint vertical half-lines r, s so that:

- $\mathrm{pr}_2(r)$ is bounded below and $\mathrm{pr}_2(s)$ is bounded above;
- r meets Z only at it starting point r_0, which verifies $\mathrm{pr}_2(r_0) = \max \mathrm{pr}_2(Z)$;
- s meets Z only at it starting point s_0, which verifies $\mathrm{pr}_2(s_0) = \min \mathrm{pr}_2(Z)$;
- Y is contained in the closure of the connected component of $\mathbb{R}^2\setminus s \cup Z \cup r$ whose first projection is unbounded to the right.

Note that for any continua Z of $\mathbb{R}^2$ there always exists such two half-lines r, s, and that $s \cup Z \cup r$ defines a unique connected component $\mathcal{R}$ whose first projection is unbounded to the right and a unique connected component $\mathcal{L}$ whose first projection is unbounded to the left. The analogous definition can be given for *K-virtually to the left*. Before presenting a proof for the proposition we first need an additional lemma.

LEMMA 2.3. *Assume we have two sequences of planar continua $(Y_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ so that Y_n is a_n-virtually to the left of L_n with $a_n \to_n \infty$, and $\mathcal{L}' = \mathrm{Fill}(\lim_H \pi(Y_n))$, $\mathcal{L} = \mathrm{Fill}(\lim_H \pi(L_n))$ are annular continua. Then $\mathrm{Fill}(\mathcal{L}') \preceq \mathrm{Fill}(\mathcal{L})$.*

Proof. Suppose that $\mathrm{Fill}(\mathcal{L}') \npreceq \mathrm{Fill}(\mathcal{L})$ for an absurd. Then we can construct a curve $\Gamma : [0, +\infty) \to \mathbb{A}$ whose image is contained in $\mathcal{U}^+(\mathcal{L})$, starting at a point $x_0 \in \mathcal{L}'$ and so that $\Gamma(t) \to_{t\to+\infty} +\infty$. Thus we can take a lift $\tilde{\Gamma}$ of Γ starting at a lift $\tilde{x}_0$ of x_0, which is contained in $\mathcal{U}^+(\tilde{\mathcal{L}})$. Moreover, we can assume that $\mathrm{pr}_2(\tilde{\Gamma})$ is bounded.

On the other hand, we can consider vertical integer translations Y_n' of the elements Y_n so that $Y_n' \cap B(\tilde{x}_0, \varepsilon_n) \neq \emptyset$ with $\varepsilon_n \to_n 0$. We claim that this implies the existence of $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ some integer vertical translation L_n' of L_n must meet $B(\tilde{x}_0, \varepsilon_n)$: for this pick n_0 so that a_n is larger that $\operatorname{diam}(\mathrm{pr}_2(\tilde{\Gamma})) + 2\varepsilon_n$. Thus by taking L_n' for all $n \geq n_0$ so that Y_n' is a_n-virtually to the left of L_n', as Y_n' is contained in the region to the left of $r \cup L_n' \cup s$ (r, s half-lines of the definition of virtually to the left) with $(r \cup s) \cap (\tilde{\Gamma} \cup B(\tilde{x}_0, \varepsilon_n)) = \emptyset$, we must have

$$(\tilde{\Gamma} \cup B(\tilde{x}_0, \varepsilon_n)) \cap L_n' \neq \emptyset,$$

and we are done with the claim.

Hence, we have that $\tilde{x}_0 \in \tilde{\mathcal{L}}$, so $x_0 \in \mathcal{L}$, which concludes. □

Proof of Proposition 2.2. Assume for a contradiction that we have some $X \in \mathrm{c.c.}(\tilde{\mathcal{C}}_1 \cap \tilde{\mathcal{C}}_2)$, which does not satisfy the proposition. This implies that we can construct a sequence $(W_n)_{n\in\mathbb{N}}$ given by integer vertical translations of some elements of $(Z_n)_{n\in\mathbb{N}}$ so that:

(i) X is not contained in W_n, for any $n \in \mathbb{N}$;

(ii) X is n-centered with respect to W_n.

We will arrive at a contradiction from this situation. In the situation considered above for $\mathcal{C}_1$, $\mathcal{C}_2$ and A, by taking sub-sequences, either the following situation or the symmetric one holds: for every $n \in \mathbb{N}$ there exists a point $x_n \in X \setminus W_n$, which is n-virtually to the right of W_n.

Let us assume this situation, for the complementary one the symmetric argument works. In this context we consider the set $s \cup W_n \cup r$ as in the definition of *virtually to the right*, and its *right* component $\mathcal{R}$ with $x_n \in \mathcal{R}$. We claim the existence of a sequence of continua $L_n \subset \tilde{C}_1$ verifying:

(1) $L_n \cap B(x_n, 1/n) \neq \emptyset$;

(2) $L_n \subset \mathcal{R}$;

(3) $\mathrm{diam}(\mathrm{pr}_2(L_n)) > n/2 - 1$.

For this, we take a reference line $\Gamma : (-\infty, 0] \to \mathcal{U}^-(C_1)$ from $-\infty$ to $B(\pi(x_n), 1/n)$ and lift it to a line $\tilde{\Gamma}$ in $\mathbb{R}^2$ with image in $U^- = \pi^{-1}(\mathcal{U}^-(C_1))$. We have that $\tilde{\Gamma} \cap W_n = \emptyset$ (abusing notation by calling the line and its image with the same name), so $\mathrm{diam}(\mathrm{pr}_2(\tilde{\Gamma} \cap \mathcal{R})) > n/2$. Moreover $\tilde{\Gamma} \cap \mathcal{R}$ is in a different connected component of $\mathcal{R} \setminus \tilde{C}_1$ than $U^+ = \pi^{-1}(\mathcal{U}^+(C_1))$, in the space $\mathcal{R}$. This implies that some connected component of $\tilde{C}_1 \cap \mathcal{R}$ separates Γ from U^+ in $\mathcal{R}$. Such a connected component, contains a continuum L_n as claimed.

As the L_n constructed are in $\mathcal{R}$, the right region of $s \cup W_n \cup r$, we have the existence of a continuum $W_n' \subset W_n$, which is $n/6$ virtually to the left of L_n, with $\mathrm{diam}(\mathrm{pr}_2(W_n')) \to +\infty$: otherwise, we can construct another line Γ' joining $-\infty$ to L_n with $\Gamma' \cap (s \cup W_n \cup r) = \emptyset$, which contradicts $L_n \subset \mathcal{R}$.

Taking subsequences, the same argument that appears in the proof of Lemma 2.1 implies that $\lim_H \pi(W_n') = \mathcal{L}' \subset A$ and $\lim_H \pi(L_n) = \mathcal{L} \subset \mathcal{C}_1$ are both continua, with $\mathrm{Fill}(\mathcal{L})$, $\mathrm{Fill}(\mathcal{L}')$ contained in $\mathcal{C}_1$ and A respectively, being annular continua.

Thus, we can apply Lemma 2.3 and obtain $\mathrm{Fill}(\mathcal{L}') \preceq \mathrm{Fill}(\mathcal{L}) = \mathcal{C}_1$, which under the considered situation implies $\mathcal{C}_1 \subset A$, a contradiction. □

We call any region $\mathrm{cl}[\mathcal{U}^+(C_1) \cap \mathcal{U}^-(C_2)]$, where C_1, C_2 are as considered in the situation **(S)**, a bunch. An annular continuum A is strongly contained in a bunch $\mathcal{B} = \mathrm{cl}[\mathcal{U}^+(C_1) \cap \mathcal{U}^-(C_2)]$ if it is as in **(S)**.

COROLLARY 2.4. *Assume $\hat{f} \in \mathrm{homeo}_0(\mathbb{A})$ lifts a toral homeomorphisms in $f \in \mathrm{homeo}_0(\mathbb{T}^2)$. Further assume that $\mathcal{B}$ is a bunch, A is an annular continuum strongly contained in $\mathcal{B}$, and $Z \subset \mathbb{R}^2$ is a planar continuum with $\pi(Z) \subset A$ so that $f^n(\pi_{\mathbb{T}^2}(Z)) \subset p(A)$ for infinitely many positive integers n. Then, Z cannot contain two points having different rotation vectors for a planar lift F of f.*

Proof. Fix a non-empty connected component X of $\tilde{C}_1 \cap \tilde{C}_2$, where $\mathrm{cl}[\mathcal{U}^+(C_1) \cap \mathcal{U}^-(C_2)]$ and $\tilde{C}_1, \tilde{C}_2$ lifts $\mathcal{C}_1, \mathcal{C}_2$. Assume for a contradiction that Z contains points having different rotation vectors.

Then $\mathrm{pr}_2(F^n(Z)) \to +\infty$, so we have by Proposition 2.2 that the number of integer copies of X contained in $F^n(Z)$ must be unbounded in n. This is imposible for the lift F of a toral homeomorphism f and a planar continuum Z. □

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In view of this corollary, we now want the following result.

PROPOSITION 2.5. *Assume an annular continuum $A \subset \mathbb{A}$ is strongly contained in a bunch generated by the circloids $\mathcal{C}_1$ and $\mathcal{C}_2$. Then, if $z_1, z_2 \in A$ are any two points there exists a continuum $Z \subset \tilde{A}$ so that $\pi^{-1}(z_1) \cap Z \neq \emptyset$ and $\pi^{-1}(z_2) \cap Z \neq \emptyset$.*

Proof. Fix two lifts $z_1', z_2' \in \tilde{A}$ of z_1, z_2, respectively. We have that both connected components of $\tilde{A}$ containing z_1', z_2', respectively, are unbounded sets of $\mathbb{R}^2$, otherwise A fails to be an annular continua. Then there exists two sequences of planar continua $(Z_n^1)_{n\in\mathbb{N}}$ and $(Z_n^2)_{n\in\mathbb{N}}$ so that:

- $Z_n^i \subset \tilde{A}$ or all $i = 1, 2,\ n \in \mathbb{N}$;
- $z_1' \in Z_n^1$ and $z_2' \in Z_n^2$ for all $n \in \mathbb{N}$;
- $\mathrm{diam}(\mathrm{pr}_2)(Z_n^i) \to \infty$ for $i = 1, 2$.

If X is any connected component of $\mathcal{C}_1 \cap \mathcal{C}_2$ we have due to Proposition 2.2 that for some positive integer n_0 both numbers $\nu(X, Z_n^1)$ and $\nu(X, Z_n^2)$ are non-zero. As X must be contained in A we are done. □

3. *Proof of Theorem 1.1*

In light of the result [**KPS16**], which forbids the existence of an extension of an irrational rotation with a rotation set as in the statement of Theorem 1.1, in order to conclude is enough to prove the following intermediate result.

THEOREM 3.1. *Assume that for a lift F of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ we have $\rho(F) = \{\alpha\} \times [\rho^-, \rho^+]$, where $\rho^- < \rho^+$, $\alpha \in \mathbb{Q}^c$, and that f has the horizontal bounded deviation property. Then, some finite cover of f is an extension of an irrational rotation.*

Thus, by the mentioned result [**KPS16**], this cannot exist. Our goal now is to prove this last result.

We start by summarizing the constructions in [**Jäg09**, **JT17**]. Fix $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ so that for some lift F we have $\rho(F) = \{\alpha\} \times [\rho^-, \rho^+]$ with $\rho^- < \rho^+$, $\alpha \in \mathbb{Q}^c$, and that f has the horizontal bounded deviation property. In the mentioned article the authors find a family of circloids $\{\mathcal{C}_r\}_{r\in\mathbb{R}}$ of $\mathbb{A}$ having the following properties related to a finite cover of f, which we keep calling f (and $\hat{f} : \mathbb{A} \to \mathbb{A}$ to its lift):

(1) $\mathcal{C}_r \preceq \mathcal{C}_s$ whenever $r \leq s$;
(2) $\mathcal{C}_r \subset B(\pi(\{r\} \times \mathbb{R}), \kappa)$ for some uniform constant κ;
(3) $\hat{f}(\mathcal{C}_r) = \mathcal{C}_{r+\alpha}$;
(4) $\mathrm{p}(\mathcal{C}_r)$ is a circloid in $\mathbb{T}^2$ for all $r \in \mathbb{R}$;
(5) $f^n(\mathrm{p}(\mathcal{C}_r)) \neq \mathrm{p}(\mathcal{C}_r)$ for every $r \in \mathbb{R}$ and every positive integer n.

The key result in [**Jäg09**, **JT17**] (see also [**JP15**]), which allows the construction of a semiconjugacy between f and an irrational rotation of angle α, is the following.

THEOREM 3.2. *Assume that for some $r_0 \in \mathbb{R}$ we have that $f^n(p(\mathcal{C}_{r_0})) \cap f^m(p(\mathcal{C}_{r_0})) = \emptyset$ whenever $n \neq m$. Then f is an extension of an irrational rotation.*

Thus, in order to prove Theorem 1.1, it is enough to see that for some $r \in \mathbb{R}$ the circloid $\pi(\mathcal{C}_r)$ is free. We assume from now on that for some $r \in \mathbb{R}$ the circloid $\mathcal{C}_r$ is not free, and construct an absurd throughout this section.

As $\mathcal{C}_r$ is not free, we have that $\mathcal{C}_r \cap \mathcal{C}_s \neq \emptyset$ for some $s \in \mathbb{R}$. We assume $s > r$ (for the symmetric case the same proof works). Thus, due to properties 1 and 5, we have a bunch $\mathcal{B} = \mathrm{cl}[\mathcal{U}^+(\mathcal{C}_r) \cap \mathcal{U}^-(\mathcal{C}_s)]$.

Furthermore, due to property 3 we have for some n_1 and some n_2 that $f^{n_i}(\mathrm{p}(\mathcal{C}_r))$ is strongly contained in $\mathrm{p}(\mathcal{B})$ for $i = 1, 2$. This implies that we have for some $r' < s'$ the following

$$\mathcal{C}_r \preceq \mathcal{C}_{r'} \preceq \mathcal{C}_{s'} \preceq \mathcal{C}_s.$$

Let $\mathcal{B}' = \mathrm{cl}[\mathcal{U}^+(\mathcal{C}_{r'}) \cap \mathcal{U}^-(\mathcal{C}_{s'})]$. Then, due to property 5, $\mathcal{B}'$ is strongly contained in the bunch $\mathcal{B}$. Moreover, again due to property 3 and property 2, we have:

(i) $\mathrm{p}(\mathcal{B}'), \ldots, \mathrm{p}(\hat{f}^{j_0}(\mathcal{B}'))$ covers $\mathbb{T}^2$, for some $j_0 \in \mathbb{N}$;

(ii) $f^n(\mathrm{p}(\mathcal{B}'))$ is strongly contained in $\mathrm{p}(\mathcal{B})$ for every n contained in a syndetic set $\mathcal{I} \subset \mathbb{N}$.

Property (i) implies that we can find in any lift $\tilde{\mathcal{B}}'$ two points b^-, b^+ having rotation vectors (α, ρ^+) and (α, ρ^-), respectively. Furthermore, as $\mathcal{B}''$ is strongly contained in $\mathcal{B}$, Proposition 2.5 allows us to find a continuum $Z \subset \tilde{\mathcal{B}}''$ containing points in the equivalence class of b^- and of b^+. But this situation together with point (ii) yields a contradiction to Corollary 2.4.

Therefore, we obtain the desired absurd, which proves Theorem 3.1 and so Theorem 1.1.

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REFERENCES

[BdCH16] P. Boyland, A. de Carvalho and T. Hall. New rotation sets in a family of torus homeomorphisms. *Invent. Math.* **204**(3) (2016), 895–937.

[CT18] P. L. Calvez and F. A. Tal. Forcing theory for transverse trajectories of surface homeomorphisms. *Invent. Math.* **212** (2018), 619–729.

[Dáv18] P. Dávalos. On annular maps of the torus and sublinear diffusion. *J. Inst. Math. Jussieu* **17**(4) (2018), 913–978.

[FM90] J. Franks and M. Misiurewicz. Rotation sets of toral flows. *Proc. Amer. Math. Soc.* **109**(1) (1990), 243–249.

[Fra88] J. Franks. Recurrence and fixed points of surface homeomorphisms. *Ergod. Th. & Dynam. Sys.* **8*** (1988), 99–107 Charles Conley Memorial Issue.

[Fra89] J. Franks. Realizing rotation vectors for torus homeomorphisms. *Trans. Amer. Math. Soc.* **311**(1) (1989), 107–115.

[Jäg09] T. Jäger. Linearization of conservative toral homeomorphisms. *Invent. Math.* **176**(3) (2009), 601–616.

[JP15] T. Jäger and A. Passeggi. On torus homeomorphisms semiconjugate to irrational rotations. *Ergod. Th. & Dynam. Sys.* **35** (2015), 2114–2137.

[JT17] T. Jäger and F. A. Tal. Irrational rotation factors for conservative torus homeomorphisms. *Ergod. Th. & Dynam. Sys.* **37**(5) (2017), 1537–1546.

[Koc16] A. Kocsard. On the dynamics of minimal homeomorphisms of $\mathbb{T}^2$ which are not pseudo-rotations. *Preprint*, 2016, arXiv:1611.03784.

[KPS16] K. Passeggi and Sambarino. The Franks–Misiurewicz conjecture for extensions of irrational rotations. *Preprint*, 2016, arXiv:1611.05498.

[Kwa95] J. Kwapisz. A toral diffeomorphism with a nonpolygonal rotation set. *Nonlinearity* **8**(4) (1995), 461–476.

https://doi.org/10.1017/etds.2019.8 Published online by Cambridge University Press

[**LM91**] J. Llibre and R. S. MacKay. Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergod. Th. & Dynam. Sys.* **11**(1) (1991), 115–128.

[**MZ89**] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. Lond. Math. Soc. (2)* **40**(3) (1989), 490–506.

[**MZ91**] M. Misiurewicz and K. Ziemian. Rotation sets and ergodic measures for torus homeomorphisms. *Fund. Math.* **137**(1) (1991), 45–52.