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# Graph Subspaces and the Spectral Shift Function

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*Abstract.* We obtain a new representation for the solution to the operator Sylvester equation in the form of a Stieltjes operator integral. We also formulate new sufficient conditions for the strong solvability of the operator Riccati equation that ensures the existence of reducing graph subspaces for block operator matrices. Next, we extend the concept of the Lifshits-Krein spectral shift function associated with a pair of self-adjoint operators to the case of pairs of admissible operators that are similar to self-adjoint operators. Based on this new concept we express the spectral shift function arising in a perturbation problem for block operator matrices in terms of the angular operators associated with the corresponding perturbed and unperturbed eigenspaces.

## 1 Introduction

The spectral analysis of operator block matrices is an important issue in operator theory and mathematical physics. The search for invariant subspaces, the problem of block diagonalization, the analytic continuation of the compressed resolvents into unphysical sheets of the spectral parameter plane as well as the study of trace formulas attracted considerable attention in the past due to numerous applications to various problems of quantum mechanics, control theory, magnetohydrodynamics, and areas of mathematical physics (see [2], [3], [28], [32], [41], [42], [44], [48], [54], [56] and references cited therein).

In this work we restrict ourselves to the study of self-adjoint operator block matrices of the form

$$\mathbf{H} = \begin{pmatrix} A_0 & B_{01} \\ B_{10} & A_1 \end{pmatrix} \tag{1.1}$$

acting in the orthogonal sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of separable Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. The entries $A_i$, $i = 0, 1$, are assumed to be self-adjoint operators in $\mathcal{H}_i$ on domains $\mathrm{dom}(A_i)$. The off-diagonal elements $B_{ij}\colon \mathcal{H}_j \to \mathcal{H}_i$, $i = 0, 1$, $j = 1 - i$, $B_{01} = B_{10}^*$, are assumed to be bounded operators. Under these assumptions the matrix $\mathbf{H}$ is a self-adjoint operator in $\mathcal{H}$ on $\mathrm{dom}(\mathbf{H}) = \mathrm{dom}(A_0) \oplus \mathrm{dom}(A_1) = \mathrm{dom}(\mathbf{A})$ where $\mathbf{A} = \mathrm{diag}\{A_0, A_1\}$. We also use the notation

$$\mathbf{H} = \mathbf{A} + \mathbf{B} \quad \text{where} \quad \mathbf{B} = \begin{pmatrix} 0 & B_{01} \\ B_{10} & 0 \end{pmatrix}.$$

In the circle of ideas concerning the block diagonalization problem for block operator matrices (1.1) the existence of invariant graph subspaces plays a crucial role.

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Recall that a subspace $\mathcal{G}_i$, $i = 0$ or $i = 1$, is said to be a *graph subspace* of $\mathcal{H}$ associated with the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ if it is the graph of a (bounded) operator $Q_{ji}$, $j = 1 - i$, mapping $\mathcal{H}_i$ to $\mathcal{H}_j$.

The existence of a reducing graph subspace for a block operator matrix (1.1) is equivalent to the existence of a bounded off-diagonal strong solution $\mathbf{Q}$ to the operator Riccati equation

$$\mathbf{QA} - \mathbf{AQ} + \mathbf{QBQ} = \mathbf{B} \tag{1.2}$$

having the form

$$\mathbf{Q} = \begin{pmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{pmatrix}, \quad Q_{10} = -Q_{01}^*. \tag{1.3}$$

Following a tradition in operator theory we call $Q_{01}$ and $Q_{10}$ (and even the total block matrix $\mathbf{Q}$) the *angular* operators.

Given a strong solution (1.3) to the equation (1.2), the operator matrix $\mathbf{H} = \mathbf{A}+\mathbf{B}$ has invariant graph subspaces $\mathcal{G}_0 = \{x \in \mathcal{H} : P_{\mathcal{H}_1}x = Q_{10}P_{\mathcal{H}_0}x\}$ and $\mathcal{G}_1 = \{x \in \mathcal{H} : P_{\mathcal{H}_0}x = Q_{01}P_{\mathcal{H}_1}x\}$ where $P_{\mathcal{H}_i}$ denote the orthogonal projections in $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ onto the channel subspaces $\mathcal{H}_i$, $i = 0, 1$. As a consequence, $\mathbf{H}$ can be block diagonalized

$$(\mathbf{I}+\mathbf{Q})^{-1}\mathbf{H}(\mathbf{I}+\mathbf{Q}) = \mathbf{A} + \mathbf{BQ} = \begin{pmatrix} A_0 + B_{01}Q_{10} & 0 \\ 0 & A_1 + B_{10}Q_{01} \end{pmatrix}$$

by the similarity transformation generated by the operator $\mathbf{I}+\mathbf{Q}$. Under these circumstances the block-diagonalization problem for $\mathbf{H}$ by a unitary transformation admits an "explicit" solution,

$$\mathbf{U}^*\mathbf{HU} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}, \tag{1.4}$$

where $\mathbf{U}$ is the unitary operator from the polar decomposition $\mathbf{I}+\mathbf{Q} = \mathbf{U}\,|\mathbf{I}+\mathbf{Q}|$, and the diagonal entries $H_i$, $i = 0, 1$, are self-adjoint operators similar to $A_0 + B_{01}Q_{10}$ and $A_1 + B_{10}Q_{01}$, respectively.

Therefore, typical problems of *qualitative* perturbation theory, such as the existence of the invariant graph subspaces as well as a possibility of the block diagonalization, can be reduced to purely analytic questions concerning the solvability of related operator Riccati (and Sylvester) equations. Extensive bibliography is devoted to the subject. Not pretending to be complete we refer to [1], [2], [3], [4], [5], [6], [8], [9], [10], [18], [19], [20], [21], [47], [48], [52], [53], [54], [55], [56], [57], [63]. Notice that the Riccati equations with operator coefficients, often unbounded, are also a basic tool in the optimal control theory (see [15], [17], [32], [42], [65]) (however, the optimal control Riccati equations are usually associated with non-self-adjoint operator matrices of the form (1.1)).

An intriguing problem of *quantitative* perturbation theory is the study of the relationship between geometrical characteristics of rotations of the invariant subspaces and the accompanying shifts of the spectrum under a given perturbation. It is the development of the quantitative perturbation theory for self-adjoint block operator matrices that is the main goal of the present paper.

In this context, the most important numerical quantitative spectral characteristics is the Lifshits-Krein spectral shift function [36], [37], [38], [39], [40], [45], [46]. Detailed reviews of results on the spectral shift function and its applications were published by Birman and Yafaev [13], [14], [69] and by Birman and Pushnitskii [12]. For many more references the interested reader can consult [23], [24], [26], [27], [58], [60]. For recent results we refer to [25], [35], [59], [61], [62], and [64].

The spectral shift function $\xi(\lambda; \mathbf{H}, \mathbf{A})$ associated with the pair $(\mathbf{H}, \mathbf{A})$ of self-adjoint operators is usually introduced by the trace formula

$$\operatorname{tr}\big(\varphi(\mathbf{H}) - \varphi(\mathbf{A})\big) = \int_{\mathbb{R}} d\lambda \varphi'(\lambda)\xi(\lambda; \mathbf{H}, \mathbf{A}). \tag{1.5}$$

The trace formula (1.5) holds for a rather extensive class of functions $\varphi\colon \mathbb{R} \to \mathbb{C}$, including the class $C_0^\infty(\mathbb{R})$ of infinitely differentiable functions with a compact support, provided that the self-adjoint operators $\mathbf{H}$ and $\mathbf{A}$ are resolvent comparable, that is, the difference of their resolvents is a trace class operator.

In case of the block operator matrices the quantitative spectral analysis outlined above has a series of specific features. In particular, if the matrix $\mathbf{H}$ admits a block diagonalization as in (1.4), one might expect the validity of the following splitting representation for the spectral shift function

$$\xi(\lambda; \mathbf{H}, \mathbf{A}) = \xi(\lambda; H_0, A_0) + \xi(\lambda; H_1, A_1). \tag{1.6}$$

However, a certain difficulty in this way is that the spectral shift function associated with a pair of self-adjoint operators is not *stable* with respect to unitary transformations of its operator arguments. That is, if $\mathbf{U}$ is a unitary operator, the representation

$$\xi(\lambda; \mathbf{U}^*\mathbf{H}\mathbf{U}, \mathbf{A}) = \xi(\lambda; \mathbf{H}, \mathbf{A}) \tag{1.7}$$

fails to hold in general, even if both terms in (1.7) are well-defined (see Example 2).

One of the main goals of the present paper is to extend the concept of the spectral shift function to pairs of *admissible* (similar to self-adjoint) operators (see Definition 4.4) followed by the proof of the splitting formula (1.6) as well as the proof of its "non-self-adjoint" version

$$\xi(\lambda, \mathbf{H}, \mathbf{A}) = \xi(\lambda, A_0 + B_{01}Q_{10}, A_0) + \xi(\lambda, A_1 + B_{10}Q_{01}, A_1) \tag{1.8}$$

in the Hilbert-Schmidt class perturbation theory.

It is worth mentioning that the splitting formula (1.8) connects a purely spectral characteristics of the perturbation, the spectral shift function $\xi(\lambda, \mathbf{H}, \mathbf{A})$, with the geometry of mutual disposition of the invariant graph subspaces of the operator matrix $\mathbf{H}$ determined by the angular operator $\mathbf{Q}$ (provided that the reducing graph subspaces for $\mathbf{H}$ exist).

The plan of the paper is as follows.

In Section 2 we compare different representations for the solutions of the operator Sylvester equation (2.3) and obtain new representations for its strong solution based on the operator Stieltjes integrals approach. These are the representations (2.26) and (2.28).

In Section 3 we extend our key result of Section 2 (Theorem 2.14) to the case of the operator Riccati equation

$$QA - CQ + QBQ = D \tag{1.9}$$

with self-adjoint (possibly unbounded) $A$ and $C$ and bounded $B$ and $D$. One of our main results (see Theorem 3.6) provides a series of new sufficient conditions that imply the weak or strong solvability of (1.9). We prove, in particular, that if the operators $A$ and $C$ are bounded and

$$\sqrt{\|B\|\,\|D\|} < \frac{1}{\pi}\operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\},$$

then (1.9) has even an operator solution. This result is optimal in the following sense: in case where $D = B^*$ the best possible constant $c$ in the inequality

$$\|B\| < c\operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\}$$

that implies the solvability of (1.9) lies within the interval $[\frac{1}{\pi}, \sqrt{2}]$ (see Remark 3.11).

In Section 4 we introduce the concept of a spectral shift function for the pairs of admissible operators which are similar to self-adjoint (see Definitions 4.4 and 4.7). We relate our general concept of the spectral shift function associated with pairs of operators similar to self-adjoint to the one based on the perturbation determinant approach originally suggested by Adamjan and Langer in the case of trace class perturbations [1].

In Section 5 we discuss invariant graph subspaces for block operator matrices and link their existence with the existence of strong solutions to the corresponding Riccati equations (Lemma 5.3 and Theorem 6.1).

In Section 6, under rather general assumptions we prove the splitting formulas (1.6) and (1.8) (Theorem 6.1).

Section 7 is devoted to a detailed study of the case where the spectra of the diagonal entries $A_0$ and $A_1$ of the operator matrix $\mathbf{H}$ are separated. Based on the results of Section 3 we prove one of the central results of the present paper (Theorem 7.13 and Corollary 7.15) concerning the validity of the splitting formulas (1.6), (1.8) in case of Hilbert-Schmidt perturbations $\mathbf{B}$: if the perturbation $\mathbf{B}$ is sufficiently small in a certain sense (see Hypotheses 7.1 and 7.2) and the operators $\mathbf{H} = \mathbf{A} + \mathbf{B}$ and $\mathbf{A}$ are resolvent comparable, then

(i) the splitting formulas (1.6) and (1.8) hold;

(ii) the following equalities are valid

$$\xi(\lambda; H_0, A_0) = \xi(\lambda; A_0 + B_{01}Q_{10}, A_0) = 0, \quad \text{for a.e. } \lambda \in \operatorname{spec}(A_0)$$

$$\xi(\lambda; H_1, A_1) = \xi(\lambda; A_1 + B_{10}Q_{10}, A_1) = 0, \quad \text{for a.e. } \lambda \in \operatorname{spec}(A_1).$$

## 2 Sylvester Equation

The principal purpose of this section is to introduce a new representation for the solution $X$ of the operator Sylvester equation

$$XA - CX = Y.$$

We also discuss and compare the known representation theorems for solution $X$. For a detail exposition and introduction to the subject we refer to the papers [9], [10], [21], [47], [57], [63] and references therein.

In the following $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denotes the Banach space of linear bounded operators between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. By $\mathcal{B}_p(\mathcal{H}, \mathcal{K})$, $p \geq 1$, we understand the standard Schatten-von Neumann ideals of $\mathcal{B}(\mathcal{H}, \mathcal{K})$. For $\mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\mathcal{B}_p(\mathcal{H}, \mathcal{H})$ we use the corresponding shorter notation $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_p(\mathcal{H})$. The $\mathcal{B}_p(\mathcal{H}, \mathcal{K})$-norm of a bounded operator $T$ acting from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\|T\|_p$.

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, recall the concept of symmetric normed ideals of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, following [29].

***Definition 2.1*** A two-sided ideal $\mathcal{S} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called a *symmetric normed ideal* of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ if it is closed with respect to a norm $|||\cdot|||$ on $\mathcal{S}$ which has the following properties:

(i) if $T \in \mathcal{S}$, $K \in \mathcal{B}(K)$, $H \in \mathcal{B}(\mathcal{H})$, then $KTH \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $|||KTH||| \leq \|K\|\,|||T|||\,\|H\|$;

(ii) if $T$ is rank one then $|||T||| = \|T\|$.

For technical reasons we also assume that

(iii) *if* $T_n \in \mathcal{S}$ *with* $\sup_n |||T_n||| < \infty$, *and if* $T_n \to A$ *in the weak operator topology, then* $A \in \mathcal{S}$ *and* $|||A||| \leq \sup_n |||T_n|||$.

Recall that if $\mathcal{K} = \mathcal{H}$ then for any symmetric normed ideal $\mathcal{S}$ possessing the properties (i)–(iii) and being different from $\mathcal{B}(\mathcal{H})$, the following holds true:

$$\mathcal{B}_1(\mathcal{H}) \subset \mathcal{S} \subset \mathcal{B}_\infty(\mathcal{H}).$$

The symmetric norm on $\mathcal{B}_\infty(\mathcal{H})$ coincides with the operator norm in $\mathcal{B}(\mathcal{H})$.

Following [52], we recall the concept of a norm with respect to the spectral measure of a self-adjoint operator.

***Definition 2.2*** Let $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a bounded operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ and let $\{E_C(\lambda)\}$ be the spectral family of a self-adjoint (not necessarily bounded) operator $C$ acting in the Hilbert space $\mathcal{K}$. Introduce

$$\|Y\|_{E_C} = \Big(\sup_{\{\delta_k\}} \sum_k \|E_C(\delta_k)Y\|^2\Big)^{1/2}, \tag{2.1}$$

where the supremum is taken over a finite (or countable) system of mutually disjoint Borel subsets $\{\delta_k\}$, $\delta_k \cap \delta_l = \varnothing$, if $k \neq l$. The number $\|Y\|_{E_C}$ is called the *$E_C$-norm of the operator $Y$*. For $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ the $E_C$-norm $\|Z\|_{E_C}$ is defined as $\|Z\|_{E_C} = \|Z^*\|_{E_C}$.

One easily checks that if the norm $\|Y\|_{E_C}$ is finite one has

$$\|Y\| \leq \|Y\|_{E_C}.$$

If, in addition, $Y$ is a Hilbert-Schmidt operator, then

$$\|Y\|_{E_C} \leq \|Y\|_2, \quad Y \in \mathcal{B}_2(\mathcal{H}, \mathcal{K}), \tag{2.2}$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm in $\mathcal{B}_2(\mathcal{H}, \mathcal{K})$.

***Definition 2.3*** Let $A$ and $C$ be densely defined possibly unbounded closed operators in the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. A bounded operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be a *weak solution* of the Sylvester equation

$$XA - CX = Y, \quad Y \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \tag{2.3}$$

if

$$\langle XAf, g\rangle - \langle Xf, C^*g\rangle = \langle Yf, g\rangle \quad \text{for all } f \in \mathrm{dom}(A) \text{ and } g \in \mathrm{dom}(C^*). \tag{2.4}$$

A bounded operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be a *strong solution* of the Sylvester equation (2.3) if

$$\mathrm{ran}\,(X|_{\mathrm{dom}(A)}) \subset \mathrm{dom}(C), \tag{2.5}$$

and

$$XAf - CXf = Yf \quad \text{for all } f \in \mathrm{dom}(A). \tag{2.6}$$

Finally, a bounded operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be an *operator solution* of the Sylvester equation (2.3) if

$$\mathrm{ran}\,(X) \subset \mathrm{dom}(C),$$

the operator $XA$ is bounded on $\mathrm{dom}(XA) = \mathrm{dom}(A)$, and the equality

$$\overline{XA} - CX = Y \tag{2.7}$$

holds as an operator equality, where $\overline{XA}$ denotes the closure of $XA$.

Along with the Sylvester equation (2.3) we also introduce the dual equation

$$ZC^* - A^*Z = Y^*, \tag{2.8}$$

for which the notion of weak, strong, and operator solutions is defined in a way analogous to that in Definition 2.3.

It is easy to see that if one of the equations (2.3) or (2.8) has a weak solution then so does the other one.

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***Lemma 2.4*** *Let $A$ and $C$ be densely defined possibly unbounded closed operators in the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a weak solution to the Sylvester equation (2.3) if and only if the operator $Z = -X^*$ is a weak solution to the dual Sylvester equation (2.8).*

**Proof** According to Definition 2.3 an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a weak solution to (2.3) if (2.4) holds. Meanwhile, (2.4) implies

$$-\langle X^*C^*g, f\rangle + \langle X^*g, A^*f\rangle = \langle Y^*g, f\rangle \quad \text{for all } g \in \operatorname{dom}(C^*) \text{ and } f \in \operatorname{dom}(A).$$

Thus, by Definition 2.3 the operator $Z = -X^*$ is a weak solution to the dual Sylvester equation (2.8). The converse statement is proven in a similar way. ■

The following result, first proven by M. Krein in 1948, gives an "explicit" representation for a unique solution of the Sylvester equation $XA - CX = Y$, provided that the spectra of the operators $A$ and $C$ are disjoint and one of them is a bounded operator. (Later, this result was independently obtained by Y. Daleckii [18] and M. Rosenblum [63]).

***Lemma 2.5*** *Let $A$ be a possibly unbounded densely defined closed operator in the Hilbert space $\mathcal{H}$ and $C$ a bounded operator in the Hilbert space $\mathcal{K}$ such that*

$$\operatorname{spec}(A) \cap \operatorname{spec}(C) = \varnothing$$

*and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then the Sylvester equation (2.3) has a unique operator solution*

$$X = \frac{1}{2\pi \mathrm{i}} \int_\gamma d\zeta (C - \zeta)^{-1} Y (A - \zeta)^{-1}, \tag{2.9}$$

*where $\gamma$ is a union of closed contours in the complex plane with total winding numbers* 0 *around* $\operatorname{spec}(A)$ *and* 1 *around* $\operatorname{spec}(C)$ *and the integral converges in the norm operator topology. Moreover, if $Y \in \mathcal{S}$ for some symmetric ideal $\mathcal{S} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ with the norm $|||\cdot|||$, then $X \in \mathcal{S}$ and*

$$|||X||| \le (2\pi)^{-1} |\gamma| \sup_{\zeta \in \gamma} \|(C - \zeta)^{-1}\| \, \|(A - \zeta)^{-1}\| \, |||Y|||,$$

*where $|\gamma|$ denotes the length of the contour $\gamma$.*

If $A$ and $C$ are unbounded densely defined closed operators, even with separated spectra, that is, $\operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\} > 0$, then the Sylvester equation (2.3) may not have bounded weak solutions (see [57] for a counterexample). Nevertheless, under some additional assumptions equation (2.3) is still weakly solvable.

The next statement is a generalization of Lemma 2.5 to the case of unbounded operators, a result first proven by Heinz [30].

***Lemma 2.6*** *Let $A - \frac{d}{2}I$ and $-C - \frac{d}{2}I$, $d > 0$, be maximal accretive operators in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then the Sylvester equation (2.3) has a unique weak solution*

$$X = \int_0^{+\infty} dt \mathrm{e}^{Ct} Y \mathrm{e}^{-At}, \tag{2.10}$$

*where the integral is understood in the weak operator topology. Moreover, if $Y \in \mathcal{S}$ for some symmetric ideal $\mathcal{S} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ with the norm $|||\cdot|||$, then $X \in \mathcal{S}$ and*

$$|||X||| \leq \frac{1}{d} |||Y|||.$$

If both $A$ and $C$ are self-adjoint operators with separated spectra one still has a statement regarding the existence and uniqueness of a weak solution with no additional assumptions.

***Theorem 2.7*** *Let $A$ and $C$ be self-adjoint operators in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and*

$$d = \operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\} > 0. \tag{2.11}$$

*Then the Sylvester equation (2.3) has a unique weak solution*

$$X = \int_{-\infty}^{\infty} e^{\mathrm{i}tC} Y e^{-\mathrm{i}tA} f_d(t)\, dt, \tag{2.12}$$

*where the integral is understood in the weak operator topology. Here $f_d$ denotes any function in $L^1(\mathbb{R})$, continuous except at zero, such that*

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}sx} f_d(s)\, ds = \frac{1}{x} \quad \text{whenever } |x| \geq \frac{1}{d}. \tag{2.13}$$

*Moreover, if $Y \in \mathcal{S}$ for some ideal $\mathcal{S} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ with a symmetric norm $|||\cdot|||$, then $X \in \mathcal{S}$ and*

$$|||X||| \leq \frac{c}{d} |||Y|||, \tag{2.14}$$

*where*

$$c = \frac{\pi}{2} \tag{2.15}$$

*and estimate (2.14) with the constant $c$ given by (2.15) is sharp. In particular, the estimate (2.14), (2.15) holds for any $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, that is,*

$$\|X\| \leq \frac{\pi}{2d} \|Y\|. \tag{2.16}$$

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***Remark 2.8*** Theorem 2.7 with the following bounds for the best possible constant $c$ in (2.14)

$$\sqrt{\frac{3}{2}} \leq c \leq 2 \tag{2.17}$$

has been proven in [9]. From [9] one can also learn that the best possible constant in (2.14) admits the following estimate from above

$$c \leq \inf\Big\{ \|f\|_{L^1(\mathbb{R})} : f \in L^1(\mathbb{R}),\, \hat{f}(x) = \frac{1}{x}, |x| \geq 1 \Big\}, \tag{2.18}$$

where

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-\mathrm{i}sx} f(s)\, ds, \quad x \in \mathbb{R}.$$

The fact that the infimum in (2.18) equals $\pi/2$ goes back to B. Sz.-Nagy and A. Strausz (*cf.* [66]). The proof of the fact that the value $c = \pi/2$ is sharp is due to R. McEachin [51].

The discussion of existence of strong solutions to the Sylvester equation needs some technical tools from the Stieltjes theory of integration. We briefly recall the main concepts and results of this theory (see [3], [7], [52], and references therein).

***Definition 2.9*** Let $[a, b) \subset \mathbb{R}$, $-\infty < a < b < +\infty$. Assume that $C$ is a self-adjoint possibly unbounded operator in $\mathcal{K}$ and $\{E_C(\mu)\}_{\mu\in\mathbb{R}}$ is its spectral family.

The operator-valued function

$$F\colon [a, b) \to \mathcal{B}(\mathcal{K}, \mathcal{H})$$

is said to be *uniformly* (resp. strongly, weakly) *integrable* from the right over the spectral measure $dE_C(\mu)$ on $[a, b)$ if the limit

$$\int_a^b F(\mu)\, dE_C(\mu) = \lim_{\max_{k=1}^n |\delta_k^{(n)}| \to 0} \sum_{k=1}^n F(\zeta_k) E_C(\delta_k^{(n)}) \tag{2.19}$$

exists in the uniform (resp. strong, weak) operator topology. Here, $\delta_k^{(n)} = [\mu_{k-1}, \mu_k)$ and $|\delta_k^{(n)}| = \mu_k - \mu_{k-1}$, $k = 1, 2, \dots, n$, where $a = \mu_0 < \mu_1 < \cdots < \mu_n = b$ is a partition of the interval $[a, b)$, and $\zeta_k \in \delta_k^{(n)}$. The limit value (2.19), if it exists, is called the *right Stieltjes integral* of the operator-valued function $F$ over the measure $dE_C(\mu)$ on $[a, b)$.

Similarly, the function

$$G\colon [a, b) \to \mathcal{B}(\mathcal{H}, \mathcal{K})$$

is said to be *uniformly* (resp. strongly, weakly) *integrable* from the left over the measure $dE_C(\mu)$ on $[a, b)$, if there exists the limit

$$\int_a^b dE_C(\mu) G(\mu) = \lim_{\max\limits_{k=1}^{n} |\delta_k^{(n)}| \to 0} \sum_{k=1}^n E_C(\delta_k^{(n)}) G(\zeta_k) \tag{2.20}$$

in the uniform (resp. strong, weak) operator topology. The corresponding limit value (2.20), if it exists, is called the *left Stieltjes integral* of the operator-valued function $G$ over the measure $dE_C(\mu)$ on $[a, b)$.

***Lemma 2.10*** ([52], Lemma 10.5) *An operator-valued function $F(\mu)$,*

$$F\colon [a, b) \to \mathcal{B}(\mathcal{K}, \mathcal{H}),$$

*is integrable in the weak (uniform) operator topology over the measure $dE_C(\mu)$ on $[a, b)$ from the left if and only if the function $[F(\mu)]^*$ is integrable in the weak (uniform) operator topology over the measure $dE_C(\mu)$ on $[a, b)$ from the right and then*

$$\left[\int_a^b F(\mu)\, dE_C(\mu)\right]^* = \int_a^b dE_C(\mu)[F(\mu)]^*. \tag{2.21}$$

***Remark 2.11*** In general, the convergence of one of the integrals (2.21) in the strong operator topology only implies the convergence of the other one in the weak operator topology.

Some sufficient conditions for the integrability of an operator-valued function $F(\mu)$ over a finite interval in the uniform operator topology are available. For instance, we have the following statement.

***Lemma 2.12*** ([3], Lemma 7.2 and Remark 7.3) *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $C$ be a self-adjoint operator in $\mathcal{K}$. Assume that the operator-valued function $F$, $F\colon [a, b) \to \mathcal{B}(\mathcal{K}, \mathcal{H})$, satisfies the Lipschitz condition*

$$\|F(\mu_2) - F(\mu_1)\| \le c|\mu_2 - \mu_1| \quad \text{for any } \mu_1, \mu_2 \in [a, b) \tag{2.22}$$

*for some constant $c > 0$. Then the operator-valued function $F$ is right-integrable on $[a, b)$ with respect to the spectral measure $dE_C(\mu)$ in the sense of the uniform operator topology.*

The improper weak, strong, or uniform right (left) integrals $\int_a^b F(\mu)\, dE_C(\mu)$ $\bigl(\int_a^b dE_C(\mu)G(\mu)\bigr)$ with infinite lower and/or upper bounds ($a = -\infty$ and/or $b = +\infty$) are understood as the limits, if they exist, of the integrals over finite intervals in the corresponding topologies. For example,

$$\int_{-\infty}^{\infty} dE_C(\mu)G(\mu) = \lim_{a\downarrow-\infty b\uparrow\infty} \int_a^b dE_C(\mu)G(\mu).$$

We also use the notations

$$\int_{\operatorname{spec}(C)} dE_C(\mu)G(\mu) = \int_{-\infty}^{+\infty} dE_C(\mu)G(\mu)$$

and

$$\int_{\operatorname{spec}(C)} F(\mu)\, dE_C(\mu) = \int_{-\infty}^{+\infty} F(\mu)\, dE_C(\mu).$$

***Lemma 2.13*** ([52], Lemma 10.7) *Let an operator-valued function* $F\colon \operatorname{spec}(C) \to \mathcal{B}(\mathcal{H})$ *be bounded*

$$\|F\|_\infty = \sup_{\mu\in\operatorname{spec}(C)} \|F(\mu)\| < \infty,$$

*and admit a bounded extension from* $\operatorname{spec}(C)$ *to the whole real axis* $\mathbb{R}$ *which satisfies the Lipschitz condition (2.22). If the* $E_C$*-norm* $\|Y\|_{E_C}$ *of the operator* $Y \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *is finite, then the integrals*

$$\int_{\operatorname{spec}(C)} dE_C(\mu) Y F(\mu) \quad \text{and} \quad \int_{\operatorname{spec}(C)} F(\mu) Y^*\, dE_C(\mu)$$

*exist in the uniform operator topology. Moreover, the following bounds hold*

$$\Big\| \int_{\operatorname{spec}(C)} dE_C(\mu) Y F(\mu) \Big\| \le \|Y\|_{E_C} \cdot \|F\|_\infty, \tag{2.23}$$

$$\Big\| \int_{\operatorname{spec}(C)} F(\mu) Y^*\, dE_C(\mu) \Big\| \le \|Y\|_{E_C} \cdot \|F\|_\infty. \tag{2.24}$$

Now we are ready to state the key result of this section: if either $A$ or $C$ is self-adjoint, then a strong solution to the Sylvester equation, if it exists, can be represented in the form of an operator Stieltjes integral.

***Theorem 2.14*** *Let $A$ be a possibly unbounded densely defined closed operator in the Hilbert space $\mathcal{H}$ and $C$ a self-adjoint operator in the Hilbert space $\mathcal{K}$. Let $Y \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and suppose that $A$ and $C$ have separated spectra, i.e.,*

$$\operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\} > 0. \tag{2.25}$$

*Then the following statements are valid.*

(i) *Assume that the Sylvester equation (2.3) has a strong solution $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then $X$ is a unique strong solution to (2.3) and it can be represented in the form of the Stieltjes integral*

$$X = \int_{\operatorname{spec}(C)} E_C(d\mu) Y (A-\mu)^{-1}, \tag{2.26}$$

*which converges in the sense of the strong operator topology in $\mathcal{B}(\mathcal{H},\mathcal{K})$.*
*Conversely, if the Stieltjes integral (2.26) converges in the strong operator topology, then $X$ given by (2.26) is a strong solution to (2.3).*

(ii) *Assume that the dual Sylvester equation*

$$ZC - A^* Z = Y^* \tag{2.27}$$

*has a strong solution $Z \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Then $Z$ is a unique strong solution to (2.27) and it can be represented in the form of the Stieltjes operator integral*

$$Z = -\int_{\operatorname{spec}(C)} (A^*-\mu)^{-1} Y^* E_C(d\mu), \tag{2.28}$$

*which converges in the sense of the strong operator topology in $\mathcal{B}(\mathcal{K},\mathcal{H})$.*
*Conversely, if the operator Stieltjes integral in (2.28) converges in the strong operator topology, then $Z$ given by (2.28) is a strong solution to (2.27).*

**Proof** (i) Assume that the Sylvester equation (2.3) has a strong solution $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$, that is, (2.5) and (2.6) hold. Let $\delta$ be a finite interval such that $\delta \cap \operatorname{spec}(C) \neq \varnothing$ and $\mu_\delta \in \delta \cap \operatorname{spec}(C)$. Applying to both sides of (2.6) the spectral projection $E_C(\delta)$, a short computation yields

$$E_C(\delta)XAf - \mu_\delta E_C(\delta)Xf = E_C(\delta)Yf + E_C(\delta)(C-\mu_\delta)Xf \tag{2.29}$$

for any $f \in \operatorname{dom}(A)$. Since $\mu_\delta \in \delta \cap \operatorname{spec}(C)$, by (2.25) one concludes that $\mu_\delta$ belongs to the resolvent set of the operator $A$. Hence, (2.29) implies

$$E_C(\delta)X = E_C(\delta)Y(A-\mu_\delta)^{-1} + (C-\mu_\delta)E_C(\delta)X(A-\mu_\delta)^{-1}. \tag{2.30}$$

Next, let $[a,b)$ be a finite interval and $\{\delta_k\}$ a finite system of mutually disjoint intervals such that $[a,b) = \cup_k \delta_k$. For those $k$ such that $\delta_k \cap \operatorname{spec}(C) \neq \varnothing$ pick a point $\mu_{\delta_k} \in \delta_k \cap \operatorname{spec}(C)$. Using (2.30) one obtains

$$\begin{aligned}\sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing} E_C(\delta_k)X = &\sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing} E_C(\delta_k)Y(A-\mu_{\delta_k})^{-1}\\ &+\sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing}(C-\mu_{\delta_k})E_C(\delta_k)X(A-\mu_{\delta_k})^{-1}.\end{aligned} \tag{2.31}$$

The left hand side of (2.31) can be computed explicitly:

$$\sum_{\delta_k\cap\operatorname{spec}(C)\neq\varnothing} E_C(\delta_k)X = E_C\big([a,b)\cap\operatorname{spec}(C)\big)X = E_C\big([a,b)\big)X. \tag{2.32}$$

The first term on the r.h.s. of (2.31) is the integral sum for the Stieltjes integral (2.26). More precisely, since $(A-\mu)^{-1}$ is analytic in a complex neighborhood of $[a,b]\cap\operatorname{spec}(C)$, by Lemma 2.12 one infers

$$\underset{\max_k|\delta_k|\to 0}{n\text{-}\lim}\sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing} E_C(\delta_k)Y(A-\mu_{\delta_k})^{-1} = \int_{[a,b)\cap\operatorname{spec}(C)} E_C(d\mu)Y(A-\mu)^{-1}. \tag{2.33}$$

The last term on the right hand side of (2.31) vanishes

$$\underset{\max_k|\delta_k|\to 0}{n\text{-}\lim}\sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing}(C-\mu_{\delta_k})E_C(\delta_k)X(A-\mu_{\delta_k})^{-1} = 0. \tag{2.34}$$

This can be seen as follows. For any $f \in \mathcal{H}$ we have the estimate

$$
\begin{aligned}
\Big\| \sum_{\delta_k \cap \operatorname{spec}(C) \neq \varnothing} & (C - \mu_{\delta_k}) E_C(\delta_k) X (A - \mu_{\delta_k})^{-1} f \Big\|^2 \\
&= \Big\langle \sum_{k:\delta_k \cap \operatorname{spec}(C) \neq \varnothing} (A^* - \mu_{\delta_k})^{-1} X^* (C - \mu_{\delta_k})^2 E_C(\delta_k) X (A - \mu_{\delta_k})^{-1} f,\ f \Big\rangle \\
&\le \sum_{\delta_k \cap \operatorname{spec}(C) \neq \varnothing} |\delta_k|^2 \|X\|^2 \|(A - \mu_{\delta_k})^{-1}\|^2 \|f\|^2 \\
&\le (b-a) \|X\|^2 \|f\|^2 \max_k |\delta_k| \sup_{\mu \in [a,b) \cap \operatorname{spec}(C)} \|(A-\mu)^{-1}\|^2 .
\end{aligned}
$$

Here we have used the estimate

$$
\|(C - \mu_{\delta_k})^2 E_C(\delta_k)\| = \Big\| \int_{\delta_k} (\mu - \mu_{\delta_k})^2 E_C(d\mu) \Big\| \le \sup_{\mu \in \delta_k} (\mu - \mu_{\delta_k})^2 \le |\delta_k|^2 .
$$

Passing to the limit $\max_k |\delta_k| \to 0$ in (2.31), by (2.32)–(2.34) one concludes that for any finite interval $[a,b)$

$$
E_C\big([a,b)\big) X = \int_{[a,b) \cap \operatorname{spec}(C)} E_C(d\mu) Y (A - \mu)^{-1} . \tag{2.35}
$$

Since

$$
\operatorname*{s-lim}_{\substack{a \to -\infty \\ b \to +\infty}} E_C\big([a,b)\big) X = X,
$$

(2.35) implies (2.26), which, in particular, proves the uniqueness of a strong solution to the Riccati equation (2.3).

In order to prove the converse statement of (i), assume that the Stieltjes integral on the r.h.s. part of (2.35) converges as $a \to -\infty$ and $b \to +\infty$ in the strong operator topology. Denote the resulting integral by $X$. Then, (2.35) holds for any finite $a$ and $b$. This implies that for any $f \in \operatorname{dom}(A)$ we have

$$
\begin{aligned}
C E_C\big([a,b)\big) X f - E_C\big([a,b)\big) X A f &= \int_{[a,b) \cap \operatorname{spec}(C)} E_C(d\mu) Y (A - \mu)^{-1} (\mu - A) f \\
&= - \int_{[a,b) \cap \operatorname{spec}(C)} E_C(d\mu) Y f = - E_C\big([a,b)\big) Y f .
\end{aligned}
$$

Hence,

$$
C E_C\big([a,b)\big) X f = E_C\big([a,b)\big) X A f - E_C\big([a,b)\big) Y f \quad \text{for any } f \in \operatorname{dom}(A) \tag{2.36}
$$

and $C E_C\big([a,b)\big) X f$ converges to $XAf + Yf$ as $a \to -\infty$ and $b \to +\infty$. Therefore,

$$
\sup_{a<b} \Big\| C E_C\big([a,b)\big) X f \Big\|^2 = \sup_{a<b} \int_{[a,b) \cap \operatorname{spec}(C)} \mu^2 \, d\langle E_C X f, X f \rangle < \infty
$$

and, hence,

$$Xf \in \operatorname{dom}(C). \tag{2.37}$$

Then (2.36) can be rewritten in the form

$$E_C\big([a,b)\big)CXf = E_C\big([a,b)\big)XAf - E_C\big([a,b)\big)Yf, \quad a<b. \tag{2.38}$$

Combining (2.37) and (2.38) proves that $X$ is a strong solution to the Sylvester equation (2.3).

(ii) Assume that the dual Sylvester equation (2.8) has a strong solution $Z \in \mathcal{B}(\mathcal{K},\mathcal{H})$. As in the proof of (i), choose a finite interval $\delta \subset \mathbb{R}$ such that $\delta \cap \operatorname{spec}(C) \neq \varnothing$. Since $E_C(\delta)\mathcal{K} \subset \operatorname{dom}(C)$, we have $ZE_C(\delta)f \in \operatorname{dom}(A^*)$ for any $f \in \mathcal{K}$ by the definition of a strong solution. Take a point $\mu_\delta \in \delta \cap \operatorname{spec}(C)$. It follows from (2.25) that $\mu_\delta \notin \operatorname{spec}(A^*)$. As in the proof of (i), it is easy to check the validity of the representation

$$ZE_C(\delta)f = -(A^*-\mu_\delta)^{-1}Y^*E_C(\delta)f - (A^*-\mu_\delta)^{-1}Z(C-\mu_\delta)E_C(\delta)f, \tag{2.39}$$

which holds for all $f \in \mathcal{K}$.

Next, let $[a,b)$ be a finite interval and $\{\delta_k\}$ a finite system of mutually disjoint intervals such that $[a,b) = \cup_k \delta_k$. For those $k$ such that $\delta_k \cap \operatorname{spec}(C) \neq \varnothing$ pick a point $\mu_{\delta_k} \in \delta_k \cap \operatorname{spec}(C)$. Using (2.39) one then finds that

$$\begin{aligned} ZE_C\big([a,b)\big)f = &-\sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing} (A^*-\mu_{\delta_k})^{-1}Y^*E_C(\delta_k)f \\ &-\sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing} (A-\mu_{\delta_k})^{-1}Z(C-\mu_{\delta_k})E_C(\delta_k)f. \end{aligned} \tag{2.40}$$

The equality (2.34) implies

$$\underset{\max_k |\delta_k|\to 0}{n\text{-}\lim} \sum_{k:\delta_k\cap\operatorname{spec}(C)\neq\varnothing} (A^*-\mu_{\delta_k})^{-1}Z(C-\mu_{\delta_k})E_C(\delta_k) = 0. \tag{2.41}$$

Thus, passing in (2.40) to the limit as $\max_k|\delta_k| \to 0$ one infers that

$$-\int_{[a,b)\cap\operatorname{spec}(C)} (A^*-\mu)^{-1}Y^*E_C(d\mu)f = ZE_C\big([a,b)\big)f. \tag{2.42}$$

Since for any $f \in \mathcal{K}$

$$\lim_{\substack{a\to-\infty\\ b\to+\infty}} ZE_C\big([a,b)\big)f = Z,$$

one concludes that the integral on the r.h.s. part of (2.28) converges as $a \to -\infty$ and $b \to +\infty$ in the strong operator topology and (2.28) holds, which gives a unique strong solution to the dual Sylvester equation (2.27).

In order to prove the converse statement of (ii), assume that there exists the strong operator limit

$$Z = \operatorname*{s\text{-lim}}_{\substack{a\to-\infty\\ b\to+\infty}} \int_{[a,b)\cap\operatorname{spec}(C)} (A^* - \mu)^{-1} Y^* E_C(d\mu), \quad Z \in \mathcal{B}(\mathcal{K}, \mathcal{H}). \tag{2.43}$$

For any finite $a$ and $b$ such that $a < b$ we have

$$ZE_C([a,b) = -\int_{\operatorname{spec}(C)\cap[a,b)} (A^* - \mu)^{-1} Y^* E_C(d\mu). \tag{2.44}$$

By (2.25) any point $\zeta \in \operatorname{spec}(C)$ belongs to the resolvent set of the operator $A$ and, hence, to the one of $A^*$. Picking such a $\zeta$, $\zeta \in \operatorname{spec}(C)$, the operator (2.44) can be split into two parts

$$ZE_C\big([a,b)\big) = J_1(a,b) + J_2(a,b), \tag{2.45}$$

where

$$J_1(a,b) = -(A^* - \zeta)^{-1} Y^* E_C\big([a,b)\big), \tag{2.46}$$

$$J_2(a,b) = +(A^* - \zeta)^{-1} \int_{\operatorname{spec}(C)\cap[a,b)} (\zeta - \mu)(A^* - \mu)^{-1} Y^* E_C(d\mu). \tag{2.47}$$

Using the functional calculus for the self-adjoint operator $C$ one obtains

$$J_2(a,b)f = -(A^* - \zeta)^{-1} \Big( \int_{\operatorname{spec}(C)\cap[a,b)} (A^* - \mu)^{-1} Y^* E_C(d\mu) \Big) (C - \zeta) f, \qquad \text{for } f \in \operatorname{dom}(C).$$

Thus, for $f \in \operatorname{dom}(C)$ one concludes that

$$\begin{aligned}
Zf &= \lim_{\substack{a\to-\infty\\ b\to+\infty}} ZE_C\big([a,b)\big) f \\
&= \lim_{\substack{a\to-\infty\\ b\to+\infty}} J_1(a,b)f + \lim_{\substack{a\to-\infty\\ b\to+\infty}} J_2(a,b)f \\
&= -(A^* - \zeta)^{-1} Y^* f \\
&\qquad - (A^* - \zeta)^{-1} \Big( \int_{\operatorname{spec}(C)} (A^* - \mu)^{-1} Y^* E_C(d\mu) \Big) (C - \zeta) f.
\end{aligned}$$

That is,

$$Zf = -(A^* - \zeta)^{-1} Y^* f + (A^* - \zeta)^{-1} Z (C - \zeta) f, \quad f \in \operatorname{dom}(C), \tag{2.48}$$

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since

$$(2.49)\quad \int_{\operatorname{spec}(C)} (A^*-\mu)^{-1}Y^*E_C(d\mu) = \operatorname*{s\text{-}lim}_{\substack{a\to-\infty\\ b\to+\infty}} \int_{\operatorname{spec}(C)\cup[a,b)} (A^*-\mu)^{-1}Y^*E_C(d\mu) = Z$$

by (2.43). It follows from (2.48) that $Zf \in \operatorname{dom}(A^*)$ for any $f \in \operatorname{dom}(C)$ and, thus,

$$(2.50)\qquad \operatorname{ran}(Z|_{\operatorname{dom}(C)}) \subset \operatorname{dom}(A^*).$$

Applying $A^* - \zeta$ to the both sides of the resulting equality (2.48) one infers that $Z$ is a strong solution to the dual Sylvester equation (2.27) which completes the proof. ■

***Corollary 2.15*** *Assume the hypothesis of Theorem 2.14. Assume, in addition, that the Sylvester equations (2.3) has a strong solution $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then $Z = -X^*$ is a unique weak solution to the dual Sylvester equation (2.8). Vice versa, if $Z \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is a strong solution of the dual Sylvester equation (2.8), then $X = -Z^*$ is a unique weak solution to the equation (2.3).*

***Remark 2.16*** The proofs of parts (i) and (ii) of Theorem 2.14 are slightly different in flavour owing to the fact that the operation of taking the adjoint is not continuous in the strong operator topology. Hence, in general, we are not able to state that the strong convergence of the Stieltjes integral in (2.26) implies the strong convergence of that in (2.28) and vice versa (*cf.* Remark 2.11).

For the sake of completeness we also present a "weak" version of Theorem 2.14.

***Theorem 2.17*** *Assume the hypothesis of Theorem 2.14. Then the following statements are equivalent.*

(i) *The Sylvester equations (2.3) has a weak solution $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$.*
(ii) *There exists the weak limit*

$$(2.51)\qquad X = \operatorname*{s\text{-}lim}_{\substack{a\to-\infty\\ b\to+\infty}} \int_{\operatorname{spec}(C)\cap[a,b)} E_C(d\mu)Y(A-\mu)^{-1}.$$

(iii) *The dual Sylvester equation (2.8) has a weak solution $Z = -X^* \in \mathcal{B}(\mathcal{K},\mathcal{H})$.*
(iv) *There exists the weak limit*

$$(2.52)\qquad Z = -\operatorname*{s\text{-}lim}_{\substack{a\to-\infty\\ b\to+\infty}} \int_{\operatorname{spec}(C)\cap[a,b)} (A^*-\mu)^{-1}Y^*E_C(d\mu).$$

The statement below concerns the existence of strong and even operator solutions to the Sylvester equation.

***Lemma 2.18*** *Assume the hypothesis of Theorem 2.14. Assume, in addition, that the condition*

$$\sup_{\mu\in\operatorname{spec}(C)} \|(A-\mu)^{-1}\| < \infty \tag{2.53}$$

*holds and the operator $Y$ has a finite $E_C$-norm, that is,*

$$\|Y\|_{E_C} < \infty. \tag{2.54}$$

*Then the Sylvester equations (2.3) and (2.8) have unique strong and, hence, unique weak solutions $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ given by (2.26) and $Z \in \mathcal{B}(\mathcal{K},\mathcal{H})$ given by (2.28), respectively, and, moreover, $Z = -X^*$. In representations (2.26) and (2.28) the Stieltjes integrals exist in the sense of the uniform operator topology.*

*Assume, in addition, that*

$$\sup_{\mu\in\operatorname{spec}(C)} \|\mu(A-\mu)^{-1}\| < \infty. \tag{2.55}$$

*Then*

$$\operatorname{ran}(X) \subset \operatorname{dom}(C), \tag{2.56}$$

$$\operatorname{ran}(Z) \subset \operatorname{dom}(A), \tag{2.57}$$

*and, thus, $X$ and $Z$ appear to be operator solutions to (2.3) and (2.8), respectively.*

**Proof** By (2.53), (2.54), and Lemma 2.13 the operator Stieltjes integrals in (2.26) and (2.28) can be understood in the operator norm topology. Thus, $X$ given (2.26) and $Y$ given by (2.28) are unique strong solutions to the Sylvester equations (2.3) and (2.8) by Theorem 2.14. Therefore, the operators $X$ and $Z$ are unique weak solutions and $Z = -X^*$ by Theorem 2.17.

In order to prove (2.56) it suffices to note that under conditions (2.55) and (2.54) for any $f \in \mathcal{H}$ and for any $a, b \in \mathbb{R}$, $a < b$, due to (2.26) the following estimate holds

$$\begin{aligned}\int_{[a,b)\cap\operatorname{spec}(C)} \mu^2\, d\langle E_C X f, X f\rangle &= \big\| C E_C\big([a,b)\big) X f\big\|^2 \\ &= \int_{[a,b)\cap\operatorname{spec}(C)} \Big\langle Y^* E_C(d\mu) Y \mu(A-\mu)^{-1} f, \mu(A-\mu)^{-1} f\Big\rangle \\ &\le \|Y\|_{E_C}^2 \Big(\sup_{\mu\in\operatorname{spec}(C)} \|\mu(A-\mu)^{-1}\|\Big)^2 \|f\|^2.\end{aligned}$$

Thus,

$$\int_{\operatorname{spec}(C)} \mu^2 \langle E_C X f, X f\rangle < \infty,$$

which proves that $Xf \in \operatorname{dom}(C)$ and, hence, the inclusion (2.56) is proven.

It remains to prove the inclusion (2.57). Given $\zeta \in \operatorname{spec}(C)$, we represent $ZE_C\big([a,b)\big)$ for some finite $a, b \in \mathbb{R}$, $a < b$, in the form (2.45) where $J_1(a,b)$ and $J_2(a,b)$ are just the same ones as in (2.46) and (2.47), respectively. Under condition (2.55), by Theorem 2.13 one concludes that the operator Stieltjes integral in (2.47) converges as $a \to -\infty$ and $b \to +\infty$ in the uniform operator topology to some operator $M \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then, from (2.45) one learns that for any $f \in \mathcal{K}$

$$\begin{aligned} Zf &= \lim_{\substack{a\to-\infty\\ b\to+\infty}} ZE_C\big([a,b)\big) f \\ &= \lim_{\substack{a\to-\infty\\ b\to+\infty}} J_1(a,b)f + \lim_{\substack{a\to-\infty\\ b\to+\infty}} J_2(a,b)f \\ &= -(A^* - \zeta)^{-1} Y^* f + (A^* - \zeta)^{-1} M f \end{aligned}$$

and, thus, $Zf \in \operatorname{dom}(A)$ which proves (2.57).

The proof is complete. ■

***Remark 2.19*** If the operator $A$ is self-adjoint, then the strong solution of the Sylvester equation, if it exists, can be represented in the form of the repeated Stieltjes integral

$$X = \int_{\operatorname{spec}(C)} dE_C(\mu) Y \int_{\operatorname{spec}(A)} \frac{dE_A(\lambda)}{\lambda - \mu}. \tag{2.58}$$

If, in addition, $Y$ is a Hilbert-Schmidt operator, then the repeated integral (2.58) can also be represented in the form of the double Stieltjes integral

$$X = \int\!\!\int_{\operatorname{spec}(C)\times\operatorname{spec}(A)} \frac{dE_C(\mu) Y\, dE_A(\lambda)}{\lambda - \mu}, \tag{2.59}$$

where the integral (2.59) can be understood as the $\mathcal{B}_2$-norm limit of the integral sums of the Lebesgue type. It is also worth to mention that by a theorem by Birman and Solomjak [11] under condition (2.25) we have the estimate

$$\|X\|_2 \le \frac{1}{d}\|Y\|_2, \tag{2.60}$$

where $d = \operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\}$. Moreover, the estimate (2.60) is sharp in the class of Hilbert-Schmidt operators.

***Remark 2.20*** If $Y$ is a Hilbert-Schmidt operator, inequality (2.60) is a considerable improvement of the more general estimate (2.14), the latter being sharp only in the class of all symmetric normed ideals. We also remark that if $A$ is self-adjoint and (2.54) holds, then (2.26) implies the estimate

$$\|X\|_{E_C} \le \frac{1}{d}\|Y\|_{E_C}. \tag{2.61}$$

## 3 Riccati Equation

The goal of this section is to develop an approach for solving the operator Riccati equations based on an applications of Banach's Fixed Point Principle for transformations of operator spaces. Putting aside the discussion of the purely geometric approach suggested and developed by Davis and Kahan [19], [20] and by Adams [6] as well as the one based on the factorization technique for operator holomorphic functions by Markus and Matsaev [49], [50] (see also [41] [52], [54], and [67]) we concentrate ourselves on applications of a purely analytic approach based on the representation theorems of Section 2.

***Definition 3.1*** Assume that $A$ and $C$ are possibly unbounded densely defined closed operators in the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, while $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $D \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

A bounded operator $Q \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be a *weak solution* of the Riccati equation

$$QA - CQ + QBQ = D \tag{3.1}$$

if

$$\langle QAf, g\rangle - \langle Qf, C^*g\rangle + \langle QBQf, g\rangle = \langle Df, g\rangle \qquad \text{for all } f \in \mathrm{dom}(A) \text{ and } d \in \mathrm{dom}(C^*).$$

A bounded operator $Q \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be a *strong solution* of the Riccati equation (3.1) if

$$\mathrm{ran}\,(Q|_{\mathrm{dom}(A)}) \subset \mathrm{dom}(C), \tag{3.2}$$

and

$$QAf - CQf + QBQf = Df \quad \text{for all } f \in \mathrm{dom}(A). \tag{3.3}$$

Finally, a bounded operator $Q \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be an *operator solution* of the Riccati equation (3.1) if

$$\mathrm{ran}\,(Q) \subset \mathrm{dom}(C),$$

the operator $QA$ is bounded on $\mathrm{dom}(QA) = \mathrm{dom}(A)$ and the equality

$$\overline{QA} - CQ + QBQ = D \tag{3.4}$$

holds as an operator equality, where $\overline{QA}$ denotes the closure of $QA$.

Along with the Riccati equation (3.1) we also introduce the dual equation

$$KC^* - A^*K + KB^*K = D^*, \tag{3.5}$$

for which the notion of weak, strong, and operator solutions is defined in a way analogous to that in Definition 3.1.

***Example 1*** **(The Friedrichs model** [22]**)** Given a nonempty open Borel set $\Delta \subset \mathbb{R}$, let $\mathcal{H} = \mathbb{C}$ and $\mathcal{K} = L^2(\Delta)$. Let $A = 0$ in $\mathcal{H}$ and let $C$ be the multiplication operator in $\mathcal{K}$,

$$(Cf)(\mu) = \mu f(\mu)$$

on

$$\operatorname{dom}(C) = \left\{ f \in L^2(\Delta) : \int_\Delta d\mu (1+\mu^2)|f(\mu)|^2 < \infty \right\},$$

$B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and, finally, $D = B^* \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

By Riesz representation theorem

$$Bf = \langle f, b \rangle = \int_\Delta d\mu\, f(\mu)\overline{b(\mu)}, \quad f \in \mathcal{K},$$

for some essentially bounded function $b \in \mathcal{K} = L^2(\Delta)$, and hence

$$(D\zeta)(\mu) = \overline{b(\mu)}\zeta, \quad \zeta \in \mathbb{C},$$

since $D = B^*$.

Under the assumptions of this example a bounded operator $Q \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a weak solution to the Riccati equation (3.1) if and only if $Q$ has the form

$$(Q\zeta)(\mu) = q(\mu)\zeta, \quad \zeta \in \mathbb{C}, \tag{3.6}$$

where $q$ is an essentially bounded function, and

$$-(\mu q)(\mu) + \langle q, b \rangle q(\mu) = b(\mu) \quad \text{for a.e. } \mu \in \Delta. \tag{3.7}$$

Moreover, any weak solution $Q$ appears to be a strong solution, that is, any essentially bounded function $q$ satisfying (3.7) belongs to $\operatorname{dom}(C)$.

Solving (3.7) with respect to $q$ one concludes that the Riccati equation (3.7) has a weak/strong solution if and only if

$$\text{there exists a } w \in \mathbb{R} \text{ such that } \frac{b(\,\cdot\,)}{\cdot - w} \in L^2(\Delta) \tag{3.8}$$

and

$$w + \int_\Delta d\mu \frac{|b(\mu)|^2}{\mu - w} = 0. \tag{3.9}$$

If conditions (3.8) and (3.9) hold for some $w \in \mathbb{R}$, then the solution $Q$ has the form (3.6), where

$$q(\mu) = \frac{b(\mu)}{w - \mu}, \quad \mu \in \mathbb{R}, \tag{3.10}$$

and

$$w = \langle q, b \rangle.$$

The next assertion is a direct corollary of Lemma 2.4.

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***Lemma 3.2*** *Let $A$ and $C$ be densely defined possibly unbounded closed operators in the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$, and $D \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then $Q \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a weak solution to the Riccati equation (3.1) if and only if $K = -Q^*$ is a weak solution to the dual Riccati equation (3.5).*

Throughout the remaining part of the section we assume the following hypothesis.

***Hypothesis 3.3*** *Assume that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $A$ and $C$ are possibly unbounded self-adjoint operators on domains $\operatorname{dom}(A)$ in $\mathcal{H}$ and $\operatorname{dom}(C)$ in $\mathcal{K}$, respectively. Also assume that $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $D \in \mathcal{B}(\mathcal{H},\mathcal{K})$.*

The representation theorems of Section 2 for solutions of the Sylvester equation are a source for iteration schemes which allow one to prove solvability of Riccati equations by using fixed point theorems. Here we present two of such schemes for the search for strong or weak solutions to the Riccati equation.

***Theorem 3.4*** *Assume Hypothesis 3.3. Then the following statements hold true.*

(i) *Assume, in addition to Hypothesis 3.3, that*

$$\operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\} > 0.$$

*Then $Q \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a weak solution to the Riccati equation (3.1) if and only if it is a solution to the equation*

$$Q = \int_{-\infty}^{\infty} e^{\mathrm{i}tC}(D - QBQ)e^{-\mathrm{i}tA} f_d(t)\,dt, \tag{3.11}$$

*where $f_d$ is a summable function satisfying (2.13) and the integral in (3.11) exists in the sense of the weak operator topology in $\mathcal{B}(\mathcal{H},\mathcal{K})$.*

(ii) *Assume, in addition to Hypothesis 3.3, that*

$$\operatorname{dist}\{\operatorname{spec}(A + BQ), \operatorname{spec}(C)\} > 0. \tag{3.12}$$

*Then $Q \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a strong (weak) solution to the Riccati equation (3.1) if and only if $Q$ is a solution of the equation*

$$Q = \int_{\operatorname{spec}(C)} E_C(d\mu) D (A + BQ - \mu)^{-1}, \tag{3.13}$$

*where the operator Stieltjes integral exists in the sense of the strong (weak) operator topology in $\mathcal{B}(\mathcal{H},\mathcal{K})$.*

(iii) *Assume, in addition to Hypothesis 3.3, that $K \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and*

$$\operatorname{dist}\{\operatorname{spec}(A - KB^*), \operatorname{spec}(C)\} > 0. \tag{3.14}$$

*Then the operator $K$ is a strong (weak) solution to the dual Riccati equation (3.5) if and only if $K$ satisfies the equation*

$$K = -\int_{\operatorname{spec}(C)} (A - KB^* - \mu)^{-1} D^* E_C(d\mu), \tag{3.15}$$

*where the operator Stieltjes integral exists in the sense of the strong (weak) operator topology.*

**Proof** (i) The operator $Q$ is a weak solution to (3.1) if and only if $Q$ is a weak solution to the equation

$$QA - CQ = Y,$$

where

$$Y = D - QBQ.$$

Applying Theorem 2.7 completes the proof of (i).

(ii) The operator $Q$ is a strong solution to (3.1) if and only if $Q$ is a strong solution to the equation

$$Q\tilde{A} - CQ = D,$$

where

$$\tilde{A} = A + BQ.$$

Applying Theorem 2.14 (i) completes the proof of (ii).

(iii) The operator $K$ is a strong solution to (3.5) if and only if $K$ is a strong solution to the equation

$$KC - \hat{A}K = D^*,$$

where

$$\hat{A} = A - KB^*.$$

Applying Theorem 2.14 (ii) completes the proof of (iii).

The proof is complete. ■

The following statement is a direct consequence of Lemma 2.18.

***Theorem 3.5*** *Assume Hypothesis 3.3 and let $D$ have a finite norm with respect to the spectral measure of the operator $C$, that is,*

$$\|D\|_{E_C} < \infty. \tag{3.16}$$

*Assume, in addition, that an operator $Q \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a weak solution of the Riccati equation (3.1) such that*

$$\operatorname{dist}\{\operatorname{spec}(A + BQ), \operatorname{spec}(C)\} > 0, \tag{3.17}$$

*and that the condition*

$$\sup_{\mu \in \operatorname{spec}(C)} \|(A + BQ - \mu)^{-1}\| < \infty \tag{3.18}$$

*holds.*

*Then $Q$ is a strong solution to (3.1) and the operator $K = -Q^*$ is a strong solution to the dual Riccati equation (3.5).*

*The strong solutions $Q$ and $K$ admit the representations*

$$Q = \int_{\operatorname{spec}(C)} E_C(d\mu) D (A + BQ - \mu)^{-1}, \tag{3.19}$$

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$$K = -\int_{\operatorname{spec}(C)} (A - KB^* - \mu)^{-1} D^* E_C(d\mu), \tag{3.20}$$

*where the operator Stieltjes integrals exist in the sense of the uniform operator topology. Hence, the operators Q and K have finite $E_C$-norm and the following bound holds true*

$$\|K\|_{E_C} = \|Q\|_{E_C} \le \|D\|_{E_C} \sup_{\mu \in \operatorname{spec}(C)} \|(A + BQ - \mu)^{-1}\|. \tag{3.21}$$

*If, in this case, instead of (3.18) the following condition holds*

$$\sup_{\mu \in \operatorname{spec}(C)} \|\mu(A + BQ - \mu)^{-1}\| < \infty, \tag{3.22}$$

*then*

$$\operatorname{ran}(Q) \subset \operatorname{dom}(C)$$

*and*

$$\operatorname{ran}(K) \subset \operatorname{dom}(A)$$

*and, hence, the strong solutions Q and K appear to be operator solutions to the Riccati equations (3.1) and (3.5), respectively.*

In the case where the spectra of the operators $A$ and $C$ are separated, under additional "smallness" assumptions upon the operators $B$ and $D$ we are able to prove the existence of fixed points for mappings given by (3.11) and (3.13).

***Theorem 3.6*** *Assume Hypothesis 3.3 and suppose that*

$$B \neq 0.$$

*Also assume that*

$$d = \operatorname{dist}\{\operatorname{spec}(A), \operatorname{spec}(C)\} > 0. \tag{3.23}$$

*Then:*

(i) *If the inequality holds*

$$\sqrt{\|B\|\,\|D\|} < \frac{d}{\pi}, \tag{3.24}$$

*then the Riccati equation (3.1) has a unique weak solution in the ball*

$$\left\{ Q \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : \|Q\| < \frac{d}{\pi \|B\|} \right\}.$$

*The weak solution Q satisfies the estimate*

$$\|Q\| \le \frac{1}{\|B\|} \left( \frac{d}{\pi} - \sqrt{\frac{d^2}{\pi^2} - \|B\|\,\|D\|} \right). \tag{3.25}$$

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*In particular, if*

$$\|B\| + \|D\| < \frac{2}{\pi}\, d, \tag{3.26}$$

*then the weak solution $Q$ is a strict contraction, that is,*

$$\|Q\| < 1.$$

(ii) *If the operator $D$ has a finite $E_C$-norm and the inequality*

$$\sqrt{\|B\|\,\|D\|_{E_C}} < \frac{d}{2} \tag{3.27}$$

*holds, then the Riccati equation (3.1) has a unique strong solution in the ball*

$$\left\{ Q \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : \|Q\| < \|B\|^{-1}\left( d - \sqrt{\|B\|\,\|D\|_{E_C}} \right) \right\}. \tag{3.28}$$

*The strong solution $Q$ has a finite $E_C$-norm and one has the estimate*

$$\|Q\|_{E_C} \le \frac{1}{\|B\|}\left( \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|\,\|D\|_{E_C}} \right). \tag{3.29}$$

*In particular, if*

$$\|B\| + \|D\|_{E_C} < d, \tag{3.30}$$

*then the strong solution $Q$ is a strict contraction in both the uniform operator and $E_C$-norm topologies, that is,*

$$\|Q\| \le \|Q\|_{E_C} < 1.$$

**Proof** The proof is based on an application of Banach's Fixed Point Theorem.

(i) Let $f \in L^1(\mathbb{R})$ be a continuous function on $\mathbb{R}$ except at zero such that

$$\hat{f}(s) = \int_{\mathbb{R}} e^{-\mathrm{i}st} f(t)\, dt = \frac{1}{s} \quad \text{whenever } |s| \ge 1.$$

Introducing the function

$$f_d(t) = f(dt), \quad t \in \mathbb{R},$$

by Theorem 3.4 (i) any fixed point of the map $F(Q)$ given by

$$F(Q) = \int_{-\infty}^{\infty} e^{\mathrm{i}tC}(D - QBQ)e^{-\mathrm{i}tA} f_d(t)\, dt, \quad Q \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \tag{3.31}$$

where the improper Riemann integral is understood in the weak sense, is a weak solution to the Riccati equation (3.1). Taking into account that

$$\|f_d\|_{L^1(\mathbb{R})} = \frac{\|f\|_{L^1(\mathbb{R})}}{d},$$

from (3.31) one concludes that

$$\|F(Q)\| \le \frac{\|f\|_{L^1(\mathbb{R})}}{d}(\|D\| + \|B\|\,\|Q\|^2), \quad Q \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \tag{3.32}$$

and

$$\|F(Q_1) - F(Q_2)\| \le \frac{\|f\|_{L^1(\mathbb{R})}}{d}\|B\|(\|Q_1\| + \|Q_2\|)(\|Q_1 - Q_2\|), \qquad Q_1, Q_2 \in \mathcal{B}(\mathcal{H}, \mathcal{K}). \tag{3.33}$$

Clearly, $F$ maps the ball $\mathcal{O}_r = \{Q \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : \|Q\| \le r\}$ into itself whenever

$$\frac{\|f\|_{L^1(\mathbb{R})}}{d}(\|D\| + \|B\|\, r^2) \le r$$

and $F$ is a strict contraction of the ball $\mathcal{O}_r$ whenever

$$\frac{2\|f\|_{L^1(\mathbb{R})}\|B\|}{d}\, r < 1.$$

Since the extremal problem for the Fourier transform, which is to find the infimum of $\|f\|_{L^1}$ over all functions $f \in L^1(\mathbb{R})$ such that $\hat{f}(s) = 1/s$ for $|s| \ge 1$, has the solution (*cf.* Remark 2.8)

$$\inf\{\|f\|_{L^1(\mathbb{R})} : f \in L^1(\mathbb{R}), \hat{f}(s) = 1/s \text{ whenever } |s| \ge 1\} = \frac{\pi}{2},$$

one concludes that $F$ maps the ball $\mathcal{O}_r = \{Q \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : \|Q\| \le r\}$ into itself whenever

$$\frac{\pi}{2d}(\|D\| + \|B\| r^2) \le r \tag{3.34}$$

and $F$ is a strict contraction of $\mathcal{O}_r$ whenever

$$\frac{\pi\|B\|}{d} r < 1. \tag{3.35}$$

Solving inequalities (3.34) and (3.35) one concludes that if the radius $r$ of the ball $\mathcal{O}_r$ is within the bounds

$$\frac{d}{\pi\|B\|} - \sqrt{\frac{d^2}{\pi^2\|B\|^2} - \frac{\|D\|}{\|B\|}} \le r < \frac{d}{\pi\|B\|}, \tag{3.36}$$

then $F$ is a strictly contractive mapping of the ball $\mathcal{O}_r$ into itself. Applying Banach's Fixed Point Theorem proves assertion (i).

(ii) Given $r \in (0, d\|B\|^{-1})$, under Hypothesis (3.23) we have the identity

$$(A + BQ - \mu)^{-1} = \big(I + (A - \mu)^{-1} BQ\big)^{-1} (A - \mu)^{-1}, \quad \mu \in \operatorname{spec}(C),\ Q \in \mathcal{O}_r, \tag{3.37}$$

which implies the estimate

$$\sup_{\mu\in\operatorname{spec}(C)} \|(A+BQ-\mu)^{-1}\| \le \sup_{\mu\in\operatorname{spec}(C)} \frac{1}{1-\|(A-\mu)^{-1}\|\,\|B\|\,\|Q\|}\,\|(A-\mu)^{-1}\|$$
$$\le \frac{1}{1-\frac{\|B\|r}{d}}\,\frac{1}{d} = \frac{1}{d-\|B\|r}, \tag{3.38}$$

whenever $Q \in \mathcal{O}_r$.

Since (3.38) holds and the operator $D$ has a finite $E_C$-norm, the mapping

$$F(Q) = \int_{\operatorname{spec}(C)} E_C(d\mu)D(A+BQ-\mu)^{-1},$$

where the integral is understood in the strong sense, is well defined on the domain

$$\operatorname{dom}(F) = \mathcal{O}_r.$$

Since for $Q \in \mathcal{O}_r$ one clearly has the estimate

$$\operatorname{dist}\{\operatorname{spec}(A+BQ), \operatorname{spec}(C)\} \ge d - \|B\|r > 0,$$

any fixed point of the map $F$ is a strong solution to the Riccati equation (3.1) by Theorem 3.4(ii).

Using (3.38) we have the following two estimates

$$\|F(Q)\| \le \|F(Q)\|_{E_C} \le \|D\|_{E_C} \sup_{\mu\in\operatorname{spec}(C)} \|(A+BQ-\mu)^{-1}\|$$
$$\le \frac{\|D\|_{E_C}}{d-\|B\|r}, \quad Q \in \mathcal{O}_r, \tag{3.39}$$

and

(3.40)

$$\|F(Q_1)-F(Q_2)\| \le \|F(Q_1)-F(Q_2)\|_{E_C}$$
$$= \Big\| \int_{\operatorname{spec}(C)} E_C(d\mu)D(A+BQ_1-\mu)^{-1}B(Q_2-Q_1)(A+BQ_2-\mu)^{-1} \Big\|_{E_C}$$
$$\le \frac{\|D\|_{E_C}}{(d-\|B\|r)^2}\|Q_2-Q_1\|, \quad Q_1, Q_2 \in \mathcal{O}_r.$$

Clearly, by (3.39) $F$ maps the ball $\mathcal{O}_r$ into itself whenever

$$\frac{\|D\|_{E_C}}{d-\|B\|r} \le r \tag{3.41}$$

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and by (3.40) $F$ is a strict contraction on $\mathcal{O}_r$ whenever

$$\frac{\|D\|_{E_C}}{(d - \|B\|r)^2} < 1. \tag{3.42}$$

Solving inequalities (3.41) and (3.42) simultaneously, one concludes that if the radius of the ball $\mathcal{O}_r$ is within the bounds

$$\frac{1}{\|B\|}\left(\frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|\,\|D\|_{E_C}}\right) \le r < \frac{1}{\|B\|}\left(d - \sqrt{\|B\|\,\|D\|_{E_C}}\,\right), \tag{3.43}$$

then $F$ is a strictly contracting mapping of the ball $\mathcal{O}_r$ into itself. Applying Banach's Fixed Point Theorem we infer that equation (3.13) has a unique solution in any ball $\mathcal{O}_r$ whenever $r$ satisfies (3.43). Therefore, the fixed point does not depend upon the radii satisfying (3.43) and hence it belongs to the smallest of these balls. This observation proves the estimate

$$\|Q\| \le \frac{1}{\|B\|}\left(\frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|\,\|D\|_{E_C}}\right). \tag{3.44}$$

Finally, using (3.39), for the fixed point $Q$ one obtains the estimate

$$\|Q\|_{E_C} = \|F(Q)\|_{E_C} \le \frac{\|D\|_{E_C}}{d - \|B\|\,\|Q\|}. \tag{3.45}$$

Then (3.44) yields

$$\|Q\|_{E_C} \le \frac{\|D\|_{E_C}}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \|B\|\,\|D\|_{E_C}}} = \frac{1}{\|B\|}\left(\frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|\,\|D\|_{E_C}}\right),$$

which completes the proof. ∎

***Remark 3.7*** Part (ii) of the theorem extends results obtained in [55], [56], and [52]. In case where the self-adjoint operator $C$ is bounded, $D$ is a Hilbert-Schmidt operator, $B$ is bounded, and $A$ is possibly unbounded densely defined closed non-self-adjoint operator, the solvability of the equation (3.19) under condition (3.23) has recently been studied in [4].

***Remark 3.8*** Under the hypotheses (3.23) and (3.24) or (3.27) the fixed point $Q$ depends continuously (in the operator norm) upon the operators $B$ and $D$, which follows from a result (see, *e.g.*, [33] Chapter XVI, Theorem 3) concerning the continuity of the mapping in Banach's Fixed Point Theorem with respect to a parameter.

***Remark 3.9*** In general, hypothesis (3.23) in Theorem 3.6 can not be omitted. In order to see this assume that $\Delta = \mathbb{R}$ in Example 1 and, thus, (3.23) does not hold. Assume, in addition, that the function $b(\,\cdot\,)$ in this example is a strictly positive continuous function. Then the necessary condition (3.8) for the solvability of the Riccati (3.7) is violated.

In order to complete the discussion of the results of Theorem 3.6 we need the following illustrative statement based on Example 1.

***Lemma 3.10*** *Assume the hypothesis of Example 1 for*

$$\Delta = (-\infty, -d) \cup (d, +\infty)$$

*and some $d > 0$.*

*If $b \in L^2(\Delta)$ and*

$$\|b\| \leq \sqrt{2}d, \tag{3.46}$$

*then the Riccati equation (3.7) has a weak/strong solution. Moreover, the constant $\sqrt{2}$ in (3.46) is sharp.*

**Proof** Under the hypothesis (3.46) we have the inequalities

$$\int_{-\infty}^{-d} d\mu \frac{|b(\mu)|^2}{d-\mu} < \frac{\|b\|^2}{2d} \leq d \quad \text{and} \quad \int_{d}^{+\infty} d\mu \frac{|b(\mu)|^2}{\mu+d} < \frac{\|b\|^2}{2d} \leq d.$$

The Herglotz function

$$f(w) = w + \int_\Delta d\mu \frac{|b(\mu)|^2}{\mu - w}$$

is a strictly increasing continuous function on $(-d, d)$ and

$$f(-d-0) = \lim_{\varepsilon \downarrow 0} f(-d-\varepsilon) \leq -d + \int_{d}^{+\infty} d\mu \frac{|b(\mu)|^2}{\mu+d} < 0,$$

$$f(d+0) = \lim_{\varepsilon \downarrow 0} f(d+\varepsilon) \geq d + \int_{-\infty}^{-d} d\mu \frac{|b(\mu)|^2}{\mu-d} > 0,$$

not withstanding the possibility for the one-sided limits $f(-d-0)$ and $f(d+0)$ to turn into $-\infty$ and $+\infty$, respectively. Therefore, the equation

$$f(w) = 0$$

has a unique root $w_0 \in (-d, d)$, the function

$$q(\mu) = \frac{b(\mu)}{\mu - w_0}, \quad \mu \in \Delta,$$

is an element of $L^2(\Delta)$, and, hence, the Riccati equation (3.7) has a weak/strong solution, since the existence criterion (3.8), (3.9) is satisfied.

In order to prove that the constant $\sqrt{2}$ in the upper bound (3.46) is sharp, it suffices to show that for any $c > 1$ there exists a function $b \in L^2(\Delta)$ such that

$$\|b\| = \sqrt{2}cd$$

and the Riccati equation (3.7) has no solutions $q \in L^2(\Delta)$.

Let $\omega \in L^1(\mathbb{R}_+)$ be a positive continuous function on $[0,\infty)$ such that

$$\int_0^\infty \omega(t)\,dt = 1.$$

Given $\varepsilon > 0$, introduce the functions

$$\omega_\varepsilon(t) = \varepsilon^{-1}\omega(t/\varepsilon), \quad t \geq 0,$$

and

$$\varphi_\varepsilon(\mu) = \begin{cases} \arctan(d+\mu)\omega_\varepsilon^{1/2}(d-\mu), & \mu \leq -d, \\ \omega_\varepsilon^{1/2}(\mu-d), & \mu \geq d. \end{cases} \tag{3.47}$$

One infers

$$\lim_{\varepsilon\downarrow 0} \|\varphi_\varepsilon\|^2 = 1$$

and

$$\lim_{\varepsilon\downarrow 0}\left(\int_{-\infty}^{-d} \frac{|\varphi_\varepsilon(\mu)|^2}{\mu+d}\,d\mu + \int_d^{+\infty} \frac{|\varphi_\varepsilon(\mu)|^2}{\mu+d}\,d\mu\right) = \frac{1}{2d}. \tag{3.48}$$

Hence for any $c > 1$, one can find an $\varepsilon_0 > 0$ such that the following inequality holds

$$\|\varphi_{\varepsilon_0}\|^{-2}\left(\int_{-\infty}^{-d} \frac{|\varphi_{\varepsilon_0}(\mu)|^2}{\mu+d}\,d\mu + \int_d^{+\infty} \frac{|\varphi_{\varepsilon_0}(\mu)|^2}{\mu+d}\,d\mu\right) > \frac{1}{2dc^2}. \tag{3.49}$$

Introducing

$$b(\mu) = \sqrt{2}cd\frac{\varphi_{\varepsilon_0}(\mu)}{\|\varphi_{\varepsilon_0}\|}, \quad \mu \in \Delta = (-\infty,-d] \cup [d,\infty),$$

one obviously concludes that

$$\|b\| = \sqrt{2}cd.$$

Meanwhile, (3.49) implies the estimate

$$\int_{-\infty}^{-d} \frac{|b(\mu)|^2}{\mu+d}\,d\mu + \int_d^{+\infty} \frac{|b(\mu)|^2}{\mu+d}\,d\mu > d.$$

Therefore, the Herglotz function $f(w)$ given by

$$f(w) = w + \int_{-\infty}^{-d} \frac{|b(\mu)|^2}{\mu-w}\,d\mu + \int_d^{+\infty} \frac{|b(\mu)|^2}{\mu-w}\,d\mu$$

does not vanish on $[-d,d)$ (note that $f(w) \to +\infty$ as $w \uparrow d$) and hence (3.9) is violated for all $w \in [-d,d)$. Since $b(\cdot)$ is a continuous function and it does not vanish on $(-\infty,d) \cup [d,\infty)$, the condition (3.8) is violated for all $w \in (-\infty,d) \cup [d,\infty)$. Hence, the Riccati equation (3.7) has no weak/strong solutions in this case since the existence criterion (3.8), (3.9) is violated. ∎

***Remark 3.11*** The result of Lemma 3.10 combined with that of Theorem 3.6 shows the following.

(i) There is a constant $c > 0$ such that the conditions (3.23) and

$$\|B\| < c\,\mathrm{dist}\{\mathrm{spec}(A), \mathrm{spec}(C)\}, \tag{3.50}$$

imply the existence of a weak solution to the Riccati equation

$$QA - CQ + QBQ = B^*.$$

(ii) In general, the "smallness" requirement on $B$ (3.50) can not be omitted (*cf.* (3.24) and (3.27)).

(iii) The sharp value of the constant $c$ in (3.50) is within the bounds

$$\frac{1}{\pi} \le c \le \sqrt{2}.$$

## 4 The Spectral Shift Function

The main purpose of this section is to recall the concept of the spectral shift function [36], [37], [38], [39], [40], [45], [46] associated with a pair of self-adjoint operators and to extend this concept to the case of pairs of closed operators that are similar to self-adjoint operators.

The spectral shift function $\xi(\lambda, H, A)$ for a pair of self-adjoint operators $(H, A)$ in a Hilbert space $\mathcal{H}$ is usually associated with the Lifshits-Krein trace formula

$$\mathrm{tr}\big(\varphi(H) - \varphi(A)\big) = \int_{\mathbb{R}} d\lambda \varphi'(\lambda)\xi(\lambda, H, A). \tag{4.1}$$

The trace formula (4.1) holds for a wide class of functions $\varphi\colon \mathbb{R} \to \mathbb{C}$, including $C_0^\infty(\mathbb{R})$, provided that the self-adjoint operators $H$ and $A$ are resolvent comparable in the sense that

$$(H - z)^{-1} - (A - z)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad \mathrm{Im}(z) \neq 0. \tag{4.2}$$

The trace formula (4.1) determines the spectral shift function up to an arbitrary complex constant. This constant may, however, be chosen in such a way that makes the spectral shift function to be real-valued.

In case of trace class perturbations, *i.e.*, if

$$\overline{H - A} \in \mathcal{B}_1(\mathcal{H}),$$

the additional requirement that

$$\xi(\,\cdot\,, H, A) \in L^1(\mathbb{R})$$

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determines the spectral shift function uniquely. Being chosen in this way, the spectral shift function $\xi(\lambda, H, A)$ can be computed by Krein's formula via the perturbation determinant

$$\xi(\lambda, H, A) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \arg\det\big((H - \lambda - i\varepsilon)(A - \lambda - i\varepsilon)^{-1}\big) \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{4.3}$$

In the case of resolvent comparable perturbations (4.2), the spectral shift function can be computed via the generalized perturbation determinant [38], [39]

$$\Delta_{H/A}(z) = \det\big((H + i)^{-1}(H - z)(A - z)^{-1}(A + i)\big), \quad \operatorname{Im}(z) \neq 0$$

by the representation

$$\xi(\lambda, H, A) = \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0}\big(\arg \Delta_{H/A}(\lambda + i\varepsilon) - \arg \Delta_{H/A}(\lambda - i\varepsilon)\big) \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{4.4}$$

Here the branch of $\arg \Delta_{H/A}(z)$ is arbitrary for $\operatorname{Im}(z) > 0$, but for $\operatorname{Im}(z) < 0$ the argument of $\Delta_{H/A}(z)$ is fixed by the condition $\arg \Delta_{H/A}(-i) = 0$. In this case

$$\int_{\mathbb{R}} d\lambda \frac{|\xi(\lambda, H, A)|}{\lambda^2 + 1} < \infty. \tag{4.5}$$

The spectral shift function $\xi(\lambda, H, A)$ in (4.4) is determined up to an integer and there is in general no natural way to choose this integer uniquely[1] [39]. Moreover, the requirement of continuity of the spectral shift function $\xi(\lambda, H, A)$ (as an element of the weighted space $L^1\big(\mathbb{R}; (1 + \lambda^2)^{-1}\, d\lambda\big)$ with respect to the operator parameters $H$ or $A$ (in an appropriate operator topology) leads to the conclusion: the spectral shift function $\xi(\lambda, H, A)$ can not be introduced uniquely as a function of the pair $(H, A)$. Given a continuous path of operators $H_t$, $t \in [0, 1]$ connecting the end points $A$ and $H$ from the same linearly connected component (in the metric space of self-adjoint operators that are resolvent comparable with $A$, equipped with the metric $\rho(H, H') = \|(H - i)^{-1} - (H' - i)^{-1}\|_{\mathcal{B}_1(\mathcal{H})}$ ), the function $\xi(\lambda, H, A)$ should be considered to be either a (path-dependent) homotopy invariant, or to be a path independent but multi-valued (modulo $\mathbb{Z}$) function of the spectral parameter $\lambda$ (see [14], [62], and [69] for details). In either case, the spectral shift function is uniquely introduced modulo $\mathbb{Z}$ in such a way that for any pairs $(H, A)$, $(H, \tilde{H})$, and $(\tilde{H}, A)$ of self-adjoint operators $A$, $H$, and $\tilde{H}$ in $\mathcal{H}$, satisfying (4.2), the following chain rule holds

$$\xi(\lambda, H, A) = \xi(\lambda, H, \tilde{H}) + \xi(\lambda, \tilde{H}, A) \pmod{\mathbb{Z}} \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{4.6}$$

The extension of a concept of the spectral shift function to the case of operators similar to self-adjoint needs additional considerations.

We start with a definition of a zero trace commutator class.

[1] Notice that the trace formula (4.1) itself determines the spectral shift function up to an arbitrary constant. The condition (4.5) does not change matters and only the relation (4.4) normalizes the spectral shift function up to an integer constant.

***Definition 4.1*** Let $\mathcal{A}(\mathcal{H})$ be the set of all bounded operators $V \in \mathcal{B}(\mathcal{H})$ possessing the property:

$$\operatorname{tr}(VR - RV) = 0 \tag{4.7}$$

whenever $R \in \mathcal{B}(\mathcal{H})$ and

$$VR - RV \in \mathcal{B}_1(\mathcal{H}). \tag{4.8}$$

The set $\mathcal{A}(\mathcal{H})$ is called the *zero trace commutator class*.

In the case of an infinite-dimensional Hilbert space $\mathcal{H}$ (4.8) does not imply (4.7) in general. For example, let $P$ be a one-dimensional orthogonal projection. Then there is a partial isometry $S$ such that $SS^* = I$ and $S^*S = I - P$. Taking $R = S^*$ and $V = S$ one obtains $VR - RV = P \in \mathcal{B}_1(\mathcal{H})$, but $\operatorname{tr}(VR - RV) = 1$, and, thus, (4.7) fails to hold despite (4.8) holds true. Therefore, the zero trace commutator class $\mathcal{A}(\mathcal{H})$ is a proper subset of $\mathcal{B}(\mathcal{H})$ if the Hilbert space $\mathcal{H}$ is infinite-dimensional.

***Lemma 4.2*** *Assume that $R, V \in \mathcal{B}(\mathcal{H})$ and at least one of the following conditions holds:*

(i) *$V \in \mathcal{B}_1(\mathcal{H})$;*
(ii) *$VR$ and $RV$ are trace class operators;*
(iii) *$V$ is a normal operator and $R \in \mathcal{B}_2(\mathcal{H})$;*
(iv) *$V$ is a self-adjoint operator and $R \in \mathcal{B}_\infty(\mathcal{H})$;*
(v) *$V$ is a self-adjoint operator having no absolutely continuous spectral subspaces;*
(vi) *$V$ is a normal operator with purely point spectrum;*
(vii) *$R \in \mathcal{B}_p(\mathcal{H})$ and $V \in \mathcal{B}_q(\mathcal{H})$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

*Then* $\operatorname{tr}(VR - RV) = 0$ *whenever $VR - RV$ is a trace class operator.*

***Remark 4.3*** The part (i) is obvious. The part (ii) follows from Lidskii's theorem. The statement (iii) is due to G. Weiss [68]. Assertion (iv) has been proven by J. Helton and R. Howe [31]. The part (v) immediately follows from a result by R. W. Carey and J. D. Pincus [16] which states that any self-adjoint operator having no absolutely continuous spectral subspace is the sum of an operator with purely point spectrum and a trace class one with arbitrary small trace norm[2]. The results (vi) and (vii) have recently been proven by V. Lauric and C. M. Pearcy [43].

Lemma 4.2 shows that the zero trace commutator class $\mathcal{A}(\mathcal{H})$ is a rather rich set. In particular, $\mathcal{A}(\mathcal{H})$ contains all the trace class operators, that is,

$$\mathcal{B}_1(\mathcal{H}) \subset \mathcal{A}(\mathcal{H}).$$

More generally, any operator of the form

$$\hat{V} = V + T, \quad V \in \mathcal{A}(\mathcal{H}),\ T \in \mathcal{B}_1(\mathcal{H}),$$

[2]We are indebted to Vadim Kostrykin who has attracted our attention to this fact.

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is an element of $\mathcal{A}(\mathcal{H})$. The class $\mathcal{A}(\mathcal{H})$ also contains all normal bounded operators $V$ with purely point spectrum and all self-adjoint bounded operators having no absolutely continuous spectrum and, therefore, in this case, if $V \in \mathcal{A}(\mathcal{H})$ and $V$ has a bounded inverse, then $V^{-1} \in \mathcal{A}(\mathcal{H})$ as well.

***Definition 4.4*** Let $H$ be a possibly unbounded densely defined closed operator in $\mathcal{H}$ on $\operatorname{dom}(H)$ with $\operatorname{spec}(H) \subset \mathbb{R}$. The operator $H$ is said to be *admissible* if there exists a self-adjoint operator $\hat{H}$ such that

(i) $H$ is similar to $\hat{H}$, *i.e.*,

$$H = V^{-1}\hat{H}V \quad \text{on } \operatorname{dom}(H) = V^{-1}\big(\operatorname{dom}(\hat{H})\big)$$

for some $V \in \mathcal{A}(\mathcal{H})$ such that $V^{-1} \in \mathcal{B}(\mathcal{H})$;

(ii) $H$ and $\hat{H}$ are resolvent comparable, *i.e.*,

$$(H - z)^{-1} - (\hat{H} - z)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad \operatorname{Im}(z) \neq 0. \tag{4.9}$$

We will call the operator $\hat{H}$ a self-adjoint representative of the admissible operator $H$.

Clearly, any self-adjoint operator is admissible. Moreover, an admissible operator may have different self-adjoint representatives.

***Lemma 4.5*** *Let $H$ be an admissible operator and $\hat{H}$ any self-adjoint representative of $H$. Then*

$$\operatorname{tr}\big((H - z)^{-1} - (\hat{H} - z)^{-1}\big) = 0, \quad \operatorname{Im}(z) \neq 0. \tag{4.10}$$

**Proof** By the definition of an admissible operator, the difference of the resolvents of $H$ and $\hat{H}$ is a trace class operator and the following representation holds for some $V \in \mathcal{A}(\mathcal{H})$ such that $V^{-1} \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}(H - z)^{-1} - (\hat{H} - z)^{-1} &= V^{-1}(\hat{H} - z)^{-1}V - (\hat{H} - z)^{-1} \\ &= [V^{-1}(\hat{H} - z)^{-1}]V - V[V^{-1}(\hat{H} - z)^{-1}] \in \mathcal{B}_1(\mathcal{H}), \quad \operatorname{Im}(z) \neq 0,\end{aligned} \tag{4.11}$$

which implies (4.10), since $V \in \mathcal{A}(\mathcal{H})$. ■

***Corollary 4.6*** *Let $H$ be an admissible operator in $\mathcal{H}$ and $\hat{H}_1$ and $\hat{H}_2$ its self-adjoint representatives from Definition 4.4. Then $\hat{H}_1$ and $\hat{H}_2$ are resolvent comparable and*

$$\xi(\lambda; \hat{H}_1, \hat{H}_2) = 0 \pmod{\mathbb{Z}} \quad \textit{for a.e. } \lambda \in \mathbb{R}, \tag{4.12}$$

*where $\xi(\lambda; \hat{H}_1, \hat{H}_2)$ is the spectral shift function associated with the pair $(\hat{H}_1, \hat{H}_2)$ of self-adjoint operators.*

Now we are ready to extend the concept of the spectral shift function to the case of pairs of admissible operators.

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***Definition 4.7*** Let $(H, A)$ be a pair of resolvent comparable admissible operators in $\mathcal{H}$ and $(\hat{H}, \hat{A})$ a pair of their self-adjoint representatives from Definition 4.4. Define the spectral shift function $\xi(\lambda; H, A)$ associated with the pair $(H, A)$ by

$$\xi(\lambda; H, A) = \xi(\lambda; \hat{H}, \hat{A}) \pmod{\mathbb{Z}} \quad \text{for a.e. } \lambda \in \mathbb{R},$$

where $\xi(\lambda; \hat{H}, \hat{A})$ is the spectral shift function associated with the pair $(\hat{H}, \hat{A})$ of self-adjoint operators.

The result of Corollary 4.6 combined with the chain rule (4.6) for the pairs of self-adjoint operators shows that the spectral shift function associated with a pair $(H, A)$ of resolvent comparable admissible operators is well-defined modulo $\mathbb{Z}$, that is, it is independent of the choice of the self-adjoint representatives $\hat{H}$ and $\hat{A}$ for the operators $H$ and $A$, respectively. In particular, we have arrived at the following result.

***Lemma 4.8*** *Assume that $\hat{H}$ and $\hat{A}$ are self-adjoint operators and $V, V^{-1} \in \mathcal{B}(\mathcal{H})$ such that*

$$V \in \mathcal{A}(\mathcal{H}), \tag{4.13}$$

$$(\hat{H} - z)^{-1} - (\hat{A} - z)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad \operatorname{Im}(z) \neq 0, \tag{4.14}$$

$$(V^{-1}\hat{H}V - z)^{-1} - (\hat{A} - z)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad \operatorname{Im}(z) \neq 0, \tag{4.15}$$

*then the* stability property *holds*

$$\xi(\lambda; V^{-1}\hat{H}V, \hat{A}) = \xi(\lambda; \hat{H}, \hat{A}) \pmod{\mathbb{Z}} \quad \textit{for a.e. } \lambda \in \mathbb{R}. \tag{4.16}$$

The next example shows that the requirements (4.14) and (4.15) by themselves do not imply (4.16), if condition (4.13) is violated.

***Example 2*** Let $H$ be the closure of the operator $H_0 = -\frac{d^2}{dx^2}$ on $L^2(\mathbb{R})$ initially defined on the domain $\operatorname{dom}(H_0) = C_0^\infty(\mathbb{R})$ and $\hat{H}$ the operator which acts in $L^2\big((-\infty, 0)\big) \oplus L^2\big((0, \infty)\big)$ and corresponds to the Dirichlet boundary condition at zero. The difference $(H - z)^{-1} - (\hat{H} - z)^{-1}$, $\operatorname{Im}(z) \neq 0$, is rank one and, therefore, $H$ is a resolvent comparable perturbation of $\hat{H}$. The operators $H$ and $\hat{H}$ are obviously unitary equivalent and, therefore, there exists a unitary operator $V$ such that $\hat{H} = V^*HV$. The spectral shift function associated with the pair $(\hat{H}, H)$ is known [27] to be a half on the essential spectrum and zero otherwise,

$$\xi(\lambda, \hat{H}, H) = \frac{1}{2}\chi_{[0,\infty)}(\lambda) \pmod{\mathbb{Z}} \quad \text{for a.e. } \lambda \in \mathbb{R},$$

where $\chi_\Delta(\lambda)$ denotes the characteristic function of the Borel set $\Delta$. Therefore,

$$0 = \xi(\lambda, H, H) \neq \xi(\lambda, V^*HV, H) = \frac{1}{2}\chi_{[0,\infty)}(\lambda) \tag{4.17}$$

on a set of positive Lebesgue measure. Representation (4.17) shows that the stability property (4.16) for the spectral shift function does not hold in this case.

The concept of a spectral shift function associated with a pair of admissible operators turns out to be rather useful in the context of not only additive but also multiplicative theory of perturbations. The following theorem illustrates such an application to the multiplicative theory of perturbations in case where the spectral shift function can be computed via the perturbation determinant. The corresponding representation appears to be an immediate analog of Krein's formula (4.3) in the self-adjoint case. The precise statement is as follows.

***Theorem 4.9*** *Let $A$ be a possibly unbounded self-adjoint operator in $\mathcal{H}$ with domain* $\mathrm{dom}(A)$*, $B = B^*$ a trace class self-adjoint operator,*

$$B \in \mathcal{B}_1(\mathcal{H}), \tag{4.18}$$

*and $V$ a bounded operator with a bounded inverse such that*

$$I - V \in \mathcal{B}_1(\mathcal{H}). \tag{4.19}$$

*Assume, in addition, that*

$$\mathrm{ran}\,(I - V) \subset \mathrm{dom}(A), \tag{4.20}$$

*the domain* $\mathrm{dom}(A)$ *is $V$-invariant*

$$V\,\mathrm{dom}(A) = \mathrm{dom}(A), \tag{4.21}$$

*and the commutator $AV - VA$, initially defined on* $\mathrm{dom}(A)$*, is a closable operator and its closure is a trace class operator, that is,*

$$\overline{AV - VA} \in \mathcal{B}_1(\mathcal{H}). \tag{4.22}$$

*Then for the operator $H$ defined by*

$$H = V^{-1}(A + B)V \quad \textit{on} \quad \mathrm{dom}(H) = \mathrm{dom}(A) \tag{4.23}$$

*the following holds true.*

(i) *The operator $H$ is admissible. Moreover, the spectral shift function $\xi(\lambda; H, A)$ is well defined and*

$$\xi(\lambda; H, A) = \xi(\lambda; A + B, A) \pmod{\mathbb{Z}}, \quad \textit{for a.e. } \lambda \in \mathbb{R}. \tag{4.24}$$

(ii)

$$(H - z)(A + B - z)^{-1} - I \in \mathcal{B}_1(\mathcal{H}), \quad \mathrm{Im}(z) \neq 0, \tag{4.25}$$

*and, hence, the perturbation determinant*

$$D_{H/(A+B)}(z) = \det\big((H - z)(A + B - z)^{-1}\big), \quad \mathrm{Im}(z) \neq 0,$$

*is well defined and, moreover,*

$$D_{H/(A+B)}(z) = 1.$$

(iii) *The perturbation determinant* $D_{H/A}(z)$ *is well defined*

$$D_{H/A}(z) = \det\bigl((H-z)(A-z)^{-1}\bigr), \quad \operatorname{Im}(z) \neq 0,$$

*and the spectral shift function for the admissible pair* $(H, A)$ *can be computed via the perturbation determinant as follows*

$$\xi(\lambda; H, A) = \pi^{-1} \lim_{\varepsilon\downarrow 0} \arg D_{H/A}(\lambda + \mathrm{i}\varepsilon) \pmod{\mathbb{Z}} \quad \textit{for a.e. } \lambda \in \mathbb{R}. \tag{4.26}$$

**Proof** (i) Hypothesis (4.19) implies that

(a) $V \in \mathcal{A}(\mathcal{H})$

and

(b) the operators $H$ and $A + B$ are resolvent comparable.

Thus, $H$ is an admissible operator. By (4.18) the operator $A + B$ is a trace class perturbation of $A$ and hence $H$ and $A$ are resolvent comparable. Therefore, (4.24) holds by the definition of the spectral shift function for a pair of resolvent comparable admissible operators, which proves (i).

(ii) We start with the representation

$$(A + B - z)V(A + B - z)^{-1} = I + W(z), \quad \operatorname{Im}(z) \neq 0,$$

where

$$W(z) = (A + B - z)(V - I)(A + B - z)^{-1}, \quad \operatorname{Im}(z) \neq 0, \tag{4.27}$$

makes sense by (4.20). By (4.27)

$$\begin{aligned} W(z) = V - I &+ (AV - VA)(A + B - z)^{-1} \\ &+ (BV - VB)(A + B - z)^{-1}, \quad \operatorname{Im}(z) \neq 0, \end{aligned}$$

which proves that

$$W(z) \in \mathcal{B}_1(\mathcal{H}), \quad \operatorname{Im}(z) \neq 0, \tag{4.28}$$

by (4.18), (4.19) and (4.22). Therefore, the Fredholm determinant of the operator $(A + B - z)V(A + B - z)^{-1}$ is well defined and

$$\begin{aligned} &\det\bigl((A+B-z)V(A+B-z)^{-1}\bigr) \\ &\quad = \det\bigl(I + (A+B-z)(V-I)(A+B-z)^{-1}\bigr), \quad \operatorname{Im}(z) \neq 0. \end{aligned} \tag{4.29}$$

Since (4.20) holds, the operator $(A+B-z)(V-I)$ is well defined on the whole Hilbert space $\mathcal{H}$ as a closed operator being the product of two closed operators. Hence, $(A + B - z)(V - I)$ is bounded by the Closed Graph Theorem. In particular, the following representation holds

$$(A + B - z)^{-1}[(A + B - z)(V - I)] = V - I. \tag{4.30}$$

Using (4.29), (4.30), and the fact that

$$\det(I+ST)=\det(I+TS),\quad ST, TS\in\mathcal{B}_1(\mathcal{H}),$$

one proves

$$\det\big((A+B-z)V(A+B-z)^{-1}\big)=\det(V),\quad \operatorname{Im}(z)\neq 0. \tag{4.31}$$

Further, using definition (4.23) of $H$ one computes

$$(H-z)(A+B-z)^{-1}=V^{-1}(A+B-z)V(A+B-z)^{-1}),\quad \operatorname{Im}(z)\neq 0, \tag{4.32}$$

which by (4.27) and (4.28) proves (4.25). Moreover, (4.32) and (4.31) yield

$$\begin{aligned}\det\big((H-z)(A+B-z)^{-1}\big) &= \det(V^{-1})\det\big((A+B-z)\,V(A+B-z)^{-1}\big)\\ &= \det(V^{-1})\det(V)=1,\quad \operatorname{Im}(z)\neq 0,\end{aligned}$$

which completes the proof of (ii).

(iii) One infers

$$(H-z)(A-z)^{-1}=(H-z)(A+B-z)^{-1}(A+B-z)(A-z)^{-1},\quad \operatorname{Im}(z)\neq 0.$$

Hence

$$(H-z)(A-z)^{-1}-I\in\mathcal{B}_1(\mathcal{H}),\quad \operatorname{Im}(z)\neq 0,$$

by (4.25) and the fact that

$$(A+B-z)(A-z)^{-1}-I\in\mathcal{B}_1(\mathcal{H}),\quad \operatorname{Im}(z)\neq 0,$$

since $B\in\mathcal{B}_1(\mathcal{H})$, which proves that the perturbation determinant $D_{H/A}(z)$ is well defined. Moreover,

$$D_{H/A}(z)=D_{H/(A+B)}(z)D_{(A+B)/A}(z)=D_{(A+B)/A}(z),\quad \operatorname{Im}(z)\neq 0. \tag{4.33}$$

By Krein's formula (4.3) we have

$$\xi(\lambda;A+B,A)=\pi^{-1}\lim_{\varepsilon\downarrow 0}\arg D_{(A+B)/A}(\lambda+i\varepsilon)\pmod{\mathbb{Z}},$$

and hence (4.26) holds by (4.33). ■

***Remark 4.10*** The idea of introducing the spectral shift function associated with a pair of operators similar to self-adjoint operators via the perturbation determinant (in the framework of the trace class perturbations theory) goes back to V. Adamjan and H. Langer [1]. The proof of Theorem 4.9 contains some fragments of their original reasoning.

## 5 Graph Subspaces and Block Diagonalization of Operator Matrices

In this section we collect some results related to existence of invariant graph subspaces of a linear operator and to the closely related problem of block diagonalization of the operator in terms of such subspaces.

First, we recall the definition of a graph subspace.

***Definition 5.1*** Let $\mathcal{N}$ be a closed subspace of a Hilbert space $\mathcal{H}$ and $Q \in \mathcal{B}(\mathcal{N}, \mathcal{N}^{\perp})$. The set

$$\mathcal{G}(\mathcal{N}, Q) = \{x \in \mathcal{H} : P_{\mathcal{N}^{\perp}} x = Q P_{\mathcal{N}} x\}$$

is called the *graph subspace* of $\mathcal{H}$ associated with the pair $(\mathcal{N}, Q)$, where $P_{\mathcal{N}}$ and $P_{\mathcal{N}^{\perp}}$ denote the orthogonal projections onto $\mathcal{N}$ and $\mathcal{N}^{\perp}$, respectively.

It is easy to check that

$$\mathcal{G}(\mathcal{N}, Q)^{\perp} = \mathcal{G}(\mathcal{N}^{\perp}, -Q^*). \tag{5.1}$$

From the analytic point of view, the search for invariant/reducing graph subspaces for a linear self-adjoint operator in $\mathcal{H}$ is equivalent to the problem of solving the operator Riccati equations studied in details in Section 3.

We adopt the following hypothesis in the sequel.

***Hypothesis 5.2*** *Assume that the Hilbert space $\mathcal{H}$ is decomposed into the orthogonal sum of two orthogonal subspaces*

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \tag{5.2}$$

*the self-adjoint operator $\mathbf{H}$ reads with respect to the decomposition (5.2) as a $2 \times 2$ operator block matrix*

$$\mathbf{H} = \begin{pmatrix} A_0 & B_{01} \\ B_{10} & A_1 \end{pmatrix}, \tag{5.3}$$

*where $A_i$, $i = 0, 1$, are self-adjoint operators in $\mathcal{H}_i$ with domains* $\operatorname{dom}(A_i)$ *while $B_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, $j = 1 - i$, are bounded operators and $B_{10} = B_{01}^*$. Thus,*

$$\mathbf{H} = \mathbf{A} + \mathbf{B}, \tag{5.4}$$

$$\operatorname{dom}(\mathbf{H}) = \operatorname{dom}(\mathbf{A}), \tag{5.5}$$

*where $\mathbf{A}$ is the diagonal self-adjoint operator,*

$$\mathbf{A} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \tag{5.6}$$

$$\operatorname{dom}(\mathbf{A}) = \operatorname{dom}(A_0) \oplus \operatorname{dom}(A_1),$$

*and the operator $\mathbf{B} = \mathbf{B}^*$ is an off-diagonal bounded operator*

$$\mathbf{B} = \begin{pmatrix} 0 & B_{01} \\ B_{10} & 0 \end{pmatrix}. \tag{5.7}$$

We start with a criterion of existence of the invariant graph subspaces $\mathcal{G}(\mathcal{H}_i, Q_{ji})$ $(Q_{ji} \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j))$, $i = 0, 1$, $j = 1-i$, associated with the $2\times 2$ block decomposition (5.3) of a self-adjoint operator $\mathbf{H}$.

***Lemma 5.3*** *Assume Hypothesis 5.2. The graph subspace $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$ for some $Q_{ji} \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$, $i = 0, 1$, $j = 1 - i$, is a reducing subspace for the operator $\mathbf{H}$ if and only if the operator Riccati equation*

$$\mathbf{QA} - \mathbf{AQ} + \mathbf{QBQ} = \mathbf{B}, \tag{5.8}$$

*has a strong solution $\mathbf{Q}$ which reads with respect to the decomposition (5.2) as*

$$\mathbf{Q} = \begin{pmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{pmatrix} \tag{5.9}$$

*with*

$$Q_{01} = -Q_{10}^*. \tag{5.10}$$

**Proof** If $\mathbf{Q}$ given by (5.9), (5.10) is a strong solution of (5.8), this means that

$$\operatorname{ran}(\mathbf{Q}|_{\operatorname{dom}(\mathbf{A})}) \subset \operatorname{dom}(\mathbf{A}), \tag{5.11}$$

and

$$\mathbf{QA}f - \mathbf{AQ}f + \mathbf{QBQ}f = \mathbf{B}f \quad \text{for any } f \in \operatorname{dom}(\mathbf{A}). \tag{5.12}$$

Under hypotheses (5.9), (5.10), and (5.11) we have the inclusions

$$\operatorname{ran}(Q_{ji}|_{\operatorname{dom}(A_i)}) \subset \operatorname{dom}(A_j), \quad i = 0, 1, \ j = 1 - i. \tag{5.13}$$

Moreover, the Riccati equation (5.12) splits into the pair of equations

$$Q_{ji}A_i f - A_j Q_{ji} f + Q_{ji} B_{ij} Q_{ji} f = B_{ji} f \\ \text{for all } f \in \operatorname{dom}(A_i), \quad i = 0, 1, \ j = 1 - i. \tag{5.14}$$

Rewriting these equations in the form

$$Q_{ji}(A_i + B_{ij}Q_{ji})f = (B_{ji} + A_j Q_{ji})f \quad \text{for all } f \in \operatorname{dom}(A_i) \tag{5.15}$$

one immediately observes that (5.15) combined with (5.13) is equivalent to invariance of the subspaces $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$, $i = 0, 1$, $j = 1 - i$, for the operator $\mathbf{H}$. In turn, (5.10) implies the invariance of the subspace $\mathcal{G}_i^\perp = \mathcal{G}(\mathcal{H}_j, -Q_{ji}^*)$, $i = 0, 1$, $j = 1 - i$, for $\mathbf{H}$, which proves the lemma. ■

***Remark 5.4*** Example 1 shows that, in general, the Riccati equations (5.14) are not always solvable and, thus, the invariant graph subspaces may not always exist either.

If the operator block matrix $\mathbf{H}$ has reducing graph subspaces, then the block diagonalization problem can be solved explicitly.

***Theorem 5.5*** *Assume Hypothesis 5.2. Assume, in addition, that the graph subspaces $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$ for some $Q_{ji} \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$, $i = 0, 1$, $j = 1 - i$, satisfying (5.10) are reducing subspaces for the operator $\mathbf{H}$. Then:*

(i) *The operator $\mathbf{V} = \mathbf{I} + \mathbf{Q}$ with $\mathbf{Q}$ given by (5.9), (5.10) has a bounded inverse.*

(ii) *The operator $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ is block diagonal with respect to the decomposition (5.2). That is,*

$$\mathbf{V}^{-1}\mathbf{H}\mathbf{V} = \begin{pmatrix} A_0 + B_{01}Q_{10} & 0 \\ 0 & A_1 + B_{10}Q_{01} \end{pmatrix} \tag{5.16}$$

*where*

$$\operatorname{dom}(A_i + B_{ij}Q_{ji}) = \operatorname{dom}(A_i), \quad i = 0, 1, \ j = 1 - i. \tag{5.17}$$

(iii) *The operator $\mathbf{U}^*\mathbf{H}\mathbf{U}$, where $\mathbf{U}$ is the unitary operator from the polar decomposition $\mathbf{V} = \mathbf{U}|\mathbf{V}|$, is block-diagonal with respect to the decomposition (5.2). That is,*

$$\mathbf{U}^*\mathbf{H}\mathbf{U} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} \tag{5.18}$$

*with*

$$H_i = (I_{\mathcal{H}_i} + Q_{ji}^*Q_{ji})^{1/2}(A_i + B_{ij}Q_{ji})(I_{\mathcal{H}_i} + Q_{ji}^*Q_{ji})^{-1/2}, \quad i = 0, 1, \ j = 1 - i, \tag{5.19}$$

$$\operatorname{dom}(H_i) = (I_{\mathcal{H}_i} + Q_{ji}^*Q_{ji})^{1/2}\big(\operatorname{dom}(A_i)\big), \tag{5.20}$$

*where $I_{\mathcal{H}_i}$ stands for the identity operator in $\mathcal{H}_i$.*

**Proof** (i) By (5.10) $\mathbf{Q}^* = -\mathbf{Q}$ and, thus, the spectrum of $\mathbf{Q}$ is a subset of the imaginary axis. This means that zero does not belong to the spectrum of $\mathbf{V} = \mathbf{I} + \mathbf{Q}$ and, hence, $\mathbf{V}$ has a bounded inverse.

(ii) Since by (i) $\mathbf{V}$ has a bounded inverse, (5.16) is equivalent to the representation

$$\mathbf{H}\mathbf{V} = \mathbf{V}\begin{pmatrix} A_0 + B_{01}Q_{10} & 0 \\ 0 & A_1 + B_{10}Q01 \end{pmatrix},$$

which, in turn, taking into account (5.17), is equivalent to the Riccati equation (5.8). Then, applying Lemma 5.3, the validity of (5.16)–(5.17) is equivalent to the fact that the graph subspaces $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$, $i = 0, 1$, $j = 1 - i$, are reducing subspaces.

(iii) Taking into account (5.10), by inspection one gets

$$\mathbf{V}\mathbf{V}^* = \mathbf{V}^*\mathbf{V} = \begin{pmatrix} I_0 + Q_{01}Q_{01}^* & 0 \\ 0 & I_1 + Q_{10}Q_{10}^* \end{pmatrix}. \tag{5.21}$$

Since $\mathbf{V} = \mathbf{U}|\mathbf{V}|$ and $|\mathbf{V}| = (\mathbf{V}\mathbf{V}^*)^{1/2}$, the validity of (5.18)–(5.20) is an immediate consequence of (5.16)–(5.17). ∎

## 6 Invariant Graph Subspaces and Splitting of the Spectral Shift Function

It is convenient to study spectral properties of the perturbed block operator matrix $\mathbf{H}$ not only in terms of the perturbation $\mathbf{B} = \mathbf{H} - \mathbf{A}$ in itself, but also in terms of the angular operator $\mathbf{Q}$ associated with the reducing graph subspaces, provided that they exists. The next (conditional) result throws light upon the *quantitative* aspects of the perturbation theory for block operator matrices in this context.

***Theorem 6.1*** *Assume Hypothesis 5.2 and let the Riccati equation (5.8) have a strong solution* $\mathbf{Q}$ *of the form (5.9). Assume, in addition, that*

(i) $\mathbf{Q}$ *is a Hilbert-Schmidt operator,*
(ii) $\mathbf{BQ}(\mathbf{A} - z)^{-1}$ *is a trace class operator for* $\operatorname{Im}(z) \neq 0$,
(iii) $\mathbf{H}$ *and* $\mathbf{A}$ *are resolvent comparable.*

*Then* $A_i + B_{ij}Q_{ji}$, $i = 0, 1$, $j = 1 - i$, *are admissible operators. Moreover,* $A_i + B_{ij}Q_{ji}$ *and* $A_i$, $i = 0, 1$, $j = 1 - i$, *are resolvent comparable. For the spectral shift function* $\xi(\lambda, \mathbf{H}, \mathbf{A})$ *associated with the pair of self-adjoint operators* $(\mathbf{H}, \mathbf{A})$ *we have the decomposition*

$$\begin{aligned}(6.1)\qquad \xi(\lambda; \mathbf{H}, \mathbf{A}) &= \xi(\lambda; A_0 + B_{01}Q_{10}, A_0)\\ &\quad + \xi(\lambda; A_1 + B_{10}Q_{01}, A_1) \pmod{\mathbb{Z}}, \quad \textit{for a.e. } \lambda \in \mathbb{R}.\end{aligned}$$

*In particular, the operator matrix* $\mathbf{H}$ *can be block diagonalized by a unitary transformation (5.2)*

$$\mathbf{U}^*\mathbf{H}\mathbf{U} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix},$$

*where* $\mathbf{U}$ *is the unitary operator from the polar decomposition*

$$\mathbf{I} + \mathbf{Q} = \mathbf{U}|\mathbf{I} + \mathbf{Q}|,$$

*and*

$$(6.2)\qquad \xi(\lambda; \mathbf{H}, \mathbf{A}) = \xi(\lambda; H_0, A_0) + \xi(\lambda; H_1, A_1) \pmod{\mathbb{Z}}, \quad \textit{for a.e. } \lambda \in \mathbb{R}.$$

**Proof** By Theorem 6.1 (i) the normal operator $\mathbf{V} = \mathbf{I} + \mathbf{Q}$ has a bounded inverse. Due to the assumption (i) the spectrum of $\mathbf{V}$ is purely point. Thus, by Lemma 4.2(vi)

$$(6.3)\qquad \mathbf{V} \in \mathcal{A}(\mathcal{H}),$$

where $\mathcal{A}(\mathcal{H})$ is the zero trace commutator class introduced by Definition 4.1. By Theorem 5.5(ii) one concludes

$$(6.4)\qquad \mathbf{V}^{-1}\mathbf{H}\mathbf{V} = \mathbf{A} + \mathbf{BQ}.$$

Therefore, since by hypothesis (ii) the operator $\mathbf{BQ}$ is a relatively trace class perturbation of $\mathbf{A}$, one concludes that the operators $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ and $\mathbf{A}$ are resolvent comparable.

By condition (iii) $\mathbf{H}$ and $\mathbf{A}$ are also resolvent comparable, and, therefore, by (6.3) the operator $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ is admissible with the self-adjoint representative $\mathbf{H}$. Thus, the stability property holds

$$\xi(\lambda; \mathbf{V}^{-1}\mathbf{H}\mathbf{V}, \mathbf{A}) = \xi(\lambda; \mathbf{H}, \mathbf{A}) \pmod{\mathbb{Z}}, \quad \text{for a.e. } \lambda \in \mathbb{R}, \tag{6.5}$$

by the definition of the spectral shift function for resolvent comparable admissible operators.

Next, let $\mathbf{V} = \mathbf{U}|\mathbf{V}|$ be the polar decomposition of $\mathbf{V}$. By Theorem 5.5(iii) the operator $\mathbf{V}\mathbf{V}^*$ is diagonal with respect to the decomposition (5.2). Using representation (5.21) one infers that $\mathbf{V}\mathbf{V}^* - \mathbf{I}$ is a trace class operator, since $\mathbf{Q}$ is the Hilbert-Schmidt operator by the hypothesis. Therefore,

$$|\mathbf{V}| - \mathbf{I} \in \mathcal{B}_1(\mathcal{H}), \tag{6.6}$$

where $|\mathbf{V}| = (\mathbf{V}\mathbf{V}^*)^{1/2}$, and, hence, $|\mathbf{V}| \in \mathcal{A}(\mathcal{H})$ by (6.6). The operator $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ is similar to the self-adjoint operator $\mathbf{U}^*\mathbf{H}\mathbf{U}$:

$$\mathbf{V}^{-1}\mathbf{H}\mathbf{V} = |\mathbf{V}|(\mathbf{U}^*\mathbf{H}\mathbf{U})|\mathbf{V}|^{-1}. \tag{6.7}$$

Using (6.6) and (6.7), one concludes that $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ and $\mathbf{U}^*\mathbf{H}\mathbf{U}$ are resolvent comparable. Therefore, taking into account that $|\mathbf{V}| \in \mathcal{A}(\mathcal{H})$ one infers that $\mathbf{U}^*\mathbf{H}\mathbf{U}$ is a self-adjoint representative of the admissible operator $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ and, hence,

$$\xi(\lambda; \mathbf{V}^{-1}\mathbf{H}\mathbf{V}, \mathbf{A}) = \xi(\lambda; \mathbf{U}^*\mathbf{H}\mathbf{U}, \mathbf{A}) \pmod{\mathbb{Z}}, \quad \text{for a.e. } \lambda \in \mathbb{R}, \tag{6.8}$$

by Lemma (4.8). By Theorem 5.5(iii) the operator $\mathbf{U}^*\mathbf{H}\mathbf{U}$ is diagonal with respect to decomposition (5.2)

$$\mathbf{U}^*\mathbf{H}\mathbf{U} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix},$$

where $H_i$, $i = 0, 1$, are self-adjoint operators in the Hilbert spaces $\mathcal{H}_i$, $i = 0, 1$, introduced by (5.19) and (5.17). Since $\mathbf{U}^*\mathbf{H}\mathbf{U}$ is a block-diagonal operator, by additivity of the spectral shift function associated with a pair of self-adjoint operators with respect to direct sum decompositions (which follows from the definition of the spectral shift function by the trace formula (4.1)) one obtains that

$$\xi(\lambda; \mathbf{U}^*\mathbf{V}\mathbf{U}, \mathbf{A}) = \sum_{i=0}^{1} \xi(\lambda; H_i, A_i). \tag{6.9}$$

By Theorem 5.5(ii) the operator $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ is diagonal with respect to the decomposition (5.2)

$$\mathbf{V}^{-1}\mathbf{H}\mathbf{V} = \begin{pmatrix} A_0 + B_{01}Q_{10} & 0 \\ 0 & A_1 + B_{10}Q_{01} \end{pmatrix},$$

where $A_i + B_{ij}Q_{ji}$, $i = 0, 1$, $j = 1 - i$, are operators similar to self-adjoint operators $H_i$ given by (5.19):

$$H_i = (I_{\mathcal{H}_i} + Q_{ji}^*Q_{ji})^{1/2}(A_i + B_{ij}Q_{ji})(I_{\mathcal{H}_i} + Q_{ji}^*Q_{ji})^{-1/2}, \quad i = 0, 1, \ j = 1 - i.$$

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Here $Q_{ij}$, $i = 0, 1$, $j = 1 - i$, are the entries in the matrix representation for the operator $\mathbf{Q}$

$$\mathbf{Q} = \begin{pmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{pmatrix}.$$

By hypothesis (i) $\mathbf{Q}$ is a Hilbert-Schmidt operator, which proves that

$$(I_{\mathcal{H}_i} + Q_{ji}^* Q_{ji})^{1/2} - I_{\mathcal{H}_i} \in \mathcal{B}_1(\mathcal{H}_i) \quad i = 0, 1,\ j = 1 - i.$$

Therefore, the operators $A_i + B_{ij} Q_{ji}$, $i = 0, 1$, $j = 1 - i$, are admissible with the self-adjoint representatives $H_i$. Since $\mathbf{V}^{-1}\mathbf{H}\mathbf{V}$ and $\mathbf{A}$ are resolvent comparable, so $A_i + B_{ij} Q_{ji}$ and $A_i$, $i = 0, 1$, $j = 1 - i$, are. Hence, we have the following representation by Lemma 4.8

$$\xi(\lambda; H_i, A_i) = \xi(\lambda; A_i + B_{ij} Q_{ji}, A_i), \quad (\mathrm{mod}\ \mathbb{Z}), \qquad \text{for a.e. } \lambda \in \mathbb{R},\ i = 0, 1,\ j = 1 - i. \tag{6.10}$$

Combining (6.5), (6.8), (6.9), and (6.10) proves (6.1). ■

***Remark 6.2*** If the operator $\mathbf{Q}$ is a trace class operator, the conditions (ii) and (iii) hold automatically. Therefore, they are redundant in this case.

## 7 Further Properties of the Spectral Shift Function

Throughout this section we assume that the spectra of the main diagonal entries $A_0$ and $A_1$ of the operator matrix (5.3) are separated. More specifically, we will adopt one of the three following hypotheses.

***Hypothesis 7.1*** *Assume Hypothesis 5.2 and suppose that the separation condition*

$$\operatorname{dist}\{\operatorname{spec}(A_0), \operatorname{spec}(A_1)\} = d > 0 \tag{7.1}$$

*holds true. Assume, in addition, that $B_{10}$ has a finite norm with respect to the spectral measure of $A_0$ or/and $A_1$ and, moreover,*

$$\|B_{01}\| \min\{\|B_{01}\|_{E_{A_1}}, \|B_{01}\|_{E_{A_0}}\} < \frac{d^2}{4}. \tag{7.2}$$

***Hypothesis 7.2*** *Assume Hypothesis 5.2 and suppose that the separation condition (7.1) holds true. Assume, in addition, that both operators $A_0$ and $A_1$ are bounded and*

$$\|B_{01}\| < \frac{d}{\pi}. \tag{7.3}$$

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***Hypothesis 7.3*** *Assume Hypothesis 5.2. Assume, in addition, that the operator $A_0$ is semibounded from above,*

$$A_0 \le a_0 < +\infty,$$

*the operator $A_1$ is semibounded from below,*

$$A_1 \ge a_1 > -\infty,$$

*and*

$$a_0 < a_1.$$

***Theorem 7.4*** *Assume Hypothesis 7.1. Then the block operator matrix $\mathbf{H}$ has two (orthogonal to each other) reducing graph subspaces $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$, $i = 0, 1$, $j = 1 - i$, associated with angular operators $Q_{ji} \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$ such that*

$$Q_{10} = -Q_{01}^*$$

*and*

$$\|B_{ij}Q_{ji}\| \le \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|_{01} \min\{\|B_{01}\|_{E_{A_0}}, \|B_{01}\|_{E_{A_1}}\}} < \frac{d}{2}, \quad i = 0, 1,\ j = 1 - i. \tag{7.4}$$

*Moreover, the graph subspaces $\mathcal{G}_i$, $i = 0, 1$, are the spectral subspaces of $\mathbf{H}$ and $\mathcal{G}_0 \oplus \mathcal{G}_1 = \mathcal{H}$.*

**Proof** Assume, for definiteness, that the operator $B_{10}$ has a finite norm with respect to the spectral measure of the diagonal entry $A_1$ of $\mathbf{H}$ and the inequality holds

$$\|B_{01}\|\,\|B_{10}\|_{E_{A_1}} < \frac{d^2}{4}. \tag{7.5}$$

Recall that by definition $\|B_{10}\|_{E_{A_1}} = \|B_{10}^*\|_{E_{A_1}}$ and hence $\|B_{10}\|_{E_{A_1}} = \|B_{01}\|_{E_{A_1}}$.

By Theorem 3.6(ii) the Riccati equation

$$QA_0 - A_1Q + QB_{01}Q = B_{10} \tag{7.6}$$

has a unique strong solution $Q \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$. Therefore, the dual Riccati equation

$$KA_1 - A_0K + KB_{10}K = B_{01} \tag{7.7}$$

has a unique strong solution $K \in \mathcal{B}(\mathcal{H}_1)$ by Theorem 3.5, and, moreover, $K = -Q^*$. Introducing the notations $Q_{10} = Q$ and $Q_{01} = K$, equations (7.6) and (7.7) can be rewritten in the form

$$Q_{ji}A_i - A_jQ_{ji} + Q_{ji}B_{ij}Q_{ji} = B_{ji}, \quad i = 0, 1,\ j = 1 - i. \tag{7.8}$$

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Therefore, the Riccati equation (5.8) has a strong solution of the form (5.9). Applying Lemma 5.3 one proves that the subspaces $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$, $i = 0, 1$, $j = 1 - i$, are reducing subspaces for $\mathbf{H}$, which proves the first assertion of the theorem under hypothesis (7.5).

In the case where $B_{10}$ has a finite norm with respect to the spectral measure of the diagonal entry $A_0$ and the inequality

$$\|B_{01}\| \, \|B_{10}\|_{E_{A_0}} < \frac{d^2}{4}$$

holds, the proof can be performed in an analogous way.

Applying Theorem 3.6(ii) (equation (3.29)) proves estimate (7.4) which, in turn, proves that

$$\operatorname{dist}\{\operatorname{spec}(A_0 + B_{01}Q_{10}), \operatorname{spec}(A_1 + B_{10}Q_{01})\} > 0.$$

The last assertion of the theorem is a corollary of Theorem 5.5.

The proof is complete. ■

***Remark 7.5*** Under Hypothesis 7.1, if

$$\sup_{\mu \in \operatorname{spec}(A_j)} \|\mu(A_i + B_{ij}Q_{ji} - \mu)^{-1}\| < \infty$$

for some $i = 0, 1$, $j = 1 - i$, then the strong solutions of the Riccati equations (7.6), (7.7) turn out to be the operator solutions by Theorem 3.5.

Under Hypothesis 7.2 one has a similar result.

***Theorem 7.6*** *Assume Hypothesis 7.2. Then the block operator matrix* $\mathbf{H}$ *has two (orthogonal to each other) reducing graph subspaces* $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$, $i = 0, 1$, $j = 1 - i$, *associated with the strictly contractive angular operators* $Q_{ji} \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$, $\|Q_{ji}\| < 1$, *such that*

$$Q_{10} = -Q_{01}^*.$$

*Moreover, the graph subspaces* $\mathcal{G}_i$, $i = 0, 1$, *are the spectral subspaces of* $\mathbf{H}$ *and* $\mathcal{G}_0 \oplus \mathcal{G}_1 = \mathcal{H}$.

**Proof** The proof is analogous to that of Theorem 7.4. The only difference is that now we refer to part (i) of Theorem 3.6, since for bounded $A_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 0, 1$. the concepts of the weak, strong and operator solutions of the Riccati equations (7.8) coincide. ■

The following statement has been proven in [3].

***Theorem 7.7*** *Assume Hypothesis 7.3. Then for any* $B_{01} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ *and* $B_{10} = B_{01}^*$ *the open interval* $(a_0, a_1)$ *appears to be a spectral gap for* $\mathbf{H}$. *At the same time the spectral subspaces of the operator* $\mathbf{H}$ *corresponding to the intervals* $(-\infty, a_0]$ *and*

*$[a_1, +\infty)$ admit representation in the form of graph subspaces associated with the pairs $(\mathcal{H}_0, Q_{10})$ and $(\mathcal{H}_1, Q_{01})$ for some $Q_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, $i = 0, 1$, $j = 1 - i$. That is,*

$$\operatorname{ran}\Big(E_{\mathbf{H}}\big((-\infty, a_0]\big)\Big) = \mathcal{G}(\mathcal{H}_0, Q_{10}) \tag{7.9}$$

*and*

$$\operatorname{ran}\Big(E_{\mathbf{H}}\big([a_1, +\infty)\big)\Big) = \mathcal{G}(\mathcal{H}_1, Q_{01}), \tag{7.10}$$

*where $E_{\mathbf{H}}(\Delta)$ denotes the spectral projection of $\mathbf{H}$ associated with the Borel set $\Delta \subset \mathbb{R}$. The angular operators $Q_{ij}$ are strict contractions, $\|Q_{ij}\| < 1$, possessing the property $Q_{10} = -Q_{10}^*$.*

*Moreover, the projections $E_{\mathbf{H}}\big((-\infty, a_0]\big)$ and $E_{\mathbf{H}}\big([a_1, +\infty)\big)$ can be expressed in terms of the operator $Q = Q_{01} = Q_{10}^*$ in the following way*

$$E_{\mathbf{H}}\big((-\infty, a_0]\big) = \begin{pmatrix} (I_0 + QQ^*)^{-1} & -(I_0 + QQ^*)^{-1}Q \\ -Q^*(I_0 + QQ^*)^{-1} & Q^*(I_0 + QQ^*)^{-1}Q \end{pmatrix}$$

*and*

$$E_{\mathbf{H}}\big([a_1, +\infty)\big) = \begin{pmatrix} Q(I_1 + Q^*Q)^{-1}Q^* & Q(I_1 + Q^*Q)^{-1} \\ (I_1 + Q^*Q)^{-1}Q^* & (I_1 + Q^*Q)^{-1} \end{pmatrix}.$$

***Corollary 7.8*** *Assume Hypothesis 7.3. Then for any $B_{01} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ and $B_{10} = B_{01}^*$ the Riccati equation*

$$Q_{10}A_0 - A_1Q_{10} + Q_{10}B_{01}Q_{10} = B_{10} \tag{7.11}$$

*has a strong contractive solution $Q_{10} \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$, $\|Q_{10}\| < 1$, and $Q_{01} = -Q_{10}^*$ is the strong solution to the dual Riccati equation*

$$Q_{01}A_1 - A_0Q_{01} + Q_{01}B_{10}Q_{01} = B_{01}. \tag{7.12}$$

*For the spectra of the operators $A_0 + B_{01}Q_{10}$ with $\operatorname{dom}(A_0 + B_{01}Q_{10}) = \operatorname{dom}(A_0)$ and $A_1 + B_{10}Q_{01}$ with $\operatorname{dom}(A_1 + B_{10}Q_{01}) = \operatorname{dom}(A_1)$ the following inclusions hold true:*

$$\operatorname{spec}(A_0 + B_{01}Q_{10}) \subset (-\infty, a_0] \quad \textit{and} \quad \operatorname{spec}(A_1 + B_{10}Q_{01}) \subset [a_1, +\infty). \tag{7.13}$$

**Proof** Any spectral subspace for $\mathbf{H}$ is its reducing subspace. Thus, by Theorem 7.7 the subspaces (7.9) and (7.10) are reducing graph subspaces for $\mathcal{H}$. Then Lemma 5.3 implies that the angular operators $Q_{01}$ and $Q_{10}$ from the r.h.s. parts of formulas (7.9) and (7.10) are strong solutions to equations (7.11) and (7.12), respectively. A proof of (7.13) can be found in [3]. ■

***Remark 7.9*** Under Hypothesis 7.3 the case where one of the self-adjoint operators $A_0$ or $A_1$ is bounded has been treated first in [2]. Recently this case has been revisited in [4] where sufficient conditions implying uniqueness of the strictly contractive solutions to the operator Riccati equations have been found.

***Lemma 7.10*** *Assume at least one of the Hypotheses 7.1, 7.2, and 7.3. Then the block operator matrix*

$$\mathbf{H}_t = \mathbf{A} + t\mathbf{B}, \quad t \in [0,1]$$

*has two (orthogonal to each other) reducing graph subspaces*

$$\mathcal{G}\big(\mathcal{H}_i, \mathrm{Q}_{ji}(t)\big), \quad i = 0,1,\ j = 1-i,\ t \in [0,1],$$

*associated with angular operators* $\mathrm{Q}_{ji}(t) \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$ *which continuously depend on* $t \in [0,1]$ *in the norm of the space* $\mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$*. In addition, under Hypothesis 7.1 the following holds true:*

$$\begin{aligned}\|B_{ij}\mathrm{Q}_{ji}(t)\| &\le \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|_{01} \min\{\|B_{01}\|_{E_{A_0}}, \|B_{01}\|_{E_{A_1}}\}} \\ &< \frac{d}{2}, \quad i = 0,1,\ j = 1-i,\ t \in [0,1].\end{aligned}$$

*Under Hypotheses 7.2 or 7.3 the operators* $\mathrm{Q}_{ji}(t)$ *are strict contractions,*

$$\|\mathrm{Q}_{ji}(t)\| < 1, \quad i = 0,1,\ j = 1-i,\ t \in [0,1].$$

**Proof** Under Hypothesis 7.1 or 7.2 this assertion is an immediate consequence of Theorems 7.4 or 7.6 respectively, and Remark 3.8.

Therefore, assume Hypothesis 7.3. Since the operator $\mathbf{B}$ is bounded, and the interval $(a_0, a_1)$ does not contain points of the spectrum of $\mathbf{H}_t$ for all $t \in \mathbb{R}$, by a result by Heinz [30] (see also [34], Theorem 5.12) the spectral projection

$$E(t) = E_{\mathbf{H}_t}\big((-\infty, a_0]\big), \quad t \in \mathbb{R}$$

continuously depends on $t \in \mathbb{R}$ in the uniform operator topology. By Theorem 7.7 the projection $E(t)$ admits matrix representation with respect to the direct sum of the Hilbert spaces $\mathcal{H}_0 \oplus \mathcal{H}_1$

$$E(t) = \begin{pmatrix} (I_0 + Q_tQ_t^*)^{-1} & -(I_0 + Q_tQ_t^*)^{-1}Q_t \\ -Q_t^*(I_0 + Q_tQ_t^*)^{-1} & Q_t^*(I_0 + Q_tQ_t^*)^{-1}Q_t \end{pmatrix}, \quad t \in \mathbb{R},$$

where $Q_t = \mathrm{Q}_{01}(t)$, $t \in \mathbb{R}$. In particular, the continuity of the family $\{E(t)\}_{t\in\mathbb{R}}$ implies the continuity of the families of operators $\{(I_0 + Q_tQ_t^*)^{-1}\}_{t\in\mathbb{R}}$ and $\{(I_0 + Q_tQ_t^*)^{-1}Q_t\}_{t\in\mathbb{R}}$ in the uniform operator topology of the spaces $\mathcal{B}(\mathcal{H}_0)$ and $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$, respectively. Since the family $\{(I_0+Q_tQ_t^*)^{-1}\}_{t\in\mathbb{R}}$ is continuous, the family $\{(I_0+Q_tQ_t^*)\}_{t\in\mathbb{R}}$ is also continuous. Multiplying the operator $(I_0+Q_tQ_t^*)^{-1}Q_t$ by $I_0 + Q_tQ_t^*$ from the left proves the continuity of the angular operators $Q_t$ as a function of $t$ in the uniform operator topology. Recalling now that $\mathrm{Q}_{10}(t) = -\mathrm{Q}_{01}(t)^* = -Q_t^*$ proves the continuity of the family $\mathrm{Q}_{ij}(t)$, $i = 0,1$, $j = 1-i$, as a function of the parameter $t \in \mathbb{R}$ in the uniform operator topology. The proof is complete. ■

To a large extent, the angular operator $\mathbf{Q}$, being a strong solution to the Riccati equation (5.8), inherits some properties of the operator $\mathbf{B}$. For instance, if $\mathbf{B}$ belongs to a symmetric ideal, so does $\mathbf{Q}$, provided that the certain spectra separation conditions are fulfilled for $A_0$ and $A_1$. In fact, we have the following result (for simplicity, formulated using the scale of Schatten-von Neumann ideals).

***Theorem 7.11*** *Assume Hypothesis 5.2 and let the Riccati equation (5.8) have a strong solution* $\mathbf{Q}$ *of the form (5.9) with respect to the decomposition* $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. *Assume, in addition, that either condition (7.1) is valid or the condition*

$$\operatorname{dist}\{\operatorname{spec}(A_i + B_{ij}Q_{ji}), \operatorname{spec}(A_j)\} > 0 \quad \textit{for some } i, j = 0, 1,\ i \neq j, \tag{7.14}$$

*holds. Then if* $\mathbf{B} \in \mathcal{B}_p(\mathcal{H})$ *for some* $p \geq 1$, *then* $\mathbf{Q} \in \mathcal{B}_p(\mathcal{H})$.

**Proof** We recall that the strong solvability of the Riccati equation (5.8) under constraint (5.9) is equivalent to the strong solvability of the following pair of equations

$$Q_{ji}A_i - A_iQ_{ji} = B_{ji} - Q_{ji}B_{ij}Q_{ji}, \quad i = 0, 1,\ j = 1 - i. \tag{7.15}$$

Therefore, the assumption $\mathbf{B} \in \mathcal{B}_p(\mathcal{H})$ for some $p \geq 1$ implies $B_{ij} \in \mathcal{B}_p(\mathcal{H}_j, \mathcal{H}_i)$ $i = 0, 1$, $j = 1 - i$. Hence, the r.h.s. of (7.15) is an element of the space $\mathcal{B}_p(\mathcal{H}_i, \mathcal{H}_j)$. Under hypothesis (7.1) one concludes that $Q_{ji} \in \mathcal{B}_p(\mathcal{H}_i, \mathcal{H}_j)$ by Theorem 2.7 (in particular by estimate (2.14)), and, thus, $\mathbf{Q} \in \mathcal{B}_p(\mathcal{H})$, since (5.9) holds.

Further, assume that (7.14) holds for some $i = 0, 1$, $j = 1 - i$. By Theorem (5.5) the operator $A_i + B_{ij}Q_{ji}$, $i = 0, 1$, $j = 1 - i$, is similar to a self-adjoint operator $H_i$. That is, the representation holds

$$A_i + B_{ij}Q_{ji} = VH_iV^{-1}, \quad i = 0, 1,\ j = 1 - i, \tag{7.16}$$

for some $V_i \in \mathcal{B}(\mathcal{H}_i)$ such that $V_i^{-1} \in \mathcal{B}(\mathcal{H}_i)$ (see (5.19)). Therefore, (7.15) can be rewritten in the form

$$Q_{ji}V_iH_iV_i^{-1} - A_jQ_{ji} = B_{ji}$$

and, hence, the operator $X_{ji} = Q_{ji}V_i$ is a strong solution to the Sylvester equation

$$X_{ji}H_i - A_jX_{ji} = B_{ji}V_i, \quad i = 0, 1,\ j = 1 - i.$$

By (7.14) and (7.16) one infers

$$\operatorname{dist}\{\operatorname{spec}(A_0 + B_{01}Q_{10}), \operatorname{spec}(A_1)\} > 0.$$

Meanwhile, the assumption $\mathbf{B} \in \mathcal{B}_p(\mathcal{H})$ for some $p \geq 1$ implies $B_{ji} \in \mathcal{B}_p(\mathcal{H}_i, \mathcal{H}_j)$ and, hence, $B_{ji}V_i \in \mathcal{B}_p(\mathcal{H}_i, \mathcal{H}_j)$, $i = 0, 1$, $j = 1 - i$. Applying Theorem 2.7 once more, one deduces that $X_{ji} \in \mathcal{B}_p(\mathcal{H}_i, \mathcal{H}_j)$. Hence, $Q_{ji} = T_{ji}V_i^{-1} \in \mathcal{B}_p(\mathcal{H}_i, \mathcal{H}_j)$, $i = 0, 1$, $j = 1 - i$. Finally, by (5.9) one concludes that $\mathbf{Q} \in \mathcal{B}_p(\mathcal{H})$.

The proof is complete. ■

In what follows we need one abstract result of a topological nature.

***Lemma 7.12*** *Let $L_t$, $t \in [0,1]$ be a one-parameter family of self-adjoint operators such that $L_t$ and $L_0$ are resolvent comparable for all $t \in [0,1]$ and the difference $(L_t - z)^{-1} - (L_0 - z)^{-1}$, $\operatorname{Im}(z) \neq 0$, is a continuous function of $t \in [0,1]$ in the trace class topology. Assume, in addition, that*

$$[a,b] \cap \operatorname{spec}(L_t) = \varnothing \quad \text{for all } t \in [0,1]$$

*for some $a, b \in \mathbb{R}$, $a < b$. Then, for the unique family of the spectral shift functions $\xi(\,\cdot\,; L_t, L_0)$ continuous in $t \in [0,1]$ in the topology of the weighted space $L^1(\mathbb{R}; (1+\lambda^2)^{-1})$ with the weight $(1+\lambda^2)^{-1}$, one has*

$$\xi(\lambda; L_t, L_0) = 0 \quad \text{for a.e. } \lambda \in [a,b],\ t \in [0,1]. \tag{7.17}$$

**Proof** The existence of the one-parameter family of the spectral shift functions $\xi(\,\cdot\,; L_t, L_0)$, $t \in [0,1]$ that is continuous in the topology of the weighted space $L^1\big(\mathbb{R}; (1+\lambda^2)^{-1}\big)$ is proven in [69]. Next, since $[a,b]$ belongs to the spectral gap of $L_t$ for any $t \in [0,1]$, the spectral shift function $\xi(\lambda; L_t, L_0)$ is a constant $n(t) \in \mathbb{Z}$ a.e. on the interval $[a,b]$. Integrating the difference $n(t) - n(s)$ over $\lambda \in [a,b]$ with the weight $(1+\lambda^2)^{-1}$ yields the estimate

$$|n(t) - n(s)| \leq \frac{\|\xi(\,\cdot\,; L_t, L_0) - \xi(\,\cdot\,; L_s, L_0)\|_{L^1(\mathbb{R};(1+\lambda^2)^{-1})}}{\arctan(b) - \arctan(a)}, \quad t, s \in [0,1],$$

which proves that $n(t)$ is a continuous integer-valued function of $t \in [0,1]$. Since $n(0) = 0$, it follows that $n(t) = 0$ for all $t \in [0,1]$. ∎

Now we are prepared to present the main result of the paper.

***Theorem 7.13*** *Assume Hypothesis 5.2 and at least one of Hypotheses 7.1, 7.2, and 7.3. Then the Riccati equation (5.8) has a strong solution $\mathbf{Q} \in \mathcal{B}(\mathcal{H})$ of the form*

$$\mathbf{Q} = \begin{pmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{pmatrix}, \quad Q_{10} = -Q_{01}^* \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1),$$

*written with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, and hence the operator $\mathbf{H}$ has reducing graph subspaces $\mathcal{G}_i = \mathcal{G}(\mathcal{H}_i, Q_{ji})$, $i = 0, 1$, $j = 1 - i$. If $\mathbf{H}$ and $\mathbf{A}$ are resolvent comparable and $\mathbf{B}$ is a Hilbert-Schmidt operator, then $A_i + B_{ij}Q_{ji}$, $i = 0, 1$, $j = 1 - i$, are admissible operators. Moreover, $A_i + B_{ij}Q_{ji}$ and $A_i$, $i = 0, 1$, $j = 1 - i$, are resolvent comparable. For the spectral shift function $\xi(\lambda, \mathbf{H}, \mathbf{A})$ associated with the pair of self-adjoint operators $(\mathbf{H}, \mathbf{A})$ one has the decomposition*

$$\begin{aligned} \xi(\lambda; \mathbf{H}, \mathbf{A}) = {} & \xi(\lambda; A_0 + B_{01}Q_{10}, A_0) \\ & + \xi(\lambda; A_1 + B_{10}Q_{01}, A_1) \pmod{\mathbb{Z}}, \quad \text{for a.e. } \lambda \in \mathbb{R}. \end{aligned} \tag{7.18}$$

*Moreover, the spectral shift functions $\xi(\lambda; A_i + B_{ij}Q_{ji}, A_i)$ associated with the pairs $(A_i + B_{ij}Q_{ji}, A_i)$, $i = 0, 1$, $j = 1 - i$, can be chosen in such a way that*

$$\xi(\lambda; A_i + B_{ij}Q_{ji}, A_i) = 0 \quad \text{for a.e. } \lambda \in \operatorname{spec}(A_j),\ i = 0, 1,\ j = 1 - i. \tag{7.19}$$

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**Proof** Under the assumptions of the theorem the existence of a strong solution $\mathbf{Q} \in \mathcal{B}(\mathcal{H})$ of the Riccati equation (5.8) is guaranteed by Lemma 5.3 and Theorem 7.4, Theorem 7.6 or Corollary 7.8. Since, by hypothesis, $\mathbf{B} \in \mathcal{B}_2(\mathcal{H})$, one infers $\mathbf{Q} \in \mathcal{B}_2(\mathcal{H})$ by Theorem 7.11. Thus, the assumption (i) of Theorem 6.1 holds. Therefore, $\mathbf{BQ}$ is a trace class operator, and hence the assumption (ii) of Theorem 6.1 holds. The assumption (iii) of Theorem 6.1 holds by hypothesis and, therefore, $A_i + B_{ij}Q_{ji}$, $i = 0, 1$, $j = 1 - i$, are admissible operators, $A_i + B_{ij}Q_{ji}$ and $A_i$, $i = 0, 1$, $j = 1 - i$, are resolvent comparable and the decomposition (7.18) takes place by Theorem 6.1.

Introducing the family $\mathbf{H}_t = \mathbf{A} + t\mathbf{B}$, $t \in [0, 1]$, by Lemma 7.10 one infers the existence of the operators $\mathrm{Q}_{ij}(t) \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$ that continuously depend on $t \in [0, 1]$ in the topology of the space $\mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$ and are such such that $\mathbf{H}_t$, $t \in [0, 1]$ has reducing graph subspaces

$$\mathcal{G}_i(t) = \mathcal{G}\big(\mathcal{H}_i, \mathrm{Q}_{ji}(t)\big), \quad i = 0, 1, \; j = 1 - i, \; t \in [0, 1].$$

Therefore, by Lemma 5.3 the Riccati equation

$$\mathbf{Q}_t\mathbf{A} - \mathbf{A}\mathbf{Q}_t + \mathbf{Q}_t(t\mathbf{B})\mathbf{Q}_t = t\mathbf{B}, \quad t \in [0, 1], \tag{7.20}$$

has a strong solution $\mathbf{Q}_t$ which reads with respect to the decomposition (5.2) as

$$\mathbf{Q}_t = \begin{pmatrix} 0 & \mathrm{Q}_{01}(t) \\ \mathrm{Q}_{10}(t) & 0 \end{pmatrix}, \quad t \in [0, 1], \tag{7.21}$$

and $\mathrm{Q}_{ji}(t) = -[\mathrm{Q}_{ij}(t)]^*$, $t \in [0, 1]$. Hence, each entry $\mathrm{Q}_{ji}(t)$, $t \in [0, 1]$, in (7.21) is a strong solution of the Riccati equation

$$\mathrm{Q}_{ji}(t)A_i - A_j\mathrm{Q}_{ji}(t) = tB_{ji} - t\mathrm{Q}_{ji}(t)B_{ij}\mathrm{Q}_{ji}(t), \quad t \in [0, 1]. \tag{7.22}$$

Since $\mathrm{Q}_{ji}(t)$ is continuous in the norm operator topology, the r.h.s. of (7.22) depends continuously on $t \in [0, 1]$ in the topology of the space $\mathcal{B}_2(\mathcal{H}_i, \mathcal{H}_j)$. Therefore, by Theorem 2.7 (estimate (2.14)) the path $\mathrm{Q}_{ji}(t)$, $t \in [0, 1]$, is continuous in the topology of the space $\mathcal{B}_2(\mathcal{H}_j, \mathcal{H}_i)$, and, thus, the family $\{tB_{ij}\mathrm{Q}_{ji}(t)\}_{t\in[0,1]}$, $i = 0, 1$, $j = 1 - i$, is continuous in the topology of the space $\mathcal{B}_1(\mathcal{H}_i, \mathcal{H}_j)$.

Clearly, the map

$$t \to (A_i + tB_{ij}\mathrm{Q}_{ji}(t) - z)^{-1} - (A_i - z)^{-1}) \in \mathcal{B}_1(\mathcal{H}_i), \qquad t \in [0, 1], \; i = 0, 1, \; \mathrm{Im}(z) \neq 0. \tag{7.23}$$

is continuous in the topology of the space $\mathcal{B}_1(\mathcal{H}_i)$, $i = 0, 1$. Taking into account that the family $\mathrm{Q}_{ji}^*(t)\mathrm{Q}_{ji}(t)$ is continuous in the topology of $\mathcal{B}_1(\mathcal{H}_i)$, $i = 0, 1$, and introducing the self-adjoint representatives of the admissible operators $A_i + tB_{ij}\mathrm{Q}_{ji}(t)$, $i = 0, 1$, $j = 1 - i$, $t \in [0, 1]$,

$$\mathrm{H}_i(t) = [I_{\mathcal{H}_i} + \mathrm{Q}_{ji}^*(t)\mathrm{Q}_{ji}(t)]^{1/2}\big(A_i + tB_{ij}\mathrm{Q}_{ji}(t)\big)\,[I_{\mathcal{H}_i} + \mathrm{Q}_{ji}^*(t)\mathrm{Q}_{ji}(t)]^{-1/2}, \tag{7.24}$$

$t \in [0,1]$, one concludes that the map

$$t \to [\mathsf{H}_i(t) - z)^{-1} - (A_i - z)^{-1}] \in \mathcal{B}_1(\mathcal{H}_i), \quad t \in [0,1], \tag{7.25}$$

is also continuous in the topology of $\mathcal{B}_1(\mathcal{H}_i)$, $i = 0, 1$.

Let

$$\Delta_i = \begin{cases} \{\lambda : \operatorname{dist}\{\lambda, \operatorname{spec}(A_i)\} > d/2\}, & \text{if Hypothesis 7.1 holds,} \\ \{\lambda : \operatorname{dist}\{\lambda, \operatorname{spec}(A_i)\} > d/\pi\}, & \text{if Hypothesis 7.2 holds,} \\ \mathbb{R} \setminus \overline{\text{convex hull}\big(\operatorname{spec}(A_i)\big)}, & \text{if Hypothesis 7.3 holds,} \end{cases}$$

$i = 0, 1$. Obviously

$$\operatorname{spec}(A_j) \subset \Delta_i, \quad i = 0, 1, \ j = 1 - i. \tag{7.26}$$

Our claim is that $\Delta_i$, $i = 0, 1$, belongs to the resolvent set of $\mathsf{H}_i(t)$, $i = 0, 1$, for all $t \in [0,1]$, that is,

$$\Delta_i \cap \operatorname{spec}\big(\mathsf{H}_i(t)\big) = \varnothing, \quad i = 0, 1, \ t \in [0,1]. \tag{7.27}$$

Under Hypothesis 7.3 the statement (7.27) is a consequence of Theorem 7.7 (equation (7.13)).

Assume, therefore, either Hypotheses 7.1 or Hypotheses 7.2.

Under Hypothesis 7.1, applying Theorem 7.4 one obtains the following uniform bounds

$$\|tB_{ij}\mathsf{Q}_{ji}(t)\| < \frac{d}{2}, \quad t \in [0,1], \ i = 0, 1, \ j = 1 - i.$$

Thus, one concludes that

$$\big\{\lambda : \operatorname{dist}\{\lambda, \operatorname{spec}(A_i)\} > d/2\big\} \cap \operatorname{spec}\big(A_i + tB_{ij}\mathsf{Q}_{ji}(t)\big) = \varnothing$$
$$\text{for all } t \in [0,1], \ i = 0, 1, \ j = 1 - i.$$

Under Hypothesis 7.2 the operator $\mathsf{Q}_{ji}(t)$, $i = 0, 1$, $j = 1 - i$, $t \in [0,1]$, is a strict contraction by Theorem 7.6. Therefore,

$$\|tB_{ij}\mathsf{Q}_{ji}(t)\| < \frac{d}{\pi}, \quad t \in [0,1], \ i = 0, 1, \ j = 1 - i,$$

and

$$\big\{\lambda : \operatorname{dist}\{\lambda, \operatorname{spec}(A_i)\} > d/\pi\big\} \cap \operatorname{spec}\big(A_i + tB_{ij}\mathsf{Q}_{ji}(t)\big) = \varnothing$$
$$\text{for all } t \in [0,1], \ i = 0, 1, \ j = 1 - i.$$

By (7.24) the operators $\mathsf{H}_i(t)$ and $A_i + tB_{ij}\mathsf{Q}_{ji}(t)$, $i = 0, 1$, $j = 1 - i$, $t \in [0,1]$, are similar to each other, which proves (7.27) under Hypotheses 7.1 or/and 7.2.

Applying Lemma 7.12 one proves that there is a family of spectral shift functions $\xi(\,\cdot\,;\mathsf{H}_i(t),A_i)\}_{t\in[0,1]}$, $i = 0,1$, continuous in the topology of the weighted space $L^2(\mathbb{R};(1+\lambda^2)^{-1})$ such that

$$\xi\big(\lambda;\mathsf{H}_i(t),A_i\big) = 0 \quad \text{for a.e. } \lambda\in[a_i,b_i],\ t\in[0,1],\ i=0,1, \tag{7.28}$$

for any interval $[a_i,b_i]\subset\Delta_i$, $i=0,1$. By (7.25) the operators $(\mathsf{H}_i(t)$ and $A_i$, $i=0,1$, $t\in[0,1]$, are resolvent comparable and, hence, by Lemma 4.8 one has the representation

$$\xi\big(\lambda;A_i+tB_{ij}\mathsf{Q}_{ji}(t),A_i\big) = \xi\big(\lambda;\mathsf{H}_i(t),A_i\big) \quad \text{for a.e. } \lambda\in\mathbb{R},\ t\in[0,1],\ i=0,1,$$

since $\mathsf{H}_i(t)$ are self-adjoint representatives of the admissible operators $A_i+tB_{ij}\mathsf{Q}_{ji}(t)$, $i=0,1$, $j=1-i$, $t\in[0,1]$. It follows that the spectral shift functions $\xi(\lambda;A_i+B_{ij}Q_{ji},A_i)$ associated with the pairs $(A_i+B_{ij}Q_{ji},A_i)$ $i=0,1$, $j=1-i$, can be chosen in such a way that for any interval $[a_i,b_i]\subset\Delta_i$, $i=0,1$,

$$\xi(\lambda;A_i+B_{ij}Q_{ji},A_i)=0 \quad \text{for a.e. } \lambda\in[a_i,b_i]\subset\Delta_i,\ i=0,1,\ j=1-i, \tag{7.29}$$

which, in particular, implies assertion (7.19), since (7.26) holds. ■

***Remark 7.14*** Assertion (6.2) under Hypothesis 7.3 in the case where $\mathbf{B}$ is a trace class operator has been proven by Adamjan and Langer [1]. Therefore, the main result of the paper [1] in its part related to the existence of the spectral shift function and to the validity of the representation (6.2) is a particular case of our more general considerations.

***Corollary 7.15*** *Assume the hypothesis of Theorem 7.13. Then*

(i) *the operator matrix* $\mathbf{H}$ *can be block-diagonalized by a unitary transformation (5.2)*

$$\mathbf{U}^*\mathbf{H}\mathbf{U} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix},$$

*where* $\mathbf{U}$ *is the unitary operator from the polar decomposition*

$$\mathbf{I}+\mathbf{Q} = \mathbf{U}|\mathbf{I}+\mathbf{Q}|;$$

(ii) *for the spectral shift function* $\xi(\lambda;\mathbf{H},\mathbf{A})$ *the following splitting formula holds*

$$\xi(\lambda;\mathbf{H},\mathbf{A}) = \xi(\lambda;H_0,A_0)+\xi(\lambda;H_1,A_1) \pmod{\mathbb{Z}}, \quad \textit{for a.e. } \lambda\in\mathbb{R};$$

(iii) *the spectral shift functions* $\xi(\lambda;H_i,A_i)$, $i=0,1$, *can be chosen in such a way that*

$$\xi(\lambda;H_i,A_i)=0 \quad \textit{for a.e. } \lambda\in\operatorname{spec}(A_{1-i}),\ i=0,1. \tag{7.30}$$

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