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NONGAUSSIAN LIMIT DISTRIBUTIONS OF LACUNARY TRIGONOMETRIC SERIES

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ABSTRACT. It is a well known fact that for rapidly increasing n_k the sequence $(\cos n_k x)_{k=1}^{\infty}$ behaves like a sequence of independent random variables; in particular $N^{-1/2}\sum_{k\leq N}\cos n_k x$ has a limiting Gaussian distribution as $N\longrightarrow\infty$. Under a certain critical speed (actually $n_k\sim e^{\sqrt{k}}$) this result breaks down and $(\cos n_k x)_{k=1}^{\infty}$ becomes strongly dependent. The purpose of this paper is to investigate the asymptotic behavior of normed sums $a_N^{-1}\sum_{k\leqq N}\cos n_k x$ in the strongly dependent domain; specifically, we construct a large class of nongaussian limit distributions of such sums.

1. **Introduction.** The purpose of the present paper is to study the Central Limit Problem for the trigonometric system $\{\sin nx, \cos nx\}_{n=1}^{\infty}$. More exactly, we are going to study the limit distribution of normalized sums

$$\frac{1}{a_N}\sum_{k\leq N}\cos n_k x - b_N \tag{1.1}$$

where $\{a_N\}$, $\{b_N\}$ are suitable numerical sequences and $\{n_k\}$ is an increasing sequence of positive integers. To fix the ideas, the probability space for this limit distribution problem will be $\big((0,2\pi),\mathcal{B},(2\pi)^{-1}\lambda\big)$ where $\mathcal{B}$ is the Borel σ-field in $(0,2\pi)$ and λ is the Lebesgue measure. However, the results of our paper remain valid if $(0,2\pi)$ is replaced by any finite interval (a,b) and $(2\pi)^{-1}\lambda$ is replaced by any probability measure on (a,b), absolutely continuous with respect to the Lebesgue measure.

From the point of view of probability theory, the trigonometric system $\{\sin nx, \cos nx\}_{n=1}^{\infty}$ is a strongly dependent sequence of r.v.'s whose asymptotic properties, studied in harmonic analysis, are essentially different from those of "typical" stochastic systems investigated in probability theory. However, it has been known for a long time that for rapidly increasing n_k the sequences $\{\cos n_k x\}_{k=1}^{\infty}$, $\{\sin n_k x\}_{k=1}^{\infty}$ behave like sequences of i.i.d.r.v.'s. By classical results of Salem-Zygmund [9], Erdős-Gál [4] and Weiss [14] if

$$n_{k+1}/n_k \geq q > 1 \tag{1.2}$$

then

$$\lim_{N\longrightarrow\infty}(2\pi)^{-1}\lambda\left\{0\leq x\leq 2\pi : \sum_{k\leq N}\cos n_k x < t\sqrt{N/2}\right\} = (2\pi)^{-1/2}\int_{-\infty}^{t} e^{-u^2/2}\,du \tag{1.3}$$

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and

$$\overline{\lim_{N\to\infty}}\,(N\log\log N)^{-1/2}\sum_{k\le N}\cos n_k x = 1 \qquad a.e. \tag{1.4}$$

A much stronger result was proved by Philipp and Stout [7] who showed that under (1.2) the partial sum process $S(t) = S(t,x) = \sum_{k\le t}\cos n_k x$, $(t \ge 0)$ is nearly Wiener in the sense that without changing its distribution it can be redefined on a suitable probability space, together with a Wiener process $W(t)$ such that

$$S(t) = W(t/2) + O(t^{1/2-\delta}) \qquad a.s. \tag{1.5}$$

for some $\delta > 0$. (1.5) implies not only (1.3) and (1.4) but it extends a large class of limit theorems of i.i.d.r.v.'s for the trigonometric sequence $\{\cos n_k x\}$. (For some typical consequences of (1.5) see [7].)

Erdős was the first to extend the central limit theorem (1.3) for a large class of sequences $\{n_k\}$ growing slower than exponentially. To formulate his result let us say, given positive numerical sequences $\{a_N\}$, $\{b_N\}$, that $a_N \succ b_N$ if $a_{N+1}/a_N \ge b_{N+1}/b_N$ for $N \ge N_0$. Then Erdős' result can be stated as follows:

THEOREM (SEE [3]). *Let $\{n_k\}$ be a sequence of positive integers satisfying*

$$n_k \succ e^{c_k\sqrt{k}} \quad \textit{for some } c_k \uparrow \infty \tag{1.6}$$

Then $\cos n_k x$ satisfies the central limit theorem (1.3). *On the other hand, for each $c > 0$ there exists a sequence $\{n_k\}$ of positive integers satisfying*

$$n_k \succ e^{c\sqrt{k}} \tag{1.7}$$

such that the central limit theorem (1.3) *is false (and in fact under no choice of a_N, b_N can* (1.1) *converge to a normal law).*

In [1], [10], [11], [13] it is proved that under slight additional assumptions on c_k, (1.6) implies not only (1.3) but also the law of the iterated logarithm (1.4) and the a.s. invariance principle (1.5). These results show that at the critical speed $n_k \approx e^{c\sqrt{k}}$ the probabilistic behavior of $\{\cos n_k x\}$ undergoes a fundamental change: from nearly independent it turns into strongly dependent. While the nearly independent case is much studied and well understood, very little is known in the strongly dependent domain. In particular, no nongaussian limit distribution of normed sums (1.1) is known, not only in the "critical strip" i.e. for sequences $\{n_k\}$ growing with speed $e^{c\sqrt{k}}$ or slightly slower, but for any lacunary sequence $\{n_k\}$ i.e. under $n_{k+1} - n_k \longrightarrow +\infty$ †. It is not known, either, what asymptotic law replaces the law of the iterated logarithm in the strongly dependent

† Erdős and Fortet noted (see [6], p. 646) that if $\{n_k\}$ consists of the numbers 2^k and 2^k+1 $(k = 1,2,\ldots)$, arranged in increasing order, then $N^{-1/2}\sum_{k\le N}\cos n_k x$ converges weakly to a mixture of normal distributions. No other type of limit distribution of sums (1.1) is known even in the nonlacunary case.

case. The difficulties in the strongly dependent domain are number-theoretic and related to the diophantine equation

$$\pm n_{i_1} \pm n_{i_2} \pm \cdots \pm n_{i_p} = 0 \qquad (1 \le i_1, \dots i_p \le N) \tag{1.8}$$

whose number of solutions $A(N,p)$ is, as is well known, closely connected with the asymptotic behaviour of $\sum_{k\le N} \cos n_k x$ (see e.g. [3],[4],[9]). For rapidly increasing n_k (e.g. under (1.6)) it is not hard to get an asymptotic formula for $A(N,p)$; this is independent of the number theoretic properties of n_k and leads easily to results like (1.3), (1.4) (see e.g. [3],[4]). For slowly increasing n_k the situation changes radically: in this case $A(N,p)$ depends strongly on the arithmetic properties of n_k and (1.8) leads to deep number-theoretic problems even for simple sequences like $n_k = [k^r]$, $r > 1$ or $n_k = [\exp(k^\alpha)]$, $\alpha > 0$. (See, in this respect, the remarks in [2], § 2.) Hence to determine the class $\mathcal{F}$ of all possible limit distributions of normed sums (1.1) seems to be a very difficult problem. The purpose of this paper is to construct, by ad hoc arguments, a few nongaussian limit distributions of the class $\mathcal{F}$. We shall exhibit (Theorem 1) a one-parameter class of nonstable infinitely divisible distributions belonging to $\mathcal{F}$ and we shall also prove (Theorem 2) that $\mathcal{F}$ contains nonnormal stable distributions. The sequences $\{n_k\}$ leading to these limit distributions will be lacunary, in fact they satisfy either (1.7) or $n_k \succ e^{\varepsilon_k \sqrt{k}}$ for some slowly decreasing sequence $\varepsilon_k \searrow 0$. Using an observation of Erdős and Fortet (see [6], p. 646) these results imply also (Theorem 3) that together with the nongaussian limit distributions of Theorem 1 and 2, some of their randomized (mixed) versions belong also to $\mathcal{F}$, at least in the sense of "partial attraction" i.e. when we require the convergence of (1.1) only along a sequence of integers N. The simplest example of such a randomized limit distribution is the Cauchy distribution with random scaling constant.

We now formulate our results in detail. For Theorem 1, we need some notation. Given any measurable sets A, B on the real line, set $\lambda_A(B) = \lambda(A \cap B)$. Now set

$$F(t) = \lambda_{(0,\infty)}(x : \sin x/x \ge t) \qquad t > 0$$
$$G(t) = \lambda_{(0,\infty)}(x : \sin x/x \le -t) \qquad t > 0$$

Clearly $F(t) \le t^{-1}$, $G(t) \le t^{-1}$ $(t > 0)$; further, $F(t)$ and $G(t)$ are continuous, nonincreasing, $F(t) = G(t) = 0$ for $t \ge 1$ and $\lim_{t\searrow 0} F(t) = \lim_{t \searrow 0} G(t) = +\infty$. Lemma 2 below will show that $F(t) \sim G(t) \sim (\pi t)^{-1}$ as $t \searrow 0$.

THEOREM 1. *Given any $A > 0$, there exists a sequence $\{n_k\}$ of integers such that $n_k \succ e^{\sqrt{k/4A}}$ and*

$$\frac{1}{a_N} \sum_{k \le N} \cos n_k x - b_N \longrightarrow Q \text{ in distribution as } N \to \infty \tag{1.9}$$

where

$$a_N = \sqrt{2/A} \cdot N^{1/2}, \quad b_N = O(1) \tag{1.10}$$

and Q is the infinitely divisible distribution with characteristic function

$$\exp\left\{\int_{R\setminus\{0\}}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dL(x)\right\} \tag{1.11}$$

where

$$L(x)=L_A(x)=\begin{cases}-\frac{1}{A\pi}\int_{x/A}^{1}\frac{F(t)}{t}\,dt & \text{if } 0<x\le A\\ \frac{1}{A\pi}\int_{-x/A}^{1}\frac{G(t)}{t}\,dt & \text{if } -A\le x<0\\ 0 & \text{if } |x|>A\end{cases} \tag{1.12}$$

THEOREM 2. *Given any sequence $\varepsilon_k \searrow 0$ there exists a sequence $\{n_k\}$ of integers such that $n_k \succ e^{\varepsilon_k\sqrt{k}}$ and (1.9) holds where Q is the Cauchy distribution with characteristic function* $\exp(-c|t|)$, *$b_N=O(1)$ and $a_N=\sqrt{N}L(N)$ where L is a slowly varying function with* $\lim_{N\to\infty}L(N)=0$.

Note that Theorem 1 yields a limit distribution $Q=F_A$ of normed sums (1.1) for each $A>0$. In Section 2 we shall prove that

$$\lim_{A\to 0}F_A(x\sqrt{A}/2)=\Phi(x),\qquad \lim_{A\to\infty}F_A(x)=C_1(x) \tag{1.13}$$

where $C_1(x)$ is the Cauchy distribution function. Thus the class $\{F_A, A>0\}$ is a one-parameter continuous "path" connecting the normal and Cauchy distributions. The normal limit in the first relation of (1.13) is heuristically explained by the fact that for $A\to 0$ the speed relation $n_k\succ e^{\sqrt{k/4A}}$ in Theorem 1 approaches (in some sense) the speed relation (1.6) implying a normal limit for (1.1). Similarly the second relation of (1.13) and the speed relations for $\{n_k\}$ in Theorems 1 and 2 show that Theorem 2 is, in a formal sense, the limit of Theorem 1 as $A\to+\infty$.

In connection with Theorem 1 we also note that by the criterion ([8], Theorem 8) the limit distributions F_A have finite moments of all order for any $A>0$.

THEOREM 3. *Given any sequence $\varepsilon_k\searrow 0$ there exists a sequence $n_1<n_2<\cdots$ of positive integers satisfying $n_k\ge e^{\varepsilon_k\sqrt{k}}$ for $k\ge k_0$ such that for some numerical sequence $\{d_N\}$ we have, for N running through a suitable increasing sequence $\{N_j\}$ of integers,*

$$\frac{1}{d_N}\sum_{k\le N}\cos n_k x\longrightarrow Q^* \text{ in distribution} \tag{1.14}$$

where

$$Q^*(x)=\frac{1}{2\pi}\int_0^{2\pi}C_1\left(\frac{x}{2\cos t}-c\right)dt \tag{1.15}$$

for some real constant c. That is, Q^* is a mixture of (noncentered) Cauchy distributions. The result remains valid if in (1.15) *we replace C_1 by F_A or replace* $2\cos t$ *by* $\sum_{j\in H}2\cos jt$ *where H is any finite set of positive integers.*

In contrast to the sequences $\{n_k\}$ in Theorems 1 and 2 the sequence $\{n_k\}$ proving Theorem 3 will not be lacunary i.e. it satisfies $\underline{\lim}_{k\to\infty}(n_{k+1}-n_k)<+\infty$. We also note that if c is a limit point of the bounded sequence b_N in Theorem 2 and $b_{N_j} \longrightarrow c$ as $j \longrightarrow \infty$ then the limit relation (1.14) holds if N runs through the sequence ρN_j for some integer $\rho \geq 1$ and with the above c in (1.15). Thus whether the whole sequence in (1.14) converges weakly or it has different limit distributions along different subsequences depends on whether b_N in Theorem 2 converges or not.

We do not know how typical the examples given by Theorems 1–3 are for the behaviour of $\{\cos n_k x\}_{k=1}^{\infty}$ in the strongly dependent domain. It is easily seen that together with Φ, the Cauchy distribution C_1 and the limit distribution F_A appearing in Theorem 1, all finite convolutions $\Phi * C_1 * F_{A_1} * \cdots * F_{A_r}$ belong also to $\mathcal{F}$. But we do not know, e.g., if $\mathcal{F}$ contains any stable distribution other than Φ and C_1. Neither do we know how large is the class of sequences $\{n_k\}$ leading, e.g., to a limiting Cauchy distribution in (1.9). Certainly, this class cannot be too large: from the results and remarks in [2], pp. 162–163 it follows that if $\{n_k\}$ is the sequence in Theorem 2 (or in fact $\{n_k\}$ is any sequence satisfying $n_k \succ e^{k^\alpha}$ for some $\alpha > 0$) then one can not only find sequences $n_k^* \sim n_k$ such that $\{\cos n_k^* x\}_{k=1}^{\infty}$ satisfies the central limit theorem (1.3) but, in some sense, almost all sequences $n_k^* \sim n_k$ have this property. Thus the Cauchy distribution (and all nongaussian limit distributions of normed sums (1.1)) are, in some sense, "exceptional".

2. **Proofs.** As we mentioned in Section 1, $n_k \succ e^{c\sqrt{k}}$ does not imply, for any $c > 0$, the central limit theorem (1.3). An example showing this was given, without proof, by Erdős [3] and verified later, in a slightly modified form, by Takahashi [12]. The sequences proving our theorems will have a structure similar to those in [3] and [12], they will namely consist of long "regular" blocks separated by large jumps. To evaluate the limit distributions belonging to such sequences we will need some simple lemmas describing the asymptotic tail behaviour of the Dirichlet kernel.

LEMMA 1. *Let $n \geq 1$ be an odd integer. Then for any $t \geq 2$ we have*

$$\lambda_{(0,2\pi)}\Big(x : \frac{\sin nx}{\sin x} \geq t\Big) = \frac{4}{n} F\Big(\frac{t}{n}(1 - \Theta_1 t^{-2})\Big) \tag{2.1}$$

$$\lambda_{(0,2\pi)}\Big(x : \frac{\sin nx}{\sin x} \leq -t\Big) = \frac{4}{n} G\Big(\frac{t}{n}(1 - \Theta_2 t^{-2})\Big) \tag{2.2}$$

where $0 \leq \Theta_1, \Theta_2 \leq 1$.

PROOF. Clearly, the inequality $\sin nx / \sin x \geq t$ can hold only if $|\sin x| \leq 1/t$ i.e. if x belongs to one of the four intervals $[0, f(t)]$, $[\pi - f(t), \pi]$, $[\pi, \pi + f(t)]$, $[2\pi - f(t), 2\pi]$ where $f(t) = \sin^{-1} 1/t$. Set

$$I_t^{(n)} = \lambda_{(0,f(t))}\Big(x : \frac{\sin nx}{\sin x} \geq t\Big)$$

Evidently $1/t \leq f(t) \leq 2/t$ and thus for $0 \leq x \leq f(t)$ we have $1 \geq \sin x / x \geq 1 - x^2/6 \geq 1 - 1/t^2$. Hence observing that $y > 0$, $\sin y / y \geq t/n$ imply $y \leq n/t \leq nf(t)$, we get

$$\frac{1}{n} F\Big(\frac{t}{n}\Big) = \frac{1}{n} \lambda_{(0,nf(t))}\Big(y : \frac{\sin y}{y} \geq \frac{t}{n}\Big) = \lambda_{(0,f(t))}\Big(x : \frac{\sin nx}{nx} \geq \frac{t}{n}\Big) \leq I_t^{(n)}$$

$$\begin{aligned}(2.3)\qquad &\leq \lambda_{(0,f(t))}\Big(x : \frac{\sin nx}{nx} \geq \frac{t}{n}(1-t^{-2})\Big)\\ &= \frac{1}{n}\lambda_{(0,nf(t))}\Big(y : \frac{\sin y}{y} \geq \frac{t}{n}(1-t^{-2})\Big) \leq \frac{1}{n}F\Big(\frac{t}{n}(1-t^{-2})\Big)\end{aligned}$$

Since n is odd, $I_t^{(n)}$ does not change if in its definition the interval $\big(0,f(t)\big)$ is replaced by $\big(\pi - f(t), \pi\big)$, $\big(\pi, \pi + f(t)\big)$ or $\big(2\pi - f(t), 2\pi\big)$. Thus, (2.3) implies (2.1); (2.2) follows similarly.

LEMMA 2. *For $0 < t \leq 1$ we have*

$$F(t) = (\pi t)^{-1} + O(1) \tag{2.4}$$

$$G(t) = (\pi t)^{-1} + O(1) \tag{2.5}$$

PROOF. Clearly, $\psi(x) = \sin x / x$ is positive in the intervals $\big(2j\pi, (2j+1)\pi\big)$, negative in the intervals $\big((2j+1)\pi, (2j+2)\pi\big)$ $(j = 0, 1, \ldots)$ and its maximum M_j over the interval $[2j\pi, (2j+1)\pi]$ satisfies $1/\big((2j+1)\pi\big) \leq M_j \leq 1/(2j\pi)$. (In particular $M_1 > M_2 > \cdots$.) Assume, without loss of generality, that $0 < t < M_1$ and let $k \geq 1$ denote the largest integer such that $M_k > t$. The equation $\psi(x) = t$ has one solution $x = \rho$ in $(0, \pi)$ and for each $1 \leq j \leq k$ it has two solutions $x_1^{(j)} < x_2^{(j)}$ in the interval $\big(2j\pi, (2j+1)\pi\big)$ which we shall write as $x_1^{(j)} = 2j\pi + p_j$, $x_2^{(j)} = (2j+1)\pi - q_j$. Thus

$$F(t) = \rho + \sum_{j=1}^{k}(\pi - p_j - q_j) \tag{2.6}$$

By the bounds given above for M_j we have

$$2k\pi t \leq 1 \leq (2k+3)\pi t \tag{2.7}$$

For $1 \leq j \leq k-1$ (2.7) implies $\psi(x_1^{(j)}) = \psi(x_2^{(j)}) = t \leq (2k\pi)^{-1} \leq \big((2j+1/2)\pi\big)^{-1} = \psi\big((2j+1/2)\pi\big)$ and thus from the graph of ψ we see that $x_1^{(j)} \leq (2j+1/2)\pi \leq x_2^{(j)}$ i.e. $p_j \leq \pi/2$, $q_j \leq \pi/2$. Hence $p_j = \sin^{-1}(tx_1^{(j)}) = \sin^{-1}(2j\pi t + \Theta t) = (2\pi t)^{-1}[\int_{2j\pi t}^{(2j+2)\pi t}\sin^{-1}x\,dx + 2\pi t\Theta'\Delta_j]$ for $1 \leq j \leq k-1$ where $0 \leq \Theta \leq \pi$, $|\Theta'| \leq 1$ and $\Delta_j = \sin^{-1}(2j+2)\pi t - \sin^{-1} 2j\pi t$. The same formula applies for q_j and thus using (2.6), $p_k + q_k \leq \pi$, $\int_0^1 \sin^{-1}x\,dx = \pi/2 - 1$ and $|2k\pi t - 1| \leq 3\pi t$, (2.4) follows. (2.5) can be proved similarly.

Using Lemma 1 and Lemma 2 (note that Lemma 2 remains trivially true for $t \geq 1$) we get the following

COROLLARY. *Let $n \geq 1$ be an odd integer. Then for any $t \geq 2$ we have*

$$\lambda_{(0,2\pi)}\Big(x : \frac{\sin nx}{\sin x} \geq t\Big) = \frac{4}{\pi t}\big(1 + O(t^{-2}) + O(t/n)\big)$$

$$\lambda_{(0,2\pi)}\Big(x : \frac{\sin nx}{\sin x} \leq -t\Big) = \frac{4}{\pi t}\big(1 + O(t^{-2}) + O(t/n)\big)$$

where the constants implied by the O are absolute.

To prove Theorem 1, let $d_k = 2^{ak^2}$, $q_k = d_{k+1}/d_k$, $m_k = [Ak]$ and $M_k = \sum_{i=1}^k m_i$ where the integer $a \geq 1$ is chosen so large that

$$4\pi m_k^2 d_k / d_{k+1} \leq 2^{-k} \qquad (k \geq 1) \tag{2.9}$$

Let $I_k = \{2d_k, 4d_k, 6d_k, \ldots, 2m_k d_k\}$; clearly the sets I_k, $k = 1, 2, \ldots$ are disjoint. Define the sequence $\{n_k\}$ by

$$\{n_k\} = \bigcup_{j=1}^{\infty} I_j$$

We show that $\{\cos n_k x\}_{k=1}^{\infty}$ satisfies the requirements of Theorem 1. Clearly $M_k \sim Ak^2/2$ and thus if $M_{k-1} < j < M_k$ then setting $i = j - M_{k-1}$ we have

$$\begin{aligned} n_{j+1}/n_j = 1 + 1/i \geq 1 + 1/m_k \geq 1 + 1/(Ak) &\geq 1 + 1/(2\sqrt{AM_{k-1}}) \\ \geq 1 + 1/(2\sqrt{Aj}) &\geq \exp\big(1/(3\sqrt{Aj})\big) \\ \geq \exp\big((2\sqrt{A})^{-1}\big(\sqrt{j+1} - \sqrt{j}\big)\big) &\qquad (j \geq j_0) \end{aligned} \tag{2.10}$$

Also, for $j = M_k$ we have $n_{j+1}/n_j = d_{k+1}/(m_k d_k) \geq 2$ by (2.9) and thus $n_k \succ e^{\sqrt{k/4A}}$. Set

$$\begin{aligned} X_k &= \sum_{j=M_{k-1}+1}^{M_k} \cos n_j x, \\ f_k(x) &= \sum_{j=1}^{m_k} \cos 2jx = \frac{\sin(2m_k+1)x}{2\sin x} - \frac{1}{2} \end{aligned} \tag{2.11}$$

Let further $\rho_k(x)$ $(0 \leq x \leq 2\pi)$ be the function which equals $2\pi j/q_k$ provided $2\pi j/q_k \leq \widehat{d_k x} < 2\pi(j+1)/q_k$ for some integer $0 \leq j \leq q_k - 1$ where $\hat{t}$ denotes the residue of t (mod 2π). Clearly, $\rho_k(x)$ is constant on each interval $[2\pi j/d_{k+1}, 2\pi(j+1)/d_{k+1})$ $(0 \leq j \leq d_{k+1} - 1)$ and is periodic with period $2\pi/d_k$. Hence, the functions $\rho_k(x)$, $k = 1, 2, \ldots$ are independent random variables over the probability space $\big((0, 2\pi), \mathcal{B}, (2\pi)^{-1}\lambda\big)$. Moreover,

$$|\rho_k(x) - \widehat{d_k x}| \leq 2\pi/q_k \qquad (0 \leq x \leq 2\pi) \tag{2.12}$$

Let now $Y_k = f_k\big(\rho_k(x)\big)$, then using $X_k = f_k(d_k x) = f_k(\widehat{d_k x})$, (2.9), (2.12), $|f_k'(x)| \leq 2m_k^2$ and the mean value theorem we get

$$|X_k - Y_k| \leq 4\pi m_k^2 d_k / d_{k+1} \leq 2^{-k} \qquad (k \geq 1) \tag{2.13}$$

Also, Y_k are independent r.v.'s over $\big((0, 2\pi), \mathcal{B}, (2\pi)^{-1}\lambda\big)$. Hence denoting the probability measure in this space alternatively by P we get, using again $X_k = f_k(d_k x)$ and (2.11),

$$\begin{aligned} P(X_k \geq t) &= \frac{1}{2\pi}\lambda_{(0,2\pi)}\Big(x : \frac{\sin(2m_k+1)d_k x}{\sin d_k x} \geq 2t + 1\Big) \\ &= \frac{1}{2\pi}\lambda_{(0,2\pi)}\Big(x : \frac{\sin(2m_k+1)x}{\sin x} \geq 2t + 1\Big) \end{aligned} \tag{2.14}$$

Similarly

$$P(X_k \le -t) = \frac{1}{2\pi}\lambda_{(0,2\pi)}\Big(x : \frac{\sin(2m_k+1)x}{\sin x} \le -2t+1\Big) \tag{2.15}$$

LEMMA 3. *For any $c > 0$ we have*

$$\lim_{n\to\infty}\sum_{k=1}^{n} P(Y_k \ge cn) = \begin{cases} \frac{1}{A\pi}\int_{c/A}^{1}\frac{F(t)}{t}\,dt & \text{if } c \le A \\ 0 & \text{if } c > A \end{cases} \tag{2.16}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} P(Y_k < -cn) = \begin{cases} \frac{1}{A\pi}\int_{c/A}^{1}\frac{G(t)}{t}\,dt & \text{if } c \le A \\ 0 & \text{if } c > A \end{cases} \tag{2.17}$$

PROOF. By $|X_k| \le m_k$ and (2.13) we have $|Y_k| \le Ak+1$ and thus $P(Y_k \ge cn)$ can differ from zero only if $k \ge (cn-1)/A$. Now if $c > A$ then for $n \ge n_0$ the last inequality cannot hold for any $1 \le k \le n$ and thus the sum in (2.16) equals 0. Assume now $c \le A$ and fix an integer k with $(cn-1)/A \le k \le n$. Then setting $H(x) = x^{-1}F\big(c/(Ax)\big)$ and using (2.13), (2.14), Lemma 1 and the uniform continuity of $F(x)$ in the interval $[c/A, +\infty)$ we get by simple calculations

$$\begin{aligned} P(Y_k \ge cn) &= \frac{1}{2\pi}\lambda_{(0,2\pi)}\left(x : \frac{\sin(2m_k+1)x}{\sin x} \ge 2cn + O(1)\right) \\ &= \frac{2}{\pi\big(2Ak+O(1)\big)}F\left(\frac{2cn+O(1)}{2Ak+O(1)}\big(1+O(n^{-2})\big)\right) \\ &= \frac{1}{Ak\pi}\left[F\Big(\frac{cn}{Ak}\Big)+o(1)\right] + O(n^{-2}) = \frac{1}{A\pi n}H\Big(\frac{k}{n}\Big) + o(n^{-1}) \end{aligned} \tag{2.18}$$

where we made repeated use of the fact that k/n is between given positive bounds. The constants implied by the O depend on c and A and the terms $o(1)$ and $o(n^{-1})$ are uniform for $(cn-1)/A \le k \le n$. Adding (2.18) for $(cn-1)/A \le k \le n$ we get that the sum in (2.16) is $(A\pi)^{-1}\int_{c/A}^{1} H(x)\,dx + o(1)$ which proves (2.16) since $\int_{c/A}^{1} H(x)\,dx = \int_{c/A}^{1} F(t)/t\,dt$. (2.17) can be proved similarly.

Observe now that by (2.13), (2.14), (2.15), Lemma 1 and the trivial estimates $F(t) \le t^{-1}$, $G(t) \le t^{-1}$ we have

$$P(|Y_k| \ge t) \le \frac{1}{2\pi}\lambda_{(0,2\pi)}\Big(x : \Big|\frac{\sin(2m_k+1)x}{\sin x}\Big| \ge 2t-3\Big) \le \frac{3}{t} \quad (k \ge 1, t \ge 3) \tag{2.19}$$

Thus denoting the distribution function of Y_k by F_k, an integration by parts yields $\int_{-l}^{l} x^2\,dF_k(x) \le 9l \quad (k \ge 1, l \ge 3)$ and consequently

$$\lim_{\varepsilon\to 0}\overline{\lim_{n\to\infty}}\sum_{k=1}^{n} n^{-2}\int_{-\varepsilon n}^{\varepsilon n} x^2\,dF_k(x) = 0 \tag{2.20}$$

Hence applying a classical limit theorem ([5], § 25, Theorem 1) for the triangular array $Y_{nk} = Y_k/n$, $1 \le k \le n$, $n = 1, 2, \ldots$ we get, using Lemma 3,

$$n^{-1}\sum_{k=1}^{n} Y_k - c_n \xrightarrow{\mathcal{D}} Q \tag{2.21}$$

where Q is the infinitely divisible distribution with characteristic function (1.11) where $L(x) = L_A(x)$ is defined by (1.12) and

$$c_n = n^{-1} \sum_{k=1}^{n} \int_{-n}^{n} x\, dF_k(x) + \gamma \tag{2.22}$$

where γ is a real constant, depending on the sequence $\{Y_k\}$. (The role of γ is to guarantee that there is no term ict in the exponential in (1.11).) Next we note that by (2.13), (2.14), (2.15) and the Corollary we have

$$\begin{aligned} P(Y_k \geq t) - P(Y_k \leq -t) &= \frac{1}{2\pi} \frac{4}{\pi(2t + O(1))} \big(1 + O(t^{-2}) + O(t/m_k)\big) \\ &\quad - \frac{1}{2\pi} \frac{4}{\pi(2t + O(1))} \big(1 + O(t^{-2}) + O(t/m_k)\big) \\ &= O(t^{-2}) + O(m_k^{-1}) \end{aligned}$$

for $t \geq 2$ where the constants in the O are absolute. Thus for $l \geq 2$ we get, using also (2.19),

$$\left| \int_{-l}^{l} t\, dF_k(t) \right| \leq \left| \int_0^l \big(1 - F_k(t) - F_k(-t)\big)\, dt \right| + O(1) = O(l/m_k) + O(1). \tag{2.23}$$

By $|Y_k| \leq m_k + 1$ the first integral in (2.23) does not change for $l \geq m_k + 1$ and thus (2.23) yields $\int_{-l}^{l} t\, dF_k(t) = O(1)$ for all $k \geq 1$, $l \geq 1$. Hence $c_n = O(1)$ by (2.22). Now (2.13) and (2.21) yield

$$n^{-1} \sum_{k=1}^{n} X_k - c_n \xrightarrow{\mathcal{D}} Q. \tag{2.24}$$

Set $S_N = \sum_{j \leq N} \cos n_j x$ and define the sequences $\{a_N\}$ and $\{b_N\}$ by

$$\begin{aligned} a_N &= k \text{ for } M_k \leq N < M_{k+1} \quad k = 1, 2, \dots \\ b_N &= c_k \text{ for } M_k \leq N < M_{k+1} \quad k = 1, 2, \dots \end{aligned}$$

By (2.24) the relation

$$a_N^{-1} S_N - b_N \xrightarrow{\mathcal{D}} Q \tag{2.25}$$

holds along the indices $N = M_k$ and by $M_k \sim Ak^2/2$ and $c_n = O(1)$ we have $a_N \sim \sqrt{2/A} \cdot N^{1/2}$ and $b_N = O(1)$. Now for $M_k \leq N < M_{k+1}$ a_N and b_N remain constant and the L_2 norm of $S_N/a_N - S_{M_k}/a_{M_k}$ is $k^{-1}\big((N - M_k)/2\big)^{1/2} \leq k^{-1} m_{k+1}^{1/2} = O(k^{-1/2})$. Thus (2.25) holds for the whole sequence and clearly it remains valid if we replace a_N by $\sqrt{2/A} \cdot N^{1/2}$. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. We use the same construction for $\{n_k\}$ as in the proof of Theorem 1 except that $m_k = [Ak]$ is replaced by $m_k = [k\omega_k]$ where $\omega_k \uparrow \infty$ so slowly that $\omega_{k^3}/\omega_k \longrightarrow 1$ (and consequently ω_k is slowly varying). Keeping the same notations as in the proof of Theorem 1, we have

$$M_k \sim k^2\omega_k/2 \tag{2.26}$$

Similarly to (2.10), we get by simple calculations

$$\begin{aligned} n_{j+1}/n_j &\geq 1 + 1/(2\sqrt{j\omega_j}) \geq \exp\left((2\sqrt{\omega_j})^{-1}\left(\sqrt{j+1} - \sqrt{j}\right)\right) \\ &\geq \exp\left(\sqrt{(j+1)/4\omega_{j+1}} - \sqrt{j/4\omega_j}\right) \qquad (j \geq j_0) \end{aligned}$$

and thus $n_k \succ e^{\sqrt{k/4\omega_k}}$. Instead of Lemma 3 we now have

LEMMA 4. *For any $c > 0$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n} P(Y_k \geq cn) = \lim_{n\to\infty}\sum_{k=1}^{n} P(Y_k < -cn) = (\pi^2 c)^{-1} \tag{2.27}$$

To see this, let $k(n)$ denote the smallest k such that $k\omega_k \geq n$; clearly $k(n) = n\varepsilon_n$ where $\varepsilon_n \longrightarrow 0$. Split the first sum in (2.27) into two parts according as $1 \leq k < n\varepsilon_n^{1/2}$ or $n\varepsilon_n^{1/2} \leq k \leq n$, respectively. By (2.19) the first sum is o(1) and for each k in the second sum we have $k\omega_k \geq n\varepsilon_n^{-1/2}$. Hence using (2.13), (2.14), (2.15) and the Corollary after Lemma 2 we see that each term of the second sum is $\left(\pi^2 c(n + O(1))\right)^{-1}\left(1 + O(n^{-2}) + O(\varepsilon_n^{1/2})\right)$ where the constants in the O depend only on c. Since the second sum has $n\left(1 + o(1)\right)$ terms, the first limit in (2.27) is $(\pi^2 c)^{-1}$. The second limit can be handled similarly.

From Lemma 4 we conclude, as in the proof of Theorem 1, that (2.21) holds where the characteristic function of Q is given by (1.11) with $L(x) = -1/(\pi^2 x)$ $(x \neq 0)$. That is, Q is a Cauchy distribution with characteristic function $\exp(-\pi^{-1}|t|)$ (see [5], § 34). The centering sequence c_n is again given by (2.22) and using (2.26) the argument leading to (1.10) yields in the present case $a_N \sim \sqrt{2N/\omega_N}$, $b_N = O(1)$, completing the proof.

REMARK. We prove now relation (1.13) describing the limiting behaviour of F_A as $A \longrightarrow 0$ and $A \longrightarrow \infty$. Clearly, $F_A(x\sqrt{A})$ is the limit distribution of the left side of (2.21) with Y_k and c_n replaced by $Y_k/\sqrt{A}$ and $c_n/\sqrt{A}$, respectively. Hence applying the same limit theorem we used to deduce (2.21) it follows that the characteristic function $\psi_A(t)$ of $F_A(x\sqrt{A})$ has form (1.11) with $L(x) = \hat{L}_A(x) = L_A(x\sqrt{A})$ where $L_A(x)$ is defined by (1.12). A simple calculation shows that $\psi_A(t)$ can be rewritten as

$$\exp\left\{\int_{R\setminus\{0\}}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\,dH_A(x)\right\} \tag{2.28}$$

where

$$H_A(x) = \begin{cases} \int_0^x (A\pi)^{-1}(1+t^2)^{-1} tF(t/\sqrt{A})\,dt & \text{if } x \geq 0 \\ \int_x^0 (A\pi)^{-1}(1+t^2)^{-1} tG(-t/\sqrt{A})\,dt & \text{if } x < 0 \end{cases} \tag{2.29}$$

By Lemma 2 the integrands in (2.29) have a finite limit as $t \longrightarrow 0$ and thus $H_A(x)$ is a bounded, continuous, nondecreasing function, constant for $x \geq \sqrt{A}$ and $x \leq -\sqrt{A}$. This shows also that the integral in (2.28) does not change if we replace the domain of integration by R. Now

$$H_A(\lambda) = \frac{1}{\pi}\int_0^{\lambda/\sqrt{A}} \frac{yF(y)}{1+Ay^2}\,dy \longrightarrow \frac{1}{\pi}\int_0^1 yF(y)\,dy \text{ as } A \searrow 0 \text{ if } \lambda > 0$$

and

$$H_A(\lambda) \longrightarrow -\frac{1}{\pi}\int_0^1 yG(y)\,dy \text{ as } A \searrow 0 \text{ if } \lambda < 0.$$

Thus by a standard convergence theorem for infinitely divisible distributions ([5], § 19, Theorem 1) $F_A(x\sqrt{A})$ converges, as $A \searrow 0$, to the normal distribution with mean zero and variance $\sigma^2 = \pi^{-1}\int_0^1 y\big(F(y)+G(y)\big)\,dy$. Integrating by parts and using Lemma 2 we get

$$\sigma^2 = -\frac{1}{2\pi}\Big\{\int_0^1 y^2\,dF(y) + \int_0^1 y^2\,dG(y)\Big\} = \frac{1}{2\pi}\int_0^\infty \Big(\frac{\sin x}{x}\Big)^2 dx = \frac{1}{4}$$

completing the proof of the first relation of (1.13). The second relation of (1.13) can be proved similarly.

PROOF OF THEOREM 3. Let $j_1 < j_2 < \cdots < j_r$ be positive integers and $\psi(t) = 2\sum_{\nu=1}^r \cos j_\nu t$; let further $\{n_k\} = \{n_k^{(A)}\}$ be the sequence in Theorem 1. Clearly, if the quantity a used in the proof of Theorem 1 is large enough then $n_k - n_{k-1} > 2j_r$ for all $k \geq 1$ and thus the numbers $n_k \pm j_\nu$ $(k = 1,2,\ldots,\nu = 1,2,\ldots,r)$ are all different. Let $\{n_k^*\}_{k=1}^\infty$ be the sequence consisting of these numbers, arranged in increasing order; let further $\{a_N\}$, $\{b_N\}$ be the sequences in Theorem 1 and $\{N_j\}$ an increasing sequence of positive integers such that $b_{N_j} \longrightarrow c$ for some real c. We show that

$$a_N^{-1}\sum_{k\leq 2Nr}\cos n_k^* x \xrightarrow{\mathcal{D}} Q^* \tag{2.30}$$

along the sequence $\{N_j\}$ where

$$Q^*(x) = \frac{1}{2\pi}\int_0^1 F_A\Big(\frac{x}{\psi(t)} - c\Big)\,dt \tag{2.31}$$

The same holds if $\{n_k\}$, $\{a_N\}$, $\{b_N\}$ are taken from Theorem 2 and in (2.31) we replace F_A by C_1. To prove our statement we first note that the sequence $\zeta_N = a_N^{-1}\sum_{k\leq N}\cos n_k x - b_N$ is mixing i.e. for any measurable function φ on $(0,2\pi)$ we have

$$P(\zeta_N < y, \varphi < z) \longrightarrow F_A(y)P(\varphi < z) \quad \text{as } N \longrightarrow \infty \tag{2.32}$$

for any $y \in H$, $z \in H$ where H is a dense set on the real line and $P = (2\pi)^{-1}\lambda$ on $(0,2\pi)$. To verify this, consider a diadic interval $(a,b) = \big(2\pi j2^{-\nu}, 2\pi(j+1)2^{-\nu}\big) \subset (0,2\pi)$ and note that by $d_k = 2^{ak^2}$ the distribution of $\rho_k(x)$ and thus of Y_k (cf. the proof of Theorem 1) remain unchanged for $k \geq k_0$ if we replace the probability space $\big((0,2\pi), \mathcal{B}, (2\pi)^{-1}\lambda\big)$ by

$\big((a,b),\mathcal{B},(b-a)^{-1}\lambda\big)$. Similarly, for $k\geq k_0$ ρ_k and thus Y_k remain independent r.v.'s over the probability space $\big((a,b),\mathcal{B},(b-a)^{-1}\lambda\big)$ as well. Hence (2.21) and thus the conclusion of Theorem 1 will also hold over (a,b) and thus (2.32) holds if φ is the indicator function of any diadic interval. By approximation, (2.32) holds for any measurable φ. Now

$$S^*_{2Nr} =: \sum_{i\leq 2Nr}\cos n^*_i x = \sum_{k\leq N}\sum_{\nu=1}^{r}\{\cos(n_k-j_\nu)x+\cos(n_k+j_\nu)x\} = \psi(x)\sum_{k\leq N}\cos n_k x.$$

Hence if $\{N_j\}$ denotes an increasing sequence of positive integers such that $b_{N_j}\longrightarrow c$ then

$$a_N^{-1}S^*_{2Nr} = (\zeta_N+c)\psi(x)+o(1)$$

along $\{N_j\}$ and (2.30) follows upon using (2.32) and noting that the function Q^* in (2.31) is the distribution function of $\xi\cdot\eta$ where ξ and η are independent r.v.'s, ξ is distributed as $\psi(t)$ over $(0,2\pi)$ and η has distribution function $F_A(t-c)$. Finally we note that if the starting sequence $\{n_k\}$ of our proof satisfies $n_k\succ e^{\delta_k\sqrt{k}}$ for some $\{\delta_k\}$ then $\{n^*_k\}$ satisfies $n^*_k\geq$const$\cdot\exp(\delta_{k/3r}\sqrt{k/3r})$ for $k\geq 1$. This completes the proof of Theorem 3.

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