Ergod. Th. & Dynam. Sys., (2024), **44**, 859–887

doi:10.1017/etds.2023.33

Equidistribution for matings of quadratic maps with the modular group

V. MATUS DE LA PARRA

Department of Mathematics, University of Rochester, New York, USA
(e-mail: vmatusde@ur.rochester.edu)

(*Received* 15 *February* 2022 *and accepted in revised form* 29 *January* 2023)

Abstract. We study the asymptotic behavior of the family of holomorphic correspondences $\{\mathcal{F}_a\}_{a\in\mathcal{K}}$, given by

$$\left(\frac{az+1}{z+1}\right)^2+\left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right)+\left(\frac{aw-1}{w-1}\right)^2=3.$$

It was proven by Bullet and Lomonaco [Mating quadratic maps with the modular group II. *Invent. Math.* **220**(1) (2020), 185–210] that $\mathcal{F}_a$ is a mating between the modular group $\mathrm{PSL}_2(\mathbb{Z})$ and a quadratic rational map. We show for every $a \in \mathcal{K}$, the iterated images and preimages under $\mathcal{F}_a$ of non-exceptional points equidistribute, in spite of the fact that $\mathcal{F}_a$ is weakly modular in the sense of Dinh, Kaufmann, and Wu [Dynamics of holomorphic correspondences on Riemann surfaces. *Int. J. Math.* **31**(05) (2020), 2050036], but it is not modular. Furthermore, we prove that periodic points equidistribute as well.

Key words: equidistribution, holomorphic correspondences, weak modularity
2020 Mathematics Subject Classification: 37F05 (Primary); 11F32, 32H99 (Secondary)

1. *Introduction*

In 1965, Brolin [3] studied asymptotic properties of polynomials $P(z) \in \mathbb{C}[z]$ of degree bigger than or equal to 2. He proved the existence of a probability measure for which the preimages $P^{-n}(z_0)$ at time n of any point $z_0 \in \mathbb{C}$ (with at most one exception) asymptotically equidistribute, as n tends to infinity. In 1983, Freire, Lopes, and Mañé [20] and Ljubich [24] independently proved the generalization to rational maps of degree at least 2 on the Riemann sphere. These results have been generalized to different settings. For instance, see [18, §1.4] and the references therein for higher dimensions, and [19, 22] for the non-Archimedean setting.

The equidistribution properties of holomorphic correspondences have also attracted considerable interest. Roughly speaking, a *holomorphic correspondence* on a complex manifold X is a multivalued map induced by a formal sum $\Gamma = \sum n_i\Gamma(i)$ of complex varieties $\Gamma(i) \subset X \times X$ of the same dimension. The multivalued map sends z to w if (z, w) belongs to some $\Gamma(i)$ (see §2.1). Let $d(F)$ denote the number of pre-images of a generic point under F. We call this number the *topological degree* of F, just as in the case of rational maps. We study the existence of a Borel probability measure μ on X that has the property that for all but at most finitely many $z_0 \in X$,

$$\frac{1}{d(F)^n}(F^n)_*\delta_{z_0} \to \mu,$$

as $n \to \infty$. Here, $(F^n)_*$ denotes the push-forward operator associated to F^n. In [13], Dinh studied the case of polynomial correspondences whose Lojasiewicz exponent is strictly bigger than 1, in which case we always have that $d(F^{-1}) < d(F)$. The case where $d(F) = d(F^{-1})$ is open but some subcases are known. For instance, Clozel, Oh, and Ullmo [9] proved equidistribution for irreducible modular correspondences, Clozel and Otal [10] proved it for exterior modular correspondences, and Clozel and Ullmo [11] for those that are self-adjoint. On the other hand, Dinh, Kaufmann, and Wu proved in [15] that if such F is not weakly modular, then the statement holds for both F and F^{-1}. Modular correspondences are weakly modular, but the reverse containment does not hold. On a different classification, Bharali and Sridharan [1] proved equidistribution for correspondences with $d(F) \geq d(F^{-1})$ having a repeller in the sense of [26]. We will study a 1-parameter family in the gap between weak-modularity and modularity, and for which the result in [1] does not apply.

Our object of study is the family of correspondences $\{\mathcal{F}_a\}_{a\in\mathcal{K}}$ on the Riemann sphere $\widehat{\mathbb{C}}$, where $\mathcal{F}_a$ is given in affine coordinates by

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3, \tag{1}$$

and $\mathcal{K}$ is the *Klein combination locus* defined in §2.2. This family was studied by Bullett *et al* in [4–6, 8]. In [4], Bullett and Lomonaco proved that there is a two sided restriction f_a of $\mathcal{F}_a$ that is hybrid equivalent to a quadratic rational map P that has a fixed point with multiplier 1. We refer the reader to [25] for conjugacy of parabolic-like mappings. Moreover, for the parameters for which the Julia set of P is connected, we have that $\mathcal{F}_a$ is a mating between the rational map P and the modular group $\mathrm{PSL}_2(\mathbb{Z})$. This generalizes a previous result by Bullett and Penrose [6]. The correspondence $\mathcal{F}_a$ has two homeomorphic copies of the filled Julia set K of P_A, denoted $\Lambda_{a,-}$ and $\Lambda_{a,+}$, and they satisfy that $\mathcal{F}_a^{-1}(\Lambda_{a,-}) = \Lambda_{a,-}$ and $\mathcal{F}_a(\Lambda_{a,+}) = \Lambda_{a,+}$. These are called the *backward* and *forward limit set*, respectively (see §3.2).

The following theorem states that this family does not fit the conditions for any of the equidistribution results listed above (see §1.1).

THEOREM 1.1. *For every $a \in \mathcal{K}$, we have that:*

(1) *$\mathcal{F}_a$ is a weakly modular correspondence that is not modular; and*
(2) *$\partial\Lambda_{a,-}$ is not a repeller for $\mathcal{F}_a$.*

Furthermore, we prove that $\mathcal{F}_a$ satisfies a property that is stronger than weak-modularity (see Remark 3.4).

The purpose of this paper is to show that equidistribution holds for the family $\{\mathcal{F}_a\}_{a\in\mathcal{K}}$. Put

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

$$\mathcal{E}_a := \begin{cases} \varnothing & \text{if } a \neq 5, \\ \{-1, 2\} & \text{if } a = 5. \end{cases}$$

We prove the following equidistribution theorem.

THEOREM 1.2. *Let $a \in \mathcal{K}$. There exist two Borel probability measures μ_+ and μ_- on $\widehat{\mathbb{C}}$, with* $\operatorname{supp}(\mu_+) = \partial\Lambda_{a,+}$ *and* $\operatorname{supp}(\mu_-) = \partial\Lambda_{a,-}$, *such that for every $z_0 \in \widehat{\mathbb{C}} \setminus \mathcal{E}_a$,*

$$\frac{1}{2^n}(\mathcal{F}_a^n)_*\delta_{z_0} \to \mu_+ \quad \textit{and} \quad \frac{1}{2^n}(\mathcal{F}_a^{-n})_*\delta_{z_0} \to \mu_-,$$

weakly, as $n \to \infty$.

In later work [12], we prove that the measures μ_+ and μ_- maximize entropy: the metric entropy in [31] yields equality for the half-variational principle with the topological entropy in [17].

Let F be a holomorphic correspondence on X with graph Γ. Denote by $\Gamma^{(n)}$ the graph of F^n and by

$$\mathfrak{D}_X := \{(z, z) | z \in X\}$$

the diagonal in $X \times X$. Then the set of *periodic points of F of period n* is defined as the set

$$\operatorname{Per}_n(F) := \pi_1(\Gamma^{(n)} \cap \mathfrak{D}_X),$$

and for $z \in \operatorname{Per}_n(F)$, we define the *multiplicity of z as a periodic point of F of order n* to be the number $\nu_{\pi_1|_{\Gamma^{(n)} \cap \mathfrak{D}_X}}(z, z)$, defined in §2.1.

Another source of motivation is whether or not periodic points equidistribute. In [24], Ljubich showed that this is the case for rational maps of degree bigger than or equal to 2, where periodic points are counted either with or without multiplicity. The equidistribution of periodic points is also studied in [2, 16, 19] in the case of maps, and in [13, 14] in the case of correspondences. We prove this holds for the family $\{\mathcal{F}_a\}_{a \in \mathcal{K}}$ as well.

THEOREM 1.3. *For $a \in \mathcal{K}$,*

$$\frac{1}{|\operatorname{Per}_n(\mathcal{F}_a)|} \sum_{z \in \operatorname{Per}_n(\mathcal{F}_a)} \delta_z \quad \textit{and} \quad \frac{1}{2^{n+1}} \sum_{z \in \operatorname{Per}_n(\mathcal{F}_a)} \nu_{\pi_1|_{\Gamma_a^{(n)} \cap \mathfrak{D}_{\widehat{\mathbb{C}}}}} \delta_z$$

are both weakly convergent to $\frac{1}{2}(\mu_- + \mu_+)$, as $n \to \infty$.

In §5, we define the set $\hat{P}_n^{\Gamma}$ of *superstable parameters of order n*. Combining the main results of [5, 29], we obtain a homeomorphism $\Psi : \mathcal{M} \to \mathcal{M}_\Gamma$ between the Mandelbrot set $\mathcal{M}$ and the connectedness locus $\mathcal{M}_\Gamma$ of the family $\{\mathcal{F}_a\}_{a \in \mathcal{K}}$. These results, together with the equidistribution result in [23], yield the following theorem.

THEOREM 1.4. *In $\mathcal{M}_\Gamma$, superstable parameters equidistribute with respect to $\Psi^* m_{\mathrm{BIF}}$, that is,*

$$\lim_{n \to \infty} \frac{1}{2^{n-1}} \sum_{a \in \hat{P}_n^{\Gamma}} \delta_a = \Psi^* m_{\mathrm{BIF}}.$$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

1.1. *Notes and references.* There is a bigger family studied by Bullett and Harvey in [8], given by replacing the right-hand side of equation (1) by $3k$, where $k \in \mathbb{C}$. For these correspondences, it is also possible to define limit sets $\Lambda_{a,k,-}$ and $\Lambda_{a,k,+}$ analogous to those in the case where $k = 1$. In [1], Bharali and Sridharan show how their equidistribution result applies to these correspondences in the case where $\Lambda_{a,k,-}$ is a repeller. Parameters for which this is the case exist from the results in [8]. However, we prove in Theorem 1.1 part (2) that this is never the case when $k = 1$ and $a \in \mathcal{K}$.

1.2. *Organization.* The structure of this paper is as follows. In §2.1, we give an introduction to holomorphic correspondences and their action on Borel measures. In §2.2, we introduce the correspondences $\mathcal{F}_a$ given by equation (1). We define critical values and find those of $\mathcal{F}_a$ in §2.3, and define Klein combination pair and the Klein combination locus $\mathcal{K}$ in §2.4. In §3.1, we define modular and weakly modular correspondences, and prove part (1) of Theorem 1.1. To prove that $\mathcal{F}_a$ is weakly modular, we use the decomposition $\mathcal{F}_a = \mathrm{J}_a \circ \mathrm{Cov}_0^{\mathrm{Q}}$, given in [4], into a certain involution J_a composed with the deleted covering correspondence $\mathrm{Cov}_0^{\mathrm{Q}}$ described in §2.2, and construct the measures in the definition of weakly modular using the symmetry of the graph of $\mathrm{Cov}_0^{\mathrm{Q}}$. The fact that $\mathcal{F}_a$ is not modular follows from the fact that Borel measures assigning positive measure to non-empty open sets are not invariant by $\mathcal{F}_a$. In §3.2, we define the limit sets $\Lambda_{a,-}$ and $\Lambda_{a,+}$, and prove part (2) of Theorem 1.1 by showing that the parabolic fixed point in $\partial\Lambda_{a,-}$ violates the definition of a repeller. In §4, we describe the exceptional set of the two-sided restriction f_a and the set $\mathrm{Per}_n(\mathcal{F}_a)$ of periodic points of period n. Finally, in §5, we use the description of f_a given in [4] together with the results in [20, 24] to prove Theorems 1.2 and 1.3 about asymptotic equidistribution of images, preimages, and periodic points. We finish with the proof of Theorem 1.4 about equidistribution of special points in the modular Mandelbrot set $\mathcal{M}_\Gamma$.

2. *Preliminaries*

2.1. *Holomorphic correspondences.* Let X be a compact Riemann surface and let $\pi_j : X \times X \to X$ be the canonical projection to the jth coordinate, $j = 1, 2$. We say that a formal sum $\Gamma = \sum_i n_i \Gamma(i)$ is a *holomorphic 1-chain* on $X \times X$ if its support $\operatorname{supp} \Gamma := \bigcup_i \Gamma(i)$ is a subvariety of $X \times X$ of pure dimension 1 whose irreducible components are exactly the $\Gamma(i)$, and the n_i are non-negative integers. We say that the $\Gamma(i)$ are the *irreducible components of* Γ.

Let $\Gamma = \sum_i n_i \Gamma(i)$ be a holomorphic 1-chain satisfying that for $j = 1, 2$ and every i such that $n_i > 0$, the restriction $\pi_j|_{\Gamma(i)}$ of the canonical projection $X \times X \to X$ to the irreducible component $\Gamma(i)$ is surjective. The chain Γ induces a multivalued map F from X to itself by

$$F(z) := \bigcup_i \pi_2|_{\Gamma(i)}(\pi_1|_{\Gamma(i)}{}^{-1}(z)).$$

The multivalued map F is called a *holomorphic correspondence* and it is said to be *irreducible* if $\sum_i n_i = 1$. We say that $\Gamma_F := \Gamma$ is the *graph* of the holomorphic correspondence F. Let $\iota : X \times X \to X \times X$ be the involution $(z, w) \mapsto (w, z)$. We can define

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

the *adjoint correspondence* F^{-1} of F by the relation $F^{-1}(z) := \bigcup_i \pi_1|_{\Gamma(i)}(\pi_2|_{\Gamma(i)}{}^{-1}(z))$, which is a holomorphic correspondence, whose graph is the holomorphic 1-chain $\Gamma_F^{-1} = \sum_i n_i \iota(\Gamma(i))$.

In [30], Stoll introduced a notion of multiplicity that will be useful for this paper. Let M be a quasi-projective variety and N a smooth quasi-projective variety. If $g : M \to N$ is regular, and $a \in M$, then we say that a neighborhood U of a is *distinguished with respect to g and a* if $\overline{U}$ is compact and $g^{-1}(g(a)) \cap \overline{U} = \{a\}$. Such neighborhoods exist if and only if $\dim_a g^{-1}(g(a)) = 0$ and, in this case, they form a base of neighborhoods. If U is distinguished with respect to g and a, then put $\mu_g(z, U) := |g^{-1}(g(z)) \cap U|$. It can be shown that $\hat{\nu}_g(a) := \limsup_{z \to a} \mu_g(z, U)$ does not depend on the distinguished neighborhood U, and the maps n_b defined on N by $a \mapsto \sum_{b \in g^{-1}(a)} \hat{\nu}_g(b)$ are constant in each component.

Suppose that $g : M \to N$ is a finite and surjective regular map, with M and N as above. Stoll proved in [30] that $\hat{\nu}_g(a)$ generalizes the notion of multiplicity of g at a and whenever φ is a continuous function with compact support in M, the map

$$a \mapsto \sum_{b \in g^{-1}(a)} \hat{\nu}_g(b)\varphi(b)$$

is continuous.

To study dynamics, we proceed to define the *composition* of two holomorphic correspondences F and G with associated holomorphic 1-chains $\Gamma_F = \sum_i n_i \Gamma_F(i)$ and $\Gamma_G = \sum_j m_j \Gamma_G(j)$, respectively. For each i and j, let $A_{i,j}$ be the image of the projection $p_{i,j} : (\Gamma_G(j) \times \Gamma_F(i)) \cap \{x_2 = x_3\} \hookrightarrow X \times X$ that forgets the second and third coordinates, i.e.,

$$A_{i,j} = \{(z, w) \in X \times X | \text{ there exists } x \in X \text{ such that } (z, x) \in \Gamma_G(j), \text{ and } (x, w) \in \Gamma_F(i)\}.$$

Let $\{\Gamma(i, j, k)\}_{k=1}^{N(i,j)}$ be the irreducible components of $A_{i,j}$. Observe that since $\Gamma_G(j)$ and $\Gamma_F(i)$ are both quasi-projective, and so is $\{x_2 = x_4\} \subset X^4$, then $p_{i,j}$ is a regular map from the quasi-projective variety $(\Gamma_G(j) \times \Gamma_F(i)) \cap \{x_2 = x_3\}$ to the smooth quasi-projective variety $X \times X$. Then we have that the map $a \mapsto n_{p_{i,j}|_{\Gamma(i,j,k)}}(a)$ defined on $\Gamma(i, j, k)$ is constant. Therefore, $\eta_{i,j,k} := n_{p_{i,j}|_{\Gamma(i,j,k)}}(a)$ denotes the number of $x \in X$ such that $((z, x), (x, w)) \in \Gamma_G(j) \times \Gamma_F(i)$, for a generic point $(z, w) \in \Gamma(i, j, k)$. Define the composition $F \circ G$ as the holomorphic correspondence determined by the holomorphic 1-chain

$$\Gamma_{F \circ G} := \sum_{i,j} \sum_{k=1}^{N(i,j)} n_i m_j \eta_{i,j,k} \Gamma(i, j, k).$$

Note that the supp $\Gamma_{F \circ G} = \bigcup_{i,j} A_{i,j}$.

Set $d(F) := \sum_i n_i \deg(\pi_2|_{\Gamma(i)})$. We have that $d(F \circ G) = d(F)d(G)$. Thus, in particular, for every integer $n \geq 1$, we have that $d(F^n) = (d(F))^n$. We call $d(F)$ the *topological degree* of F, and it corresponds to the number of preimages of a generic point under F.

If F is an irreducible holomorphic correspondence over X with graph Γ and $\varphi : X \to \mathbb{C}$ is a continuous function, then

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

$$F^*\varphi(z) := \sum_{(z,w)\in \pi_1|_\Gamma^{-1}(z)} \hat{\nu}_{\pi_1|_\Gamma}(z,w)\varphi(w)$$

is continuous as well, see [11, Lemma 1.1]. Now let F be a holomorphic correspondence that is not necessarily irreducible, with graph $\Gamma_F = \sum_i n_i\Gamma(i)$. We denote by F_i the holomorphic correspondence induced by $\Gamma(i)$ and we put

$$\nu_{F_i}(z,w) = \begin{cases} \hat{\nu}_{\pi_1|_{\Gamma(i)}} & \text{if } (z,w)\in\Gamma(i), \\ 0 & \text{otherwise,} \end{cases}$$

and $\nu_F := \sum_i n_i \nu_{F_i}$. Then for every continuous function $\varphi : X \to \mathbb{C}$, the map

$$\begin{aligned} z \mapsto \sum_{w\in F(z)} \nu_F(z,w)\varphi(w) &= \sum_{w\in f(z)} \bigg(\sum_i n_i \nu_{F_i}(z,w)\bigg)\varphi(w) \\ &= \sum_i n_i F_{i*}\varphi(w) \end{aligned}$$

is also continuous.

The holomorphic correspondence F induces an action F_* on finite Borel measures μ by duality, namely $\langle F_*\mu, \varphi\rangle := \langle \mu, F^*\varphi\rangle$, called the *push-forward operator* and the resultant measure $F_*\mu$ is the *push-forward measure of* μ *under* F. We define as well the action $F^* := (F^{-1})_*$, called the *pull-back operator* and the resultant measure $F^*\mu$ is called the *pull-back measure of* μ *under* F. This action on measures agrees with the action on points

$$F^*\delta_z := \sum_{w\in F(z)} \nu_F(z,w)\delta_w,$$

where δ_z is the Dirac delta at z.

To see this, note that for every continuous function $\varphi : X \to \mathbb{C}$,

$$\begin{aligned} \bigg\langle \sum_{w\in F(z)} \nu_F(z,w)\delta_w, \varphi \bigg\rangle &= \sum_{w\in F(z)} \nu_F(z,w)\langle \delta_w, \varphi\rangle \\ &= \sum_{w\in F(z)} \nu_F(z,w)\varphi(w) \\ &= \int \bigg(\sum_{w\in F(\zeta)} \nu_F(\zeta,w)\varphi(w)\bigg) d\delta_z(\zeta) \\ &= \langle \delta_z, F_*\varphi\rangle \\ &= \langle F^*\delta_z, \varphi\rangle. \end{aligned}$$

2.2. *The family* $\{\mathcal{F}_a\}_a$. Let $Q(z)\in\mathbb{C}[z]$ be a nonlinear polynomial. The *deleted covering relation of* Q on $\mathbb{C}\times\mathbb{C}$ is defined by $w \in \mathrm{Cov}_0^Q(z)$ if and only if

$$P_Q(z,w) := \frac{Q(z)-Q(w)}{z-w} = 0. \tag{2}$$

Note that the denominator 'deletes' the obvious association of z with itself in the equation $Q(z) = Q(w)$.

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

In this section, we will identify $\widehat{\mathbb{C}}$ with the complex projective line when it is convenient to work with homogeneous coordinates $(z : w)$.

PROPOSITION 2.1. *Put $Q(z) := z^3 - 3z$. The closure of the relation in equation (2) is an irreducible quasiprojective complex variety Γ_0 of $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ of dimension 1. Moreover, the projections $\pi_1|_{\Gamma_0}$ and $\pi_2|_{\Gamma_0}$ are both surjective and of degree 2.*

Proof. Note that $P_Q(z, w) = z^2 + zw + w^2 - 3$ and consider $P_Q(z, w)$ as a single variable polynomial in $(\mathbb{C}[z])[w]$. Then its discriminant $-3z^2 + 12$ is not a square in $\mathbb{C}[z]$. Therefore, $P_Q(z, w)$ is an irreducible polynomial, and hence

$$\mathcal{Z} := \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid P_Q(z, w) = 0\}$$

is an irreducible subvariety of $\mathbb{C} \times \mathbb{C}$.

Now we want to describe the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ in $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$. Observe that if we fix $z \in \mathbb{C}$, then $\lim_{w\to\infty} P_Q(z, w) = \infty$, and if we fix $w \in \mathbb{C}$, then $\lim_{z\to\infty} P_Q(z, w) = \infty$. Therefore, there are no points of the form (z, ∞) or (∞, w) in $\overline{\mathcal{Z}}$. Given $R > 0$, let $z \in \mathbb{C}$ be such that $|z| = R$. Observe that $P_Q(z, \cdot) \in \mathbb{C}[w]$ is non-constant and therefore has at least one root in $\mathbb{C}$. Let $w \in \mathbb{C}$ be a root. Then $(z, w) \in \mathcal{Z}$ and we have that $|w^3 - 3w| = |Q(w)| = |Q(z)| = |z^3 - 3z| \geq R^3 - 3R$. By taking $R \to \infty$, we get that $(z, w) \to (\infty, \infty)$. Therefore, $\overline{\mathcal{Z}} = \mathcal{Z} \cup \{(\infty, \infty)\}$. In particular, $\overline{\mathcal{Z}}$ extends the relation given by equation (2) from $\mathbb{C} \times \mathbb{C}$ to $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$.

Take the homogenization

$$T(z, x, w, y) := z^2y^2 + zxwy + x^2w^2 - 3x^2y^2$$

of $P_Q(z, w)$, so $P_Q(z, w) = T(z, 1, w, 1)$, and note that $T(\lambda_1 z, \lambda_1 x, \lambda_2 w, \lambda_2 y) = \lambda_1^2\lambda_2^2 T(z, x, w, y)$. Thus, for the closed subvariety

$$\Gamma_0 := \{((z : x), (w : y)) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \mid T(z, x, w, y) = 0\}$$

of $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$, we have that $\Gamma_0 = \overline{\mathcal{Z}}$. To prove that Γ_0 is irreducible, note that each of its irreducible components intersecting $\mathbb{C} \times \mathbb{C}$ must be a closed subset of Γ_0 containing $\Gamma_0 \cap (\mathbb{C} \times \mathbb{C}) = \mathcal{Z}$, and therefore it is Γ_0 itself. Thus, Γ_0 has only one irreducible component and hence it is irreducible.

We proceed to show Γ_0 has dimension 1. Observe that the polynomial $T(z, x, w, y)$ is irreducible in $\mathbb{C}[z, x, w, y]$, as whenever $S(z, x, w, y) | T(z, x, w, y)$ in $\mathbb{C}[z, x, w, y]$, then $S(z, 1, w, 1) | P_Q(z, w)$ in $\mathbb{C}[z, w]$. Therefore, the zero set $Z(T) \subset \mathbb{C}^2 \times \mathbb{C}^2$ of T is an irreducible hypersurface of $\mathbb{C}^2 \times \mathbb{C}^2$, and hence it has codimension 1. Now let $p : \mathbb{C}^2 \setminus \{(0, 0)\} \to \widehat{\mathbb{C}}$ be the projection sending $(z, x) \mapsto (z : x)$. Note that in the chart

$$U_1 = \{(z, w) \in \mathbb{C}^2 \setminus \{(0, 0)\} | w \neq 0\},$$

the map p is simply $(z, w) \mapsto (z/w : 1)$, and in the chart

$$U_2 = \{(z, w) \in \mathbb{C}^2 \setminus \{(0, 0)\} | z \neq 0\},$$

it becomes $(z, w) \mapsto (1 : w/z)$. Let $\hat{p} : (\mathbb{C}^2 \setminus \{(0, 0)\}) \times (\mathbb{C}^2 \setminus \{(0, 0)\}) \to \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ be the map defined by $\hat{p}((z, x), (w, y)) := (p(z, x), p(w, y))$. Then $\hat{p}|_{Z(T)} : Z(T) \to \Gamma_0$ is a

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

regular map between irreducible varieties, and $\hat{p}|_{Z(T)}$ has constant fiber dimension equal to 2, as T is homogeneous in (z, x) and in (w, y). Therefore,

$$\dim \Gamma_0 = \dim Z(T) - \dim \hat{p}|_{Z(T)}{}^{-1}(z, w) = 1.$$

Finally, observe that the polynomial equation (2) has at least one and at most two solutions for every $z \in \mathbb{C}$, and by symmetry, the same holds for $w \in \mathbb{C}$. Note as well that ∞ is in correspondence with and only with itself. Thus, the projections $\pi_1|_{\Gamma_0}$ and $\pi_2|_{\Gamma_0}$ are both surjective. Moreover, $P_Q(1, 1) = P_Q(1, -2) = 0$, and hence $P_Q(1, w)$ has exactly two solutions. Therefore, $\deg(\pi_1|_{\Gamma_0}) = \deg(\pi_2|_{\Gamma_0}) = 2$. □

Remark 2.2. Proposition 2.1 holds for a large class of polynomials $Q(z)$. Observe that no polynomial can have $Q(z) = Q(\infty)$ for a finite number z. Therefore, following the proof of Proposition 2.1, we conclude that to get an irreducible holomorphic correspondence, it suffices to prove that $P_Q(z, w)$ is irreducible over $\mathbb{C}$. This holds under fairly general conditions. For instance, this is the case when Q is indecomposable and not linearly related to either z^n or a Chebyshev polynomial (see [**21**]).

On the other hand, note that whenever $Q = R \circ S$ with R and S of degree greater than 1, then $P_S(z, w)$ divides $P_Q(z, w)$, and therefore $P_Q(z, w)$ is reducible.

Proposition 2.1 says that Γ_0 is the graph of an irreducible holomorphic correspondence, where Γ_0 is a quasi-projective variety and $\pi_1|_{\Gamma_0} : \Gamma_0 \to \widehat{\mathbb{C}}$ is a finite and surjective morphism over $\mathbb{C}$, and hence we can use our definition of pull-back and push-forward operators induced by the correspondence on finite measures. We call this correspondence the *deleted covering correspondence of Q*, denoted by $\mathrm{Cov}_0^{\mathrm{Q}}$ as well. That is, $\mathrm{Cov}_0^{\mathrm{Q}}$ is the holomorphic correspondence on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ such that $\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}} = \Gamma_0$. From now on, we always consider $Q(z) = z^3 - 3z$.

Now take $a \in \mathbb{C} \setminus \{1\}$ and let $\mathrm{J}_a : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the involution

$$\mathrm{J}_a(z) := \frac{(a+1)z - 2a}{2z - (a+1)}.$$

The composition of $\mathrm{Cov}_0^{\mathrm{Q}}$ with the involution J_a is again an irreducible holomorphic correspondence $\mathcal{F}_a := \mathrm{J}_a \circ \mathrm{Cov}_0^{\mathrm{Q}}$ on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ with graph Γ_a for which we can use the pull-back and push-forward operators above as well. Note that $d(\mathcal{F}_a) = d(\mathcal{F}_a^{-1}) = 2$ and $\mathcal{F}_a^{-1} = \mathrm{Cov}_0^{\mathrm{Q}} \circ \mathrm{J}_a = \mathrm{J}_a \circ \mathcal{F}_a \circ \mathrm{J}_a$, since $\mathrm{Cov}_0^{\mathrm{Q}\,-1} = \mathrm{Cov}_0^{\mathrm{Q}}$ and $\mathrm{J}_a^{-1} = \mathrm{J}_a$.

Set $\phi_a(z) := (az + 1)/(z + 1)$. Then $\phi_a^{-1} \circ \mathcal{F}_a \circ \phi_a$ is a holomorphic correspondence on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ that, restricted to $(\widehat{\mathbb{C}} \setminus \{-1\}) \times (\widehat{\mathbb{C}} \setminus \{1\})$, induces the relation given by equation (1). Thus, $(z, w) \in (\widehat{\mathbb{C}} \setminus \{-1\}) \times (\widehat{\mathbb{C}} \setminus \{1\})$ satisfies equation (1) if and only if $w \in \mathcal{F}_a(z)$ [**4**, Lemma 3.1].

Observe that $\mathcal{F}_a^{-1}(1) = \mathrm{Cov}_0^{\mathrm{Q}}(\mathrm{J}_a(1)) = \mathrm{Cov}_0^{\mathrm{Q}}(1) = \{1, -2\}$, independent of $a \in \mathbb{C} \setminus \{1\}$. In particular, $1 \in \mathrm{Per}_1(\mathcal{F}_a)$ and we say that 1 is a *fixed point* of $\mathcal{F}_a$.

2.3. *Critical values.* In this section, we will discuss what parts of the graph of $\mathcal{F}_a$ are locally the graph of a holomorphic function, by defining and finding all critical values and ramification points of Γ_a. This will be used in §4 to find the exceptional set.

Definition 2.3. Let Γ be the graph of an irreducible holomorphic correspondence on $\widehat{\mathbb{C}}$, and put

$$A_j(\Gamma) := \{\alpha \in \Gamma \mid \text{for all open neighborhoods } W \text{ of } \alpha, \pi_j|_{W\cap\Gamma} \text{ is not injective}\},$$

for $j = 1, 2$ and $B_j(\Gamma) := \pi_j(A_j(\Gamma))$.

We extend the definition to holomorphic 1-chains $\Gamma = \sum_i n_i \Gamma(i)$ by

$$A_j(\Gamma) := \bigcup_i A_j(\Gamma(i)) \quad \text{and} \quad B_j(\Gamma) := \pi_j(A_j(\Gamma)).$$

We call $A_2(\Gamma)$ the set of *ramification points* of the holomorphic correspondence associated to Γ, and $B_2(\Gamma)$ the set of its *critical values*.

Note that $A_1(\Gamma) = \iota(A_2(\Gamma^{-1}))$ and $A_2(\Gamma) = \iota(A_1(\Gamma^{-1}))$, where $\iota : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ is the involution $\iota(z, w) = (w, z)$.

Suppose Γ is the graph of a holomorphic correspondence on $\widehat{\mathbb{C}}$. Let $g : \Omega \to \widehat{\mathbb{C}}$ be a holomorphic function defined on a domain Ω of $\widehat{\mathbb{C}}$, whose graph $\mathrm{Gr}(g)$ is contained in one of the irreducible components $\Gamma(i)$ of Γ. If $a \in \Omega$ is a critical point for g, then $(a, g(a)) \in A_2(\Gamma(i)) \subset A_2(\Gamma)$ and therefore $g(a) \in B_2(\Gamma)$, i.e., the critical value $\pi_2|_{\Gamma(i)}(\alpha) = g(a)$ of the function g is a critical value for the holomorphic correspondence associated to Γ, as well.

On the other hand, if $\alpha \notin A_1(\Gamma)$, then there exists a holomorphic function $g : \Omega \to \widehat{\mathbb{C}}$ defined on a neighborhood Ω of $a = \pi(\alpha)$, such that $(a, g(a)) = \alpha$, and $\mathrm{Gr}(g) \subset \Gamma(i)$ for some i. If in addition $\alpha \in A_2(\Gamma)$, then g is not locally injective at a, and therefore $g'(a) = 0$. Therefore, a is a critical point of g and $\pi_2(\alpha) = g(a)$ is a critical value of g.

If we denote by $\mathrm{CritPt}(g)$ the set of critical points of g, and by

$$\mathrm{CritVal}(g) := \{g(a) \mid a \in \mathrm{CritPt}(g)\}$$

the set of critical values of g, then we get a motivation for the name ‘critical values’ in Definition 2.3 by the containment

$$B_2(\Gamma) \setminus B_1(\Gamma) \subset \bigcup_i \bigcup_{\mathrm{Gr}(g)\subset\Gamma(i)} \mathrm{CritVal}(g),$$

where the first union runs over the irreducible components of Γ and the second union runs over all the holomorphic functions $g : \Omega \to \widehat{\mathbb{C}}$ whose graph $\mathrm{Gr}(g)$ is contained in $\Gamma(i)$.

PROPOSITION 2.4. *For every $a \in \mathbb{C} \setminus \{1\}$, we have that*

$$A_1(\Gamma_a) = \left\{\left(\infty, \frac{a+1}{2}\right), \left(-2, 1\right), \left(2, \frac{3a+1}{3+a}\right)\right\}, \tag{3}$$

and

$$A_2(\Gamma_a) = \left\{\left(\infty, \frac{a+1}{2}\right), \left(1, \frac{4a+2}{a+5}\right), \left(-1, \frac{2}{3-a}\right)\right\}. \tag{4}$$

As a consequence, $B_1(\Gamma_a) = \{\infty, -2, 2\}$ *and* $B_2(\Gamma_a) = \{(a+1)/2, (4a+2)/(a+5), 2/(3-a)\}$. *Moreover,* Γ_a *is smooth at all points except* $(\infty, (a+1)/2)$.

To prove this proposition, we first prove the following lemma.

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

LEMMA 2.5. *For each $a \in \mathbb{C} \setminus \{1\}$,*

$$A_1(\Gamma_{\mathrm{Cov}_0^Q}) = \{(\infty, \infty), (-2, 1), (2, -1)\}$$

and

$$A_2(\Gamma_{\mathrm{Cov}_0^Q}) = \{(\infty, \infty), (1, -2), (-1, 2)\}.$$

Thus, $B_1(\Gamma_{\mathrm{Cov}_0^Q}) = \{\infty, -2, 2\}$ and $B_2(\Gamma_{\mathrm{Cov}_0^Q}) = \{\infty, 1, -1\}$.

Proof. Differentiating the equation $P_Q(z, w) = 0$ with respect to w, we get that $\partial_w P_Q(z, w) = z + 2w$ vanishes if and only if $w = -z/2$. Also, $P_Q(z, -z/2) = 0$ if and only if $z = \pm 2$. Therefore, dw/dz exists on $\mathbb{C} \setminus \{-2, 2\}$. Thus, by the implicit function theorem, for every $(z, w) \in \Gamma_{\mathrm{Cov}_0^Q}$ such that $z \in \mathbb{C} \setminus \{-2, 2\}$, there exists a domain Ω containing z and a holomorphic function $g : \Omega \to \widehat{\mathbb{C}}$ such that $g(z) = w$ and $\mathrm{Gr}(g) = \Gamma_{\mathrm{Cov}_0^Q} \cap U$, for some open neighborhood U of (z, w). In addition, the function g will be locally injective at z if $\partial_z P_Q(z, w) = 2z + w$ is non-zero. Therefore, if $(z, w) \in \Gamma_{\mathrm{Cov}_0^Q}$ satisfies that both z and w are different from ± 2, then $(z, w) \notin A_2(\Gamma_{\mathrm{Cov}_0^Q})$.

On the other hand, observe that the only points $(z, w) \in \Gamma_{\mathrm{Cov}_0^Q}$ with $z = \pm 2$ are $(-2, 1)$ and $(2, -1)$. In particular, $w \neq \pm 2$, and by the symmetry of the above argument, we can use the implicit function theorem to obtain a neighborhood U of w and a function $g : U \to \widehat{\mathbb{C}}$ satisfying $\mathrm{Gr}(g) \subset \Gamma_{\mathrm{Cov}_0^Q}$ and $g(w) = z$, and such that $\pi_2|_{\Gamma_{\mathrm{Cov}_0^Q}}$ is injective in the neighborhood $(g(U) \times U) \cap \Gamma_{\mathrm{Cov}_0^Q}$ of (z, w). This proves that neither $(-2, 1)$ or $(2, -1)$ belong to $A_2(\Gamma_{\mathrm{Cov}_0^Q})$. Since the only points $(z, w) \in \Gamma_{\mathrm{Cov}_0^Q}$ with $z = \pm 1$ are $(1, -2)$ and $(-1, 2)$, and since $\Gamma_{\mathrm{Cov}_0^Q} \setminus (\mathbb{C} \times \mathbb{C}) = \{(\infty, \infty)\}$, we have that $A_2(\Gamma_{\mathrm{Cov}_0^Q})$ is contained in $\{(\infty, \infty), (1, -2), (-1, 2)\}$.

We will check that for every neighborhood W of (∞, ∞), $(1, -2)$, and $(-1, 2)$, we have that $\pi_2|_{W \cap \Gamma_{\mathrm{Cov}_0^Q}}$ is not injective. Let W_1, W_2, and W_3 be open neighborhoods of (∞, ∞), $(1, -2)$, and $(-1, 2)$, respectively. Then there exists $T > 0$ such that for every $0 < t < T$,

$$\left(\frac{1}{2}\left(\pm\sqrt{3}\sqrt{-\frac{1}{t^2} - \frac{4}{t}} + \frac{1}{t} + 2\right), -2 - \frac{1}{t}\right) \in \Gamma_{\mathrm{Cov}_0^Q} \cap W_1,$$

$$\left(\frac{1}{2}(\pm\sqrt{3}\sqrt{-t^2 - 4t} + t + 2), -2 - t\right) \in \Gamma_{\mathrm{Cov}_0^Q} \cap W_2,$$

and

$$\left(\frac{1}{2}(\pm\sqrt{3}\sqrt{4t - t^2} + t - 2), 2 - t\right) \in \Gamma_{\mathrm{Cov}_0^Q} \cap W_3.$$

We conclude that $A_2(\Gamma_{\mathrm{Cov}_0^Q}) = \{(\infty, \infty), (1, -2), (-1, 2)\}$, and by the symmetry of Cov_0^Q, $A_1(\Gamma_{\mathrm{Cov}_0^Q}) = \{(\infty, \infty), (-2, 1), (2, -1)\}$. We conclude that

$$B_2(\Gamma_{\mathrm{Cov}_0^Q}) = \{\infty, -2, 2\} = B_1(\Gamma_{\mathrm{Cov}_0^Q}). \qquad \square$$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

Proof of Proposition 2.4. Since $\mathcal{F}_a = \mathrm{J}_a \circ \mathrm{Cov}_0^{\mathrm{Q}}$ and J_a is an involution, we have that

$$A_j(\Gamma_a) = \{(z, \mathrm{J}_a(w)) : (z, w) \in A_j(\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}})\},$$

for $j = 1, 2$. Using Lemma 2.5, this gives us equations (3) and (4), and thus $B_1(\Gamma_a) = \{\infty, -2, 2\}$ and

$$B_2(\Gamma_a) = \{\mathrm{J}_a(\infty), \mathrm{J}_a(-2), \mathrm{J}_a(2)\} = \left\{\frac{a+1}{2}, \frac{4a+2}{a+5}, \frac{2}{3-a}\right\}.$$

In addition, observe that locally, Γ_a is either a function on z or on w for all $(z, w) \in \Gamma_a$ with $z \neq \infty$, then the only point that can be irregular is $(\infty, (a+1)/2)$. Indeed, this point is irregular, as the curve given by the points

$$\left(\frac{1}{2}\left(\pm\sqrt{3}\sqrt{-\frac{1}{t^2} - \frac{4}{t}} + \frac{1}{t} + 2\right), -2 - \frac{1}{t}\right) \in \Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}}$$

self-intersects at (∞, ∞) with an angle of $2\pi/3$. In other words, there are two functions $z(w)$ which intersect with different derivatives, which makes (∞, ∞) an irregular point of $\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}}$. Thus, passing through the involution J_a, we get that $(\infty, (a+1)/2)$ is an irregular point of Γ_a. □

Remark 2.6. The correspondence $\mathrm{Cov}_0^{\mathrm{Q}}$, and hence $\mathcal{F}_a$, sends open sets to open sets. Indeed, let $U \subset \widehat{\mathbb{C}}$ be open and take $w_0 \in \mathrm{Cov}_0^{\mathrm{Q}}(U)$. Then there exists $z_0 \in U$ for which $(z_0, w_0) \in \Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}}$. We will prove that $\mathrm{Cov}_0^{\mathrm{Q}}(U)$ is open by showing that in all the cases, $w_0 \in \mathrm{int}(\mathrm{Cov}_0^{\mathrm{Q}}(U))$.

- Suppose $(z_0, w_0) \notin A_1(\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}})$. By Lemma 2.5 and since $(\mathrm{Cov}_0^{\mathrm{Q}})^{-1}(\infty) = \{\infty\}$, we have that $w_0 \neq \infty$. Moreover, there exists a holomorphic function $g : \Omega \to \mathbb{C}$ on an open subset $\Omega \subset U$, and $(z_0, w_0) \in \mathrm{Gr}(g) \subset \Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}}$. Furthermore, $\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}}$ is irreducible and $\mathrm{Cov}_0^{\mathrm{Q}}$ is not constant, so g is not constant. Thus, g is open and then $w_0 \in g(\Omega) \subset \mathrm{int}(\mathrm{Cov}_0^{\mathrm{Q}}(U))$.
- Now suppose $(z_0, w_0) \notin A_2(\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}})$. By Lemma 2.5 and since $\mathrm{Cov}_0^{\mathrm{Q}}(\infty) = \{\infty\}$, then $z_0 \neq \infty$ and there exists a holomorphic function $\tilde{g} : \widetilde{\Omega} \to \mathbb{C}$ on an open subset $\widetilde{\Omega} \subset \mathbb{C}$ so that $(z_0, w_0) \in \iota(\mathrm{Gr}(\tilde{g})) \subset \Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}}$. Since $\tilde{g}$ is continuous, then $\tilde{g}^{-1}(U)$ is open and $w_0 \in \tilde{g}^{-1}(U) \subset \mathrm{int}(\mathrm{Cov}_0^{\mathrm{Q}}(U))$.
- Finally, if $(z_0, w_0) \in A_1(\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}}) \cap A_2(\Gamma_{\mathrm{Cov}_0^{\mathrm{Q}}})$, then Lemma 2.5 implies that $(z_0, w_0) = (\infty, \infty)$. For each $r > 0$, put $U_r := \{|z| > r\} \cup \{\infty\}$. To show that $\infty \in \mathrm{int}(\mathrm{Cov}_0^{\mathrm{Q}}(U))$, we will show that for every $R > \sqrt{3}$, $U_{2R} \subset \mathrm{Cov}_0^{\mathrm{Q}}(U_R)$. We proceed by the contrapositive. If $P_Q(z, w) = 0$ and $|w| \leq R$, then

$$|z|^2 = |3 - zw - w^2| \leq 3 + |z||w| + |w|^2 \leq 2R^2 + |z|R.$$

  Hence, $|z|^2 - R|z| - 2R^2 \leq 0$ and $|z| \leq 2R$. This implies that for every $R > \sqrt{3}$, we have that $\mathrm{Cov}_0^{\mathrm{Q}}(U_{2R}) \subset U_R$. By the symmetry of $\mathrm{Cov}_0^{\mathrm{Q}}$,

$$U_{2R} \subset \mathrm{Cov}_0^{\mathrm{Q}}(\mathrm{Cov}_0^{\mathrm{Q}}(U_{2R})) \subset \mathrm{Cov}_0^{\mathrm{Q}}(U_R).$$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

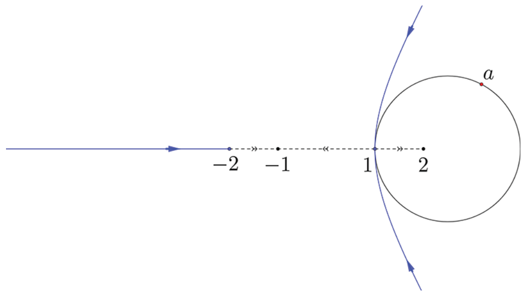

FIGURE 1. Klein combination pair for $|a-4| \leq 3$.

Moreover, if $R > \sqrt{3}$ is large enough so that $U_R \subset U$, then

$$\infty \in U_{2R} \subset \operatorname{int}(\operatorname{Cov}_0^{\mathrm{Q}}(U)).$$

Therefore, $\operatorname{Cov}_0^{\mathrm{Q}}$ sends open sets to open sets. Since the involution J_a also sends open sets to open sets, then so does $\mathcal{F}_a$.

2.4. *Klein combination pairs.* In this section, we will define the set $\mathcal{K}$ of parameters we will consider for our family.

Definition 2.7. A *fundamental domain* for an irreducible holomorphic correspondence F is an open set Δ_F that is maximal with the property that $\Delta_F \cap F(\Delta_F) = \varnothing$.

Definition 2.8. We say that a pair $(\Delta_{\operatorname{Cov}_0^{\mathrm{Q}}}, \Delta_{\mathrm{J}_a})$ of fundamental domains for $\operatorname{Cov}_0^{\mathrm{Q}}$ and J_a, respectively, is a *Klein combination pair* for $\mathcal{F}_a$ if both $\Delta_{\operatorname{Cov}_0^{\mathrm{Q}}}$ and Δ_{J_a} are simply connected domains, bounded by Jordan curves, and satisfy

$$\Delta_{\operatorname{Cov}_0^{\mathrm{Q}}} \cup \Delta_{\mathrm{J}_a} = \widehat{\mathbb{C}} \setminus \{1\}.$$

We define as well the *Klein combination locus* $\mathcal{K}$ to be the set of parameters $a \in \mathbb{C} \setminus \{1\}$ for which there exist a Klein combination pair.

In [4], for $|a-4| \leq 3$, $a \neq 1$, the authors found a Klein combination pair for $\mathcal{F}_a$, where $\Delta_{\operatorname{Cov}_0^{\mathrm{Q}}}$ is given by the right side of the curve

$$L := \left\{ \left(1 + \frac{t}{2}\right) \pm i\sqrt{3\left(t + \frac{t}{2}^2\right)} \,\middle|\, t \in [0, \infty] \right\} = \operatorname{Cov}_0^{\mathrm{Q}}((-\infty, -2]),$$

and Δ_{J_a} is given by the exterior of the circle passing through $z = 1$ and $z = a$ with diameter contained in the real line.

This pair $(\Delta_{\operatorname{Cov}_0^{\mathrm{Q}}}, \Delta_{\mathrm{J}_a})$ is composed by simply connected fundamental domains for $\operatorname{Cov}_0^{\mathrm{Q}}$ and J_a, respectively, whose boundaries are Jordan curves, smooth except from $\partial\Delta_{\operatorname{Cov}_0^{\mathrm{Q}}}$ at (∞, ∞) (see Figure 1 and [4, Proposition 3.3]). In particular,

$$\{a \in \mathbb{C} | \, |a-4| \leq 3\} \setminus \{1\} \subset \mathcal{K}.$$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

From now on, whenever $a \in \mathcal{K}$, we denote by $(\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta_{\mathrm{J}_a})$ a Klein combination pair for $\mathcal{F}_a$.

The following remark will be useful to prove that $\mathcal{F}_a$ is not modular, and later to analyze the asymptotic behavior of $\mathcal{F}_a$.

Remark 2.9. Let $a \in \mathcal{K}$.

(1) Note that $\mathcal{F}_a(\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) \subset \widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}$ and $\mathcal{F}_a((\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) \setminus \{1\}) \subset \widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{J}_a}}$. Indeed,

$$\mathcal{F}_a(1) = \mathrm{J}_a(\mathrm{Cov}_0^{\mathrm{Q}}(1)) = \mathrm{J}_a(\{1, -2\}) = \left\{1, \frac{4a+2}{a+5}\right\}.$$

From Remark 2.6, we have that $\mathrm{Cov}_0^{\mathrm{Q}}$ sends open sets to open sets. In particular, -2 cannot belong to $\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}$, as $1 \notin \mathrm{Cov}_0^{\mathrm{Q}}(\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}) = \widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}}$. By the Klein combination pair condition, this implies that $-2 \in \Delta_{\mathrm{J}_a}$, and thus $\mathrm{J}_a(-2) = (4a+2)/(a+5) \notin \Delta_{\mathrm{J}_a}$. Since we also have that $1 \notin \Delta_{\mathrm{J}_a}$, we have that

$$\mathcal{F}_a(1) \subset \widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}.$$

However, for $z \in \widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}$, $z \neq 1$, we have that $z \in \Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}$. Therefore,

$$\mathrm{Cov}_0^{\mathrm{Q}}(z) \subset \widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}} \subset \Delta_{\mathrm{J}_a},$$

and thus, since J_a is an involution fixing $\partial\Delta_{\mathrm{J}_a}$,

$$\mathcal{F}_a(z) = \mathrm{J}_a \circ \mathrm{Cov}_0^{\mathrm{Q}}(z) \subset \mathrm{J}_a(\Delta_{\mathrm{J}_a}) = \widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{J}_a}}.$$

In particular, $\Delta_{\mathrm{J}_a} \setminus \mathcal{F}_a(\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) \neq \varnothing$.

(2) Observe that $\mathcal{F}_a^{-1}(\Delta_{\mathrm{J}_a}) \subset \Delta_{\mathrm{J}_a}$ and $\mathcal{F}_a^{-1}(\overline{\Delta_{\mathrm{J}_a}}) \subset \Delta_{\mathrm{J}_a} \cup \{1\}$. Indeed, for every $z \in \Delta_{\mathrm{J}_a}$, we have that $\mathrm{J}_a(z) \in \widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{J}_a}} \subset \Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}$. From Remark 2.6 and since $(\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta_{\mathrm{J}_a})$ is a Klein combination pair, then

$$\mathcal{F}_a^{-1}(z) = \mathrm{Cov}_0^{\mathrm{Q}}(\widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{J}_a}}) \subset \mathrm{Cov}_0^{\mathrm{Q}}(\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}) \subset \widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}} \subset \Delta_{\mathrm{J}_a}.$$

Furthermore, for $w \in \overline{\Delta_{\mathrm{J}_a}}$, we have that

$$\mathcal{F}_a^{-1}(w) \subset \mathrm{Cov}_0^{\mathrm{Q}}(\mathrm{J}_a(\overline{\Delta_{\mathrm{J}_a}})) \subset \mathrm{Cov}_0^{\mathrm{Q}}(\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) \subset \mathrm{Cov}_0^{\mathrm{Q}}(\overline{\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}}) \subset \widehat{\mathbb{C}} \setminus \Delta_{\mathrm{Cov}_0^{\mathrm{Q}}} \subset \overline{\Delta_{\mathrm{J}_a}}.$$

3. *Clasification of the family* $\{\mathcal{F}_a\}_a$

In this section, we prove that $\mathcal{F}_a$ is weakly modular but not modular, for every $a \in \mathcal{K}$. In addition, we prove that it does not satisfy the required conditions for the equidistribution result [**1**, Theorem 3.5]. All together proving Theorem 1.1.

3.1. *Modularity and weak modularity.*

Definition 3.1. Let G be a connected Lie group, Λ a torsion free lattice, and K a compact Lie subgroup. Let $g \in G$ be such that $g\Lambda g^{-1} \cap \Lambda$ has finite index in Λ. The *irreducible modular correspondence* induced by g is the multivalued map F_g on $X = \Lambda \setminus G/K$ corresponding to the projection to X of the map $x \mapsto (x, gx)$ on $G \to G \times G$. Denote by Γ_g the graph of F_g. A *modular correspondence* F is a correspondence whose graph is of the form $\sum_j n_j \Gamma_{g_j}$, for Γ_{g_j} as before.

The following definition was introduced by Dinh, Kaufmann, and Wu in [15].

Definition 3.2. Let X be a compact Riemann surface and let F be a holomorphic correspondence on X with graph Γ such that $d(F) = d(F^{-1})$. We say that F is a *weakly modular correspondence* if there exist Borel probability measures μ_1 and μ_2 on X, such that

$$(\pi_1|_\Gamma)^*\mu_1 = (\pi_2|_\Gamma)^*\mu_2.$$

Remark 3.3

(1) Let F be a modular correspondence that is also a holomorphic correspondence. Then it is always the case that

$$d(F) = \sum_j n_j[\Lambda : g_j\Lambda g_j^{-1} \cap \Lambda] = \sum_j n_j[\Lambda : g_j^{-1}\Lambda g_j \cap \Lambda] = d(F^{-1}).$$

(2) With the notation in Definition 3.1, let λ be the direct image on X of the finite Haar measure on $\Lambda \setminus G$. Then, $(1/d)(F_g)F_g^*\lambda = \lambda$ and if we put $\mu_1 = \mu_2 = \lambda$, we get that

$$(\pi_1|_{\Gamma_g})^*\mu_1 = (\pi_2|_{\Gamma_g})^*\mu_2.$$

Therefore, modular correspondences are weakly modular.

(3) The measure λ above is Borel, invariant under F_g, and assigns positive measure to non-empty open sets.

Proof of Theorem 1.1 part 1. Observe that the graph $\Gamma_{\mathrm{Cov}_0^Q}$ of the correspondence Cov_0^Q is symmetric with respect to the diagonal $\mathfrak{D}_{\widehat{\mathbb{C}}} = \{(z, z) | z \in \widehat{\mathbb{C}}\}$, as $z \in \mathrm{Cov}_0^Q(w)$ if and only if $w \in \mathrm{Cov}_0^Q(z)$. Let m be any positive and finite measure on $\Gamma_{\mathrm{Cov}_0^Q}$, and $\iota : \Gamma_{\mathrm{Cov}_0^Q} \to \Gamma_{\mathrm{Cov}_0^Q}$ the involution $\iota(z, w) := (w, z)$. Take

$$m_0 := \Big(\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)^*\Big(\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)_* m + \iota^*\Big(\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)^*\Big(\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)_* m.$$

The measure m_0 is symmetric in $\Gamma_{\mathrm{Cov}_0^Q}$ in the sense that $\iota^*m' = m'$ and moreover

$$\Big(\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)^*\Big(\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)_* m_0 = \Big(\pi_2|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)^*\Big(\pi_2|_{\Gamma_{\mathrm{Cov}_0^Q}}\Big)_* m_0. \tag{5}$$

After normalizing if necessary, this proves that Cov_0^Q is weakly modular with measures $\mu_1' := (\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}})_* m_0$ and $\mu_2' := (\pi_2|_{\Gamma_{\mathrm{Cov}_0^Q}})_* m_0$. Our goal is to show there are probability measures μ_1 and μ_2 on $\widehat{\mathbb{C}}$ such that

$$(\pi_1|_{\Gamma_a})^*\mu_1 = (\pi_2|_{\Gamma_a})^*\mu_2, \tag{6}$$

where Γ_a is the graph of the correspondence $\mathcal{F}_a = \mathcal{F}_a \circ \mathrm{Cov}_0^Q$.

Put $\mu_2 := (\mathrm{J}_a)^*\mu_2' = (\mathrm{J}_a)_*\mu_2'$ and observe that by symmetry of J_a, we have that

$$(\pi_2|_{\Gamma_a})^*\mu_2 = (\pi_2|_{\Gamma_a})^*(\mathrm{J}_a)^*\mu_2' = (\mathrm{J}_a \circ \pi_2|_{\Gamma_a})^*\mu_2'. \tag{7}$$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

Now let $\widehat{\mathrm{J}_a} : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ be given by $\widehat{\mathrm{J}_a}(z, w) := (z, \mathrm{J}_a(w))$. Observe that whenever $(z, w) \in \Gamma_{\mathrm{Cov}_0^Q}$, then $(z, \mathrm{J}_a(w)) \in \Gamma_a$ and

$$(\mathrm{J}_a \circ \pi_2|_{\Gamma_{\mathrm{Cov}_0^Q}})(z, w) = \mathrm{J}_a(w) = (\pi_2|_{\Gamma_a} \circ \widehat{\mathrm{J}_a})(z, w). \tag{8}$$

On the other hand, $\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}} \circ \widehat{\mathrm{J}_a}(z, w) = \pi_1|_{\Gamma_a}(z, w)$ with multiplicity 2. This, together with equations (5), (7), and (8) yield

$$(\pi_2|_{\Gamma_a})^* \mu_2 = (\widehat{\mathrm{J}_a})^* (\pi_2|_{\Gamma_{\mathrm{Cov}_0^Q}})^* \mu_2' = (\widehat{\mathrm{J}_a})^* (\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}})^* \mu_1' = 2(\pi_1|_{\Gamma_{\mathrm{Cov}_0^Q}})^* \mu_1'. \tag{9}$$

Observe that $\mu_1 := 2\mu_1'$ and μ_2 both have 2 times the mass of μ_1 and μ_2. After normalizing, equation (9) proves that $\mathcal{F}_a$ is weakly modular, as desired.

To check $\mathcal{F}_a$ is not modular, we will prove that no Borel measure λ on $\widehat{\mathbb{C}}$ that gives positive measure to non-empty open sets can be invariant under $\mathcal{F}_a$. Suppose by contradiction that λ is such a measure satisfying $\frac{1}{2}\mathcal{F}_a^* \lambda = \lambda$. In particular, we have that $\lambda((\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) \setminus \mathcal{F}_a(\widehat{\mathbb{C}} \setminus \Delta_J)) = 0$. On the other hand, note that

$$\mathcal{F}_a(\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) = \pi_2|_{\Gamma_a}(\Gamma_a \cap ((\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) \times \widehat{\mathbb{C}}))$$

is closed in $\widehat{\mathbb{C}}$. By Remark 2.9 part (1), $(\widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{J}_a}}) \setminus \mathcal{F}_a(\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a})$ is open and non-empty, contained in $(\widehat{\mathbb{C}} \setminus \Delta_{\mathrm{J}_a}) \setminus \mathcal{F}_a(\widehat{\mathbb{C}} \setminus \Delta_J)$. This contradicts part (3) of Remark 3.3, as λ cannot assign 0 measure to open sets. □

Remark 3.4. For a (d, d) holomorphic correspondence F on compact Riemann surface X, the operator $(1/d)F^*$ acts on the space $L^2_{(1,0)}$ of $(1, 0)$-forms with L^2 coefficients. In [**15**], the authors showed that the operator norm satisfies $\|(1/d)F^*\| \le 1$, with strict inequality for non weakly modular correspondences. This strict inequality is a key factor of their equidistribution result. However, this is never the case for $F = \mathcal{F}_a$, $a \neq 1$. We claim that

$$\left\|\tfrac{1}{2}\mathcal{F}_a^*\right\| = \sup\left\{\left\|\tfrac{1}{2}\mathcal{F}_a^*\phi\right\|_{L^2} \middle| \phi \in L^2_{(1,0)}, \|\phi\|_{L^2} = 1\right\} = 1,$$

and furthermore that the supremum is attained. To prove this, we use that $\|(1/d)F^*\phi\|_{L^2} = \|\phi\|_{L^2}$ for $\phi \in L^2_{(1,0)}$ if and only if for every $U \subset X \setminus B_1(\Gamma)$ and for every pair of local branches f_1 and f_2 of F on U, the equality $f_1^*\phi = f_2^*\phi$ holds on U (see [**15**, Proposition 2.1]).

Observe that the form $\phi(z) = e^{-|Q(z)|}\, dz$ belongs to $L^2_{(1,0)}$ for Q as in §2.2. Let $U \subset \widehat{\mathbb{C}} \setminus B_1(\Gamma_{\mathrm{Cov}_0^Q})$. Then the deleted covering correspondence Cov_0^Q sends z to the values w for which $(Q(z) - Q(w))/(z - w) = 0$. Hence, any two local branches f_1 and f_2 of Cov_0^Q satisfy $f_1^*\phi(z) = Q(z) = f_2^*\phi(z)$, and thus $\|\frac{1}{2}\,{\mathrm{Cov}_0^Q}^*\| = 1$. Now note that J_a is an involution, and hence J_a^* has operator norm 1. Thus we can conclude that $\|\frac{1}{2}\mathcal{F}_a^*\| = 1$ as well, with supremum attained at ϕ.

3.2. *Limit sets.* In this section, we define limit sets and give some properties. We will also prove that the application listed in [**1**, §7] does not hold for our family of correspondences, and hence this is a new case to study equidistribution.

Remark 3.5. From Proposition 2.4, observe that $1 \notin B_1(\Gamma_a)$, and hence there is a holomorphic function g whose graph contains $(1, 1)$ and is contained in Γ_a. After the change of coordinates $\psi(z) = z - 1$, the function g has Taylor series expansion

$$g^{\psi}(z) = z + \frac{a-7}{3(a-1)} z^2 + \cdots$$

whenever $a \neq 7$, and

$$g^{\psi}(z) = z + \frac{1}{27} z^4 + \cdots$$

for $a = 7$ (see [4, Proposition 3.5]). In particular, g^{ψ} has a parabolic fixed point at 0 with multiplier 1.

In [4, Proposition 3.8], the authors showed that for each $a \in \mathcal{K}$, after a small perturbation of $\partial\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}$ and $\partial\Delta_{\mathrm{J}_a}$ around $z = 1$, we can choose a Klein combination pair $(\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta'_{\mathrm{J}_a})$ so that $\partial\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}$ and $\partial\Delta'_{\mathrm{J}_a}$ are both smooth at 1, and transverse at 1 to the line generated by the repelling direction at $z = 1$ for $a \neq 7$, and to the real axis in the case $a = 7$.

For $a \in \mathcal{K}$ and $(\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta_{\mathrm{J}_a})$ as above, we define

$$\Lambda_{a,+} := \bigcap_{n=0}^{\infty} \mathcal{F}_a^n(\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}) \quad \text{and} \quad \Lambda_{a,-} := \bigcap_{n=0}^{\infty} \mathcal{F}_a^{-n}(\overline{\Delta'_{\mathrm{J}_a}})$$

to be the *forward and backward limit set of* $\mathcal{F}_a$, respectively. These sets do not depend on the choice of the Klein combination pair $(\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta'_{\mathrm{J}_a})$ as above.

LEMMA 3.6. *Let $a \in \mathcal{K}$. We have that:*

(1) $\mathrm{J}_a(\Lambda_{a,\pm}) = \Lambda_{a,\mp}$ *and* $\mathrm{J}_a(\partial\Lambda_{a,\pm}) = \partial\Lambda_{a,\mp}$*;*

(2) $\Lambda_{a,-} \cap \Lambda_{a,+} = \{1\}$;

(3) *if $z \notin \Lambda_{a,-}$, then there exists $n \geq 1$ so that $\mathcal{F}_a^n(z) \subset \widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$, and if $z \notin \Lambda_{a,+}$, then there exists $n \geq 1$ such that $\mathcal{F}_a^{-n}(z) \subset \Delta'_{\mathrm{J}_a}$;*

(4) $\mathcal{F}_a^{-1}(\Lambda_{a,-}) = \Lambda_{a,-}$ *and* $\mathcal{F}_a^{-1}(\partial\Lambda_{a,-}) = \partial\Lambda_{a,-}$*; and*

(5) $\mathcal{F}_a(\Lambda_{a,+}) = \Lambda_{a,+}$ *and* $\mathcal{F}_a(\partial\Lambda_{a,+}) = \partial\Lambda_{a,+}$.

Proof. Since J_a is an involution sending $\overline{\Delta'_{\mathrm{J}_a}}$ and $\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$ to each other, and since $\mathrm{J}_a \circ \mathcal{F}_a^n = \mathcal{F}_a^{-n} \circ \mathrm{J}_a$, then

$$\mathrm{J}_a(\Lambda_{a,+}) = \bigcap_{n=0}^{\infty} \mathrm{J}_a \circ \mathcal{F}_a^n(\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}) = \bigcap_{n=0}^{\infty} \mathcal{F}_a^{-n} \circ \mathrm{J}_a(\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}) = \Lambda_{a,-},$$

and applying J_a to both sides, we also get $\mathrm{J}_a(\Lambda_{a,-}) = \Lambda_{a,+}$. Moreover, since J_a is continuous, note that

$$\Lambda_{a,-} \cap \Lambda_{a,+} \subset \overline{\Delta'_{\mathrm{J}_a}} \cap (\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}) = \partial\Delta'_{\mathrm{J}_a}.$$

Since $(\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta'_{\mathrm{J}_a})$ is a Klein combination pair, $\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a} \subset \Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}$, so $\partial\Delta'_{\mathrm{J}_a} \subset \overline{\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}}$.

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

If $z \in \Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}$, then $\mathrm{Cov}_0^{\mathrm{Q}}(z) \subset \widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$, and $\mathcal{F}_a(z) = \mathrm{J}_a \circ \mathrm{Cov}_0^{\mathrm{Q}}(z) \subset \Delta'_{\mathrm{J}_a}$. This is a contradiction, as $\mathcal{F}_a(z)$ must belong to $\Lambda_{a,+} \subset \widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$ as well. Again, since $(\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta'_{\mathrm{J}_a})$ is a Klein combination pair, we have that $z \in \partial\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}} \cap \partial\Delta'_{\mathrm{J}_a} = \{1\}$ and we conclude that

$$\Lambda_{a,-} \cap \Lambda_{a,+} = \{1\}.$$

We prove part (3) by the contrapositive. Suppose that for all n, there exists $w \in \mathcal{F}_a^n(z) \cap (\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a})$, then $z \in \mathcal{F}_a^n(w) \subset \mathcal{F}_a^n(\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a})$ for all n. This implies that z must belong to $\Lambda_{a,-}$. The other case is analogous.

It is immediate from the definition of $\Lambda_{a,-}$ and from part (2) of Remark 2.9 that

$$\mathcal{F}_a^{-1}(\Lambda_{a,-}) = \Lambda_{a,-}. \tag{10}$$

From this and Remark 2.6,

$$\mathcal{F}_a^{-1}(\mathrm{int}(\Lambda_{a,-})) \subset \mathrm{int}(\mathcal{F}_a^{-1}(\Lambda_{a,-})) = \mathrm{int}(\Lambda_{a,-}). \tag{11}$$

Observe that if $z \in \partial\Lambda_{a,-} \setminus \{1\} \subset \Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}$, then

$$\mathcal{F}_a(z) = \mathrm{J}_a(\mathrm{Cov}_0^{\mathrm{Q}}(z)) \subset \mathrm{J}_a(\widehat{\mathbb{C}}) \subset \mathrm{J}_a(\Delta'_{\mathrm{J}_a}) \subset \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}}.$$

Since $\Lambda_{a,-} \subset \overline{\Delta'_{\mathrm{J}_a}}$, then we conclude that

$$\Lambda_{a,-} \cap \partial\Delta'_{\mathrm{J}_a} = \{1\}. \tag{12}$$

Put $w \in \partial\Lambda_{a,-}$ and $z \in \mathcal{F}_a^{-1}(w) \subset \Lambda_{a,-}$. We will show that $z \in \partial\Lambda_{a,-}$ by the contrapositive. Suppose $w \neq 1$ and $z \in \mathrm{int}(\Lambda_{a,-})$. Then equation (12) implies that

$$w \in \mathcal{F}_a(\mathrm{int}(\Lambda_{a,-})) \cap \Delta'_{\mathrm{J}_a}.$$

From Remark 2.6, and the fact that Δ'_{J_a} is open, we have that $\mathcal{F}_a(\mathrm{int}(\Lambda_{a,-})) \cap \Delta'_{\mathrm{J}_a}$ is open. Moreover, for each $z' \in \mathrm{int}(\Lambda_{a,-})$, the set $\mathcal{F}_a(z')$ consists of a point in Δ'_{J_a} and one in $\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$. Since the one in Δ'_{J_a} is actually in $\Lambda_{a,-}$ by definition of $\Lambda_{a,-}$, then $\mathcal{F}_a(\mathrm{int}(\Lambda_{a,-})) \cap \Delta'_{\mathrm{J}_a} \subset \Lambda_{a,-}$. Then $w \in \mathrm{int}(\Lambda_{a,-})$. This proves that

$$\mathcal{F}_a^{-1}(\partial\Lambda_{a,-}) \subset \Lambda_{a,-}. \tag{13}$$

As for $w = 1$, we have that $z \in \mathcal{F}_a^{-1}(1) = \{-2, 1\}$. We know $1 \in \partial\Lambda_{a,-}$, so it suffices to show that $-2 \in \partial\Lambda_{a,-}$ as well. Let $U \subset \widehat{\mathbb{C}}$ be an open neighborhood of -2. From Remark 2.6, we have that $\mathcal{F}_a(U)$ is an open neighborhood of 1. Since $1 \in \partial\Delta'_{\mathrm{J}_a}$, and $\partial\Delta'_{\mathrm{J}_a}$ is a Jordan curve, then there exists a point $w' \in U \cap \partial\Delta'_{\mathrm{J}_a}$, $w' \neq 1$. We have that $\mathcal{F}_a(\partial\Delta'_{\mathrm{J}_a} \setminus \{1\}) \subset \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}}$, so $\mathcal{F}_a^{-1}(w')$ consists of two points, $z', z'' \in \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}}$, with $z' \in U$. We will prove that $z' \notin \Lambda_{a,-}$ by showing that $\mathcal{F}_a(z') \cap \Lambda_{a,-} = \varnothing$, and thus $-2 \in \Lambda_{a,-} \setminus \mathrm{int}(\Lambda_{a,-}) = \partial\Lambda_{a,-}$. Indeed,

$$\mathcal{F}_a(z') = \mathrm{J}_a(\{z'', \mathrm{J}_a(w')\}) = \{\mathrm{J}_a(z''), w'\} \subset \widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}.$$

From part (2) of Remark 2.9, we have that $z' \notin \Lambda_{a,-}$, and $-2 \in \partial\Lambda_{a,-}$ as desired. This, together with equations (10), (11), and (13) prove part (4).

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

We have that part (1) of Remark 2.9 together with the definition of $\Lambda_{a,+}$ prove that $\mathcal{F}_a(\Lambda_{a,+}) = \Lambda_{a,+}$. To prove the rest of part (5), we use part (4) together with the fact that $\mathcal{F}_a = \mathrm{J}_a \circ \mathcal{F}_a^{-1} \circ \mathrm{J}_a$ and $\mathrm{J}_a(\partial\Lambda_{a,\pm}) = \partial\Lambda_{a,\mp}$, as J_a is a continuous involution. Thus,

$$\mathcal{F}_a(\partial\Lambda_{a,+}) = \mathrm{J}_a \circ \mathcal{F}_a^{-1} \circ \mathrm{J}_a(\partial\Lambda_{a,+}) = \mathrm{J}_a(\mathcal{F}_a^{-1}(\partial\Lambda_{a,-})) = \mathrm{J}_a(\partial\Lambda_{a,-}) = \partial\Lambda_{a,+}. \qquad \square$$

The following definition is from [1, 26].

Definition 3.7. Let F be a holomorphic correspondence on X. We say that $\mathcal{R} \subset X$ is a *repeller* for F if there exists a set U such that $\mathcal{R}$ is contained in the interior of U, and

$$\mathcal{R} = \bigcap_{K \in \mathfrak{K}(U, F^{-1})} K,$$

where

$$\mathfrak{K}(U, F^{-1}) := \{K \subset X \mid F^{-1}(K) \subset K \text{ and } F^{-n}(U) \subset K \text{ for some } n \geq 0\}.$$

Bharali and Sridharan proved an equidistribution result similar to that in this paper [1, Theorem 3.5] for correspondences having a repeller. Moreover, in [1, §7.2], they showed that there is a set of pairs (a, k) for which their result can be applied to the correspondence

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k.$$

For these correspondences (studied by Bullett and Harvey in [8]), there is a set $\Lambda_{a,-}$ analogous to that presented here (see [7] for the general definition of limit sets). For the pair (a, k) to work for their theorem, it is crucial for $\partial\Lambda_{a,-}$ to be a repeller for the correspondence. Nevertheless, part (2) of Theorem 1.1 says this never happens for $k = 1$ and $|a - 4| \leq 3$.

Proof of Theorem 1.1 part (2). Let $U \subset \widehat{\mathbb{C}}$ contain $\partial\Lambda_{a,-}$ in its interior. We have that $1 \in \partial\Lambda_{a,-}$ and is a parabolic fixed point of the function g whose graph is contained in Γ_a, described in Remark 3.5. Take an attracting petal $\mathcal{P}$ at 1 so that $\mathcal{P} \subset U$. We first show that every $K \in \mathfrak{K}(U, \mathcal{F}_a^{-1})$ contains $\partial\Lambda_{a,-} \cup \mathcal{P}$. Indeed, for every $K \in \mathfrak{K}(U, \mathcal{F}_a^{-1})$ and some integer $n \geq 0$,

$$\mathcal{P} \subset g^{-n}(\mathcal{P}) \subset \mathcal{F}_a^{-n}(\mathcal{P}) \subset \mathcal{F}_a^{-n}(U) \subset K. \tag{14}$$

Moreover, from part (4) of Lemma 3.6, we have that

$$\partial\Lambda_{a,-} \subset \mathcal{F}_a^{-n}(U) \subset K. \tag{15}$$

Putting equations (14) and (15) together, we get that every $K \in \mathfrak{K}(U, \mathcal{F}_a^{-1})$ contains the union $\partial\Lambda_{a,-} \cup \mathcal{P}$, and therefore so does the intersection over all K. Since $\mathrm{int}(\partial\Lambda_{a,-}) = \varnothing$ and $\mathrm{int}(\mathcal{P}) \neq \varnothing$, we have that

$$\partial\Lambda_{a,-} \subsetneq \partial\Lambda_{a,-} \cup \mathcal{P} \subset \bigcap_{K \in \mathfrak{K}(U, \mathcal{F}_a^{-1})} K.$$

Since U is arbitrary, we conclude that $\partial\Lambda_{a,-}$ is not a repeller for $\mathcal{F}_a$. $\square$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

4. *Exceptional set and periodic points*

In this section, we will define a two-sided restriction of $\mathcal{F}_a$ and prove it is a proper holomorphic map of degree 2. We will find its exceptional set and that of $\mathcal{F}_a$. This will be important for the next section, as it is the set of all points that may escape from the equidistribution property given in Theorem 1.2.

The following definition is classical.

Definition 4.1. Let $f : U \to V$ be a holomorphic proper map, with U, V open, $U \subset V$. For $z \in U$, we denote by $[z]$ the equivalence class of z by the equivalence relation

$$w \sim z \Leftrightarrow \text{ there exist } n, m \in \mathbb{Z}^+ \cup \{0\},\ f^n(w) = f^n(z).$$

We say that z is *exceptional* for f if $[z]$ is finite, and we call *exceptional set* the set $\mathcal{E}$ of all points that are exceptional for f.

Put $a \in \mathcal{K}$ and denote by f_a the two-sided restriction $\mathcal{F}_a| : \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \to \Delta'_{\mathrm{J}_a}$, meaning f_a sends each $z \in \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ to the unique point in $\mathcal{F}_a(z) \cap \Delta'_{\mathrm{J}_a}$. This it is a single-valued, continuous, and holomorphic 2-to-1 map (see [4, Proposition 3.4] and Theorem 4.2 below) that extends on a neighborhood of every point $z \in \partial\mathcal{F}_a^{-1}(\Delta_{\mathrm{J}_a}) \setminus \{-2\}$. In particular, f_a extends around $z = 1$ and $f_a(1) = 1$. Since Δ'_{J_a} is open and f_a is continuous, then $\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})(= f_a^{-1}(\Delta'_{\mathrm{J}_a}))$ is open as well. Note that the analogous of part (2) of Remark 2.9 holds for Δ'_{J_a} and $\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}$ instead, using the definition of Klein pair and the fact that J_a is open. Thus, $\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \subset \Delta'_{\mathrm{J}_a}$.

The following theorem is the main result of this section.

THEOREM 4.2. *For each $a \in \mathcal{K}$, we have the following.*

(1) *The two-sided restriction $f_a : \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \to \Delta'_{\mathrm{J}_a}$ of $\mathcal{F}_a$ is holomorphic and proper, of degree 2.*

(2) *The map f_a has a critical point if and only if $2 \in \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}}$. Furthermore, in that case, we have that the critical point is -1.*

(3) *The exceptional set $\mathcal{E}_{a,-}$ of f_a is non-empty if and only if $a = 5$. In that case, $\mathcal{E}_{a,-} = \{-1\}$.*

Computing images and preimages under $\mathcal{F}_a$, we see that when $a = 5$, we have that $\circlearrowright -1 \mapsto 2 \circlearrowleft$. In the following section, it will be useful to use the full orbit of $\mathcal{E}_{a,-}$ under $\mathcal{F}_a$, meaning

$$\mathcal{E}_a := \begin{cases} \varnothing & \text{if } a \neq 5, \\ \{-1, 2\} & \text{if } a = 5. \end{cases}$$

Proof of Theorem 4.2. We first show part (1). Observe that ∞ and $\mathrm{J}_a(\infty)$ lie on opposite sides of $\partial\Delta'_{\mathrm{J}_a}$. Therefore, $(\infty, (a+1)/2) \notin (\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \times \Delta'_{\mathrm{J}_a})$. In view of Proposition 2.4, this implies that for every

$$(z_0, w_0) \in \mathrm{Gr}(f_a) = (\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \times \Delta'_{\mathrm{J}_a}) \cap \Gamma_a,$$

there there exists a neighborhood U of (z_0, w_0) such that $U \cap \Gamma_a$ is the graph of a holomorphic function either in z or in w. Therefore, $\mathrm{Gr}(f_a)$ is an open subset of

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

$\Gamma_a \setminus \{(\infty, (a+1)/2)\}$, which has no singularities. Thus, $\mathrm{Gr}(f_a)$ is a Riemann surface and f_a is holomorphic.

It is clear that $f_a^{-1}(w)$ is compact for all $w \in \Delta_{\mathrm{J}_a}$, since

$$1 \le |f_a^{-1}(w)| \le |\mathcal{F}_a^{-1}(w)| \le 2.$$

Moreover, f_a is a closed map, and therefore a proper map. Indeed, let $C \subset \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ be closed. Then $C = C' \cap \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ for some closed subset $C' \subset \widehat{\mathbb{C}}$. Observe that $f_a(C) = \mathcal{F}_a(C') \cap \Delta'_{\mathrm{J}_a}$ and that

$$\mathcal{F}_a(C') = \pi_2(\Gamma_a \cap (C' \times \widehat{\mathbb{C}})).$$

By compactness of $\widehat{\mathbb{C}}$, π_2 is closed, and since both Γ_a and $C' \times \widehat{\mathbb{C}}$ are closed subsets of $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$, then $\mathcal{F}_a(C')$ is closed. Therefore, $f_a(C)$ is closed in Δ'_{J_a} and f_a is a closed map. We conclude f_a is proper. Moreover, by definition of f_a, every preimage z of a point $w \in \Delta'_{\mathrm{J}_a}$ belongs to the domain of f_a, and $f_a(z) = w$. Therefore, f_a has degree 2, since $\mathcal{F}_a$ has two preimages of a generic point in Δ'_{J_a}.

We proceed to show part (2). Let z_0 be a critical point of f_a. Then $(z_0, f_a(z_0))$ belongs to

$$A_2(\Gamma_a) \cap (\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \times \Delta'_{\mathrm{J}_a}).$$

Observe that $\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \subset \Delta'_{\mathrm{J}_a}$ and $1 \notin \Delta'_{\mathrm{J}_a}$, so $(1, \mathrm{J}_a(-2)) \notin \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \times \Delta'_{\mathrm{J}_a}$. Since we also have that ∞ and $\mathrm{J}_a(\infty)$ lie on opposite sides of $\partial\Delta'_{\mathrm{J}_a}$ and from Proposition 2.4, it must be the case that $(z_0, f_a(z_0)) = (-1, \mathrm{J}_a(2))$. Thus, $z_0 = -1$ and $\mathrm{J}_a(2) \in \Delta'_{\mathrm{J}_a}$, and therefore $2 \in \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}}$. To prove the reverse implication, note that whenever $2 \in \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}}$, we have that $\mathrm{J}_a(2) = 2/(3-a) \in \Delta'_{\mathrm{J}_a}$, so there exists $z_0 \in \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ such that $f_a(z_0) = 2/(3-a)$, by definition of f_a. Then $z_0 = -1 \in \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ is a critical point for f_a, as f_a has degree 2 and $f_a^{-1}(2/(3-a)) = \{-1\}$. This proves that f_a has a critical point if and only if $2 \in \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}}$, and the critical point is $z_0 = -1$.

Finally, we prove part (3). Observe that if $w \in f_a^{-1}(z)$, then $[w] = [z]$, so $f_a^{-1}(\mathcal{E}_{a,-}) \subset \mathcal{E}_{a,-}$ and thus, $|f_a^{-1}(\mathcal{E}_{a,-})| \le |\mathcal{E}_{a,-}|$. In particular, all points of $\mathcal{E}_{a,-}$ must have only one preimage under f_a, and therefore, be critical. By part (2), if $\mathcal{E}_{a,-} \neq \varnothing$, then -1 must be fixed. We know $f_a(-1) = 2/(3-a)$, and therefore if $\mathcal{E}_{a,-} \neq \varnothing$, then $2/(3-a) = -1$. Solving the equation, we get that $a = 5$. On the other hand, if $a = 5$, then $2 \in \widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$ and therefore $z_0 = -1$ is a critical point for f_a that is fixed. □

Remark 4.3. Put $a \neq 1$, $|a-4| \le 3$, and $(\Delta_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta_{\mathrm{J}_a})$ the Klein combination pair from [4] described in §2.2. Away from $z = 1$, Δ'_{J_a} agrees with Δ_{J_a}. Let r be the radius of the circle $\partial\Delta_{\mathrm{J}_a}$. Since the circle passes through a and has center $(1+r)$, then $2 \in \widehat{\mathbb{C}} \setminus \overline{\Delta_{\mathrm{J}_a}}$ if and only if $r > \frac{1}{2}$. Equivalently, $a \notin \overline{B}(\frac{3}{2}, \frac{1}{2})$. Since 2 is away from $z = 1$, $2 \in \widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}}$ if and only if $a \notin \overline{B}(\frac{3}{2}, \frac{1}{2})$ as well.

- If $0 < r \le \frac{1}{2}$, then $f_a : \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \to \Delta'_{\mathrm{J}_a}$ has no critical points and hence is an unramified covering map. Since f_a is a 2-to-1 map, the Riemann–Hurwitz formula yields

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

$$\chi(\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})) = 2\chi(\Delta'_{\mathrm{J}_a}) = 2,$$

where χ denotes the Euler characteristic. Since f_a has degree 2, $\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ has at most two connected components, each of them with Euler characteristic at most 1. It follows that $\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ has exactly two connected components, each of them homeomorphic to a disk.

- If $\frac{1}{2} < r < 3$, then f_a is a ramified covering of degree 2 with one critical point at $z_0 = -1$, whose ramification index equals 2. Therefore, again by the Riemann–Hurwitz formula, we get that

$$\chi(\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})) = 2\chi(\Delta'_{\mathrm{J}_a}) - (2-1) = 1.$$

Let Ω be the component of $\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a})$ that contains -1. Then $f|_\Omega$ is not locally injective at -1, and therefore of degree 2. Therefore, $\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) = \Omega$ has one connected component, which is homeomorphic to a circle, and mapped 2-to-1 onto Δ'_{J_a}.

We finish this section by listing some properties of the periodic points of $\mathcal{F}_a$. To do so, recall that for a holomorphic correspondence F on X,

$$\mathrm{Per}_n(F) = \pi_1(\Gamma^{(n)} \cap \mathfrak{D}_X),$$

where $\Gamma^{(n)}$ is the graph of F^n and $\mathfrak{D}_X$ is the diagonal in $X \times X$. Note that

$$\mathrm{Per}_n(F) = \{z \in X | z \in F^n(z)\} = \{z \in X | z \in F^{-n}(z)\}.$$

LEMMA 4.4. *For every $a \in \mathcal{K}$, we have that:*
(1) $\mathrm{Per}_n(\mathcal{F}_a) \subset \Lambda_{a,-} \cup \Lambda_{a,+}$*;*
(2) $\mathrm{J}_a(\mathrm{Per}_n(\mathcal{F}_a)) = \mathrm{Per}_n(\mathcal{F}_a)$*;*
(3) $\mathrm{J}_a(\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,\pm}) = \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,\mp}$*; and*
(4) $\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-} = \mathrm{Per}_n(f_a)$.

Proof. To prove part (1), suppose $z \in \mathrm{Per}_n(\mathcal{F}_a)$. If $z \in \overline{\Delta'_{\mathrm{J}_a}}$, then $z \in \mathcal{F}_a^{-kn}(z) \subset \mathcal{F}_a^{-kn}(\overline{\Delta'_{\mathrm{J}_a}})$ for all $k \geq 1$. Since the sets $\mathcal{F}_a^{-k}(\overline{\Delta'_{\mathrm{J}_a}})$ are nested and their intersection is $\Lambda_{a,-}$, then $z \in \Lambda_{a,-}$. On the other hand, if $z \notin \Lambda_{a,-}$, by Lemma 3.6 part (3), there exists $m \geq 1$ so that $\mathcal{F}_a^m(z) \subset \widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$. A similar argument as above shows that for all k sufficiently large, $z \in \mathcal{F}_a^k(\widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}})$, which implies that $z \in \Lambda_{a,+}$.

Next, note that if $z \in \mathrm{Per}_n(\mathcal{F}_a)$, then $z \in \mathcal{F}_a^{-n}(z)$ and $\mathrm{J}_a(z) \in \mathrm{J}_a(\mathcal{F}_a^{-n}(z)) = \mathcal{F}_a^n(\mathrm{J}_a(z))$. Therefore, $\mathrm{J}_a(z) \in \mathrm{Per}_n(\mathcal{F}_a)$ as well. Since J_a is an involution, this shows part (2).

Recall from Remark 3.6 that J_a sends the limits sets to each other. Therefore, $\mathrm{J}_a(\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-}) = \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,+}$, and again since J_a is an involution, we also have that

$$\mathrm{J}_a(\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,+}) = \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-}.$$

This proves part (3).

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

Finally, we prove part (4). We have that for each $z \in \mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \cup \{1\}$, $\mathcal{F}_a^{-n}(z) = f_a^{-n}(z)$ and part (1) implies that all periodic points of $\mathcal{F}_a$ not in $\Lambda_{a,+}$ must belong to $\Lambda_{a,-}$. Since $z \in \mathrm{Per}_n(f_a)$ if and only if $z \in f_a^{-n}(z) = \mathcal{F}_a^{-1}(z)$, then

$$\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-} = \mathrm{Per}_n(\mathcal{F}_a) \cap (\mathcal{F}_a^{-1}(\Delta'_{\mathrm{J}_a}) \cup \{1\}) = \mathrm{Per}_n(f_a). \qquad \square$$

5. *Equidistribution of* $\{\mathcal{F}_a^n\}_n$

In this section, we prove Theorems 1.2 and 1.3 using the results from [20, 24], together with [4, 5].

Recall every rational map P_A of the form

$$P_A(z) = z + 1/z + A, \tag{16}$$

with $A \in \mathbb{C}$, has critical points ± 1 and a parabolic fixed point at ∞ with multiplier equal to 1. Now note that in the coordinate $\phi(z) = 1/z$, we have that

$$P_A^{\phi}(z) = \frac{z}{z^2 + Az + 1} = z - Az^2 + (A^2 - 1)z^3 - \cdots$$

near 0. Thus, $z = 0$ is a fixed point of P_A^{ϕ} with multiplicity 2 if $A \neq 0$, and 3 if $A = 0$. We conclude that for $A \neq 0$, the parabolic fixed point has multiplicity 1, and 3 for $A = 0$.

For $A \in \mathbb{C} \setminus \{0\}$, let Ω_A be the basin of attraction of ∞, and for $A = 0$, we make the choice $\Omega_0 = \{x + iy | x, y \in \mathbb{R}, x > 0\}$. We define the *filled Julia set* of P_A as the set

$$K_{P_A} = \widehat{\mathbb{C}} \setminus \Omega_A.$$

Remark 5.1. No periodic points of P_A live in Ω_A. Indeed, it is clear that points in Ω_A for $A \neq 0$ cannot be periodic, as their iterates converge to ∞. Moreover, it is easy to check that if $\Re(z)$ denotes the real part of $z \neq 0$, then $\Re(P_A(z)) = \Re(z)(|z|^2 + 1)/|z|^2$. Then Ω_0 is completely invariant and has no periodic points. Therefore, for all $A \in \mathbb{C}$, we have that $\mathrm{Per}_n(P_A) \subset K_{P_A}$, for all $n \geq 1$.

LEMMA 5.2. *For every $a \in \mathcal{K}$, there exists a measure μ_- with* $\mathrm{supp}(\mu_-) = \partial\Lambda_{a,-}$ *such that*

$$\frac{1}{2^n}(f_a^n)^*\delta_{z_0} \to \mu_-$$

weakly, for all $z_0 \in \Delta_{\mathrm{J}_a} \setminus \mathcal{E}_{a,-}$.

Here, $\mathcal{E}_{a,-}$ is the exceptional set of the two-sided restriction f_a of $\mathcal{F}_a$ that leaves $\Lambda_{a,-}$ invariant, described in §4.

Proof. From the previous section, we can see that the two-sided restriction f_a can extend around 1. In [4, Proposition 5.2], the authors proved the existence of closed topological disks V'_a and V_a containing $\Lambda_{a,-}$ with $V'_a = \mathcal{F}_a^{-1}(V_a) \subset V_a$ and satisfying most properties for f_a to be a parabolic-like map from V'_a and V_a. Moreover, from the proof of [4, Theorem B], there exists a neighborhood U of $\Lambda_{a,-}$, $A \in \mathbb{C}$ and a quasiconformal map $h : U \cap V'_a \cap \to h(U) \cap V_a$ such that $h \circ f = P_A \circ h$, where P_A is as in equation (16). Such conjugacy sends $\Lambda_{a,-}$ onto the filled Julia set K_{P_A} of P_A.

From [20, 24], there exists a measure $\tilde{\mu}_A$ on $\widehat{\mathbb{C}}$ supported on the Julia set $\mathcal{J}_{P_A}$ of P_A, such that for all $z_0 \in \widehat{\mathbb{C}}$ not in the exceptional set E of P_A, $(1/2^n)(P_A^n)^*\delta_{z_0}$ is weakly convergent to $\tilde{\mu}_A$.

Recall $\Lambda_{a,-}$ is the intersection of the nested compact sets $\mathcal{F}_a^{-1}(\overline{\Delta'_{\mathrm{J}_a}})$. Thus, there exists $N \in \mathbb{N}$ such that $\mathcal{F}_a^n(\overline{\Delta'_{\mathrm{J}_a}}) \subset U \cap V'_a$, for all $n \geq N$. Observe as well that $h^{-1}(E) = \mathcal{E}_{a,-}$ as elements in $\mathcal{E}_{a,-}$ must have only one preimage and h is a homeomorphism, and the analogous holds for E and h^{-1}.

Now take $z_0 \in \overline{\Delta'_{\mathrm{J}_a}} \setminus \mathcal{E}_{a,-}$. Then, $f_a^{-N}(z_0) = \mathcal{F}_a^{-N}(z_0) \subset U \cap V'_a$. Then for each $\zeta \in f_a^{-N}(z_0)$, we have that $h(\zeta) \notin E$ and $(1/2^n)(P_A^n)^*\delta_{h(\zeta)}$ is weakly convergent to $\tilde{\mu}_A$. Note that $\delta_{h(\zeta)} = h_*\delta_\zeta$, and therefore for $n \geq N$,

$$\frac{1}{2^n}(P_A^n)^*\delta_{h(\zeta)} = \frac{1}{2^n}(h^{-1} \circ P_A^n)^*\delta_\zeta = \frac{1}{2^n}(f_a^n \circ h^{-1})^*\delta_\zeta = h_*\bigg(\frac{1}{2^n}(f_a^n)^*\delta_\zeta\bigg).$$

Since the left-hand side is weakly convergent to $\tilde{\mu}_A$, then $(1/2^n)(f_a^n)^*\delta_\zeta$ is weakly convergent to $\mu_- := h^*\tilde{\mu}_A$, which is supported on $h^{-1}(\mathcal{J}_{P_A}) = \partial\Lambda_{a,-}$. Thus,

$$\frac{1}{2^n}(f_a^n)^*\delta_{z_0} = \frac{1}{2^N}\sum_{\zeta \in f_a^{-N}(z_0)} \nu_{f_a^N}(\zeta)\frac{1}{2^{n-N}}(f_a^{n-N})^*\delta_\zeta$$

is weakly convergent to μ_- as desired. $\square$

Proof of Theorem 1.2. Let μ_- be as in Lemma 5.2 and put $\mu_+ := (\mathrm{J}_a)_*\mu_-$. Clearly, $\mathrm{supp}(\mu_+) = \partial\Lambda_{a,+}$. Denote by δ_{z_0} the Dirac measure at z_0. If $z_0 \in \widehat{\mathbb{C}} \setminus (\Lambda_{a,-} \cup \mathcal{E}_a)$, there exists $n_0 \in \mathbb{Z}^+ \cup \{0\}$ such that $\mathcal{F}_a^{n_0}(z_0) \cap \overline{\Delta'_{\mathrm{J}_a}} = \varnothing$ by part (3) of Lemma 3.6. In particular, this gives us that $\mathcal{F}_a^{n_0}(z_0)$ is contained in $\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$, and therefore $\mathcal{F}_a^n(z_0) \subset \widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{J}_a}$, for all $n \geq n_0$. In addition, for every $z_j \in \mathcal{F}_a^{n_0}(z_0)$, $\mathrm{J}_a(z_j) \notin \mathcal{E}_{a,-}$. Thus, for such a z_0, we get that

$$\begin{aligned}
\frac{1}{2^n}(\mathcal{F}_a^n)_*\delta_{z_0} &= \frac{1}{2^n}\sum_{\zeta_j \in \mathcal{F}_a^{n_0}(z_0)} \nu_{\pi_1|_{\Gamma^{(n_0)}}}(z_0, \zeta_j)(\mathcal{F}_a^{n-n_0})_*\delta_{\zeta_j} \\
&= \frac{1}{2^{n_0}}\sum_{\zeta_j \in \mathcal{F}_a^{n_0}(z_0)} \nu_{\pi_1|_{\Gamma^{(n_0)}}}(z_0, \zeta_j)\frac{1}{2^{n-n_0}}(\mathcal{F}_a^{n-n_0})_*\delta_{\zeta_j} \\
&= \frac{1}{2^{n_0}}\sum_{\zeta_j \in \mathcal{F}_a^{n_0}(z_0)} \nu_{\pi_1|_{\Gamma^{(n_0)}}}(z_0, \zeta_j)\frac{1}{2^{n-n_0}}(\mathrm{J}_a \circ \mathcal{F}_a^{-(n-n_0)} \circ \mathrm{J}_a)_*\delta_{\zeta_j} \\
&= \frac{1}{2^{n_0}}\sum_{\zeta_j \in \mathcal{F}_a^{n_0}(z_0)} \nu_{\pi_1|_{\Gamma^{(n_0)}}}(z_0, \zeta_j)(\mathrm{J}_a)_*\bigg(\frac{1}{2^{n-n_0}}(\mathcal{F}_a^{n-n_0})^*\delta_{\mathrm{J}_a(\zeta_j)}\bigg) \\
&\to \frac{1}{2^{n_0}}\sum_{\zeta_j \in \mathcal{F}_a^{n_0}(z_0)} \nu_{\pi_1|_{\Gamma^{(n_0)}}}(z_0, \zeta_j)(\mathrm{J}_a)_*\mu_- = \mu_+,
\end{aligned}$$

weakly, as $n \to \infty$, where $\Gamma^{(n_0)}$ is the graph of $\mathcal{F}_a^{n_0}$.

Now observe that the two-sided restriction f_a of $\mathcal{F}_a$ sends $\Lambda_{a,-}$ to itself. Indeed, note that the sets $\mathcal{F}_a^{-n}(\overline{\Delta'_{\mathrm{J}_a}})$ are nested and if $f_a(z) \notin \mathcal{F}_a^{-n}(\overline{\Delta'_{\mathrm{J}_a}})$ for some $n \in \mathbb{Z}^+$, then $z \notin \mathcal{F}_a^{n-1}(\overline{\Delta'_{\mathrm{J}_a}}) \supset \Lambda_{a,-}$. Note as well that $-2 \in \mathcal{F}_a^{-1}(1)$ and $1 \in \Lambda_{a,-}$, so $-2 \in \Lambda_{a,-}$.

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

Define $\tilde{f}_a : \Lambda_{a,-} \to (\widehat{\mathbb{C}} \setminus \overline{\Delta'_{\mathrm{J}_a}}) \cup \{1\}$ by $\tilde{f}_a(z) = w$, where $\mathcal{F}_a(z) \setminus \{f_a(z)\} = \{w\}$ for $z \neq -2$, and $\tilde{f}_a(-2) = f_a(-2) = 1$ for $z \in \mathcal{F}_a^{-1}(1) \setminus \{1\}$. Then for $z_0 \in \Lambda_{a,-} \setminus \mathcal{E}_a$, we have that $\mathcal{F}_a(z_0) = \{f_a(z_0), \tilde{f}_a(z_0)\}$, and

$$\begin{aligned}
\frac{1}{2}(\mathcal{F}_a)_*\delta_{z_0} &= \frac{1}{2}\delta_{f_a(z_0)} + \frac{1}{2}\delta_{\tilde{f}_a(z_0)},\\
\frac{1}{4}(\mathcal{F}_a^2)_*\delta_{z_0} &= \frac{1}{4}\delta_{f_a{}^2(z_0)} + \frac{1}{4}\delta_{\tilde{f}_a\circ f_a(z_0)} + \frac{1}{2}(\mathcal{F}_a)_*\delta_{\tilde{f}_a(z_0)},\\
\frac{1}{8}(\mathcal{F}_a^3)_*\delta_{z_0} &= \frac{1}{8}\delta_{f_a{}^3(z_0)} + \frac{1}{8}\delta_{\tilde{f}_a\circ f_a{}^2(z_0)} + \frac{1}{4}(\mathcal{F}_a)_*\delta_{\tilde{f}_a\circ f_a(z_0)} + \frac{1}{2}(\mathcal{F}_a^2)_*\delta_{\tilde{f}_a(z_0)},\\
&\ \ \vdots\\
\frac{1}{2^n}(\mathcal{F}_a^n)_*\delta_{z_0} &= \frac{1}{2^n}\delta_{f_a{}^n(z_0)} + \sum_{j=1}^{n}\frac{1}{2^j}(\mathcal{F}_a^{n-j})_*\delta_{\tilde{f}_a\circ f_a{}^{j-1}(z_0)}.
\end{aligned}$$

Since $(1/2^n)\delta_{f_a{}^n(z_0)}$ has mass $1/2^n$, then it is weakly convergent to a measure with zero mass. Put

$$\mu_{n,j} := \frac{1}{2^j}(\mathcal{F}_a^{n-j})_*\delta_{\tilde{f}_a\circ f_a{}^{j-1}(z_0)} \quad \text{and} \quad \mu_n := \frac{1}{2^n}(\mathcal{F}_a^n)_*\delta_{z_0} - \frac{1}{2^n}\delta_{f_a{}^n(z_0)} = \sum_{j=1}^{n}\mu_{n,j}.$$

If we truncate the sum defining μ_n at $N < n$, we obtain a sequence $\mu_n^{(N)} = \sum_{j=1}^{N}\mu_{n,j}$ satisfying $\mu_n^{(N)} \to \sum_{j=1}^{N}(1/2^j)\mu_+ = (1-2^{-N})\mu_+$ weakly. Let $\varphi : \widehat{\mathbb{C}} \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Choose $N \in \mathbb{Z}^+$ big so that $(3/2^N)\sup|\varphi| < \varepsilon$, and $n > N$ such that

$$\left|\int \varphi\, d\mu_n^{(N)} - (1-2^{-N})\int \varphi\, d\mu_+\right| < \frac{1}{2^N}\sup|\varphi|$$

by the convergence $\mu_n^{(N)} \to (1-2^{-N})\mu_+$. Then,

$$\begin{aligned}
\left|\int \varphi\, d\mu_n^{(N)} - \int \varphi\, d\mu_+\right| &\le \left|\int \varphi\, d\mu_n^{(N)} - \left(1-\frac{1}{2^N}\right)\int \varphi\, d\mu_+\right| + \frac{1}{2^N}\left|\int \varphi\, d\mu_+\right|\\
&\le \frac{1}{2^{N-1}}\sup|\varphi|.
\end{aligned}$$

Since $\mu_n - \mu_n^{(N)} = \sum_{j=N+1}^{n}\mu_{n,j}$ has mass at most $1/2^N$, we get

$$\begin{aligned}
\left|\int \varphi\, d\mu_n - \int \varphi\, d\mu_+\right| &\le \left|\int \varphi\, d\mu_n - \int \varphi\, d\mu_n^{(N)}\right| + \left|\int \varphi\, d\mu_n^{(N)} - \int \varphi\, d\mu_+\right|\\
&\le \frac{3}{2^N}\sup|\varphi| < \varepsilon.
\end{aligned}$$

This proves that for all $\varphi : \widehat{\mathbb{C}} \to \mathbb{R}$, we have that $\int \varphi\, d\mu_n \to \int \varphi\, d\mu_+$ as $n \to \infty$ and hence $\mu_n \to \mu_+$. Since $(1/2^n)(\mathcal{F}_a^n)_*\delta_{z_0} = (1/2^n)\delta_{f_a{}^n(z_0)} + \mu_n$, we obtain that $(1/2^n)(\mathcal{F}_a^n)_*\delta_{z_0}$ is weakly convergent to μ_+, as desired.

Applying again the push-forward by J_a, we get that for every $z_0 \notin \mathcal{E}_a$, $(1/2^n)(\mathcal{F}_a^n)^*\delta_{z_0}$ is weakly convergent to μ_-. $\square$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

Now, we show the asymptotic equidistribution of periodic points of $\mathcal{F}_a$ of order n with respect to the probability measure $\frac{1}{2}(\mu_- + \mu_+)$.

Proof of Theorem 1.3. From Lemma 4.4, we have that all periodic points lie in $\Lambda_{a,-} \cup \Lambda_{a,+}$ and

$$|\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-}| = |\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,+}| = |\mathrm{Per}_n(f_a)|.$$

Denote this number by d_n. Since $1 \in \mathcal{F}_a(1)$ and Lemma 4.4 says $\Lambda_{a,-} \cap \Lambda_{a,+} = \{1\}$, then

$$(\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-}) \cap (\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,+}) = \{1\}.$$

Thus, $|\mathrm{Per}_n(\mathcal{F}_a)| = 2d_n - 1$. Since the conjugacy h between f_a and the quadratic rational map P_A sends $\Lambda_{a,-}$ onto the filled Julia set K_{P_A} of P_A, which contains all periodic points of P_A by Remark 5.1, $\mathrm{Per}_n(P_A) = h(\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-})$ and

$$|\mathrm{Per}_n(f_a)| = |\mathrm{Per}_n(P_A)| = d_n$$

as well. We have that $\lim_{n\to\infty} d_n = \infty$ (see in [24, p. 363]). Then,

$$\begin{aligned}
\frac{1}{2d_n - 1} \sum_{z \in \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-}} \delta_z &= \frac{1}{2d_n - 1} \sum_{z \in \mathrm{Per}_n(f_a)} \delta_z \\
&= h^*\bigg(\frac{1}{2d_n - 1} \sum_{z \in \mathrm{Per}_n(f_a)} \delta_{h(z)}\bigg) \\
&= h^*\bigg(\frac{d_n}{2d_n - 1} \frac{1}{d_n} \sum_{\zeta \in \mathrm{Per}_n(P_A)} \delta_\zeta\bigg),
\end{aligned}$$

which is weakly convergent to $\frac{1}{2} h^* \widetilde{\mu}_A = \frac{1}{2}\mu_-$ by the corollary to Theorem 3 in [24]. Thus,

$$\frac{1}{2d_n - 1} \sum_{z \in \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,+}} \delta_z = \mathrm{J}_a^*\bigg(\frac{1}{2d_n - 1} \sum_{\mathrm{J}_a(z) \in \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-}} \delta_{\mathrm{J}_a(z)}\bigg)$$

is weakly convergent to $\frac{1}{2}\, \mathrm{J}_a^*\, \mu_- = \frac{1}{2}\mu_+$, and

$$\frac{1}{|\mathrm{Per}_n(\mathcal{F}_a)|} \sum_{z \in \mathrm{Per}_n(\mathcal{F}_a)} \delta_z = \frac{1}{2d_n - 1} \sum_{\substack{z \in \mathrm{Per}_n(\mathcal{F}_a) \\ z \in \Lambda_{a,-}}} \delta_z + \frac{1}{2d_n - 1} \sum_{\substack{z \in \mathrm{Per}_n(\mathcal{F}_a) \\ z \in \Lambda_{a,+}}} \delta_z - \frac{1}{2d_n - 1}\delta_1$$

converges weakly to $\frac{1}{2}(\mu_- + \mu_+)$, as desired.

Finally, we proceed to show that the equidistribution still holds when counting with multiplicity. Observe that the only points of $\Lambda_{a,-}$ that are not in the interior of the topological conjugacy h are -2 and 1, and we have that -2 is not a periodic point for $\mathcal{F}_a$. We claim that for every $z_0 \in \mathrm{Per}_n(f_a) \setminus \{1\}$, we have that

$$\nu_{\pi_1|_{\mathrm{Gr}(f_a^n) \cap \mathfrak{D}_{\widehat{\mathbb{C}}}}}(z_0, z_0) = \nu_{\pi_1|_{\mathrm{Gr}(P_A^n) \cap \mathfrak{D}_{\widehat{\mathbb{C}}}}}(h(z_0), h(z_0)).$$

Indeed, the topological conjugacy h sends attracting periodic points to attracting periodic points, so if $|(f_a^n)'(z_0)| < 1$, then $|(P_A^n)'(h(z_0))| < 1$ as well. Thus, both z_0 and $h(z_0)$ have multiplicity 1 as a periodic point in this case. The same happens for $|(f_a^n)'(z_0)| > 1$.

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

Finally, if $|(f_a^n)'(z_0)| = |(P_A^n)'(h(z_0))| = 1$, Naishul's theorem [27] (later re-proven by Pérez-Marco in [28]) shows that $(f_a^n)'(z_0) = (P_A^n)'(h(z_0))$. Moreover, attracting directions are preserved under topological conjugacy, so the parabolic fixed point z_0 of $\mathcal{F}_a$ and the parabolic fixed point $h(z_0)$ of P_A have the same number of attracting directions. Therefore, both z_0 and $h(z_0)$ have the same multiplicity as periodic points of order n of f_a and P_A, respectively. Therefore,

$$\sum_{\substack{z\in \mathrm{Per}_n(\mathcal{F}_a)\cap\Lambda_{a,-}\\ z\neq 1}} \nu_{\pi_1|_{\Gamma_a^{(n)}\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(z,z) = \sum_{\substack{\zeta\in \mathrm{Per}_n(P_A)\\ \zeta\neq\infty}} \nu_{\pi_1|_{\mathrm{Gr}(P_A^n)\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(\zeta,\zeta)$$
$$= (2^n+1) - \nu_{\pi_1|_{\mathrm{Gr}(P_A^n)\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(\infty,\infty).$$

From Lemma 4.4 part (2) and given that J_a is a Mobius transformation, we have that

$$\sum_{\substack{z\in \mathrm{Per}_n(\mathcal{F}_a)\cap\Lambda_{a,+}\\ z\neq 1}} \nu_{\pi_1|_{\Gamma_a^{(n)}\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(z,z) = (2^n+1) - \nu_{\pi_1|_{\mathrm{Gr}(P_A^n)\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(\infty,\infty)$$

as well.

Since the multiplicity of a periodic point is the multiplicity as a periodic point of its minimal period and since $\phi^{-1}(0) = \infty$, we have that

$$\nu_{\pi_1|_{\mathrm{Gr}(P_A^n)\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(\infty,\infty) = \begin{cases} 3 & \text{if } A = 0,\\ 2 & \text{otherwise.}\end{cases}$$

Similarly, from the equations for g^{ψ} listed in Remark 3.5, we have that

$$\nu_{\pi_1|_{\Gamma_a^{(n)}\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(1,1) = \begin{cases} 4 & \text{if } a = 7,\\ 2 & \text{if } a\in\mathcal{K}\setminus\{7\}.\end{cases}$$

In [5, Corollary 4.3], the authors showed that for $a = 7$, the member of the family of quadratic rational maps

$$\{P_A(z) = z + 1/z + A \mid A\in\mathbb{C}\}$$

that is conjugate to f_a is P_0. Therefore, for all $a\in\mathbb{C}\setminus\{1\}$, $|a-4|\leq 3$,

$$\sum_{z\in \mathrm{Per}_n(\mathcal{F}_a)} \nu_{\pi_1|_{\Gamma_a^{(n)}\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(z,z) = 2\Big(2^n + 1 - \nu_{\pi_1|_{\mathrm{Gr}(P_A^n)\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(\infty,\infty)\Big) + \nu_{\pi_1|_{\Gamma_a^{(n)}\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(1,1)$$
$$= 2^{n+1}$$

and we have that

$$\frac{1}{2^n}\sum_{\substack{z\in \mathrm{Per}_n(\mathcal{F}_a)\cap\Lambda_{a,-}\\ z\neq 1}} \nu_{\pi_1|_{\Gamma_a^{(n)}\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(z,z)\delta_z = h^*\Bigg(\frac{1}{2^n}\sum_{\substack{\zeta\in \mathrm{Per}_n(P_A)\\ \zeta\neq\infty}} \nu_{\pi_1|_{\mathrm{Gr}(P_A^n)\cap\mathfrak{D}_{\widehat{\mathbb{C}}}}}(\zeta,\zeta)\delta_\zeta\Bigg),$$

https://doi.org/10.1017/etds.2023.33 Published online by Cambridge University Press

which also converges weakly to μ_- by [24, Theorem 3]. Using that J_a preserves multiplicities and that $\mathrm{J}_a^* \mu_- = \mu_+$, we have that

$$\frac{1}{2^n} \sum_{\substack{z \in \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,+} \\ z \neq 1}} \nu_{\pi_1|_{\Gamma_a^{(n)} \cap \mathfrak{D}_{\widehat{\mathbb{C}}}}}(z, z)\delta_z = \mathrm{J}_a^* \left(\frac{1}{2^n} \sum_{\substack{z \in \mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-} \\ z \neq 1}} \nu_{\pi_1|_{\Gamma_a^{(n)} \cap \mathfrak{D}_{\widehat{\mathbb{C}}}}}(z, z)\delta_z \right)$$

is weakly convergent to μ_+. Since $(1/2^n)\nu_{\pi_1|_{\Gamma_a^{(n)} \cap \mathfrak{D}_{\widehat{\mathbb{C}}}}}(1, 1)\delta_1$ has total mass $1/2^n$ and

$$\mathrm{Per}_n(\mathcal{F}_a) = \{1\} \cup (\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,-} \setminus \{1\}) \cup (\mathrm{Per}_n(\mathcal{F}_a) \cap \Lambda_{a,+} \setminus \{1\})$$

is a disjoint union, then

$$\frac{1}{2^{n+1}} \sum_{z \in \mathrm{Per}_n(\mathcal{F}_a)} \nu_{\pi_1|_{\Gamma_a^{(n)} \cap \mathfrak{D}_{\widehat{\mathbb{C}}}}}(z, z)\delta_z$$

is weakly convergent to $\frac{1}{2}(\mu_- + \mu_+)$. □

In what follows, we will introduce terminology and prove Theorem 1.4.

Definition 5.3. A holomorphic correspondence F with graph Γ is said to be *postcritically finite* if for all $w \in B_2(\Gamma)$, there exist $0 \leq m < n$ so that

$$F^m(w) \cap F^n(w) \neq \varnothing. \tag{17}$$

Equivalently, for every $w \in B_2(\Gamma)$, there exists $m \geq 0$ and $z \in F^m(w)$ so that $z \in \mathrm{Per}_n(F)$ for some $n \geq 1$. This definition generalizes that of postcritically finite rational maps. Indeed, if F is a rational map that is postcritically finite as a correspondence, then we have that every $w \in B_2(\Gamma)$ is pre-periodic. Since $B_2(\Gamma) = F(\mathrm{CritPt}(F))$, then every critical point is pre-periodic as well. Hence, F is postcritically finite as a rational map.

If F is a holomorphic correspondence and $z_0 \in \mathrm{Per}_n(F)$, there exists a cycle

$$(z_0, z_1, z_2, \ldots, z_{n-1}) \tag{18}$$

so that $z_i \in F(z_{i-1})$ and $z_0 \in F(z_{n-1})$.

Definition 5.4. We say that a holomorphic correspondence with graph Γ is *superstable* if there exists $\alpha \in A_2(\Gamma)$ and a cycle in equation (18) satisfying that $z_0 = \pi_1(\alpha)$ and $z_1 = \pi_2(\alpha)$.

Observe that if F is a rational map, then $\pi_1(\alpha)$ corresponds to a critical point and $\pi_2(\alpha)$ corresponds to its critical value. Therefore, the definition of superstable correspondences reduces to F having a superstable cycle, meaning a cycle containing a critical point.

In the context of the family $\{\mathcal{F}_a\}_{a \in \mathcal{K}}$, Proposition 2.4 says that

$$B_2(\Gamma_a) = \left\{ \frac{a+1}{2}, \frac{4a+2}{a+5}, \frac{2}{3-a} \right\}.$$

Observe that $(a+1)/2 = \mathrm{J}_a(\infty) \notin \Lambda_{a,-} \cup \Lambda_{a,+}$. Indeed, since ∞ is fixed by $\mathrm{Cov}_0^{\mathrm{Q}}$ and $(\Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}, \Delta'_{\mathrm{J}_a})$ is a Klein combination pair, then $\infty \in (\widehat{\mathbb{C}} \setminus \Delta'_{\mathrm{Cov}_0^{\mathrm{Q}}}) \setminus \{1\} \subset \Delta'_{\mathrm{J}_a}$, so

$J_a(\infty) \notin \Lambda_{a,-} \subset \overline{\Delta'_{J_a}}$. On the other hand, $\mathcal{F}_a^{-1}(J_a(\infty)) = \{\infty\}$ does not intersect $\Lambda_{a,+}$, so $J_a(\infty) \notin \Lambda_{a,+}$. By Lemma (4.4), $(a+1)/2$ cannot satisfy equation (17). Therefore, there are no parameters $a \in \mathcal{K}$ for which $\mathcal{F}_a$ is postcritically finite.

However, $\mathcal{F}_a^{-(n+1)}((4a+2)/(a+5)) = \mathcal{F}_a^{-n}(1) \subset \Lambda_{a,-}$ and $(4a+2)/(a+5) \in \Lambda_{a,+}$, so $(4a+2)/(a+5)$ is not periodic. Therefore, $\mathcal{F}_a$ is superstable if and only if $2/(3-a)$ is periodic. Since $\mathcal{F}_a^{-1}(2/(3-a)) = \{-1\}$ and $\mathcal{F}_a^{-1} = f_a^{-1}$, then $\mathcal{F}_a$ is superstable if and only if -1 is a critical point of f_a that is periodic. We say that the parameter $a \in \mathcal{K}$ is *superstable* whenever $\mathcal{F}_a$ is superstable.

We consider the quadratic family

$$\{p_c(z) = z^2 + c | c \in \mathbb{C}\}$$

and let $\hat{P}_n := \{c \in \mathbb{C} | p_c^n(0) = 0\}$ be the set of superstable parameters of order n. Note that $\hat{P}_n$ has 2^{n-1} points, counted with multiplicity, and that it is contained in the Mandelbrot set $\mathcal{M}$. Levin proved in [23] that

$$\lim_{n\to\infty} \frac{1}{2^{n-1}} \sum_{c\in\hat{P}_n} \delta_c$$

converges to a measure m_{BIF} on $\mathcal{M}$, which we call the *bifurcation measure*.

Let $\mathcal{M}_\Gamma$ denote the connectedness locus of the family $\{\mathcal{F}_a\}_{a\in\mathcal{K}}$ and $\mathcal{M}_1$ denote the parabolic Mandelbrot set. The conjugacy shown in [4] between f_a and P_A induces a map $a \mapsto 1 - A^2$ from $\mathcal{M}_\Gamma$ and $\mathcal{M}_1$. From [5, Main Theorem], that map is a homeomorphism. On the other hand, [29, Main Theorem] says that $\mathcal{M}_1$ is homeomorphic to $\mathcal{M}$. Moreover, both homeomorphisms constructed preserve the type of dynamics associated to the parameter (see [29, §1]). Therefore, there is a homeomorphism $\Psi : \mathcal{M} \to \mathcal{M}_\Gamma$ that gives a one-to-one correspondence between superstable parameters of $\mathcal{M}_\Gamma$ and superstable parameters of $\mathcal{M}$. Pushing the bifurcation measure forward through Ψ, and writing

$$\hat{P}_n^\Gamma = \{a \in \mathcal{K} \text{ superstable } | f_a^n(-1) = -1\},$$

we obtain the statement in Theorem 1.4.

Acknowledgements. The author would like to thank Shaun Bullett and Luna Lomonaco for a useful conversation in Santiago de Chile, and especially thank Juan Rivera-Letelier for his great patience and guidance.

References

[1] G. Bharali and S. Sridharan. The dynamics of holomorphic correspondences of P^1: invariant measures and the normality set. *Complex Var. Elliptic Equ.* **61**(12) (2016), 1587–1613.

[2] J.-Y. Briend and J. Duval. Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de $\mathbb{C}P^k$. *Acta Math.* **182**(2) (1999), 143–157.

[3] H. Brolin. Invariant sets under iteration of rational functions. *Ark. Mat.* **6**(2) (1965), 103–144.

[4] S. Bullett and L. Lomonaco. Mating quadratic maps with the modular group II. *Invent. Math.* **220**(1) (2020), 185–210.

[5] S. Bullett and L. Lomonaco. Mating quadratic maps with the modular group III: the modular Mandelbrot set. *Preprint*, 2022, arXiv:2010.04273.

- **[6]** S. Bullett and C. Penrose. Mating quadratic maps with the modular group. *Invent. Math.* **115**(3) (1994), 483–512.
- **[7]** S. Bullett and C. Penrose. Regular and limit sets for holomorphic correspondences. *Fund. Math.* **167** (2001), 111–171.
- **[8]** S. R. Bullett and W. J. Harvey. Mating quadratic maps with Kleinian groups via quasiconformal surgery. *Electron. Res. Announcements Amer. Math. Soc.* **6** (2000), 21–30.
- **[9]** L. Clozel, H. Oh and E. Ullmo. Hecke operators and equidistribution of Hecke points. *Invent. Math.* **144**(2) (2001), 327–351.
- **[10]** L. Clozel and J.-P. Otal. Unique ergodicité des correspondances modulaires. *Essays on Geometry and Related Topics*. Vol. 1. 2001, p. 2.
- **[11]** L. Clozel and E. Ullmo. Correspondances modulaires et mesures invariantes. *J. Reine Angew. Math.* **558** (2003), 47–83.
- **[12]** V. Matus de la Parra. On the dynamics of covering correspondences. Manuscript in preparation, 2022.
- **[13]** T.-C. Dinh. Distribution des préimages et des points périodiques d'une correspondance polynomiale. *Bull. Soc. Math. France* **133**(3) (2005), 363–394.
- **[14]** T.-C. Dinh. Equidistribution of periodic points for modular correspondences. *J. Geom. Anal.* **23**(3) (2013), 1189–1195.
- **[15]** T.-C. Dinh, L. Kaufmann and H. Wu. Dynamics of holomorphic correspondences on Riemann surfaces. *Int. J. Math.* **31**(05) (2020), 2050036.
- **[16]** T.-C. Dinh, V.-A. Nguyen and T. T. Truong. Equidistribution for meromorphic maps with dominant topological degree. *Indiana Univ. Math. J.* (2015), 1805–1828.
- **[17]** T.-C. Dinh and N. Sibony. Upper bound for the topological entropy of a meromorphic correspondence. *Israel J. Math.* **163**(1) (2008), 29–44.
- **[18]** T.-C. Dinh and N. Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. *Holomorphic Dynamical Systems*. Eds. G. Gentili, J. Guenot and G. Patrizio. Springer, Berlin, 2010, pp. 165–294.
- **[19]** C. Favre and J. Rivera-Letelier. Théorie ergodique des fractions rationnelles sur un corps ultramétrique. *Proc. Lond. Math. Soc. (3)* **100**(1) (2010), 116–154.
- **[20]** A. Freire, A. Lopes and R. Mañé. An invariant measure for rational maps. *Bull. Braz. Math. Soc. (N.S.)* **14**(1) (1983), 45–62.
- **[21]** M. Fried. On a conjecture of Schur. *Michigan Math. J.* **17**(1) (1970), 41–55.
- **[22]** W. Gignac. Equidistribution of preimages over nonarchimedean fields for maps of good reduction. *Ann. Inst. Fourier (Grenoble)* **64** (2014), 1737–1779.
- **[23]** G. M. Levin. Theory of iterations of polynomial families in the complex plane. *J. Soviet Math.* **52**(6) (1990), 3512–3522.
- **[24]** M. J. Ljubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergod. Th. & Dynam. Sys.* **3**(3) (1983), 351–385.
- **[25]** L. Lomonaco. Parabolic-like mappings. *Ergod. Th. & Dynam. Sys.* **35**(7) (2015), 2171–2197.
- **[26]** R. McGehee. Attractors for closed relations on compact Hausdorff spaces. *Indiana Univ. Math. J.* **41**(4) (1992), 1165–1209.
- **[27]** V. A. Naishul. Topological invariants of analytic and area-preserving mappings and their application to analytic differential equations in $\mathbb{C}^2$ and $\mathbb{C}P^2$. *Tr. Mosk. Mat. Obs.* **44** (1982), 235–245.
- **[28]** R. Pérez-Marco. Fixed points and circle maps. *Acta Math.* **179**(2) (1997), 243–294.
- **[29]** C. L. Petersen and P. Roesch. The parabolic Mandelbrot set. *Preprint*, 2021, arXiv:2107.09407.
- **[30]** W. Stoll. The multiplicity of a holomorphic map. *Invent. Math.* **2** (1966), 15–58.
- **[31]** K. J. Vivas and V. F. Sirvent. Metric entropy for set-valued maps. *Discrete Contin. Dyn. Syst. Ser. B* **27** (2022), 6589–6604.