J. Plasma Physics (2000), *vol.* 64, *part* 2, *pp.* 195–200. Printed in the United Kingdom

On the MHD analogue of the Brunt–Väisälä frequency in a magnetized plasma

ALEXANDER I. ERSHKOVICH and PETER L. ISRAELEVICH
Department of Geophysics and Planetary Sciences, Tel Aviv University, Ramat Aviv, 69978, Tel Aviv, Israel (peter@luna.tau.ac.il)

(Received 4 February 2000 and in revised form 20 April 2000)

Abstract. The MHD analogue of the Brunt–Väisälä frequency, N_B, in a magnetized, ideally conducting plasma is obtained, with the vertical component of the magnetic field, B_r, taken into account. The magnetic field vector (B_r, B_θ, B_φ) is assumed to satisfy the condition $\mathbf{B}\cdot\boldsymbol{\nabla}B \approx B_r\,dB/dr$, which holds in many cases of interest. The frequency N_B happens to depend, generally speaking, on the magnetic field orientation relative to the direction of gravity. However, for an isentropic gas, the convective instability criterion is governed by the magnetic field strength (rather than by the orientation of $\mathbf{B}$). In general, the magnetic field has a stabilizing (destabilizing) effect if B/ρ grows (decreases) along the vertical axis r. This conclusion seems not to depend on the specific magnetic field configuration.

1. Introduction

A vertically stratified atmosphere is known to be stable if $N^2 > 0$, where

$$N^2 = -g\left(\frac{1}{\rho}\frac{d\rho}{dr} + \frac{g}{c_s^2}\right); \tag{1.1}$$

N is the Brunt–Väisälä frequency (hereinafter BVF: Väisälä 1925; Brunt 1927). Here g is the gravitational acceleration, and ρ and c_s are respectively the density and the adiabatic sound velocity in hydrostatic equilibrium. It is easy to show that the stability condition $N^2 > 0$ coincides with the condition that convection is absent: $ds/dr > 0$ (where s is the entropy), which reduces to the well-known condition on the temperature gradient dT/dr (see e.g. Landau and Lifshitz 1959). For instance, for a perfect gas, it becomes

$$\frac{dT}{dr} > \left(\frac{dT}{dr}\right)_{\mathrm{ad}}, \tag{1.2}$$

where $(dT/dr)_{\mathrm{ad}} = -g/c_p$ is the adiabatic temperature gradient, (c_p is the specific heat). The stability condition (1.2) is called the Schwarzschild criterion (Schwarzschild 1906). These stability conditions are usually obtained from buoyancy-type arguments, which are confirmed by linear perturbation theory (see e.g. Eckart 1960; Lighthill 1978; Priest 1982; Gombosi 1998).

Taking into account the magnetic field in a magnetized plasma is of great interest for solar physics, the physics of magnetospheres (both planetary and stellar), and the physics of the ionosphere. The effect of horizontal magnetic field (normal to the r axis) was considered by Gilman (1970), Chen and Lykoudis (1972), Acheson (1979), Priest (1982), Ershkovich et al. (1989), McKenzie et al. (1990), and Ershkovich and Israelevich (1993).

However, in many interesting cases, the magnetic field in equilibrium is mainly vertical. This is the situation in the polar ionosphere of the Earth and other magnetized planets, in the polar regions of stellar magnetospheres (including pulsars), and in some regions of induced cometary magnetospheres. For instance, in the polar caps of a magnetic dipole, the angle between **B** and the local vertical does not exceed 10° up to a magnetic latitude of 70°. Therefore, in this paper, we shall derive the MHD analogue of the BVF for a magnetized, ideally conducting plasma, taking into account the vertical component of the magnetic field in equilibrium.

2. Stability analysis

The static MHD momentum balance is

$$-\nabla p + \mathbf{F}_L + \rho\mathbf{g} + \rho\nu\mathbf{V}_n = 0, \tag{2.1}$$

where p is the gas pressure, $\mathbf{F}_L = (\nabla\times\mathbf{B})\times\mathbf{B}/4\pi$ is the Lorentz body force, ρ is the plasma density, $\mathbf{V}_n$ is the neutral-particle velocity, and ν is the ion–neutral collision frequency. The last term in (2.1) describes the interaction between the static plasma and a neutral gas flow, such as the polar wind in planetary magnetospheres or the neutral wind escaping from a cometary nucleus (in the latter case, the gravitational acceleration $g = 0$). The force $\mathbf{F}_L$ may be represented in the form

$$\begin{aligned}\mathbf{F}_L &= -\frac{1}{8\pi}\nabla B^2 + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\,\mathbf{B}\\ &= -\frac{1}{8\pi}\nabla B^2 + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla B)\frac{\mathbf{B}}{B} + \frac{B^2}{4\pi R_c}\mathbf{n},\end{aligned} \tag{2.2}$$

where $\mathbf{n}$ is the unit vector of the principal normal to the magnetic field line (directed towards its centre of curvature) and R_c is the radius of curvature of the field line. In a spherical coordinate system (r, θ, φ),

$$\mathbf{B}\cdot\nabla B = B_r\frac{\partial B}{\partial r} + \frac{1}{r}B_\theta\frac{\partial B}{\partial\theta} + \frac{1}{r\sin\theta}B_\varphi\frac{\partial B}{\partial\varphi}. \tag{2.3}$$

We assume that $\mathbf{B}\cdot\nabla B \approx B_r\,dB/dr$. This approximation holds if the magnetic field strength B depends mainly on the radial (i.e. vertical) coordinate r. It also holds in the important case of the polar caps in the magnetic dipole field, where $B_\varphi = 0$, and the ratio $(B_r\,\partial B/\partial r)/(r^{-1}B_\theta\,\partial B/\partial\theta) = 2(1+3\cos^2\theta)/\sin^2\theta \to \infty$ on approaching the magnetic pole, $\theta \to 0$. This ratio equals 10 for $\theta = 45°$. Using (2.3), the vertical component of the Lorentz body force becomes

$$\begin{aligned}F_{L,r} &= -\frac{B^2}{4\pi}\frac{1}{B}\frac{dB}{dr} + \frac{B_r^2}{4\pi}\frac{1}{B}\frac{dB}{dr} + \frac{B^2}{4\pi R_c}n_r\\ &= -V_{A\perp}^2\frac{\rho}{B}\frac{dB}{dr} + \rho\frac{V_A^2}{R_c}n_r\end{aligned} \tag{2.4}$$

where $B_\perp^2 = B^2 - B_r^2$ (so that $\mathbf{B}_\perp = (0, B_\theta, B_\varphi)$ is the horizontal component of the magnetic field), $V_A = B/(4\pi\rho)^{1/2}$ is the Alfvén velocity, and $V_{A\perp} = B_\perp/(4\pi\rho)^{1/2}$. Hence the r component of the momentum balance (2.1) is

$$\frac{dp}{dr} = -\rho g^* - V_{A\perp}^2\frac{\rho}{B}\frac{dB}{dr}, \tag{2.5}$$

where the effective gravitational acceleration is

$$g^* = g - \frac{V_A^2}{R_c} n_r - \nu V_{n,r}, \tag{2.6}$$

and we have taken into account that $g_r = -g$.

Uniform rotation of a star or planet, with constant angular velocity $\boldsymbol{\omega}$, may be easily taken into account by working in the frame of reference rotating with the astrophysical object under consideration. In this frame of reference, statics holds ($\mathbf{v} = 0, \partial\mathbf{v}/\partial t = 0$), but the centrifugal force $\rho\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega})$ has to be added to the left-hand-side of (2.1). The vertical component of the centrifugal force reduces the gravitational force, so that inside the rigidly co-rotating plasmasphere, the gravitation acceleration g in (2.6) is to be replaced by $g - r\omega^2 \sin^2\alpha$, where α is the angle between the axis of rotation and the local vertical. Thus rotation has a stabilizing effect (Gilman 1970; Acheson 1978).

Suppose now that a plasma element having been at the level r, is displaced sporadically (and adiabatically) to the level $r + dr = r + \xi$. Its pressure adjusts to the new altitude instantaneously, and the change of the density inside the plasma element is obtained from (2.5):

$$\delta p = \left(\frac{dp}{d\rho}\right)_s \delta\rho_{\mathrm{in}} = c_s^2\, \delta\rho_{\mathrm{in}} = -\rho g^* \xi - V_{A\perp}^2 \frac{\rho}{B}\frac{dB}{dr}\xi\,. \tag{2.7}$$

We assume that the plasma is ideally conducting (despite possible ion–neutral collisions). For example, this is the case in a cometary ionosphere (Ershkovich and Israelevich 1996), although the neutral number density there is much higher than that of the plasma. In this case, the magnetic field is frozen into the plasma, and $(d/dt)(\mathbf{B}/\rho) = [(\mathbf{B}/\rho)\boldsymbol{\nabla}]\mathbf{v} = 0$. Thus $\mathbf{B}/\rho$ is conserved in the displaced plasma element. Then the last term on the right-hand side of (2.7) becomes $-V_{A\perp}^2(\delta\rho)_{\mathrm{in}}$. Thus the density change inside the displaced element, according to (2.7), is

$$(\delta\rho)_{\mathrm{in}} = -\frac{\rho g^* \xi}{c_s^2 + V_{A\perp}^2}, \tag{2.8}$$

whereas, outside this element,

$$(\delta\rho)_{\mathrm{out}} = \frac{d\rho}{dz}\xi. \tag{2.9}$$

The buoyancy (Archimedean) force equals $g^*(\delta\rho_{\mathrm{out}} - \delta\rho_{\mathrm{in}})$, whence, by using (2.8)–(2.9), we arrive at the equation of motion of the sporadically displaced plasma element in the form

$$\ddot{\xi} + N_B^2 \xi = 0, \tag{2.10}$$

with the solution $\xi \propto \exp(\pm i N_B t)$. The MHD analogue of the BVF is determined by

$$N_B^2 = -g^* \left(\frac{1}{\rho}\frac{d\rho}{dr} + \frac{g^*}{c_s^2 + V_{A\perp}^2}\right); \tag{2.11}$$

the instability growth rate, naturally, equals $|N_B|$. For horizontal and rectilinear magnetic field lines (and $V_n = 0$), $g^* = g$ and $V_{A\perp} = V_A$, and (2.11) reduces to the expression obtained by Chen and Lykoudis (1972). For example, in a magnetic dipole field, the ratio $(V_{A\perp}/V_A)^2 = 3.2 \times 10^{-2}$ and 7.7×10^{-2} at the magnetic latitudes of 70° and 60°, respectively. Hence the BVF N_B may depend critically on the magnetic field inclination, unless $c_s^2 \gg V_A^2$.

https://doi.org/10.1017/S0022377800008588 Published online by Cambridge University Press

The BVF N_B happens to depend differently on the horizontal and vertical components of the magnetic field. The effective gravitational acceleration g^* depends on the magnetic field strength B through the magnetic curvature term $V_A^2 n_r/R_c$. This term may be dominant, for instance, in a pulsar magnetosphere.

It should be kept in mind that this analysis is valid only if the plasma particle motion is not quantized in Landau levels (Landau and Lifshitz 1977) – that is if $\hbar\omega_B \ll kT$, where ω_B is the particle gyrofrequency, and $2\pi\hbar$ and k are respectively the Planck and Boltzmann constants. This condition gives a limit of the magnetic field strength: $B \ll 10^4\ T(\mathrm{K}) \approx 10^{10} - 10^{11}$ G with typical plasma temperature $T \approx 10^6 - 10^7$ K. Thus quantization of the electron motion is expected to arise in the superstrong (of order 10^{12} G) magnetic fields near pulsars, and does not occur near the magnetopause of accreting pulsars in binary systems, where the field B is of order 10^5 G.

If, however, the gravity g prevails ($g^* \approx g$) then the vertical component of the magnetic field, B_r, naturally, does not affect the motion of the plasma element, which remains vertical in the linear approximation. Of course, vorticity may change this conclusion. However, vortices are known to arise as a result of the nonlinear evolution of the convective instability, which is beyond the scope of this paper.

The BVF N_B may be represented in an alternative form if $g^*/(c_s^2+V_{A\perp}^2)$ in (2.11) is eliminated by means of (2.5), where $dp/dr = c_n^2\, d\rho/dr$ for a polytropic gas ($p \propto \rho^n$, $c_n^2 = np/\rho$):

$$\frac{g^*}{c_s^2+V_{A\perp}^2} = -\left[\frac{c_n^2}{c_s^2+V_{A\perp}^2}\frac{1}{\rho}\frac{d\rho}{dr} + \frac{V_{A\perp}^2}{c_s^2+V_{A\perp}^2}\frac{1}{B}\frac{dB}{dr}\right]. \tag{2.12}$$

Substituting (2.12) into (2.11) yields

$$N_B^2 = \frac{\rho}{B}\frac{V_{A\perp}^2}{c_s^2+V_{A\perp}^2} g^* \frac{d}{dr}\left(\frac{B}{\rho}\right) - \frac{g^*(c_s^2-c_n^2)}{c_s^2+V_{A\perp}^2}\frac{1}{\rho}\frac{d\rho}{dr}. \tag{2.13}$$

In general, the equation of state may be written in the form $p = p(\rho, s)$, whence

$$\frac{dp}{dr} = c_s^2\frac{d\rho}{dr} + \left(\frac{\partial p}{\partial s}\right)_\rho \frac{ds}{dr}.$$

Then, instead of (2.13), one obtains

$$N_B^2 = \frac{\rho}{B}\frac{V_{A\perp}^2}{c_s^2+V_{A\perp}^2} g^* \frac{d}{dr}\left(\frac{B}{\rho}\right) - \frac{g^*}{\rho(c_s^2+V_{A\perp}^2)}\left(\frac{\partial T}{\partial V}\right)_s \frac{ds}{dr}, \tag{2.14}$$

where, according to the Maxwell relation $(\partial T/\partial V)_s = -(\partial p/\partial s)_\rho < 0$, $V = 1/\rho$ is a specific volume. Naturally, for a polytropic gas, (2.14) reduces to (2.13). If $ds/dr = 0$ (note that with $B = 0$, such a gas is always unstable) the convective instability criterion is

$$g^*\frac{d}{dr}\left(\frac{B}{\rho}\right) \leqslant 0. \tag{2.15}$$

This condition depends on the magnetic field strength B (rather than on the $\mathbf{B}$ components). The instability criterion in the form

$$\frac{d}{dr}\left(\frac{B}{\rho}\right) < 0 \tag{2.16}$$

was obtained previously by Gilman and Cadez (1970) for long-wavelength perturbations using linear perturbation theory and assuming that magnetic field lines

https://doi.org/10.1017/S0022377800008588 Published online by Cambridge University Press

in equilibrium are horizontal and rectilinear. The inequality (2.15) (obtained from $N_B^2 \leqslant 0$) may sometimes become just the opposite to the inequality (2.16). For instance, in cometary ionospheres, an effective gravity is created mainly because of ion–neutral drag: $g^* \approx -\nu V_n$. Then, instead of (2.16), the instability criterion for an isentropic gas becomes

$$\frac{d}{dr}\left(\frac{B}{\rho}\right) \geqslant 0. \tag{2.17}$$

It is noteworthy that the plasma stability is governed by the magnetic field strength (rather than by the orientation vector $\mathbf{B}$) also in the case when $c_s^2 \gg V_A^2$ (see (2.11)).

In general, the magnetic field, according to (2.13) and (2.14), has a stabilizing (destabilizing) effect if B/ρ grows (decreases) with altitude r.

3. Conclusion

The Brunt–Väisälä frequency N_B in a magnetized ideally conducting plasma has been obtained using buoyancy-type arguments for the magnetic field, which satisfies the condition $\mathbf{B} \cdot \boldsymbol{\nabla} B \approx B_r\, dB/dr$. This approximation holds, for example, if the magnetic field strength B depends mainly on the vertical coordinate r. In particular, it holds in the important case of the polar caps in a magnetic dipole field (where $B_\varphi = 0$), up to a magnetic latitude of 45°. In this approximation, the BVF N_B (and hence the instability growth rate $|N_B|$), generally speaking, depend both on the horizontal and vertical components of the magnetic field. However, the instability criterion $N_B^2 \leqslant 0$, in the above approximation, is governed by the magnetic field strength B (rather than by the orientation of the vector $\mathbf{B}$), both with $ds/dr = 0$ and with $c_s^2 \gg V_A^2$. In general, the magnetic field has a stabilizing (destabilizing) effect if B/ρ grows (respectively decreases) with altitude r. This conclusion seems not to depend on the specific magnetic field configuration.

Acknowledgement

This work was supported by a grant from the Binational Science Foundation.

References

Acheson, D. J. 1979 Instability by magnetic buoyancy. *Solar Phys.* **62**, 23.

Brunt, D. 1927 The period of simple vertical oscillations in the atmosphere. *Q. J. R. Met. Soc.* **53**, 30.

Chen, C.-J. and Lykoudis, P. S. 1972 Velocity oscillations in solar plage regions. *Solar Phys.* **25**, 380.

Eckart, C. 1960 *Hydrodynamics of Oceans and Atmospheres*. Pergamon, Oxford.

Ershkovich, A. I. and Israelevich, P. L. 1993 On the convective instability of the cometary ionosphere. *Astrophys. J.* **411**, 891.

Ershkovich, A. I. and Israelevich, P. L. 1996 Plasma conductivity in a cometary ionosphere. *Astrophys. J.* **464**, 959.

Ershkovich, A. I., McKenzie, J. F. and Axford, W. I. 1989 Stability of a cometary ionosphere/ionopause determined by ion-neutral friction *Astrophys. J.* **344**, 932.

Gilman, P. A. 1970 Instability of magnetohydrostatic stellar interiors from magnetic buoyancy *Astrophys. J.* **162**, 1019.

Gilman, P. A. and Cadez, V. M. 1970 Appendix to Gilman (1970).

Gombosi, T. I. 1998 *Physics of the Space Environment*. Cambridge University Press.

Landau, L. D. and Lifshitz, E. M. 1959 *Fluid Mechanics*. Pergamon, Oxford.

Landau, L. D. and Lifshitz, E. M. 1977 *Quantum Mechanics: Nonrelativistic Theory*. Pergamon, Oxford.

Lighthill, J. 1978 *Waves in Fluids*. Cambridge University Press.

McKenzie, J. F., Ershkovich, A. I. and Axford, W. I. 1990 Stability of a cometary ionosphere/ionopause including pressure effects. *Astrophys. J.* **360**, 275.

Priest, E. R. 1982 *Solar Magnetohydrodynamics*. Reidel, Dordrecht.

Schwarzschild, K. 1906 *Gött. Nach.* **1**, 41.

Väisälä, V. 1925 Über die Wirkung der Windschwankungen auf die Pilot-beobachtungen. *Soc. Sci. Fenn. Commentat. Phys. Math.* **2**(19), 19.