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Thermodynamic formalism for Haar systems in noncommutative integration: transverse functions and entropy of transverse measures

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Abstract. We consider here a certain class of groupoids obtained via an equivalence relation (the so-called subgroupoids of pair groupoids). We generalize to Haar systems in these groupoids some results related to entropy and pressure which are well known in thermodynamic formalism. We introduce a transfer operator, where the equivalence relation (which defines the groupoid) plays the role of the dynamics and the corresponding transverse function plays the role of the *a priori* probability. We also introduce the concept of invariant transverse probability and of entropy for an invariant transverse probability, as well as of pressure for transverse functions. Moreover, we explore the relation between quasi-invariant probabilities and transverse measures. Some of the general results presented here are not for continuous modular functions but for the more general class of measurable modular functions.

Key words: Haar systems, transverse functions, transverse measures, entropy, noncommutative integration
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1. *Introduction*

Our purpose here is to extend the concepts of invariant probability, entropy and pressure from thermodynamic formalism to the setting of quasi-invariant probabilities, transverse functions and transverse measures, which are naturally defined on groupoids and Haar systems. The groupoids we consider here will be always obtained via an equivalence relation (the so-called subgroupoids of the pair groupoid according to [**37**, §3]). Most of our results are for the general class of measurable modular functions.

The results we obtain can be seen as similar to the classical results of thermodynamic formalism. We refer the reader to [**28**, **36**] for results on thermodynamic formalism and to [**11**, **18**, **29**] for results on Haar systems, groupoids and transverse measures (see also [**5**, **8**] for a strictly measure-theoretical perspective). But in any case we point out that the present work is *self-contained* for readers familiar with thermodynamic formalism.

The classical Kolmogorov–Sinai entropy is defined for probabilities which are invariant for a deterministic dynamical system. We point out that for a Haar system on a groupoid there is (in general) no underlying dynamical system. To realize that entropy depends on the *a priori* probability (as described in [**23**]) is the key issue for finding a suitable procedure to extend this formalism (of thermodynamic formalism for Hölder potentials) to Haar systems. When the alphabet is not countable (the so-called generalized XY models as considered in [**4**, **23**]) the definition of entropy via dynamical partitions is not suitable anymore and an *a priori* probability is necessary.

In the dictionary to be used here the transverse function of a Haar system is the mathematical object corresponding to the *a priori* probability and the equivalence relation (the groupoid) plays the role of the dynamics. The role of the potential is played by the modular function and, finally, the transverse measures and quasi-invariant probabilities in Haar systems play the role of the measures in thermodynamic formalism.

In §2 we introduce the main notation and definitions concerning Haar systems, including the concepts of transverse function, modular function, transverse measure and quasi-invariant probability.

Theorem 54 in [**8**] shows that Dobrushin–Lanford–Ruelle probabilities (see [**9**] for definition) are quasi-invariant probabilities for a certain class of Hölder modular functions in the case where the alphabet is finite. In §3 below we show analogous results for the case where the alphabet is a compact metric space. We consider as an example the generalized XY model, as studied in [**23**], and we show that any eigenprobability for the dual Ruelle operator is a quasi-invariant probability for the associated Haar system. We assume in this section that the modular function is just of Hölder class.

The results in the following sections are for the general class of measurable modular functions.

In §4 we consider particular modular functions and develop the main part of the paper studying Haar systems from a thermodynamic formalism point of view. We introduce the concepts of Haar invariant probabilities, Haar invariant transverse probabilities, entropy for Haar invariant transverse probabilities and pressure for transverse functions. The relation between transverse measures and quasi-invariant probabilities is presented in [**11**] (see also [**8**, §5]). In §4.2 we prove an equivalence between Haar invariant probabilities and Haar invariant transverse probabilities.

In §5 we exhibit examples and analyze the relations between the concepts introduced in this work and the classical ones for thermodynamic formalism. In §5.2, which considers a large class of dynamically defined groupoids and quasi-invariant probabilities, Rokhlin's disintegration theorem plays an important role.

We refer to [**27**, **34**, **38**] for classical results on measurable dynamics. The classical references for Haar systems when the transverse function is the counting measure are [**15**, **16**, **18**]. For the relation between quasi-invariant probabilities and Kubo–Martin–Schwinger (KMS) states of C^*-algebras (and von Neumann algebras) see [**2**, **3**, **6**–**8**, **17**, **21**, **22**, **29**, **31**–**33**]. A different kind of relation between KMS states of C^*-algebras and thermodynamic formalism is described in [**12**–**14**]. We refer the reader to [**8**] for an extensive presentation of Haar systems and non-commutative integration on groupoids obtained via an equivalence relation (some results are for dynamically defined equivalence relations).

2. *Transverse functions and transverse measures*

Consider a metric space Ω with metric d, and denote by $\mathcal{B}$ the Borel sigma-algebra on Ω. We fix an equivalence relation R on Ω, and if two points x, y are related we write $x \sim y$. We denote by $G \subset \Omega \times \Omega$ the associated *groupoid*

$$G = \{(x, y) \in \Omega \times \Omega \mid x \sim y\}$$

and by $[x] = \{y \in \Omega \mid x \sim y\}$ the class of x.

This corresponds to subgroupoids of the pair groupoid (see [**37**, §3]). These are the only kind of groupoids we will consider here.

Extreme examples of such groupoids are the cartesian product (pair groupoid) $G = \Omega \times \Omega$ when $[x] = \Omega$, for all x (where any two points are related) and the diagonal $G = \{(x, x) \mid x \in \Omega\}$ when $[x] = \{x\}$, for all x (where each point is related just with itself).

We consider over G the topology induced by the product topology on $\Omega \times \Omega$ and denote also by $\mathcal{B}$ the Borel sigma-algebra induced on G.

Definition 1. We say that G is a *measurable groupoid* if the maps

$$s(x, y) = x, \quad r(x, y) = y, \quad h(x, y) = (y, x) \quad \text{and} \quad Z((x, s), (s, y)) = (x, y),$$

are Borel measurable.

If G is a measurable groupoid, then, in particular, each class $[x]$, $x \in \Omega$, is a measurable set of Ω. In all this work we suppose that G is a measurable groupoid obtained from a general equivalence relation.

Remark 2. For each pair of objects x and y, the concept of a set of morphisms $\gamma : x \to y$ appears in the general definition of groupoids (see [**18**, page 100]). In our work the objects are the points of Ω, and, given two points $a \sim b$ in Ω, there exists a unique morphism $\gamma : a \to b$ which is represented by (a, b). It follows that $s(\gamma)$ and $r(\gamma)$ in [**18**] just correspond to the projections $s(x, y) = x$ and $r(x, y) = y$. The morphisms are not explicitly used in our work.

Definition 3. A *kernel* λ on the measurable groupoid G is a map of Ω in the space of measures on $(\Omega, \mathcal{B})$ such that:

(1) for all $y \in \Omega$, the measure λ^y has support on $[y]$;

(2) for all $A \in \mathcal{B}$, we have that $\lambda^y(A)$, as a function of y, is measurable.

In some sense, the two above items correspond to items (i) and (ii) in [**35**, Definition 5.14] (disintegration of a measure with respect to a partition). See also Theorem 46 below.

There is a subtle point in item (1) in Definition 3. An alternative definition could be: (1) for any $y \in \Omega$, we have that $\lambda^y(\Omega - [y]) = 0$. Some of the results we get here could be obtained with this alternative condition (but we will not elaborate on that).

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Definition 4. A *transverse function* ν on the measurable groupoid G is a kernel satisfying $\nu^x = \nu^y$, for any $(x, y) \in G$ (that is, $x \sim y$). We denote by $\mathcal{E}^+$ the *set of transverse functions*.

The concept of transverse function is a natural generalization of the concept of measurable non-negative functions $f : \Omega \to \mathbb{R}$ (see Remark 28 below).

We denote by $\mathcal{E}$ the set of signed transverse functions. More precisely, $\nu \in \mathcal{E}$ if the family of measures $(\nu^y)^+$ and $(\nu^y)^-$ that form, for each y, the Hann–Jordan decomposition of ν^y are both transverse functions. An important example of a signed transverse function is $\mu^y(dx) := f(x)\nu^y(dx)$. where $f : \Omega \to \mathbb{R}$ is measurable and bounded and $\nu \in \mathcal{E}^+$ satisfies $\int 1\, \nu^y(dx) = 1$, for all y.

Definition 5. The pair $(G, \hat{\nu})$, where G is a measurable groupoid and $\hat{\nu}$ is a transverse function. will be called a *Haar system*.

The convolution operator defined on the set of measurable functions $f : G \to \mathbb{C}$ (derived from the transverse function $\hat{\nu}$ of a Haar system) allows one to obtain in a natural way a von Neumann algebra as described in [**8**].

Example 6. Take $\Omega = [0, 1] \times [0, 1]$ and consider the groupoid G defined from the equivalence relation $x = (a_1, a_2) \sim y = (b_1, b_2)$ if $a_1 = b_1$. The classes can be identified as vertical lines of the unit square. They are the local unstable leaves of a Baker map (see [**8**] for a complete discussion).

Given a probability ν on $[0, 1]$ and a measurable function $\varphi : \Omega \to [0, +\infty)$, we can interpret φ as a family of density functions $\varphi^{a_1} : [a_1, \cdot] \to \mathbb{R}$, each one acting in a vertical fiber, and define a transverse function $\hat{\nu}$ which coincides with $\varphi^{a_1}\, d\nu$ in the fiber $[a_1, \cdot]$. Then $(G, \hat{\nu})$ is a Haar system.

A kernel λ is characterized by the operator

$$\lambda(f)(y) = \int f(x, y)\, \lambda^y(dx),$$

acting over λ-integrable functions $f : G \to \mathbb{R}$. Given a kernel λ and a λ-integrable function $g \geq 0$, we denote by $g\lambda$ the kernel $(g\lambda)^y(dx) = g(x, y)\lambda^y(dx)$. In this way

$$(g\lambda)(f)(y) = \int f(x, y)g(x, y)\lambda^y(dx).$$

The convolution of two kernels λ_1 and λ_2 is the kernel $\lambda_1 * \lambda_2$ satisfying

$$(\lambda_1 * \lambda_2)(f)(y) = \int f(x, y)\, (\lambda_1 * \lambda_2)^y(dx) = \iint f(s, y)\lambda_2^x(ds)\lambda_1^y(dx), \tag{1}$$

for any $\lambda_2^x(ds)\lambda_1^y(dx)$-integrable function f.

Definition 7. A *modular function* over the groupoid G is a measurable function $\delta : G \to \mathbb{R}$ such that, for any $x \in \Omega$ and any pair $y, z \in [x]$, we have that $\delta(x, y)\, \delta(y, z) = \delta(x, z)$.

Definition 8. A *transverse measure* Λ for the groupoid G and the modular function δ is a linear (this means $\Lambda(a\nu + b\mu) = a\Lambda(\nu) + b\Lambda(\mu)$ for any $\mu, \nu \in \mathcal{E}^+$ and $a, b \in \mathbb{R}$ such that $(a\nu + b\mu) \in \mathcal{E}^+$) function $\Lambda : \mathcal{E}^+ \to \mathbb{R}^+$, which satisfies the property: for each kernel

λ such that for any y we have $\lambda^y(1) = 1$, if ν_1 and ν_2 are transverse functions satisfying $\nu_1 * (\delta\lambda) = \nu_2$, it will be required that

$$\Lambda(\nu_1) = \Lambda(\nu_2). \tag{2}$$

The action of a transverse measure Λ on $\mathcal{E}^+$ can be linearly extended to $\mathcal{E}$ (this can be done by writing each ν as a sum of a negative function ν^- and a positive function ν^+).

As we will see later, the concept of transverse measure (acting on transverse functions) is a natural generalization of the classical concept of measure (acting by integration on functions) for the setting of Haar systems (see Remark 28 and Example 50).

Definition 9. Given a modular function δ, a groupoid G and a fixed transverse function $\hat{\nu}$, which is a probability for any y, we say that a probability M on Ω is *quasi-invariant* for the Haar system $(G, \hat{\nu})$ if, for any bounded measurable function $f : G \to \mathbb{R}$,

$$\iint f(y, x)\, \hat{\nu}^y(dx)\, dM(y) = \iint f(x, y)\delta(x, y)^{-1}\hat{\nu}^y(dx)\, dM(y).$$

In the Theorem 31 we exhibit, under certain hypotheses, a relation between transverse measures and quasi-invariant probabilities for modular functions in the particular form $\delta(x, y) = e^{V(y)-V(x)}$.

There are different (analogous) definitions of quasi-invariant probability. For example in [**25**, **26**] there is no mention of transverse functions and the concept is defined via Borel injections (considered as the concept of δ-invariant probability). For the existence of quasi-invariant probabilities in the measurable dynamics setting see the appendix to [**30**] or [**19**, **20**, **27**, **34**].

An interesting class of groupoids is described by [**24**, Definitions 1.9 and 1.10]. The authors called a continuous (or Lipschitz) groupoid one defined by an equivalence relation on the symbolic space $X = \{1, 2, \ldots, d\}^{\mathbb{N}}$, where, given two close elements $x, y \in X$, there is a continuous (or Lipschitz) correspondence such that one can associate elements on each of the finite classes $[x]$ and $[y]$ (which have same cardinality). In this case a kind of Ruelle operator (the Haar–Ruelle operator) can be defined and stronger properties (compare to the measurable setting we consider here) can be obtained.

Here the transverse function $\hat{\nu}$ (defining a Haar system) plays an important role. Note, however, that in the definition of transverse measure a fixed Haar system is not mentioned.

Given an equivalence relation defining a groupoid G, suppose that $[x]$ is a finite set for any $x \in \Omega$. The saturation of a measurable set $B \subset \Omega$ is the subset of G given by

$$S[B] = \bigcup_{x \in B} [x].$$

Consider the Haar system where the transverse function is the counting measure. In [**5**, §4.2] (or in [**15**]) it is shown that a probability M is quasi-invariant for some modular measurable function δ if and only if it satisfies the condition

$$M(B) = 0 \text{ implies that } M(S[B]) = 0 \quad \text{for all Borel sets } B \in \Omega. \tag{3}$$

This is a classical result.

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Example 10. Consider the example of a Haar system where $\Omega = [0, 1] \times [0, 1]$, each class is a vertical line and $\hat{\nu}^a$ is the Lebesgue measure on each line. The classes are the local unstable leaves of a nonlinear Baker map (see [**8**] for a precise definition) $F : \Omega \to \Omega$ given by

$$F(a_1, a_2) = (H(a_1, a_2), T(a_2)),$$

where $T : [0, 1] \to [0, 1]$ is a $C^{1+\alpha}$ expanding transformation (F is a simplified version of an Anosov transformation). There is an interesting relation between the Sinai–Bowen–Ruelle probability for F and the quasi-invariant probability M (see [**8**]) for the modular function δ given by

$$\delta((a_1, y_1), (a_1, y_2)) = \frac{V(a_1, y_1)}{V(a_1, y_2)} = \Pi_{n=1}^{\infty} \frac{T'(b^n(a_1, y_1))}{T'(b^n(a_1, y_2))},$$

where for each $n \geq 0$, the points $b^n(a_1, y_1)$ and $b^n(a_1, y_2)$ are, respectively, the successive n-preimages of (a_1, y_1) and (a_1, y_2) which are close together on the same vertical line (locally unstable).

3. *The inspiring model*

The purpose of this section is to present a preliminary example which can help the reader to understand why the reasoning we will pursue on the following sections is natural.

In this section we fix a compact metric space K and the associated Bernoulli space† $\Omega = K^{\mathbb{N}}$. Points $x \in \Omega$ are denoted by $x = (x_1, x_2, \ldots, x_n, \ldots)$. This is called the generalized XY model studied in [**23**] (see also [**1**, **10**]).

The groupoid $G \subset \Omega \times \Omega$ is defined from the equivalence relation $x = (x_1, x_2, x_3, \ldots) \sim y = (y_1, y_2, y_3, \ldots)$ if $x_j = y_j$, for all $j \geq 2$. Observe that $x \sim y$ means that $x = (a, y_2, y_3, \ldots)$ for some $a \in K$, which is equivalent to $\sigma(x) = \sigma(y)$, where σ is the shift map. We say that a groupoid G defined in such a way is *dynamically defined*. We will consider a more general class of dynamically defined groupoids in §5.2.

We denote by m a fixed *a priori* probability on K (with support equal to K). Consider the transverse function $\hat{\nu}$ such that, for each $y \in \Omega$ and continuous function $f : [y] \to \mathbb{R}$, we have

$$\int f(x)\hat{\nu}^y(dx) = \int f(a, y_2, y_3, \ldots)\, dm(a).$$

In the case where $K = \{1, 2, \ldots, d\}$ it is natural to take the probability m such that each point in K has m-mass equal to $1/d$, but by no means does this have to be the only choice. Similarly, when $K = S^1$ it is also natural to consider the Lebesgue probability da as the *a priori* probability (see [**4**]).

Given a Hölder function $V : \Omega \to \mathbb{R}$, the associated Ruelle operator (acting on continuous functions) is defined by

$$f \to \mathcal{L}_V(f)(x) = \int e^{V(a,x_1,x_2,\ldots)} f(a, x_1, x_2, \ldots)\, dm(a).$$

Consider a Hölder function $V : \Omega \to \mathbb{R}$ and take as modular function $\delta(x, y) = e^{V(y)-V(x)}$. Denote by c the eigenvalue and by φ the eigenfunction for the transfer (Ruelle)

† This describes the statistical mechanics system where the fiber of spins is the metric space K (which mat or may not be finite) and each site of the lattice is in $\mathbb{N}$.

operator $\mathcal{L}_V$. We denote by ρ the eigenprobability for $\mathcal{L}_V^*$, which satisfies $L_V^*(\rho) = c\rho$. In this way, if $\int \varphi \, d\rho = 1$, the probability $\mu := \varphi \, \rho$ is the equilibrium probability for V (see [**28**]). We denote by U the normalized Hölder potential

$$U = V + \log \varphi - \log(\varphi \circ \sigma) - \log(c).$$

Given $k_0 \in K$, consider fixed the point $z_0 = (k_0)^\infty = (k_0, k_0, \ldots) \in \Omega$. Since, for any continuous function $g : \Omega \to \mathbb{R}$,

$$\int g(x) \, d\mu(x) = \lim_{n\to\infty} \mathcal{L}_U^n(g) \, (z_0),$$

it follows that, for any continuous function h,

$$\int h(x) \, d\rho(x) = \int \frac{h(x)}{\varphi(x)} \, d\mu(x) = \lim_{n\to\infty} \mathcal{L}_U^n\left(\frac{h}{\varphi}\right)(z_0) = \lim_{n\to\infty} \frac{1}{c^n \varphi(z_0)} \ \mathcal{L}_V^n(h) \, (z_0).$$

Consequently,

$$\int h(x) \, d\rho(x) = \lim_{n\to\infty} \frac{\mathcal{L}_V^n(h) \, (z_0)}{\mathcal{L}_V^n(1) \, (z_0)}.$$

This kind of expression appears in [**9**]. The next result is a generalization of a similar one in [**8**, §4].

PROPOSITION 11. *Under above hypotheses and notation, the eigenprobability ρ for the dual Ruelle operator $\mathcal{L}_V^*$ is quasi-invariant for the modular function $\delta(x, y) = e^{V(y)-V(x)}$, that is, for all continuous function $f : G \to \mathbb{R}$, we have*

$$\iint f(y, x) \, \hat{\nu}^y(dx) \, d\rho(y) = \iint f(x, y)\delta(x, y)^{-1} \hat{\nu}^y(dx) \, d\rho(y).$$

Proof. In this proof we denote $dm(a_1) \, dm(a_2) \ldots dm(a_n)$ by $dm(a_1, \ldots, a_n)$. We write $y = (y_1, y_2, y_3, \ldots)$, and for $x \in [y]$ we write $x = (a, y_2, y_3, \ldots)$. Let us define two auxiliary functions

$$g_1(y) := \int f((a, y_2, y_3, \ldots), (y_1, y_2, y_3, \ldots))e^{V(a,y_2,y_3,\ldots)} \, dm(a)$$

and

$$g_2(y) := \int f((y_1, y_2, y_3, \ldots), (a, y_2, y_3, \ldots)) \, dm(a).$$

Then,

$$\begin{aligned}
&\iint f(x, y)\delta(x, y)^{-1} \hat{\nu}^y(dx) \, d\rho(y) = \iint f(x, y)e^{V(x)-V(y)} \hat{\nu}^y(dx) \, d\rho(y) \\
&= \iint f((a, y_2, y_3, \ldots), (y_1, y_2, y_3, \ldots))e^{V(a,y_2,y_3,\ldots)} \, dm(a)e^{-V(y_1,y_2,y_3,\ldots)} d\rho(y) \\
&= \int g_1(y)e^{-V(y)} \, d\rho(y) = \lim_{n\to\infty} \frac{\mathcal{L}_V^n(g_1 \cdot e^{-V}) \, (z_0)}{\mathcal{L}_V^n(1) \, (z_0)} \\
&= \lim_{n\to\infty} \frac{\int e^{S_n(V)(a_1,\ldots,a_n,z_0)} g_1(a_1, \ldots, a_n, z_0)e^{-V(a_1,\ldots,a_n,z_0)} \, dm(a_1, \ldots, a_n)}{\mathcal{L}_V^n(1) \, (z_0)} \\
&= \lim_{n\to\infty} \frac{\int e^{S_{n-1}(V)(a_2,\ldots,a_n,z_0)} g_1(a_1, \ldots, a_n, z_0) \, dm(a_1, \ldots, a_n)}{\mathcal{L}_V^n(1) \, (z_0)}
\end{aligned}$$

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$$= \lim_{n\to\infty} \frac{\iint e^{S_n(V)(a,a_2,\ldots,a_n,z_0)} f((a, a_2, \ldots, a_n, z_0), (a_1, \ldots, a_n, z_0))\, dm(a)\, dm(a_1, \ldots, a_n)}{\mathcal{L}_V^n(1)\,(z_0)}$$

$$= \lim_{n\to\infty} \frac{\iint e^{S_n(V)(a_1,a_2,\ldots,a_n,z_0)} f((a_1, a_2, \ldots, a_n, z_0), (a, a_2, \ldots, a_n, z_0))\, dm(a)\, dm(a_1, \ldots, a_n)}{\mathcal{L}_V^n(1)\,(z_0)}$$

$$= \lim_{n\to\infty} \frac{\int e^{S_n(V)(a_1,a_2,\ldots,a_n,z_0)} g_2(a_1, a_2, \ldots, a_n, z_0)\, dm(a_1, \ldots, a_n)}{\mathcal{L}_V^n(1)\,(z_0)}.$$

$$= \int g_2(y)\, d\rho(y) = \iint f((y_1, y_2, y_3, \ldots), (a, y_2, y_3 \ldots))\, dm(a)\, d\rho(y)$$

$$= \int f(y, x) \hat{\nu}^y(dx)\, d\rho(y). \qquad \square$$

We point out that the above probability ρ is not the unique quasi-stationary probability for such δ (see [**8**, end of §4]).

4. *A thermodynamic formalism point of view for Haar systems*

We now return to the analysis of general Haar systems (not necessarily as the previous generalized XY model). We consider a metric space Ω with the Borel sigma-algebra $\mathcal{B}$ and a measurable groupoid G. Throughout this section we fix a Haar system $(G, \hat{\nu})$ where the transverse function $\hat{\nu}$ (see Definition 4) satisfies $\int 1\, \hat{\nu}^y(dx) = 1$, for all y.

In the present setting the dynamical action is replaced by the equivalence relation which is described by the groupoid G. The transverse function $\hat{\nu}$ will play here the role of the *a priori* probability in the thermodynamic formalism for the generalized XY model.

4.1. *A transfer operator for Haar systems.*

We will consider modular functions of the form $\delta(x, y) = e^{V(y)-V(x)}$, where $V : \Omega \to \mathbb{R}$ is a bounded and *measurable function*. Then a probability M on Ω is quasi-invariant for the Haar system $(G, \hat{\nu})$ and V if it satisfies the property

$$\iint f(y, x)\, \hat{\nu}^y(dx)\, dM(y) = \iint f(x, y) e^{V(x)-V(y)} \hat{\nu}^y(dx)\, dM(y),$$

for all measurable and bounded functions f (see Definition 9).

Definition 12. A bounded and measurable function $V : \Omega \to \mathbb{R}$ is *Haar normalized* for the Haar system $(G, \hat{\nu})$ (or simply $\hat{\nu}$-normalized) if it satisfies

$$\int e^{V(x)}\, \hat{\nu}^y(dx) = 1 \quad \text{for all } y \in \Omega.$$

The above property corresponds in classical thermodynamic formalism to the concept of normalized potential for the Ruelle operator. Note that we do not assume that V is of Hölder class.

Definition 13. A *Haar invariant probability* for the Haar system $(G, \hat{\nu})$ will be a probability M on Ω such that, for some Haar normalized function $V : \Omega \to \mathbb{R}$,

$$\iint f(y, x)\, \hat{\nu}^y(dx)\, dM(y) = \iint f(x, y) e^{V(x)-V(y)} \hat{\nu}^y(dx)\, dM(y),$$

for all measurable bounded functions f.

Remark 14. Any Haar invariant measure is quasi-invariant.

Remark 15. In Proposition 11 the probability ρ, which is an eigenprobability for $\mathcal{L}_V^*$, is also quasi-invariant. It is necessary to assume that V is a normalized potential (for the Ruelle operator) in order to exhibit the case where the probability ρ is an invariant measure for the shift map. In this case such normalized potential V is also Haar normalized and ρ is also a Haar invariant measure for the Haar system $(G, \hat{\nu})$.

A probability M is Haar invariant and associated to the normalized function V if and only if, for any test function f, we have

$$\iint f(y, x)e^{V(x)}\, \hat{\nu}^y(dx)\, dM(y) = \iint f(x, y)e^{V(x)}\, \hat{\nu}^y(dx)\, dM(y). \tag{4}$$

Furthermore, if M is Haar invariant and associated to the normalized function V then, considering the particular case where $f(z, w) = f(w)$, we obtain from (4) that

$$\iint f(x)e^{V(x)}\hat{\nu}^y(dx)\, dM(y) = \int f(y)\, dM(y). \tag{5}$$

It follows that M is a fixed point for the operator H_V^*, defined below.

Definition 16. Given a measurable and bounded function $U : \Omega \to \mathbb{R}$, we define the operator H_U acting on measurable and bounded functions by

$$H_U(f)(y) = \int e^{U(x)} f(x)\hat{\nu}^y(dx). \tag{6}$$

If V is Haar normalized, the dual operator H_V^* restricted to the convex set of probabilities on Ω satisfies, for any measurable and bounded function f,

$$\int f\, dH_V^*(M_1) := \int e^{V(x)}\, f(x)\, \hat{\nu}^y(dx)\, dM_1(y) = \int H_V(f)\, dM_1. \tag{7}$$

The above operator H_V *is not the Ruelle operator* when one considers the particular setting of §3. If $\mathcal{L}_V$ is the Ruelle operator, then

$$H_V(f)(y) = \mathcal{L}_V(f)(\sigma(y)),$$

where σ is the shift map. We remark also that H_V is not the Haar–Ruelle operator studied in [**24**].

PROPOSITION 17. *M is Haar invariant for $(G, \hat{\nu})$ if and only if there exists a Haar normalized (measurable) function V such that M is a fixed point for the operator H_V^* defined in (7). We will call e^V a* Haar Jacobian *of M.*

Proof. It was shown above that any Haar invariant measure is a fixed point for the operator H_V^*. We now suppose that a probability M satisfies, for any measurable and bounded function F,

$$\iint e^{V(z)} F(z)\hat{\nu}^y(dz)\, dM(y) = \int F(y)\, dM(y),$$

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where V is Haar normalized. In this way we want to prove that M is Haar invariant, that is, it satisfies (4).

We begin by analyzing the left-hand side of (4). We fix a test function f and let $F(y) := \int f(y, x)e^{V(x)}\, \hat{\nu}^y(dx)$. Then

$$\begin{aligned}\iint f(y, x)e^{V(x)}\, \hat{\nu}^y(dx)\, dM(y) &= \int F(y)\, dM(y)\\ &\overset{\text{hypothesis}}{=} \iint F(z)e^{V(z)}\hat{\nu}^y(dz)\, dM(y)\\ &= \iiint f(z, x)e^{V(x)}\, \hat{\nu}^z(dx)e^{V(z)}\hat{\nu}^y(dz)\, dM(y)\\ &\overset{\hat{\nu}^z=\hat{\nu}^y}{=} \iiint f(z, x)e^{V(x)}\, \hat{\nu}^y(dx)e^{V(z)}\hat{\nu}^y(dz)\, dM(y)\\ &= \iiint f(z, x)e^{V(x)+V(z)}\, \hat{\nu}^y(dx)\hat{\nu}^y(dz)\, dM(y).\end{aligned}$$

We now apply similar computations on the right-hand side of (4):

$$\begin{aligned}\iint f(x, y)e^{V(x)}\, \hat{\nu}^y(dx)\, dM(y) &= \iiint f(x, z)e^{V(x)}\hat{\nu}^z(dx)e^{V(z)}\hat{\nu}^y(dz)\, dM(y)\\ &= \iiint f(x, z)e^{V(x)+V(z)}\, \hat{\nu}^y(dx)\hat{\nu}^y(dz)\, dM(y)\\ &\overset{x\leftrightarrow z}{=} \iiint f(z, x)e^{V(x)+V(z)}\, \hat{\nu}^y(dz)\hat{\nu}^y(dx)\, dM(y).\end{aligned}$$

It follows from Fubini's theorem that the two sides of (4) are equal. □

In the present setting—where there is no dynamics—the above result shows us that there is a natural way to obtain the analogous concept of invariant measure via a transfer operator (which is analogous to the Ruelle operator in symbolic dynamics). We observe that in this setting the operator is defined from an *a priori* measure that depends on the point y, which is the transverse function $\hat{\nu}$.

In the sequel in this section we describe some properties of the operators H_U and H_V^*.

PROPOSITION 18. *For any given measurable and bounded function U, consider the operator H_U as defined in (6) and the function $\tilde{U}(y) = \int e^{U(x)}\hat{\nu}^y(dx)$, which is constant on classes.*

(1) *If f is constant on classes then*

$$H_U(f)(y) = \tilde{U}(y)f(y).$$

(2) *The function $V := U - \log(\tilde{U})$ is Haar normalized.*

(3) *If there exists some positive eigenfunction g (for a certain eigenvalue) for the operator H_U, then $\tilde{U}$ must be constant. This constant value $\tilde{U}$ is the corresponding eigenvalue (it is also positive).*

(4) *If $\tilde{U}$ is constant and $\lambda := \tilde{U}(y)$, for all y, then a (measurable) function g is an eigenfunction for H_U if and only if g is constant on classes. In this case $H_U(g) = \lambda g$.*

Proof. We have

$$H_U(f)(y) = \int e^{U(x)} f(x)\hat{\nu}^y(dx) = \int e^{U(x)} f(y)\hat{\nu}^y(dx)$$
$$= \left[\int e^{U(x)}\hat{\nu}^y(dx)\right] f(y) = \tilde{U}(y) f(y).$$

This proves (1).

We have

$$\int e^{U(x)-\log(\tilde{U}(x))}\,\hat{\nu}^y(dx) = H_U\left(\frac{1}{\tilde{U}}\right)(y) \overset{\text{item (1)}}{=} \frac{\tilde{U}(y)}{\tilde{U}(y)} = 1.$$

This proves (2).

To prove (3), suppose that, for some measurable and bounded function $g > 0$ and real number λ, we have that $H_U(g) = \lambda g$. As $H_U(g)$ is constant on classes and $H_U(g) = \lambda g$, the function g is necessarily constant on classes too. It follows from (1) that $H_U(g) = \tilde{U} g$ and therefore $\lambda g = \tilde{U} g$. As g is positive, we obtain $\tilde{U} = \lambda$. It follows from the definition of $\tilde{U}$ that $\lambda > 0$.

Turning, finally, to (4), if g is constant on classes, then from (1) we obtain that $H_U(g) = \tilde{U} g = \lambda g$. On the other hand, if g is an eigenfunction of H_U, following the proof of (3), we obtain that g is constant on classes and, furthermore, that $H_U(g) = \lambda g$. □

In the above result the function $\tilde{U}$ plays the role of the eigenvalue of the operator H_U. The normalization procedure (obtaining a Haar normalized V from the given U) described by item (2) in the above proposition is much more simpler that the corresponding one in thermodynamic formalism (where one has to add a coboundary).

COROLLARY 19. *Suppose that V is Haar normalized. Consider the operator H_V defined by* (6). *If f is constant on classes, then $H_V(f) = f$. In particular, $H_V \circ H_V = H_V$.*

Proof. If V is normalized, then $\tilde{V} = 1$, and consequently, if f is constant on classes, then $H_V(f) = \tilde{V} f = f$. Consequently, $H_V(H_V(g)) = H_V(g)$ for any measurable and bounded function g, because $H_V(g)$ is constant on classes. □

Example 20. If we consider the groupoid defined from the equivalence relation $x \sim y$ if and only if $x = y$, then we have $[y] = \{y\}$, and, therefore $\hat{\nu}$ is trivial, that is, $\hat{\nu}^y = \delta_y$, over the set $\{y\}$. In this case, the unique Haar normalized function is $V \equiv 0$. Furthermore, for any function U we obtain that $\tilde{U} = e^U$ and $U - \log(\tilde{U}) = 0 = V$. In this model it is quite simple to see that $H_V = \text{Id}$ and consequently any probability M is fixed for H_V^*. Then we have the following statements.

(1) The fixed probability of H_V^* is not unique.
(2) If f is not constant, then $H_V^n(f) := H_V^{n-1} \circ H_V(f) = f$ does not converge to a constant (it does not converge, for instance, to any possible given $\int f\, dM$).

Analyzing equation (5), the following reasoning is natural. This equation could be solved independently for each class $[y]$, and the solutions could then be combined (adding class by class) in order to obtain a probability M over all the space Ω. Furthermore, the weight that M has in each class seems to have no relevance in order to obtain a Haar invariant measure. This remark is presented more formally in the next theorem.

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THEOREM 21. *Let V be a Haar normalized function and μ be any probability measure on Ω. There exists a unique Haar invariant probability M with Jacobian e^V and such that, for any bounded and measurable function g, constant on classes, we obtain*

$$\int g\,dM = \int g\,d\mu.$$

Proof. Let $M = H_V^*(\mu)$. Then, for any integrable function f we have

$$\int f\,dM = \int H_V(f)\,d\mu.$$

In particular, as $H_V \circ H_V = H_V$ we obtain, for any integrable function f,

$$\int H_V(f)\,dM = \int H_V(H_V(f))\,d\mu = \int H_V(f)\,d\mu = \int f\,dM.$$

This shows (see Proposition 17) that M is Haar invariant with Jacobian e^V. Furthermore, for any g constant on classes, we have

$$\int g(y)\,dM(y) \overset{M=H_V^*(\mu)}{=} \iint e^{V(x)} g(x)\,\hat{\nu}^y(dx)\,d\mu(y)$$
$$\overset{g(x)=g(y)}{=} \int g(y) \int e^{V(x)}\,\hat{\nu}^y(dx)\;d\mu(y) = \int g(y)\,d\mu(y).$$

Suppose now that M_1 and M_2 are Haar invariant measures with Jacobian e^V satisfying

$$\int g\,dM_1 = \int g\,d\mu = \int g\,dM_2,$$

for any bounded function g constant on classes. Since, for any bounded function f, the function $H_V(f)$ is constant on classes, we obtain

$$\int f\,dM_1 = \int H_V(f)\,dM_1 = \int H_V(f)\,dM_2 = \int f\,dM_2. \qquad \square$$

COROLLARY 22. *Let V be a Haar normalized function and g be a measurable and bounded function which is constant on classes. Then*

$$\sup_{y\in\Omega} g(y) = \sup\left\{\int g\,dM \mid M \textit{ is probability Haar invariant}\right\}$$
$$= \sup\left\{\int g\,dM \mid M \textit{ is probability Haar invariant with Jacobian } e^V\right\}.$$

Proof. Clearly,

$$\sup_{y\in\Omega} g(y) \geq \sup\left\{\int g\,dM \mid M \text{ is probability Haar invariant with Jacobian } e^V\right\}.$$

On the other hand, for any given $\epsilon > 0$, let $y_\epsilon \in \Omega$ be such that $g(y_\epsilon) + \epsilon > \sup_{y\in\Omega} g(y)$. Let $\mu_\epsilon = \delta_{y_\epsilon}$. From above theorem there exists a probability M_ϵ Haar invariant, with Jacobian e^V, such that

$$\int g\,dM_\epsilon = \int g\,d\delta_{y_\epsilon} = g(y_\epsilon) > \Big(\sup_{y\in\Omega} g(y)\Big) - \epsilon.$$

Taking the supremum over Haar invariant probabilities with Jacobian e^V and observing that ϵ is arbitrary, we complete the proof. $\square$

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4.2. *Transverse measures and Haar invariant probabilities.* General references on transverse measures are [**11**, **18**]. Recall that we consider fixed a certain Haar system $(G, \hat{\nu})$ where the transverse function $\hat{\nu}$ satisfies $\int 1\, \hat{\nu}^y(dx) = 1$, for all y.

In this section we study in our setting the relation between a transverse measure Λ and a Haar invariant probability M. We give a simpler and more direct discussion, under the present setting, similar to that appearing in [**8**, §5] but with some new proofs. The goal here is to prove Theorem 31 (which has a similar claim in [**8**, §5] but it will improved here). We remark that in [**8**] Haar invariant probabilities are not considered. We start by recalling the following result stated in [**8**].

PROPOSITION 23. *Given a transverse function $\hat{\nu}$, a modular function $\delta(x, y)$ and a quasi-invariant probability M, if $\hat{\nu} * \lambda_1 = \hat{\nu} * \lambda_2$, where λ_1, λ_2 are kernels, then*

$$\int \delta^{-1}\lambda_1(1)\, dM = \int \delta^{-1}\lambda_2(1)\, dM.$$

This means that

$$\iint \delta(y, x)\lambda_1^y(dx)M(y) = \iint \delta(y, x)\lambda_2^y(dx)M(y).$$

Proof. See [**8**, §5, Proposition 65]. □

As $\hat{\nu}^y$ is a probability, for any transverse function ν we obtain

$$\begin{aligned}\int f(s, y)\, \nu^y(ds) &= \int f(s, y)\nu^y(ds)\hat{\nu}^y(dx)\\ &= \int f(s, y)\nu^x(ds)\hat{\nu}^y(dx) \overset{(1)}{=} \int f(s, y)(\hat{\nu} * \nu)^y(ds).\end{aligned}$$

This shows that $\nu = \hat{\nu} * \nu$ for any transverse function ν. Consequently, if λ is any kernel such that $\nu = \hat{\nu} * \lambda$, then $\hat{\nu} * \nu = \hat{\nu} * \lambda$. Applying the above proposition, we obtain, for any quasi-invariant probability M for $\hat{\nu}$ and δ, that

$$\nu = \hat{\nu} * \lambda \quad \text{implies} \quad \iint \delta(y, x)\lambda^y(dx)\, dM(y) = \iint \delta(y, x)\nu^y(dx)\, dM(y).$$

Given the transverse function $\hat{\nu}$, a modular function $\delta(x, y)$ and an associated quasi-invariant measure M, we define (see [**8**, §5, Theorem 66] or [**11**]) a transverse measure Λ as

$$\Lambda(\nu) = \iint \delta(y, x)\lambda^y(dx)dM(y),$$

where λ is any kernel satisfying $\nu = \hat{\nu} * \lambda$.

From now on we will consider a modular function $\delta(x, y) = e^{V(y)-V(x)}$, where V is $\hat{\nu}$-normalized, and a Haar invariant probability M with Jacobian e^V. Furthermore, as in the present setting $\hat{\nu}^y$ is a probability, we can take $\lambda = \nu$ and define

$$\Lambda(\nu) = \iint e^{V(x)-V(y)}\nu^y(dx)\, dM(y). \tag{8}$$

The first result below provides an alternative expression for Λ in (8).

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PROPOSITION 24. *Suppose that M is a Haar invariant probability associated to the Jacobian e^V. Then, for any transverse function ν,*

$$\Lambda(\nu) := \iint e^{V(x)-V(y)}\nu^y(dx)\, dM(y) = \iint e^{V(x)}\nu^y(dx)\, dM(y). \tag{9}$$

Proof. As M is Haar invariant (for the fixed transverse function $\hat{\nu}$) and associated to the Jacobian e^V, if we denote $F(y) = e^{-V(y)} \int e^{V(x)}\nu^y(dx)$, then we obtain from (8) and Proposition 17 that, for any transverse function ν,

$$\begin{aligned}
\Lambda(\nu) &= \iint e^{V(x)-V(y)}\nu^y(dx)\, dM(y) = \int F(y)\, dM(y) \\
&\overset{H_V^*(M)=M}{=} \iint e^{V(x)} F(x)\, \hat{\nu}^y(dx)\, dM(y) \\
&= \iint e^{V(x)}\left[e^{-V(x)} \int e^{V(s)}\nu^x(ds)\right] \hat{\nu}^y(dx)\, dM(y) \\
&= \iiint e^{V(s)}\nu^x(ds)\, \hat{\nu}^y(dx)\, dM(y) = \iiint e^{V(s)}\nu^y(ds)\, \hat{\nu}^y(dx)\, dM(y) \\
&= \iint e^{V(s)}\nu^y(ds)\, dM(y) = \iint e^{V(x)}\nu^y(dx)\, dM(y).
\end{aligned}$$
□

Definition 25. We say that a transverse measure Λ is a *Haar invariant transverse probability* if it has modulus $\delta(x, y) = e^{V(y)-V(x)}$, where V is a Haar normalized function, and, furthermore, $\Lambda(\hat{\nu}) = 1$.

We denote by $\mathcal{M}(\hat{\nu})$ the set of all Haar invariant transverse probabilities Λ for the Haar system $(G, \hat{\nu})$.

The next result corresponds to [**8**, Theorem 66].

PROPOSITION 26. *Suppose that M is a Haar invariant probability associated to the Jacobian e^V. Then Λ, as defined by expression (9), is a Haar invariant transverse probability.*

Proof. Clearly $\Lambda(\hat{\nu}) = 1$. We want to show that Λ satisfies Definition 8 with $\delta(x, y) = e^{V(y)-V(x)}$. From (9), Λ is linear over transverse functions. Furthermore, if $\nu_1 * (\delta\lambda) = \nu_2$, with $\lambda^y(1) = 1$ for all y, and $\delta(x, y) = e^{V(y)-V(x)}$, then

$$\begin{aligned}
\Lambda(\nu_2) &= \iint e^{V(x)}\, \nu_2^y(dx)\, dM(y) = \iiint e^{V(s)}\delta(s, x)\, \lambda^x(ds)\nu_1^y(dx)\, dM(y) \\
&= \iiint e^{V(s)}e^{V(x)-V(s)}\, \lambda^x(ds)\nu_1^y(dx)\, dM(y) = \iiint e^{V(x)}\, \lambda^x(ds)\nu_1^y(dx)\, dM(y) \\
&= \iiint 1\, \lambda^x(ds)e^{V(x)}\nu_1^y(dx)\, dM(y) = \iint e^{V(x)}\, \nu_1^y(dx)\, dM(y) = \Lambda(\nu_1).
\end{aligned}$$
□

The next two results complete the study of the relation between the Haar invariant transverse probability Λ and the Haar invariant probability M.

PROPOSITION 27. *Suppose that M is Haar invariant and associated to the Jacobian e^V. Let Λ be the transverse measure as defined by expression (9). Consider the transverse* *function* $\nu^y(dx) = F(x)\hat{\nu}^y(dx)$*, where* F *is measurable and bounded. Then*

$$\Lambda(\nu) = \int F(x)\ dM(x).$$

Proof. Since M is quasi-invariant (see Remark 14), we have

$$\Lambda(\nu) := \iint e^{V(x)-V(y)}\nu^y(dx)\,dM(y) = \iint e^{V(x)-V(y)}F(x)\,\hat{\nu}^y(dx)\,dM(y)$$
$$= \iint F(y)\,\hat{\nu}^y(dx)\,dM(y) = \int F(y)\int 1\,\hat{\nu}^y(dx)\,dM(y) = \int F(y)\,dM(y). \qquad \square$$

Remark 28. The above theorem says that the transverse function ν is a more general concept than a function F and the transverse measure Λ is a more general concept than a measure M. Note that if any class of the equivalence relation is finite, then, given any transverse function ν, there exists a function F such that $\nu^y(dx) = F(x)\hat{\nu}^y(dx)$.

Remark 29. If we consider a more general density $\nu^y(dx) = F(x, y)\hat{\nu}^y(dx)$, then $\nu^y(dx)$ is a kernel but not a transverse function, except if $F(x, y) = F(x, z)$ for any $z \in [y]$. But in this case, since $x \in [y]$, we obtain that $F(x, y) = F(x, x)$, that is, F depends only on x as in the above theorem.

The next result shows us that any Haar invariant transverse probability of modulus $\delta(x, y) = e^{V(y)-V(x)}$ is of the form (9).

PROPOSITION 30. *Let* Λ *be a Haar invariant transverse probability for the modular function* $\delta(x, y) = e^{V(y)-V(x)}$*, where* V *is Haar normalized. Let us define a probability* M *on* Ω *which satisfies, for any measurable and bounded function* $F : \Omega \to \mathbb{R}$*,*

$$\int F(y)\,dM(y) := \Lambda(F(x)\hat{\nu}^y(dx)).$$

Then M *is a Haar invariant probability with Jacobian* e^V*. Furthermore, for any transverse function* ν*, we have*

$$\Lambda(\nu) = \iint e^{V(x)}\nu^y(dx)\,dM(y).$$

Proof. Let $\lambda^y(dx) := e^{V(x)}\hat{\nu}^y(dx)$. Then, since V is normalized, $\lambda^y(1) = 1$, for all y.

Claim. $(\nu * (\delta\lambda))^y(dz) = C(z)\hat{\nu}^y(dz)$, where

$$C(z) := \int e^{V(x)}\nu^z(dx) = \int e^{V(x)}\nu^y(dx) = C(y)$$

is a constant function on the class of y.

To prove the claim we consider any test function $f(x, y)$. Then, for each fixed y,

$$\int f(x, y)[\nu * (\delta\lambda)]^y(dx) = \iint f(s, y)\delta(s, x)\lambda^x(ds)\nu^y(dx)$$
$$= \iint f(s, y)e^{V(x)-V(s)}e^{V(s)}\hat{\nu}^x(ds)\nu^y(dx) = \iint f(s, y)e^{V(x)}\hat{\nu}^x(ds)\nu^y(dx)$$
$$= \int\left[\int f(s, y)\hat{\nu}^x(ds)\right]e^{V(x)}\nu^y(dx) = \int\left[\int f(s, y)\hat{\nu}^y(ds)\right]e^{V(x)}\nu^y(dx)$$

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$$= \left[\int f(s, y)\hat{\nu}^y(ds)\right]\left[\int e^{V(x)}\nu^y(dx)\right] = \left[\int f(s, y)\hat{\nu}^y(ds)\right][C(y)]$$
$$= \int f(s, y)C(y)\hat{\nu}^y(ds) = \int f(x, y)C(y)\hat{\nu}^y(dx).$$

This proves the claim.

Let M be defined by

$$\int F(y)\, dM(y) := \Lambda(F(x)\hat{\nu}^y(dx)),$$

for any measurable and bounded function F. Since Λ is linear and $\Lambda(\hat{\nu}) = 1$, we obtain that M is a probability on Ω.

As Λ is a transverse measure it follows from the claim (see Definition 8) that

$$\Lambda(\nu) = \Lambda(C(z)\hat{\nu}^y(dz)) = \int C(y)\, dM(y) = \iint e^{V(x)}\nu^y(dx)\, dM(y).$$

It remains to prove that M is Haar invariant with Jacobian e^V. Let $f : \Omega \to [0, +\infty)$ be a measurable and bounded function and define $\nu^y(dx) = f(x)\hat{\nu}^y(dx)$. It follows from the above claim that

$$(\nu * (\delta\lambda))^y(dz) = \int e^{V(x)} f(x)\hat{\nu}^z(dx)\hat{\nu}^y(dz),$$

and then, as Λ is a transverse measure, we obtain

$$\Lambda(f(x)\hat{\nu}^y(dx)) = \Lambda(\nu) = \Lambda(\nu * (\delta\lambda)) = \Lambda\left(\int e^{V(x)} f(x)\hat{\nu}^z(dx)\hat{\nu}^y(dz)\right).$$

Therefore, by definition of M, we finally obtain

$$\int f(x)\, dM(x) = \int\int e^{V(x)} f(x)\hat{\nu}^z(dx)\, dM(z),$$

which, by linearity, can extend the claim for any measurable and bounded function f. This shows that M is Haar invariant with Jacobian e^V. □

We summarize the results of this section in the following theorem.

THEOREM 31. *Let V be a Haar normalized function. There is an invertible map that associates, for each Haar invariant probability M over Ω, with Jacobian e^V, a Haar invariant transverse probability Λ of modulus $\delta(x, y) = e^{V(y)-V(x)}$. For any given M the associated Λ obtained by this map satisfies*

$$\Lambda(\nu) := \iint e^{V(x)}\nu^y(dx)\, dM(y), \qquad \nu \in \mathcal{E}^+.$$

On the other hand, given Λ, the associated M by the inverse map satisfies

$$\int F(y)\, dM(y) := \Lambda(F(x)\hat{\nu}^y(dx)) \quad \text{for any } F \text{ measurable and bounded.}$$

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4.3. *Entropy of transverse measures.* In this section we continue to fix a Haar system $(G, \hat{\nu})$ where the transverse function $\hat{\nu}$ satisfies $\int 1\, \hat{\nu}^y(dx) = 1$, for all y. We use the notation and hypotheses of Theorem 31. Recall that we denote by $\mathcal{M}(\hat{\nu})$ the set of Haar invariant transverse probabilities Λ for the Haar system $(G, \hat{\nu})$. In some sense (see Theorem 31) the set $\mathcal{M}(\hat{\nu})$ corresponds in thermodynamic formalism (ergodic theory) to the set of invariant probabilities.

We will be able to extend some concepts in ergodic theory concerning entropy to the Haar system formalism. The transverse function $\hat{\nu}$ will play the role of the *a priori* probability in [**23**] (where one can find the motivation for the definition below).

Our concept of invariant probability does not necessarily coincide with that in classical measurable dynamics.

Definition 32. We define the entropy of a Haar invariant transverse probability Λ relative to $\hat{\nu}$ (or relative to $(G, \hat{\nu})$) as

$$h_{\hat{\nu}}(\Lambda) = -\sup\{\Lambda(F(x)\hat{\nu}^y(dx)) \mid F \text{ is Haar normalized}\}.$$

If Λ has modulus $\delta(x, y) = e^{V(y)-V(x)}$, where V is Haar normalized, and M is the corresponding Haar invariant probability given in Theorem 31, then we obtain

$$h_{\hat{\nu}}(\Lambda) = -\sup\left\{\int F(x)\, dM(x) \mid F \text{ is Haar normalized}\right\}.$$

Since we define entropy just for transverse measures in the set $\mathcal{M}(\hat{\nu})$, it follows from Theorem 31 that we are defining entropy similarly for any Haar invariant probabilities.

THEOREM 33. *Suppose $\Lambda \in \mathcal{M}(\hat{\nu})$ has modulus $\delta(x, y) = e^{V(y)-V(x)}$, where V is a Haar normalized function. Then*

$$h_{\hat{\nu}}(\Lambda) = -\int V(x)\, dM(x) = -\Lambda(V\hat{\nu}), \tag{10}$$

where M is defined in Theorem 31.

Proof. This proof follows ideas from [**23**]. By construction, M is Haar invariant with Jacobian e^V. The second equality in expression (10) is a consequence of Theorem 31. In order to prove the first equality, we consider a general Haar normalized function U, and then we claim that

$$\int U(x)\, dM(x) \le \int V(x)\, dM(x). \tag{11}$$

From this inequality we obtain

$$\sup\left\{\int F(x)\, dM(x) \mid F \text{ is Haar normalized}\right\} = \int V(x)\, dM(x),$$

which proves that

$$h_{\hat{\nu}}(\Lambda) = -\int V(x)\, dM(x).$$

In order to prove (11), we again consider the operator

$$H_V(f)(y) = \int e^{V(x)} f(x)\, d\hat{\nu}^y(dx).$$

The probability M satisfies $H_V^*(M) = M$ according to Proposition 17.

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Let $u = e^{U-V}$. Then $ue^{V-U} = 1$ and, moreover, $H_V(u)(y) = 1$, for any y. It follows that

$$0 = \log(1/1) = \log\left(\frac{H_V(u)}{ue^{V-U}}\right) = \log(H_V(u)) - \log(u) + U - V$$

and

$$0 = \int \log(H_V(u))\, dM - \int \log(u)\, dM + \int U\, dM - \int V\, dM.$$

Therefore,

$$\int V\, dM - \int U\, dM = \int \log(H_V(u))\, dM - \int \log(u)\, dM$$
$$= \int \log(H_V(u))\, dM - \int H_V(\log(u))\, dM \geq 0,$$

because for any y we can consider the probability $e^{V(x)}\hat{\nu}^y(dx)$ and apply Jensen's inequality in the following way:

$$\log(H_V(u)) = \log\bigg(\int u(x)e^{V(x)}\hat{\nu}^y(dx)\bigg) \geq \int \log(u)(x)e^{V(x)}\hat{\nu}^y(dx) = H_V(\log(u)). \qquad \square$$

PROPOSITION 34. *The entropy defined above has the following properties:*

(1) $h_{\hat{\nu}}(\Lambda) \leq 0$ *for any* $\Lambda \in \mathcal{M}(\hat{\nu})$*;*

(2) $h_{\hat{\nu}}(\cdot)$ *is concave;*

(3) $h_{\hat{\nu}}(\cdot)$ *is upper semicontinuous. More precisely, if* $\Lambda_n(\nu) \to \Lambda(\nu)$*, for any transverse function* ν*, then*

$$\limsup_n h_{\hat{\nu}}(\Lambda_n) \leq h_{\hat{\nu}}(\Lambda).$$

Proof. To prove (1), just take $V = 0$,which is Haar normalized.

Turning to (2), suppose that $\Lambda = a_1\Lambda_1 + a_2\Lambda_2$, where Λ_1 and Λ_2 are Haar invariant transverse probabilities, $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$. Then

$$\begin{aligned} h_{\hat{\nu}}(\Lambda) &= -\sup\{\Lambda(F\hat{\nu}) \mid F \text{ normalized}\} \\ &= \inf\{\Lambda(-F\hat{\nu}) \mid F \text{ normalized}\} \\ &= \inf\{a_1\Lambda_1(-F\hat{\nu}) + a_2\Lambda_2(-F\hat{\nu}) \mid F \text{ normalized}\} \\ &\geq a_1 \inf\{\Lambda_1(-F\hat{\nu}) \mid F \text{ normalized}\} + a_2 \inf\{\Lambda_2(-F\hat{\nu}) \mid F \text{ normalized}\} \\ &= a_1 h_{\hat{\nu}}(\Lambda_1) + a_2 h_{\hat{\nu}}(\Lambda_1). \end{aligned}$$

Finally, to prove (3), let V be a Haar normalized function such that Λ has modulus $e^{V(y)-V(x)}$. Given any $\epsilon > 0$, we have that $\Lambda_n(-V\hat{\nu}) \leq \Lambda(-V\hat{\nu}) + \epsilon = h_{\hat{\nu}}(\Lambda) + \epsilon$, for sufficiently large n. Then, for sufficiently large n, we obtain

$$h_{\hat{\nu}}(\Lambda_n) = \inf\{\Lambda_n(-F\hat{\nu}) \mid F \text{ normalized}\} \leq \Lambda_n(-V\hat{\nu}) \leq h_{\hat{\nu}}(\Lambda) + \epsilon. \qquad \square$$

4.4. *Pressure of transverse functions.* In this section we continue to fix a Haar system $(G, \hat{\nu})$ where the transverse function $\hat{\nu}$ satisfies $\int \hat{\nu}^y(dr) = 1$, for all y.

Definition 35. We define the $\hat{\nu}$*-pressure of the transverse function* ν by

$$P_{\hat{\nu}}(\nu) = \sup_{\Lambda \in \mathcal{M}(\hat{\nu})} \{\Lambda(\nu) + h_{\hat{\nu}}(\Lambda)\}. \tag{12}$$

A transverse measure $\Lambda \in \mathcal{M}(\hat{\nu})$ which attains the supremum on the above expression will be called an *equilibrium transverse measure* for the *transverse function* ν.

In the following we denote by M_V a Haar invariant probability with Jacobian e^V. Recall from Theorem 21 that there are several such associated probabilities.

It follows from Theorems 31 and 33 that

$$P_{\hat{\nu}}(\nu) = \sup_{V\ \hat{\nu}\text{-normalized}} \ \sup_{M_V} \left[\int e^{V(x)} \nu^y(dx)\, dM_V(y) - \int V(y)\, dM_V(y) \right],$$

which, from Proposition 17, can be rewritten as

$$P_{\hat{\nu}}(\nu) = \sup_{V\ \hat{\nu}\text{-normalized}} \ \sup_{M_V} \left[\int e^{V(x)} \nu^y(dx)\, dM_V(y) - \int e^{V(x)} V(x) \hat{\nu}^y(dx)\, dM_V(y) \right].$$

The cases where ν is of the form $\nu = U(x)\ \hat{\nu}^y(dx)$ are studied below. In these particular cases it is natural to interpret ν as the function $U : \Omega \to \mathbb{R}$ and, using Theorem 31, to interpret Λ as the associated probability M on Ω.

PROPOSITION 36. *Suppose that* U *is* $\hat{\nu}$*-normalized. Consider the transverse function* $\nu = U(x)\ \hat{\nu}^y(dx)$*. Then* $P_{\hat{\nu}}(\nu) = 0$*. If* M_U *is any Haar invariant probability with Jacobian* e^U *and* Λ_U *is the associated transverse measure from Theorem 31, then* Λ_U *is an equilibrium for* ν*.*

Proof. We have

$$\begin{aligned} P_{\hat{\nu}}(\nu) &= \sup_{V\ \hat{\nu}\text{-normalized}} \ \sup_{M_V} \left[\int e^{V(x)} U(x) \hat{\nu}^y(dx)\, dM_V(y) \right. \\ &\qquad \left. - \int e^{V(x)} V(x) \hat{\nu}^y(dx)\, dM_V(y) \right] \\ &\overset{H_V^*(M_V)=M_V}{=} \sup_{V\ \hat{\nu}\text{-normalized}} \ \sup_{M_V} \left[\int U(y)\, dM_V(y) - \int V(y)\, dM_V(y) \right]. \end{aligned}$$

From (11) the last expression is non-positive and, on the other hand, taking $V = U$ as a particular V under the supremum, the last expression is equal to zero. Then the choices $V = U$ and any probability M_U attain the supremum. □

For the next result we suggest that the reader recall the claim of Proposition 18.

PROPOSITION 37. *Consider the transverse function* $\nu = U(x)\hat{\nu}^y(dx)$*, where* U *is measurable and bounded, but not necessarily* $\hat{\nu}$*-normalized. Suppose that* $\tilde{U}(y) = \int e^{U(x)}\ \hat{\nu}^y(dx)$ *is a constant function,* $\tilde{U}(y) = \lambda$ *for all* y*. Then*

$$P_{\hat{\nu}}(\nu) = \log(\lambda).$$

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Proof. From Proposition 18, the function $U - \log(\lambda)$ is normalized. Then

$$P_{\hat{\nu}}(\nu) = \sup_{V\ \hat{\nu}\text{-normalized}} \sup_{M_V} \bigg[\int e^{V(x)} U(x) \hat{\nu}^y(dx)\, dM_V(y) - \int e^{V(x)} V(x) \hat{\nu}^y(dx)\, dM_V(y) \bigg]$$
$$= \sup_{V\ \hat{\nu}\text{-normalized}} \sup_{M_V} \bigg[\int U(y)\, dM_V(y) - \int V(y)\, dM_V(y) \bigg]$$
$$= \sup_{V\ \hat{\nu}\text{-normalized}} \sup_{M_V} \bigg[\int [U - \log(\lambda)\, dM_V - \int V dM_V + \log(\lambda) \bigg] = \log(\lambda),$$

where the last equality is a consequence of (11) together with the fact that we can take also $V = U - \log(\lambda)$ under the supremum. □

We now consider the general case where $\tilde{U}$ is not constant.

PROPOSITION 38. *Consider the transverse function $\nu = U(x)\hat{\nu}^y(dx)$ where U is bounded and measurable, but not necessarily normalized. Let $\tilde{U}(y) = \int e^{U(x)}\, \hat{\nu}^y(dx)$. Then*

$$P_{\hat{\nu}}(\nu) = \sup_{y\in\Omega} [\log(\tilde{U}(y))].$$

Proof. Note that

$$P_{\hat{\nu}}(\nu) = \sup_{V\ \hat{\nu}\text{-normalized}} \sup_{M_V} \bigg[\int U(y)\, dM_V(y) - \int V(y)\, dM_V(y) \bigg]$$
$$= \sup_{V\ \hat{\nu}\text{-normalized}} \sup_{M_V} \bigg[\int [U - \log(\tilde{U})\, dM_V - \int V dM_V + \int \log(\tilde{U})\, dM_V \bigg].$$

On the one hand, from (11), as $U - \log(\tilde{U})$ is normalized, we obtain

$$P_{\hat{\nu}}(\nu) \le \sup_{V\ \hat{\nu}\text{-normalized}} \sup_{M_V} \bigg[\int \log(\tilde{U})\, dM_V \bigg] = \sup_{M \text{ Haar invariant}} \int \log(\tilde{U})\, dM.$$

On the other hand, choosing $V = U - \log(\tilde{U})$, we obtain

$$P_{\hat{\nu}}(\nu) \ge \sup_{M_{U-\log\tilde{U}}} \int \log(\tilde{U})\, dM_{U-\log\tilde{U}}.$$

Since $\tilde{U}$ is constant on classes, we invoke Corollary 22 to conclude the proof. □

Remark 39. If there exists $y_0 \in \Omega$ satisfying $\sup_{y\in\Omega} [\log(\tilde{U}(y))] = \log(\tilde{U}(y_0))$, then, taking $\mu = \delta_{y_0}$ and applying Theorem 21, there exists a Haar invariant probability $M_{U-\log(\tilde{U})}$ satisfying

$$P_{\hat{\nu}}(\nu) = \int \log(\tilde{U})\, d\delta_{y_0} = \int \log(\tilde{U})\, dM_{U-\log(\tilde{U})}.$$

In this case, if Λ is the transverse measure associated to $M_{U-\log(\tilde{U})}$ by Theorem 31, then Λ is an equilibrium for ν.

We observe that $P_{\hat{\nu}}(\cdot)$ plays the role of the Legendre transform of $-h_{\hat{\nu}}(\cdot)$. As $-h$ is convex it is natural to expect an involution, that is,

$$-h_{\hat{\nu}}(\Lambda) = \sup_{\nu}[\Lambda(\nu) - P_{\hat{\nu}}(\nu)]$$

or, equivalently,

$$h_{\hat{\nu}}(\Lambda) = \inf_{\nu}[-\Lambda(\nu) + P_{\hat{\nu}}(\nu)].$$

PROPOSITION 40. *The $\hat{\nu}$-pressure of the transverse function ν and the entropy of the Haar invariant transverse probability Λ are related by the expression:*

$$h_{\hat{\nu}}(\Lambda) = \inf_{\nu}[-\Lambda(\nu) + P_{\hat{\nu}}(\nu)].$$

Proof. We observe that for any ν, by definition of pressure,

$$h_{\hat{\nu}}(\Lambda) \le [-\Lambda(\nu) + P_{\hat{\nu}}(\nu)].$$

It follows that

$$h_{\hat{\nu}}(\Lambda) \le \inf_{\nu}[-\Lambda(\nu) + P_{\hat{\nu}}(\nu)].$$

On the other hand, if Λ has modulus $e^{V(y)-V(x)}$, where V is $\hat{\nu}$-normalized, then, taking $\mu^y(dx) = V(x)\hat{\nu}^y(dx)$, we obtain from Theorem 33 and Proposition 36 that

$$h_{\hat{\nu}}(\Lambda) = -\Lambda(\mu) \quad \text{and} \quad P_{\hat{\nu}}(\mu) = 0.$$

Therefore,

$$h_{\hat{\nu}}(\Lambda) = -\Lambda(\mu) + P_{\hat{\nu}}(\mu) \ge \inf_{\nu}[-\Lambda(\nu) + P_{\hat{\nu}}(\nu)]. \qquad \square$$

5. *Examples*

5.1. *Entropy and pressure in the XY model.*

We consider the hypotheses and notation of §3. In this case it is easy to see that

$$H_V(f)(y) = \mathcal{L}_V(f)(\sigma(y)),$$

where $\mathcal{L}_V$ is the classical Ruelle operator. We say that a bounded and measurable potential V is normalized if it satisfies $\mathcal{L}_V(1) = 1$.

PROPOSITION 41. *V is normalized if and only if V is Haar normalized.*

Proof. If V is normalized then, for any $y \in \Omega$,

$$\int e^{V(x)}\, \hat{\nu}^y(dx) = H_V(1)(y) = \mathcal{L}_V(1)(\sigma(y)) = 1,$$

which proves that V is Haar normalized.

If V is Haar normalized then, for any given $y \in \Omega$, we choose $z \in \Omega$ such that $\sigma(z) = y$. It follows that

$$L_V(1)(y) = L_V(1)(\sigma(z)) = H_V(1)(z) = 1 \quad \text{for all } y \in \Omega. \qquad \square$$

PROPOSITION 42. *If a probability measure M on Ω satisfies $\mathcal{L}_V^*(M) = M$, for some normalized Hölder function V, then it is Haar invariant.*

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Proof. From Propositions 11 and 17 we conclude that M is Haar invariant with Jacobian e^V. □

The identification of σ-invariant probabilities and Haar invariant probabilities is, in general, false. There exist Haar invariant probabilities which are not invariant for the shift map and vice versa, as the next example shows.

Example 43. Let μ be any probability measure on Ω, such that the push-forward probability ν defined from $\int f\, d\nu := \int f \circ \sigma\, d\mu$ is not invariant for the shift map†. By Theorem 21, there exists a Haar invariant probability M such that $\int f \circ \sigma\, dM = \int f \circ \sigma\, d\mu$, for any measurable and bounded function f (because $f \circ \sigma$ is constant on classes). If M were invariant for the shift map, then

$$\int f\, dM = \int f \circ \sigma\, dM = \int f \circ \sigma\, d\mu = \int f\, d\nu,$$

which is a contradiction, because ν is not invariant.

On the other hand, there are shift-invariant probabilities which are not Haar invariant for a fixed Haar system $(G, \hat{\nu})$. For instance, consider $M = \delta_{0^\infty}$, where $0^\infty = (0, 0, 0, \ldots) \in K^{\mathbb{N}} = [0, 1]^{\mathbb{N}}$. In this case, supposing by contradiction that M is Haar invariant, where, for each x, $\hat{\nu}^x$ is identified with the Lebesgue measure m on $[0, 1]$, there must be a measurable and bounded function V such that, for any measurable and bounded function f,

$$\int f(a0^\infty) e^{V(a0^\infty)}\, dm(a) = f(0^\infty).$$

This is impossible because, as functions of f, the value on the right-hand side can be easily changed without affecting the mean of the left-hand side.

PROPOSITION 44. *Let M be the equilibrium measure for a normalized function V. Let Λ be the transverse measure defined by (9). Then $h_{\hat{\nu}}(\Lambda) = -\int V\, dM$.*

Proof. The claim easily follows from Theorem 33. □

The above proposition shows that the Haar entropy is a natural generalization for Haar systems of the Kolmogorov–Sinai entropy. We refer to [**23**] for a discussion of Kolmogorov–Sinai entropy and the concept of negative entropy for the generalized XY model.

The concept of pressure is very different when considered the thermodynamic formalism setting instead of Haar systems. As an example, note that in thermodynamic formalism we have

$$P_\sigma(f \circ \sigma) = P_\sigma(f) = \sup_{\mu \text{ invariant}} \left[\int f\, d\mu + h_m(\mu) \right],$$

for any continuous function f. On the other hand, in Haar systems, if we consider a transverse function ν in the form $\nu = U\hat{\nu}$, where $U = f \circ \sigma$, then U is constant on classes. It follows that $\log(\tilde{U}) = U$ and, from Proposition 38, we obtain

$$P_{\hat{\nu}}((f \circ \sigma)\hat{\nu}) = P_{\hat{\nu}}(U\hat{\nu}) = \sup_{y \in \Omega} [U(y)] = \sup_{y \in \Omega} [f(\sigma(y))] = \sup_{z \in \Omega} [f(z)].$$

† For instance, $\mu = \delta_{(1,1,0,0,0,0\ldots)}$.

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But if we consider the transverse function $f\hat{\nu}$,

$$P_{\hat{\nu}}(f\hat{\nu}) = \sup_{y\in\Omega}\left[\log\int e^{f(x)}\,\hat{\nu}^y(dx)\right] = \sup_{y\in\Omega}\left[\log\int e^{f(a,y_2,y_3,\ldots)}\,dm(a)\right].$$

The main reason for the difference between the two kinds of pressure is in some sense described in Example 43. When we consider the Haar entropy for a different set of probabilities and then consider the pressure, as a 'Lengendre transform' of $-h$ (which is defined over this different set), it is natural to get a different meaning for pressure.

5.2. *Haar systems dynamically defined.* Assume that Ω is a complete and separable metric space and $\mathcal{B}$ denotes the Borel sigma-algebra on Ω. In this section we generalize results from §3.

Suppose that $T:\Omega\to\Omega$ is a continuous map and consider the groupoid G defined by the equivalence relation $x\sim y$ if and only if $T(x)=T(y)$. In this way any class is closed and any transverse function $\hat{\nu}$ (which is a probability on each class) can be identified as a choice of a probability m_y over the set $T^{-1}(y)$, for each $y\in\Omega$. For any measurable and bounded function $U:\Omega\to\mathbb{R}$ and transverse function $\hat{\nu}$, we define the generalized Ruelle operator

$$\mathcal{L}_U(f)(y) = \int_{T(x)=y} e^{U(x)} f(x)\nu^x(dx) = \int_{T(x)=y} e^{U(x)} f(x)\,dm_y(x),$$

where $\nu^x=\nu^z$ if and only if $T(x)=T(z)$, that is, ν^x is a probability m_y, if $x\in T^{-1}(y)$.

We say that a measurable and bounded function V is normalized if

$$\int_{T(x)=y} e^{V(x)}\nu^x(dx) = 1 \quad \text{for all } y\in\Omega.$$

PROPOSITION 45. *Under the above hypotheses and notation, suppose that an invariant probability M for T satisfies $\mathcal{L}_V^*(M)=M$, for some normalized (measurable and bounded) function V. Then $H_V^*(M)=M$, that is, M is Haar invariant.*

Proof. Observe that $H_V(f)(y)=\mathcal{L}_V(f)(Ty)$. Then

$$\begin{aligned}\int H_V(f)(y)\,dM(y) &= \int \mathcal{L}_V(f)(Ty)\,dM(y)\\ &\overset{M \text{ is } T \text{ invariant}}{=} \int \mathcal{L}_V(f)(y)\,dM(y) = \int f(y)\,dM(y).\end{aligned}$$ □

The definition of entropy $h_{\hat{\nu}}$ in this work can be applied to the case of any invariant measure M satisfying $\mathcal{L}_V^*(M)=M$, for some V normalized. Such M is associated with a transverse probability Λ by Theorem 31.

In §2 the *a priori* probability m_y is a fixed probability m independent of y in a natural way, because in that example the preimages of any point y are identified with a fixed set K, where $\Omega=K^{\mathbb{N}}$. Observe that for a general dynamic system there is not a natural identification of preimages of different points, that is, the sets $T^{-1}(y)$ and $T^{-1}(z)$ can be of quite distinct nature. One of the simplest examples of this are subshifts of finite type, where distinct points can have sets of preimages with different cardinalities. In the present general case, in contrast with the XY model (described before), it is natural to take as an *a priori* probability a general transverse function.

In contrast to Example 43, the next theorem shows that any T-invariant probability can be seen as a Haar invariant probability. Items (1) and (2) of the theorem say that μ^x is some kind of kernel, from the viewpoint of almost every point (M-a.e.) $x \in \Omega$ (see Definition 3). Item (3) says that this 'kernel' is a transverse function, and item (4) says that M is 'Haar invariant' with Jacobian $J = e^V = 1$. In the proof we use Rokhlin's disintegration theorem. A reference for this topic is [**35**, Ch. 5].

THEOREM 46. *Let Ω be a complete and separable metric space and $T : \Omega \to \Omega$ be a continuous map. Consider the groupoid G defined by the equivalence relation $x \sim z$ if and only if $T(x) = T(z)$. Then, for any fixed T-invariant probability M on Ω, there exists a family $\{\mu^x \mid x \in \Omega\}$ of probabilities on Ω satisfying the following conditions:*
(1) *μ^x has support on $[x]$ for M-a.e. $x \in \Omega$;*
(2) *for each measurable set $E \subseteq \Omega$, the map $x \to \mu^x(E)$ is measurable;*
(3) *$\mu^x = \mu^z$ for any $x, z \in \Omega$ satisfying $x \sim z$;*
(4) *$\int f(x)\, dM(x) = \int\int f(z)\, \mu^x(dz)\, dM(x)$ for any measurable and bounded function $f : \Omega \to \mathbb{R}$.*

Proof. Since Ω is a complete and separable metric space, there exists an enumerable base of open sets $A_1, A_2, A_3, \ldots$. This means that, for each point $x \in \Omega$ and open set U containing x, there exists some A_i satisfying $x \in A_i \subseteq U$. Let $\mathcal{P}$ be the partition of Ω defined in the following way: x and z belong to the same element of the partition if and only if $\chi_{A_i}(T(x)) = \chi_{A_i}(T(z))$, for any $i \in \mathbb{N}$. We observe that $\mathcal{P}$ is the partition of Ω in the classes of G, that is, two points x and z are in the same element of the partition $\mathcal{P}$ if and only if $T(x) = T(z)$. Indeed, clearly $x \sim y$ implies that x and y belong to the same element of the partition. Conversely, if $T(x) \neq T(y)$, then there exists an open set U such that $T(x) \in U$ and $T(z) \notin U$. It follows that, for some A_i, we have $T(x) \in A_i \subseteq U$ and $T(z) \notin A_i$, which proves that x and y belong to different elements of the partition.

We claim that $\mathcal{P}$ is a measurable partition. Indeed, we need only consider the partitions $\mathcal{P}_n$, $n \in \mathbb{N}$, defined in the following way: two points x and z belong to the same element of the partition $\mathcal{P}_n$ if and only if $\chi_{A_i}(T(x)) = \chi_{A_i}(T(z))$, for any $i \in \{1, \ldots, n\}$. Observe that $\mathcal{P}_n$ has 2^n elements,

$$\mathcal{P}_1 \prec \mathcal{P}_2 \prec \mathcal{P}_3 \prec \cdots ,$$

and $\mathcal{P} = \bigvee_{n \geq 1} \mathcal{P}_n$. This proves the claim.

Recall that two points x and z are on the same element of the partition $\mathcal{P}$ if and only if $T(x) = T(z)$. We denote by P_y the element of the partition $\mathcal{P}$ that contains the preimages of y, that is, $P_y = \{x \in \Omega \mid T(x) = y\}$. Observe that we can identify Ω with $\mathcal{P}$ from $y \to P_y$. We define $\pi : \Omega \to \mathcal{P}$ by the following rule: $\pi(x)$ is the element of the partition $\mathcal{P}$ that contains x. In this way $\pi(x) = P_y$ if and only if $T(x) = y$. We say that $\mathcal{Q} \subseteq \mathcal{P}$ is measurable if the set $\pi^{-1}(\mathcal{Q})$ is a measurable subset of Ω. For a given invariant probability M on Ω, we associate a probability $\hat{M}$ on $\mathcal{P}$ by

$$\hat{M}(\mathcal{Q}) := M(\pi^{-1}(\mathcal{Q})),$$

where $\mathcal{Q} \subseteq \mathcal{P}$ is measurable. Observe that, using the identification $y \to P_y$, for any given measurable subset $Q \subset \Omega$ we can associate the measurable subset $\mathcal{Q} = \{P_y \mid y \in Q\}$ of $\mathcal{P}$. Furthermore, since M is T-invariant,

$$\hat{M}(\mathcal{Q}) = M(\pi^{-1}(\mathcal{Q})) = M(\pi^{-1}(\{P_y \mid y \in Q\}) = M(T^{-1}(Q)) = M(Q).$$

As the metric space Ω is complete and separable and the partition $\mathcal{P}$ is measurable, by Rokhlin's disintegration theorem (see [**35**]), any invariant probability M admits a disintegration, which is a family of probabilities $\{m_P \mid P \in \mathcal{P}\}$ on Ω satisfying, for any measurable set $E \subset \Omega$:

(1) $m_P(P) = 1$, for $\hat{M}$-a.e. $P \in \mathcal{P}$;
(2) $P \to m_P(E)$ is measurable;
(3) $M(E) = \int m_P(E)\, d\hat{M}(P)$.

Using the identification $y \to P_y$, we obtain a family of probabilities $\{m_y \mid y \in \Omega\}$ on Ω satisfying, for any measurable set $E \subset \Omega$:

(1) $m_y(T^{-1}(y)) = 1$ for M-a.e. $y \in \Omega$;
(2) $y \to m_y(E)$ is measurable;
(3) $M(E) = \int m_y(E)\, dM(y)$.

We define, for each $x \in \Omega$, the probability $\mu^x := m_{T(x)}$, that is, $\mu^x := m_y$ if $x \in T^{-1}(y)$. By construction, $x \sim z$ implies $\mu^x = \mu^z$. As M is T-invariant, it follows from (1) that $\mu^x([x]) = 1$, for M-a.e. $x \in \Omega$. The sets $[x] = T^{-1}\{y\}$ are closed, because T is continuous, therefore μ^x has support on $[x]$ for M-a.e. $x \in \Omega$. Furthermore, for a fixed measurable set E, since $x \to y = T(x)$ and $y \to m_y(E)$ are measurable maps, we obtain that $x \to \mu^x(E)$ is measurable.

In order to conclude the proof it remains to prove item (4) of the theorem. For any measurable and bounded function $f : \Omega \to \mathbb{R}$ we have, from (3) and using the fact that M is T-invariant,

$$\begin{aligned}\int f(x)\, dM(x) &= \iint f(z)\, dm_x(z)\, dM(x) = \iint f(z)\, dm_{T(x)}(z)\, dM(x)\\ &= \iint f(z)\, \mu^x(dz)\, dM(x).\end{aligned}$$

□

In the next corollary we suppose that any class is finite and we consider the transverse function which is the counting measure on each class. We remark that it is a finite measure but not a probability. In any case, an easy normalization is sufficient to obtain a probability, that is, to replace $\sum_{T(x)=T(z)}$ by $(1/\#[x]) \sum_{T(x)=T(z)}$ and $J(z)$ by $\tilde{J}(z) := (\#[x]) \cdot J(z)$.

COROLLARY 47. *Suppose that M is a complete and separable metric space and suppose that the continuous map $T : M \to M$ is such that any point y has a finite number of preimages. Then, for any T-invariant probability M, there exists a bounded function J defined for M-a.e. $x \in \Omega$ (a Haar Jacobian of M) satisfying, for M-a.e. $x \in \Omega$, $\sum_{T(x)=T(z)} J(z) = 1$ and*

$$\int f(x)\, dM(x) = \int \sum_{T(x)=T(z)} J(z) f(z)\, dM(x).$$

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Proof. Using the notation of the proof of the above theorem, since for M-a.e. x the probability μ^x has support in the finite set $[x]$, there exists, for any such x, a function J^x defined over the class of x satisfying $\mu^x(\{z\}) = J^x(z)$, for any $z \sim x$. We define a function J almost everywhere by $J(z) = J^x(z)$ if $x \sim z$, and $\mu^x([x]) = 1$ otherwise. The images of J belong to $[0, 1]$, clearly $\sum_{T(x)=T(z)} J(z) = 1$ and, furthermore, from the above theorem and the definition of J, it follows that

$$\int f(x)\, dM(x) = \iint f(z)\, \mu^x(dz)\, dM(x) = \int \sum_{T(x)=T(z)} J(z) f(z)\, dM(x). \qquad \square$$

Example 48. If $\Omega \subseteq \{1, \ldots, d\}^{\mathbb{N}}$ is a subshift of finite type, defined from an aperiodic matrix, and M is an invariant probability for the shift map σ, then for M-a.e. $x \in \Omega$, $x = (x_0, x_1, x_2, \ldots)$, there exists

$$J(x) = \lim_{n\to\infty} \frac{M([x_0, x_1, \ldots, x_n])}{M([x_1, x_2, \ldots, x_n])}.$$

In thermodynamic formalism this function J (called the Jacobian of the measure, or sometimes the inverse of the Jacobian) is M-integrable, and for any measurable and bounded function $f : \Omega \to \mathbb{R}$ it satisfies

$$\int f(y)\, dM(y) = \int \sum_{\sigma(z)=y} J(z) f(z)\, dM(y).$$

As M is σ-invariant, for any measurable and bounded function f,

$$\int f(y)\, dM(y) = \int \sum_{\sigma(z)=\sigma(y)} J(z) f(z)\, dM(y).$$

Therefore, M is also Haar invariant with Haar Jacobian J.

The Kolmogorov–Sinai entropy of M is given by

$$h_\sigma(M) = -\int \log(J)\, dM,$$

which is compatible with the definition of Haar entropy of M, introduced in this work.

Note that in the case of the groupoid of §3 (taking V continuous and positive and assuming that the equivalence classes are finite), if ρ is an eigenprobability for the operator H_V^* associated to the eigenvalue $\lambda > 0$, then condition (3) is true for any cylinder set B. Indeed, $\lambda \int I_B d\rho = \int \mathcal{L}_V\,(I_B)(\sigma)\, d\rho$.

5.3. *Extremal cases.* In this section we suppose that Ω is measurable and consider as examples two extremal cases: (1) the case where $[x] = \Omega$, for all $x \in \Omega$; and (2) the case where $[x] = \{x\}$, for all $x \in \Omega$. We will explore in these examples the meaning of the theoretical results we have obtained. In this procedure we will recover some classical concepts which are well known in the literature. This shows that our reasoning is quite justifiable.

Example 49. Consider the case $[x] = \Omega$ for any $x \in \Omega$, that is, $x \sim y$, for any $x, y \in \Omega$. In this case, the transverse functions are the measures on Ω and we fix a probability m (that plays the role of the transverse function $\hat{\nu}$ on Ω). The Haar system (G, m) will remain fixed in this example.

A function V is Haar normalized if

$$\int e^{V(x)}\, dm(x) = 1.$$

For any function f, we have that $H_U(f)$ is constant and equal to $\int e^{U(x)} f(x)\, dm(x)$. A probability M is Haar invariant with Jacobian e^V if and only if $dM = e^V\, dm$, because $H_V^*(M) = M$ means that

$$\int f(x) e^{V(x)} dm(x) = \int f(y)\, dM(y) \quad \text{for all } f.$$

For a fixed Haar invariant probability M with Jacobian e^V, we associate the transverse measure Λ acting on measures as

$$\Lambda(\nu) = \int e^{V(x)}\, d\nu(x)\, dM(y) = \int e^{V(x)}\, d\nu(x).$$

In this way, it is more natural to consider that we associate to a Haar normalized function V the above Λ, which is the unique transverse probability for the modular function $\delta(x, y) = e^{V(y)-V(x)}$.

The entropy of Λ associated to V is

$$h_m(\Lambda) = -\int V(x)\, dM(x) = -\int V(x) e^{V(x)}\, dm(x).$$

If we denote $P(x) = e^{V(x)}$, then

$$h_m(\Lambda) = -\int P(x) \log(P(x))\, dm(x),$$

which is a classical expression for the entropy when there is no dynamics.

The pressure of a measure ν satisfies

$$P_m(\nu) = \sup_{V \text{ normalized}} \left[\int e^{V(x)}\, d\nu(x) - \int V(x) e^{V(x)}\, dm(x) \right].$$

Then

$$P_m(\nu) = \sup_{P>0,\ \int P(x)\, dm(x)=1} \left[\int P(x)\, d\nu(x) - \int P(x) \log P(x)\, dm(x) \right].$$

If $d\nu = U dm$, then

$$P_m(U\ m) = \sup_{P>0,\ \int P(x)\, dm(x)=1} \left[\int U(x)\ P(x)\, dm(x) - \int \log(P(x))\ P(x)\, dm(x) \right]$$

$$\overset{\text{Proposition 38}}{=} \log \int e^{U(x)}\, dm(x).$$

Note that $\int e^{U(x)}\, dm(x)$ (after normalization) is a classical expression for the Gibbs probability for the potential U (when there is no dynamics).

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Example 50. Suppose that $[x] = \{x\}$ for any $x \in \Omega$, that is, $x \sim y$ if and only if $x = y$. In this case any transverse function is a function ν. Indeed, for each x, we associate the class $[x] = \{x\}$, and then we assign to it a positive number $\nu(x)$. We fix as $\hat{\nu}^y$ the Dirac delta measure on $\{y\}$, for each $y \in \Omega$, that is, $\hat{\nu}$ is the constant function 1. We consider fixed the Haar system $(G, \hat{\nu})$.

The unique Haar normalized function is $V \equiv 0$ and any probability M on Ω is Haar invariant with Jacobian $e^V = 1$. For any function U, we have $\log(\tilde{U}) = U$ and $U - \log(\tilde{U}) = 0 = V$.

For each probability M, we associate a transverse measure Λ by

$$\Lambda(\nu) = \iint e^{V(x)} \nu^y(dx)\, dM(y) = \int \nu(y)\, dM(y).$$

On the other hand, as $[x] = \{x\}$, the unique modular function is $\delta(x, x) = 1$. Then any transverse measure has the above form.

The entropy of any transverse probability Λ is equal to

$$h_{\hat{\nu}}(\Lambda) = -\int V(x)\, dM(x) = 0,$$

and the pressure of a transverse function ν is given by

$$P_\mu(\nu) = \sup_{\Lambda}[\Lambda(\nu) + h_{\hat{\nu}}(\Lambda)] = \sup_{M \text{ probability}} \int \nu(y)\, dM(y) = \sup_y \ \nu(y).$$

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