

THE EVOLUTIONARY CODE CESAM: NUMERICAL TECHNIQUES

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ABSTRACT

CESAM is a consistent set of programs and routines designed for the calculations of stellar evolution. Until now it allows the computation of the evolution from PMS or ZAMS to helium flash for stellar masses of some solar mass. It is constructed in such a way that all the physics works as external routines. The numerical techniques are based on the B-spline formalism. This formalism used both for the integration of the differential equations and for 1D and 2D interpolation schemes of various tables of physical data.

I - OVERALL STRUCTURE

CESAM has been implemented by many contributors, get together in the CNRS GDR 131. The set of partial differential equations of stellar evolution is split as usual into two ordinary differential problems: the quasi-static model, a two-point boundary values problem, and an initial value problem, the evolution of the chemical composition in time. At each time step they are solved in turn until convergence.

CESAM treats the core and the envelope as a whole system, thus avoiding the usual matching, at some level, of a core and an envelope solutions obtained with different algorithms. The outer boundary conditions of the internal structure are implemented in the $P(R, L)$, $T(R, L)$, $M(R, L)$ form, for respectively the pressure, the temperature and the mass, R and L being the external radius and luminosity of the envelope; these conditions are applied deep enough in the atmosphere so that the medium is already optically thick.

II - STATIC MODEL: B-SPLINE/COLLOCATION TECHNIQUE

The equations of the two points value problem are solved using the spline/collocation technique. In this integration method, originated by De Boor (1978), the coordinates of each unknown f_j , $1 \leq j \leq n_e$, of a system of n_e ordinary differential equations of order 1 are projected as $f_j(x) = \sum_{i=1}^M f_{i,j} N_i^m(x)$ on the B-spline basis $\{N_i^m\}_{i=1}^M$ of order m , on a nodal vector - to be precised - constructed on the mesh points x_1, x_2, \dots, x_n . Then the derivatives become $f_j'(x) = \sum_{i=1}^M f_{i,j} \frac{dN_i^m(x)}{dx}$ and one has similar equations for the boundary conditions. The new unknowns, the reals $f_{i,j}$, are derived by writing that the equations are satisfied at an *ad hoc* number of so-called collocation points which are not necessarily the mesh points, as in finite difference methods. The basis being local, one has to deal with a block diagonal jacobian matrix for solving, with the Newton-Raphson method, the implicit set of stellar structure differential equations.

Moreover, as in Gaussian integration, with a judicious choice for the nodal vector and for the collocation points there is "superconvergence" at the mesh points where the order of precision of the solution is higher than the order of the B-splines. An another advantage of the method is the ability of retrieving exactly the numerical solution at any point simply by using the projections on the B-splines. That proves very convenient in

the calculation of acoustic eigenmodes.

With the B-spline formalism it is possible to deal with discontinuities without too much difficulties, that simplify the calculations with convective mixing.

During the time evolution, according to the changes of the dependent variables, an automatic mesh refinement scheme optimizes the grid by insuring constant jumps from a mesh point to the next of a so-called "mesh distribution function" (Press *et al.*, 1986).

Following Eggleton(1971) the set of dependent variables: $\ln P$, $\ln T$, R^2 , $L^{2/3}$, with respect to $M^{2/3}$ avoids the central singularity of the equations of stellar structure and it can be shown (Morel, 1992) that it also provides an optimal numerical accuracy. On the other hand, the $2/3$ power of the luminosity – which avoids a singularity of the derivative of a square root on the vicinity of the origin – is not adapted to negative values of L . With CESAM the less accurate set of dependent variables: $\ln P$, $\ln T$, M , L with respect to R can be also used; there is no central singularity but an external free limit which is managed without difficulty by the automatic mesh point allocation scheme.

For the core and the envelope (calculated as a whole) a fixed number $n = 300$ of grid points is usually used with B-splines of order $m = 2$ that corresponds to $599 = 300 \times 2 - 1$ collocation points, hence the solution is of fourth-order accuracy for the quasi-static model. For the Sun, calculations have shown that higher number of mesh points or higher order for the B-spline does not lead to significant improvement of the internal accuracy (see Table I).

Many performing algorithms used in CESAM for B-spline calculations are taken from Schumaker(1981).

The three functions $P(R, L)$, $T(R, L)$, $M(R, L)$ which serve as outer boundary conditions are deduced from an atmosphere model, together with the derivatives which are needed for the iterative Newton-Raphson resolution of the implicit equations. This atmosphere model is integrated separately from the interior stratification, again with the B-spline/collocation procedure for arbitrary given $T(\tau)$ laws.

III – EVOLUTION IN TIME OF THE CHEMICAL COMPOSITION

The stiffly stable implicit BDF scheme (Hairer and Wanner 1991) is employed for the numerical integration of the set of differential equations governing the time evolution of the chemical species. The implicit equations are solved with the iterative Newton-Raphson algorithm. The choice of this scheme is guided by its properties of stiff-stability, and also because it is strictly conservative for all linear sets of the abundances. To ensure sufficient precision, one limits the changes of the chemical species during a time step. Only formula with accuracy order up to 2 are used. Due to the presence of the convective mixing the higher order schemes are almost useless.

In convective zones one assumes that the time scale of the turbulent eddies is less than the nuclear time scale, hence the time integration is performed on the mixed abundances time derivatives. A discontinuity is set on chemical composition distribution at each limit of growing convective zones. The B-spline formalism is also used for the interpolation of the chemical composition.

IV – INTERPOLATION TECHNIQUES AND ACCURACY

Without a significant loss of accuracy, tabulations and interpolations of physical data avoid important amount of calculations and allow an improvement of the numerical stability. Interpolation programs and routines with Lagrange, B-spline and Hermite schemes in 1D and 2D have been developed consistently with the numerical techniques already outlined. CESAM uses a 2D tensorial B-spline interpolation either in $(\ln T, \ln \rho)$ (LAOL tables) or in $(T_6, \ln \rho/T_6)$ (OPAL tables) and a 1D Lagrange scheme

in chemical composition for the opacities, a 2D B-spline or Hermite scheme with respect to $(\ln P, \ln T)$ and a 1D five order polynomial fit at Chebychev points with respect to the chemical composition for the equation of state, a 1D B-spline piecewise polynomial scheme for the thermonuclear reactions rates. One advantage of interpolation with the B-spline formalism is the control of the order of accuracy by only one parameter but convenient dual basis are not available as it does for Hermite and Lagrange interpolations.

TABLE I

n	m	X	α	$\rho_c (\text{g cm}^{-3})$	$T_c (10^6 \text{K})$	X_c
300	2	0.70617	1.8982	152.676	15.5492	0.34233
		$\Delta X/X$	$\Delta \alpha/\alpha$	$\Delta \rho_c/\rho_c$	$\Delta T_c/T_c$	$\Delta X_c/X_c$
150	3	$1. \cdot 10^{-6}$	$7.7 \cdot 10^{-5}$	$7.9 \cdot 10^{-6}$	$-3.2 \cdot 10^{-5}$	$1.3 \cdot 10^{-4}$
300	3	$1. \cdot 10^{-6}$	$7.7 \cdot 10^{-5}$	$6.7 \cdot 10^{-6}$	$-1. \cdot 10^{-6}$	$6. \cdot 10^{-6}$
600	3	$2.4 \cdot 10^{-5}$	$-3.2 \cdot 10^{-6}$	$0.4 \cdot 10^{-6}$	$2.3 \cdot 10^{-5}$	$2. \cdot 10^{-4}$

The accuracy of CESAM have been only investigated for calibrated Solar models. Internal accuracy is estimated from differences between models calculated with different number of shells n and with different orders of B-spline m . A comparison of these models is given in Table I where the first line gives the initial hydrogen abundance, the mixing length, the central density, temperature and hydrogen abundance for calibrated models with $n = 300$ and $m = 2$. The other lines give the relative difference of the same quantities for the other models. Solar models computed by CESAM with simplified but fixed physics have been submitted to the SMC comparison organized by GONG, with respect to the other models submitted they present relative differences smaller than 10^{-4} (Christensen Dalsgaard, 1991; Gabriel, 1991).

A more detailed discussion of the accuracy for more realistic solar models is given in Morel *et al.* (1990, 1992).

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REFERENCES

- Christensen-Dalsgaard J. 1991, *Challenges to Theories of the Structure of Mode rate-Mass Stars*, D.O. Gough, J. Toomre (Eds), Springer Verlag, p 11-36
- De Boor C. 1978, *A Practical Guide to Splines* (Springer; third ed. 1985)
- Eggleton, P. 1971 *Mon. Not. R. astr. Soc.* **151** 351-364
- Gabriel M. 1991, *Challenges to Theories of the Structure of Mode rate-Mass Stars*, D.O. Gough, J. Toomre (Eds), Springer Verlag, p 51-55
- Hairer E., Wanner G. 1991, *Solving ordinary differential equations II*. Springer Verlag
- Morel P., Provost J., Berthomieu G. 1990, *Solar Physics* **128**, 7
- Morel P., Berthomieu G., Provost J., Lebreton Y. 1992, these proceedings
- Morel P., 1992 in preparation
- Press W.H., Flannery B.P., Teukolsky S.A., Vetterling W.T. 1986 *Numerical Recipes*, Cambridge University Press, Cambridge
- Schumaker L. 1981, *Splines Functions : Basic Theory*, John Wiley.