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Multiple solutions for semilinear elliptic equations in unbounded cylinder domains

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In this paper, we show that if $b(x) \geqslant b^\infty > 0$ in $\bar{\Omega}$ and there exist positive constants C, δ, R_0 such that

$$b(x) \geqslant b^\infty + C\exp(-\delta|z|) \quad \text{for } |z| \geqslant R_0, \quad \text{uniformly for } y \in \varpi,$$

where $x = (y, z) \in \mathbf{R}^N$ with $y \in \mathbf{R}^m$, $z \in \mathbf{R}^n$, $N = m + n \geqslant 3$, $m \geqslant 2$, $n \geqslant 1$, $1 < p < (N+2)/(N-2)$, $\omega \subseteq \mathbf{R}^m$ a bounded $C^{1,1}$ domain and $\Omega = \omega \times \mathbf{R}^n$, then the Dirichlet problem $-\Delta u + u = b(x)|u|^{p-1}u$ in Ω has a solution that changes sign in Ω, in addition to a positive solution.

1. Introduction

In this paper, we will study the existence of solutions of semilinear elliptic problem

$$\left.\begin{aligned} -\Delta u + u = b(x)|u|^{p-1}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \quad u \not\equiv 0, \end{aligned}\right\} \tag{1.1}$$

where $N = m+n \geqslant 3$, $m \geqslant 2$, $n \geqslant 1$, $1 < p < (N+2)/(N-2)$, $\omega \subseteq \mathbf{R}^m$ a bounded $C^{1,1}$ domain, $\Omega = \omega \times \mathbf{R}^n$, $b(x)$ is a positive, bounded and continuous function on $\bar{\Omega}$. Moreover, $b(x)$ satisfies assumption (H1) below.

(H1) $b(x) \geqslant b^\infty > 0$ in $\bar{\Omega}$, $b(x) \not\equiv b^\infty$ and

$$\lim_{|z|\to\infty} b(x) = b^\infty \quad \text{uniformly for } y \in \varpi.$$

It is well known that (1.1) has infinitely many solutions if Ω is bounded ($n = 0$ in our case) (see [15], and the references therein). Here, we only interest in unbounded domains ($n \geqslant 1$ in our case). If $\Omega = \mathbf{R}^n$ ($m = 0$ in our case), the existence of solutions of (1.1) has been investigated, among others, in [1–3, 6, 11, 12, 17] (where general nonlinearities are considered). In [17], Zhu has studied the multiplicity of solutions of (1.1). He has given the following result.

Assume that $N \geqslant 5$, $\lim_{|x|\to\infty} b(x) = b^\infty$, $b(x) \geqslant b^\infty$ and that there exist positive constants C, γ, R_0 such that $b(x) - b^\infty \geqslant C|x|^{-\gamma}$ for $|x| \geqslant R_0$. Then (1.1) has at least two pairs of non-trivial solutions.

His result is our particular case (see theorem 1.2). If $m \geqslant 1$, $n \geqslant 1$, that is, Ω is an unbounded cylinder, then Lions [11] used the concentration-compactness method to prove the existence of solutions concerning equation (1.1). In this paper, we use the concentration-compactness argument, due to Lions [11], to estimate the decay of solutions developed in [8], and some ideas of Cerami *et al*. [5] to prove the existence of another solution without constant sign.

Throughout this article, let $x=(y,z)$ be the generic point of $\boldsymbol{R}^N$ with $y\in\boldsymbol{R}^m$, $z\in\boldsymbol{R}^n$, $N=m+n\geqslant 3$, $m\geqslant 2$, $n\geqslant 1$, $1<p<(N+2)/(N-2)$, with ϕ the first positive eigenfunction of the Dirichlet problem $-\Delta$ in ω with eigenvalue λ_1. This paper is organized as follows. In § 2, we establish a decomposition lemma. In § 3, we establish some regularity lemmas and asymptotic behaviour of the solution of equation (1.1). In § 4, we prove some auxiliary lemmas and finally show the existence of another solution without constant sign.

We now state our main results.

THEOREM 1.1. *Assume that* $N=m+n\geqslant 3$, $m\geqslant 2$, $n\geqslant 1$, $b(x)$ *satisfies condition (H1) and there exist positive constants* C, δ, R_0 *such that*

$$b(x)\geqslant b^\infty+C\exp(-\delta|z|)\quad \text{for } |z|\geqslant R_0,\quad \text{uniformly for } y\in\varpi.$$

Then (1.1) has a solution that changes sign in unbounded cylinder domains in addition to a positive solution.

THEOREM 1.2 ($\Omega=\boldsymbol{R}^N$). *Assume that* $N\geqslant 3$, $b(x)$ *satisfies condition (H1) and there exist positive constants* C, δ, R_0 *such that*

$$b(x)\geqslant b^\infty+C\exp(-\delta|x|)\quad \text{for } |x|\geqslant R_0.$$

Then (1.1) has a solution that changes sign in $\boldsymbol{R}^N$ *in addition to a positive solution.*

2. Preliminaries and a decomposition lemma

In this paper, we always assume that Ω is an unbounded cylinder or $\boldsymbol{R}^N$ ($N\geqslant 3$), unless otherwise specified. Now we begin our discussion by giving some definitions and some known results. The energy functional of (1.1) is

$$I(u)=\frac{1}{2}\int_\Omega |\nabla u|^2+|u|^2-\frac{1}{p+1}\int_\Omega b(x)|u|^{p+1},\quad u\in H_0^1(\Omega).$$

We shall denote by u_0 the positive ground-state solution of (1.1), found in [11], if $b(x)$ satisfies condition (H1).

Consider the equation

$$\left.\begin{aligned} -\Delta u+u=b^\infty|u|^{p-1}u \quad \text{in } \Omega,\\ u>0 \quad \text{in } \Omega,\quad u\in H_0^1(\Omega),\end{aligned}\right\}\tag{2.1}$$

and its associated energy functional I^∞ defined by

$$I^\infty(u)=\frac{1}{2}\int_\Omega |\nabla u|^2+|u|^2-\frac{1}{p+1}\int_\Omega b^\infty|u|^{p+1},\quad u\in H_0^1(\Omega).$$

By [11] or [10], equation (2.1) has a ground-state solution $\bar u$.

Now we define

$$g(u)=\begin{cases}\dfrac{\int_\Omega b(x)|u|^{p+1}}{\int_\Omega |\nabla u|^2+|u|^2} & \text{if } u\in H_0^1(\Omega)\setminus\{0\},\\ 0 & \text{if } u\equiv 0,\end{cases} \tag{2.2}$$

$$\mathcal{M}_1=\{u\in H_0^1(\Omega)\mid g(u)=1\},$$
$$c_1=\inf\{I(u)\mid u\in\mathcal{M}_1\},$$
$$c_\infty=\inf\{I^\infty(u)\mid u\in H_0^1(\Omega),\ {I^\infty}'(u)u=0\}.$$

By [11], for c_1, c_∞, u_0, $\bar{u}$, we have the following,

$$\left.\begin{aligned} c_\infty&=I^\infty(\bar{u})=\sup_{t\geqslant 0}I^\infty(t\bar{u}),\\ c_1&=I(u_0)=\sup_{t\geqslant 0}I(tu_0)<c_\infty,\end{aligned}\right\} \tag{2.3}$$

provided condition (H1) holds.

We give the following decomposition lemma for later use.

PROPOSITION 2.1. *Let $\{u_k\}$ be a $(PS)_c$-sequence of I in $H_0^1(\Omega)$,*

$$\begin{aligned} I(u_k)&=c+o(1) && \text{as } k\to\infty,\\ I'(u_k)&=o(1) && \text{strongly in } H^{-1}(\Omega).\end{aligned}$$

Then there exist an integer $l\geqslant 0$, a sequence $\{x_k^i\}\subseteq \boldsymbol{R}^N$ of the form $(0,z_k^i)\in\Omega$ and functions $u,\bar{u}_i\in H_0^1(\Omega)$, $1\leqslant i\leqslant l$, such that, for some subsequence $\{u_k\}$, we have

$$u_k-\bigg(u+\sum_{i=1}^{l}\bar{u}_i(\cdot-x_k^i)\bigg)\to 0 \quad \text{as } k\to\infty,$$
$$c=I(u)+\sum_{i=1}^{l}I^\infty(\bar{u}_i),$$
$$-\Delta u+u=b(x)|u|^{p-1}u \quad \text{in } H^{-1}(\Omega),$$
$$-\Delta\bar{u}_i+\bar{u}_i=b^\infty|\bar{u}_i|^{p-1}\bar{u}_i \quad \text{in } H^{-1}(\Omega),\quad 1\leqslant i\leqslant l,$$
$$|x_k^i|\to\infty,\quad |x_k^i-x_k^j|\to\infty,\quad 1\leqslant i\neq j\leqslant l.$$

Proof. The proof can be obtained by using the arguments in [2] (also see [11, 12]). We omit the details. □

3. Asymptotic behaviour

In order to get the asymptotic behaviour of solutions of (1.1), we need the following lemmas.

LEMMA 3.1. *Let $\Omega\subseteq\boldsymbol{R}^N$ be a $C^{1,1}$ domain in $\boldsymbol{R}^N$ and let f satisfy the following condition.*

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(H2) *We have*

$$|f(u)| \leqslant C(|u| + |u|^p), \qquad 1 < p < \frac{N+2}{N-2}$$

for some positive constant C.

Let $u \in H_0^1(\Omega)$ *be a weak solution of equation* $-\Delta u + u = f(u)$. *Then* $u \in L^q(\Omega)$ *for* $q \in [2, +\infty)$.

Proof. The proof follows by the classical regularity theory based on a result of Brezis-Kato [4]. We will write it in detail for the reader's convenience.

For $s \geqslant 0, \ell \geqslant 1$, let $\varphi = \varphi_{s,\ell} = u \min\{|u|^{2s}, \ell^2\} \in H_0^1(\Omega)$. Since $u \in H_0^1(\Omega)$ is a weak solution of equation $-\Delta u + u = f(u)$. Then we have

$$\int_\Omega \nabla u \cdot \nabla \varphi = -\int_\Omega u\varphi + \int_\Omega f(u)\varphi.$$

Suppose $u \in L^{2s+p+1}(\Omega)$. Since f satisfies condition (H2), we obtain that

$$\begin{aligned}\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, \ell^2\} + 2s \int_{\{x\in\Omega \mid |u(x)|^s \leqslant \ell\}} |\nabla u|^2 |u|^{2s} \\ \leqslant \int_\Omega |u|^{2+2s} + C\int_\Omega |u|^{2+2s} + C\int_\Omega |u|^{2s+p+1} \\ \leqslant C.\end{aligned}$$

Now we conclude that

$$\int_{\{x\in\Omega \mid |u(x)|^s \leqslant \ell\}} |\nabla(|u|^{s+1})|^2 \leqslant C\int_\Omega |\nabla(u \min\{|u|^s, \ell\})|^2 \leqslant C$$

for any $\ell \geqslant 1$. Hence we may let $\ell \to \infty$ in order to derive $|u|^{s+1} \in H_0^1(\Omega)$. Note that $H_0^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega)$, so $u \in L^{((2s+2)N)/(N-2)}(\Omega)$.

Now let $s_0 = 0$ and $2s_i + p + 1 = (s_{i-1} + 1)(2N/(N-2))$ for $i = 1, 2, \ldots$. Then $u \in L^{2s_{i-1}+p+1}(\Omega)$ implies $u \in L^{2s_i+p+1}(\Omega)$. Also, it is easy to see that $s_i \to \infty$ as $i \to \infty$. Therefore, $u \in L^q(\Omega)$, $2 \leqslant q < \infty$. This completes the proof. □

Now we quote a global regularity for the unbounded $C^{1,1}$ domain Ω in [8].

LEMMA 3.2. *Let* $g \in L^2(\Omega) \cap L^q(\Omega)$ *for some* $q \in (2, \infty)$ *and* $u \in H_0^1(\Omega)$ *be a weak solution* $-\Delta u + u = g$ *in* Ω. *Then* $u \in W^{2,2}(\Omega) \cap W^{2,q}(\Omega)$.

First, we give a rough asymptotic behaviour of solution of (1.1) at infinity.

LEMMA 3.3. *Let* u *be a solution of (1.1). Then*

$$\lim_{|z|\to\infty} u(y, z) = 0 \quad \textit{uniformly for } y \in \omega.$$

Proof. Let u satisfy

$$-\Delta u + u = b(x)|u|^{p-1}u \quad \text{in } H^{-1}(\Omega).$$

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By lemma 3.1, we obtain $u \in L^q(\Omega)$ for $q \in [2, \infty)$. Hence $g(x) = b(x)|u|^{p-1}u \in L^q(\Omega)$ for $q \in [2, \infty)$. Then, by lemma 3.2, we have $u \in W^{2,q}(\Omega)$ for $q \in [2, \infty)$. By [8, lemma 2.10], $u \in C^1(\bar{\Omega})$ and there exists $C > 0$ such that, for any $r > 1$,

$$\|u\|_{L^\infty(\bar{B}_r^c)} \leqslant C\|u\|_{W^{2,N}(\bar{B}_r^c)},$$

where $\bar{B}_r^c = \{x = (y, z) \in \Omega \mid |z| > r\}$. Hence $\lim_{|z|\to\infty} u(y, z) = 0$ uniformly for $y \in \omega$. □

Finally, we give here a precise asymptotic behaviour for positive solutions of (1.1) at infinity. First, we consider the case of unbounded cylinders.

Proposition 3.4. *Let u be a positive solution of (1.1) in an unbounded cylinder $\Omega = \omega \times \boldsymbol{R}^n \subseteq \boldsymbol{R}^{m+n}$, $m \geqslant 2$, $n \geqslant 1$ and ϕ be the first positive eigenfunction of the Dirichlet problem $-\Delta\phi = \lambda_1\phi$ in ω. Then, for any $\varepsilon > 0$, there exist constants $C_\varepsilon, \tilde{C}_\varepsilon > 0$ such that*

$$\left.\begin{aligned} u(x) \leqslant C_\varepsilon \phi(y)\exp(-\sqrt{1+\lambda_1}|z|)|z|^{-(n-1)/2+\varepsilon} \\ \tilde{C}_\varepsilon \phi(y)\exp(-\sqrt{1+\lambda_1}|z|)|z|^{-(n-1)/2-\varepsilon} \leqslant u(x) \end{aligned}\right\} \quad \text{as } |z| \to \infty, \quad y \in \varpi. \tag{3.1}$$

Proof. We divide the proof into the following steps.

Step 1. First, we claim that, for any $\delta > 0$ with $0 < \delta < 1 + \lambda_1$, there exists $C > 0$ such that

$$u(x) \leqslant C\phi(y)\exp(-\sqrt{1+\lambda_1-\delta}|z|) \quad \text{as } |z| \to \infty, \quad y \in \varpi.$$

Without loss of generality, we may assume $\delta < 1$. Now, given $\delta > 0$, by lemma 3.3, we may choose R_0 large enough such that

$$b(x)u^p(x) \leqslant \delta u(x) \quad \text{for } |z| \geqslant R_0.$$

Let $q = (q_y, q_z)$, $q_y \in \partial\omega$, $|q_z| = R_0$ and B be a small ball in Ω such that $q \in \partial B$. Since $\phi(y) > 0$ for $x = (y, z) \in B$, $\phi(q_y) = 0$, $u(x) > 0$ for $x \in B$, $u(q) = 0$, by the strong maximum principle, $\partial\phi/\partial y(q_y) < 0$, $\partial u/\partial x(q) < 0$. Thus

$$\lim_{\substack{x\to q\\ |z|=R_0}} \frac{u(x)}{\phi(y)} = \frac{\partial u/\partial x(q)}{\partial\phi/\partial y(q_y)} > 0.$$

Note that $u(x)\phi^{-1}(y) > 0$ for $x = (y, z)$, $y \in \omega$, $|z| = R_0$. Thus $u(x)\phi^{-1}(y) > 0$ for $x = (y, z)$, $y \in \varpi$, $|z| = R_0$. Since $\phi(y)\exp(-\sqrt{1+\lambda_1-\delta}|z|)$ and $u(x)$ are $C^1(\overline{\omega \times \partial B_{R_0}(0)})$, if we set

$$\alpha_1 = \sup_{y\in\varpi, |z|=R_0} (u(x)\phi^{-1}(y)\exp(\sqrt{1+\lambda_1-\delta}R_0)),$$

then $\alpha_1 > 0$ and

$$\alpha_1\phi(y)\exp(\sqrt{1+\lambda_1-\delta}R_0) \geqslant u(x) \quad \text{for } y \in \varpi, \quad |z| = R_0.$$

Let

$$\Phi_1(x) = \alpha_1\phi(y)\exp(-\sqrt{1+\lambda_1-\delta}|z|) \quad \text{for } x \in \bar{\Omega}.$$

Then, for $|z| \geqslant R_0$, we have

$$\begin{aligned}\Delta(u-\Phi_1)(x)-(u-\Phi_1)(x) &= -b(x)u^p(x)+\left(\delta+\frac{\sqrt{1+\lambda_1-\delta}(n-1)}{|z|}\right)\Phi_1(x)\\ &\geqslant -\delta u(x)+\delta\Phi_1(x)\\ &= \delta(\Phi_1-u)(x).\end{aligned}$$

Hence $\Delta(u-\Phi_1)(x)-(1-\delta)(u-\Phi_1)(x) \geqslant 0$ for $|z| \geqslant R_0$.

The strong maximum principle implies that $u(x)-\Phi_1(x) \leqslant 0$ for $x=(y,z)$, $y \in \varpi$, $|z| \geqslant R_0$, and therefore we get the claim.

STEP 2. We claim that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$u(x) \leqslant C_\varepsilon \phi(y)\exp(-\sqrt{1+\lambda_1}|z|)|z|^{-(n-1)/2+\varepsilon} \quad \text{as } |z| \to \infty, \quad y \in \varpi.$$

Without loss of generality, we may assume that $0 < \varepsilon < \frac{1}{2}(n-1)$. Now, given $\varepsilon > 0$, let $m_\varepsilon = \frac{1}{2}(n-1)-\varepsilon$ and

$$h(z) = 2\varepsilon\sqrt{1+\lambda_1}|z|^{-m_\varepsilon-1}+m_\varepsilon(m_\varepsilon-n+2)|z|^{-m_\varepsilon-2}.$$

Now we choose $\delta > 0$ such that $\sqrt{1+\lambda_1} < p\sqrt{1+\lambda_1-\delta}$. Then, by step 1, there exist $R_0 > 0$, $C_1 > 0$ such that

$$u(x) \leqslant C_1\phi(y)\exp(-\sqrt{1+\lambda_1-\delta}|z|) \quad \text{for } y \in \varpi \text{ and } |z| \geqslant R_0.$$

This implies that there exists $C_2 > 0$ such that

$$b(x)u^p(x) \leqslant C_2\phi(y)\exp(-p\sqrt{1+\lambda_1-\delta}|z|) \quad \text{for } y \in \varpi \text{ and } |z| \geqslant R_0.$$

We can choose $R_1 > 0$ such that, for $|z| \geqslant R_1$,

$$h(z)\exp(-\sqrt{1+\lambda_1}|z|)-C_2\exp(-p\sqrt{1+\lambda_1-\delta}|z|) \geqslant 0.$$

As in step 1, if we set

$$\alpha_2 = \max_{y\in\varpi,|z|=R_1}(u(x)\phi^{-1}(y)\mathrm{e}^{\sqrt{1+\lambda_1}R_1}R_1^{m_\varepsilon}+1),$$

then $\alpha_2 > 0$.

Let

$$\Phi_2(x) = \alpha_2\phi(y)\exp(-\sqrt{1+\lambda_1}|z|)|z|^{-m_\varepsilon} \quad \text{for } x \in \bar{\Omega}.$$

For $x \in \Omega$, $|z| \geqslant R_1$, we have

$$\begin{aligned}&\Delta(u-\Phi_2)(x)-(u-\Phi_2)(x)\\ &\quad = -b(x)u^p(x)+h(z)\Phi_2(x)|z|^{m_\varepsilon}\\ &\quad \geqslant -C_2\phi(y)\exp(-p\sqrt{1+\lambda_1-\delta}|z|)+\alpha_2\phi(y)h(z)\exp(-\sqrt{1+\lambda_1}|z|)\\ &\quad \geqslant \phi(y)(h(z)\exp(-\sqrt{1+\lambda_1}|z|)-C_2\exp(-p\sqrt{1+\lambda_1-\delta}|z|))\\ &\quad \geqslant 0.\end{aligned}$$

Hence, by the maximum principle, we obtain that

$$\Phi_2(x) \geqslant u(x) \quad \text{for } y \in \varpi \text{ and } |z| \geqslant R_1.$$

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That is,

$$u(x) \leqslant \alpha_2 \phi(y) \exp(-\sqrt{1+\lambda_1}|z|)|z|^{-(n-1)/2+\varepsilon} \quad \text{for } y \in \varpi \text{ and } |z| \geqslant R_1.$$

STEP 3. Given $\varepsilon > 0$, let $\tilde{m}_\varepsilon = \frac{1}{2}(n-1) + \varepsilon$ and

$$g(z) = 2\varepsilon\sqrt{1+\lambda_1}|z|^{-1} + \tilde{m}_\varepsilon(\tilde{m}_\varepsilon - n + 2)|z|^{-2}.$$

We can choose $R_0 > 0$ such that $g(z) \geqslant 0$ for $|z| \geqslant R_0$. As in step 1, if we set

$$\beta = \inf_{y\in\varpi,\,|z|=R_0} (u(x)\phi^{-1}(y)\mathrm{e}^{\sqrt{1+\lambda_1}R_0} R_0^{\tilde{m}_\varepsilon}),$$

then $\beta > 0$ and

$$\beta\phi(y)\mathrm{e}^{-\sqrt{1+\lambda_1}|z|}|z|^{-\tilde{m}_\varepsilon} \leqslant u(x) \quad \text{for } y \in \varpi \text{ and } |z| = R_0.$$

Now let $\Psi(x) = \beta\phi(y)\mathrm{e}^{-\sqrt{1+\lambda_1}|z|}|z|^{-\tilde{m}_\varepsilon}$ for $x \in \bar{\Omega}$. If $x \in \Omega$, $|z| \geqslant R_0$, we have

$$\Delta(\Psi - u)(x) - (\Psi - u)(x) = g(z)\Psi(x) + b(x)u^p(x) \geqslant 0.$$

Then, by the maximum principle, we obtain that

$$u(x) \geqslant \Psi(x) \quad \text{for } y \in \varpi \text{ and } |z| \geqslant R_0.$$

That is,

$$u(x) \geqslant \beta\phi(y) \exp(-\sqrt{1+\lambda_1}|z|)|z|^{-(n-1)/2-\varepsilon} \quad \text{for } y \in \varpi \text{ and } |z| \geqslant R_0.$$

□

REMARK 3.5. For the case $b(x) \equiv b^\infty > 0$, we have that every positive solution of (2.1) has the same asymptotic behaviour as in proposition 3.4.

REMARK 3.6. From the above proof, we can deduce that $u(x)\phi^{-1}(y) > 0$ for $x = (y, z) \in \bar{\Omega}$. Hence, for any compact subset $K \subset \bar{\Omega}$, there exist $C_1, C_2 > 0$ such that $C_1\phi(y) \leqslant u(x) \leqslant C_2\phi(y)$ for $x = (y, z) \in K$.

For the case $\Omega = \boldsymbol{R}^N$ $(N \geqslant 3)$, the positive solutions of (1.1) also have a similar asymptotic behaviour at infinity.

PROPOSITION 3.7. *Let u be a positive solution of (1.1) in $\boldsymbol{R}^N$ $(N \geqslant 3)$. Then there exist $C_1, C_2 > 0$ such that*

$$C_1 \leqslant u(x)\mathrm{e}^{|x|}|x|^{(N-1)/2} \leqslant C_2 \quad \textit{for } x \in \boldsymbol{R}^N.$$

Proof. Consider the equation

$$\left.\begin{aligned} -\Delta u + u = M|u|^{p-1}u \quad \text{in } \boldsymbol{R}^N, \\ u > 0 \quad \text{in } \boldsymbol{R}^N, \quad u \in H^1(\boldsymbol{R}^N), \end{aligned}\right\} \tag{3.2}$$

where $M = \max_{x\in\boldsymbol{R}^N} b(x)$.

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We denote the unique positive solution of (2.1), (3.2) by $\bar{u}$, $\tilde{u}$, respectively (see [9]), and there exist constants $\bar{C}$, $\tilde{C}$ such that (see [2, 3, 7, 16])

$$\left.\begin{aligned} \bar{u}(x)|x|^{(N-1)/2}\mathrm{e}^{|x|} &\to \bar{C} > 0 \quad \text{as } |x| \to \infty, \\ \tilde{u}(x)|x|^{(N-1)/2}\mathrm{e}^{|x|} &\to \tilde{C} > 0 \quad \text{as } |x| \to \infty. \end{aligned}\right\} \tag{3.3}$$

By (1.1) and (2.1), we have

$$\begin{aligned} \int_{\boldsymbol{R}^N} [-\Delta(\bar{u} - u) + (\bar{u} - u)](\bar{u} - u)^- &= \int_{\boldsymbol{R}^N} (b^\infty \bar{u}^p - b(x)u^p)(\bar{u} - u)^- \\ &\leqslant \int_{\boldsymbol{R}^N} b(x)(\bar{u}^p - u^p)(\bar{u} - u)^- \\ &\leqslant 0. \end{aligned}$$

That is,

$$\|(\bar{u} - u)^-\|^2_{H^1(\boldsymbol{R}^N)} \leqslant 0, \quad \text{where } (\bar{u} - u)^- = \max\{-(\bar{u} - u), 0\}.$$

Hence $u \leqslant \bar{u}$ in $\boldsymbol{R}^N$. Similarly, $u \geqslant \tilde{u}$ in $\boldsymbol{R}^N$. By (3.3), we obtain that there exist $C_1, C_2 > 0$ such that

$$C_1 \leqslant u(x)\mathrm{e}^{|x|}|x|^{(N-1)/2} \leqslant C_2 \quad \text{for } x \in \boldsymbol{R}^N.$$

□

4. Multiplicity of solutions

Let $e_N = (0, 0, \dots, 0, 1) \in \boldsymbol{R}^N$ and let

$$\begin{aligned} \mathcal{M}_2 &= \{u \in H_0^1(\Omega) \mid g(u^+) = g(u^-) = 1\}, \\ \mathcal{N} &= \{u \in H_0^1(\Omega) \mid |g(u^+) - 1| < \tfrac{1}{2},\ |g(u^-) - 1| < \tfrac{1}{2}\}, \end{aligned}$$

where g is defined by (2.2), $u^+ = \max\{u, 0\}$ and $u^- = u^+ - u$,

$$c_2 = \inf\{I(u) \mid u \in \mathcal{M}_2\}.$$

Then we have the following lemma.

Lemma 4.1. *There exists a sequence $\{u_k\} \subset \mathcal{N}$ such that*

$$\left.\begin{aligned} I(u_k) &= c_2 + o(1) \quad \text{as } k \to \infty, \\ I'(u_k) &= o(1) \qquad\quad \text{strongly in } H^{-1}(\Omega). \end{aligned}\right\} \tag{4.1}$$

Proof. This lemma is similar to the one in [5] (or [17]) and can be proved similarly (see [17] for a detailed proof). We omit the details. □

The next lemma is the compactness result on energy level c_2.

Lemma 4.2. *Suppose that $\{u_k\} \subset \mathcal{N}$ satisfies (4.1) and*

$$0 < c_2 < c_1 + c_\infty.$$

Then $\{u_k\}$ has a subsequence converging strongly in $H_0^1(\Omega)$.

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Proof. From (4.1), it is easily to see that $\{u_k\}$ is bounded in $H_0^1(\Omega)$ and satisfies

$$\int_\Omega |\nabla u_k^\pm|^2 + |u_k^\pm|^2 - \int_\Omega b(x)|u_k^\pm|^{p+1} = o(1) \quad \text{as } k \to \infty. \tag{4.2}$$

By $\{u_k\} \subset \mathcal{N}$ and the Sobolev inequality, there exists β, independent of k, such that

$$\int_\Omega |\nabla u_k^\pm|^2 + |u_k^\pm|^2 > \beta > 0.$$

By proposition 2.1, there exist an integer $l \geqslant 0$, a sequence $\{x_k^i\} \subset \boldsymbol{R}^N$ of the form $(0, z_k^i) \in \Omega$, functions $u, \bar{u}_i \in H_0^1(\Omega)$, for $1 \leqslant i \leqslant l$, such that, for some subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$), we have

$$\begin{gathered}
\left\| u_k - u - \sum_{i=1}^{l} \bar{u}_i(\cdot - x_k^i) \right\|_{H_0^1(\Omega)} = o(1) \quad \text{as } k \to \infty, \\
c_2 = I(u) + \sum_{i=1} l I^\infty(\bar{u}_i), \\
-\Delta u + u = b(x)|u|^{p-1}u \quad \text{in } H^{-1}(\Omega), \\
-\Delta \bar{u}_i + \bar{u}_i = b^\infty |\bar{u}_i|^{p-1}\bar{u}_i \quad \text{in } H^{-1}(\Omega), \quad 1 \leqslant i \leqslant l, \\
|x_k^i| \to \infty, \quad |x_k^i - x_k^j| \to \infty \quad \text{as } k \to \infty, \quad 1 \leqslant i \neq j \leqslant l.
\end{gathered}$$

If $l \geqslant 2$, that is, $\bar{u}_1 \not\equiv 0$, $\bar{u}_2 \not\equiv 0$, then we obtain $I^\infty(\bar{u}_i) \geqslant I^\infty(\bar{u}) = c_\infty$ $(i = 1, 2)$, by (2.3), which implies a contradiction $c_2 \geqslant 2c_\infty > c_1 + c_\infty$, since $I(u) \geqslant 0$. Hence $l \leqslant 1$. Suppose $u \equiv 0$. Then $l = 1$ and

$$\|u_k - \bar{u}_i(\cdot - x_k^i)\|_{H_0^1(\Omega)} = o(1), \quad \text{as } k \to \infty.$$

From $|x_k^i| \to \infty$ and (4.2), we have

$$\int_\Omega |\nabla \bar{u}_1^\pm|^2 + |\bar{u}_1^\pm|^2 = \int_\Omega b^\infty |\bar{u}_1^\pm|^{p+1}, \quad \bar{u}_1^+ \not\equiv 0, \quad \bar{u}_1^- \not\equiv 0.$$

So

$$I^\infty(\bar{u}_1) = I^\infty(\bar{u}_1^+) + I^\infty(\bar{u}_1^-) \geqslant 2c_\infty,$$

which contradicts $c_2 < 2c_\infty$. Hence $u \not\equiv 0$. If $\{u_k\}$ does not converge strongly to u, then $\bar{u}_1 \not\equiv 0$. Again, $I^\infty(\bar{u}_1) \geqslant c_\infty$. So we get $c_2 \geqslant I(u) + I^\infty(\bar{u}_1) \geqslant c_1 + c_\infty$, a contradiction. Hence $\{u_k\}$ converges strongly to u in $H_0^1(\Omega)$. Therefore, $u \in \mathcal{M}_2$, and we complete the proof of lemma 4.2. □

Now we prove one of our main results.

THEOREM 4.3. *Assume that $b(x)$ satisfies condition (H1) and there exist positive constants C, δ, R_0 such that*

$$b(x) \geqslant b^\infty + C\exp(-\delta|z|) \quad \textit{for } |z| \geqslant R_0, \quad \textit{uniformly for } y \in \varpi.$$

Then (1.1) has a solution that changes sign in unbounded cylinder domains in addition to a positive solution.

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Proof. From lemmas 4.1 and 4.2, we can show the existence of the second solution (without constant sign) by verifying

$$c_2 < c_1 + c_\infty. \tag{4.3}$$

We do this through 'interaction computation', which is similar to that found in [5, 13, 17]. Let $\bar{u}_k = \bar{u}(x + 2ke_N)$, $u_k = \alpha u_0 - \beta \bar{u}_k$, where u_0, $\bar{u}$ are the positive ground solutions of (1.1), (2.1), respectively. The existence of u_0, $\bar{u}$ is proved in [11], provided $b(x)$ satisfies condition (H1).

Define

$$h^{\pm}(\alpha,\beta,k) = \int_\Omega |\nabla(\alpha u_0 - \beta\bar{u}_k)^{\pm}|^2 + |(\alpha u_0 - \beta\bar{u}_k)^{\pm}|^2 - \int_\Omega b(x)|(\alpha u_0 - \beta\bar{u}_k)^{\pm}|^{p+1}.$$

We have that

$$\int_\Omega |\nabla(\tfrac{1}{2}u_0)|^2 + |\tfrac{1}{2}u_0|^2 - \int_\Omega b(x)|\tfrac{1}{2}u_0|^{p+1} > 0,$$
$$\int_\Omega |\nabla(2u_0)|^2 + |2u_0|^2 - \int_\Omega b(x)|2u_0|^{p+1} < 0.$$

For k large enough,

$$\int_\Omega |\nabla(\tfrac{1}{2}\bar{u}_k)|^2 + |\tfrac{1}{2}\bar{u}_k|^2 - \int_\Omega b(x)|\tfrac{1}{2}\bar{u}_k|^{p+1} > 0,$$
$$\int_\Omega |\nabla(2\bar{u}_k)|^2 + |2\bar{u}_k|^2 - \int_\Omega b(x)|2\bar{u}_k|^{p+1} < 0.$$

Thus, by $\bar{u}(x) \to 0$ and $u_0(x) \to 0$, as $|z| \to \infty$ uniformly for $y \in \varpi$, there exists $k_0 > 0$ such that, for $k \geqslant k_0$, we have

$$\left.\begin{array}{r} h^+(\frac{1}{2},\beta,k) > 0 \\ h^+(2,\beta,k) < 0 \end{array}\right\} \quad \text{for all } \beta \in [\tfrac{1}{2}, 2],$$

$$\left.\begin{array}{r} h^-(\alpha,\frac{1}{2},k) > 0 \\ h^-(\alpha,2,k) < 0 \end{array}\right\} \quad \text{for all } \alpha \in [\tfrac{1}{2}, 2].$$

By the mean-value theorem (see [14]), there exist α^*, β^* such that $\frac{1}{2} \leqslant \alpha^*, \beta^* \leqslant 2$,

$$h^{\pm}(\alpha^*,\beta^*,k) = 0 \quad \text{for } k \geqslant k_0.$$

That is,

$$\alpha^* u_0 - \beta^* \bar{u}_k \in \mathcal{M}_2 \quad \text{for } k \geqslant k_0.$$

Hence we only need to prove

$$\sup_{1/2 \leqslant \alpha,\,\beta \leqslant 2} I(\alpha u_0 - \beta\bar{u}_k) < c_1 + c_\infty \quad \text{for } k \geqslant k_0.$$

Indeed,

$$
\begin{aligned}
I(\alpha u_0 - \beta\bar{u}_k)
&= \frac{1}{2}\int_\Omega |\nabla(\alpha u_0)|^2 + |\alpha u_0|^2 + \frac{1}{2}\int_\Omega |\nabla(\beta\bar{u}_k)|^2 + |\beta\bar{u}_k|^2 \\
&\quad - \frac{1}{p+1}\int_\Omega b(x)|\alpha u_0 - \beta\bar{u}_k|^{p+1} - \alpha\beta\int_\Omega (\nabla u_0\cdot\nabla\bar{u}_k + u_0\bar{u}_k) \\
&\leqslant I(\alpha u_0) + I^\infty(\beta\bar{u}) - \frac{\beta^{p+1}}{p+1}\int_\Omega (b(x)-b^\infty)|\bar{u}_k|^{p+1} + C_1\int_\Omega (u_0^p\bar{u}_k + \bar{u}_k^p u_0).
\end{aligned}
$$

Here, we have used the following inequality,

$$
(t-s)^{p+1} \geqslant t^{p+1} + s^{p+1} - C_1(t^p s + t s^p),
$$

for all $t \geqslant 0$, $s \geqslant 0$, where $C_1 > 0$ is some constant. Thus

$$
\begin{aligned}
\sup_{1/2\leqslant\alpha,\,\beta\leqslant 2} I(\alpha u_0 - \beta\bar{u}_k)
&\leqslant \sup_{\alpha\geqslant 0} I(\alpha u_0) + \sup_{\beta\geqslant 0} I^\infty(\beta\bar{u}) \\
&\quad - \frac{1}{2^{p+1}(p+1)}\int_\Omega (b(x)-b^\infty)|\bar{u}_k|^{p+1} + C_1\int_\Omega (u_0^p\bar{u}_k + \bar{u}_k^p u_0).
\end{aligned}
$$

Without loss of generality, we may assume that $\delta < p/(p+1)$. Now, given $\delta > 0$, we let $\varepsilon = \frac{1}{4}(n-1)$. Then, by proposition 3.4 and remarks 3.5 and 3.6, we have

$$
\begin{aligned}
\int_\Omega u_0^p\bar{u}_k &= \int_{\omega\times\{|z|\leqslant(2\delta/p)k\}} u_0^p\bar{u}_k + \int_{\omega\times\{|z|>(2\delta/p)k\}} u_0^p\bar{u}_k \\
&\leqslant C\exp\biggl(-\sqrt{1+\lambda_1}\biggl(2-\frac{2\delta}{p}\biggr)k\biggr)\biggl[\biggl(2-\frac{2\delta}{p}\biggr)k\biggr]^{-(n-1)/2+\varepsilon} \\
&\qquad + C\exp(-2\sqrt{1+\lambda_1}\delta k)\biggl(\frac{2\delta}{p}k\biggr)^{p(-(n-1)/2+\varepsilon)} \\
&\leqslant C_2\exp(-2\sqrt{1+\lambda_1}\delta k),
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega u_0\bar{u}_k^p &= \int_{\omega\times\{|z|\leqslant 2\delta k\}} u_0\bar{u}_k^p + \int_{\omega\times\{|z|>2\delta k\}} u_0\bar{u}_k^p \\
&\leqslant C\exp(-p\sqrt{1+\lambda_1}(2-2\delta)k)[(2-2\delta)k]^{p(-(n-1)/2+\varepsilon)} \\
&\qquad + C\exp(-2\sqrt{1+\lambda_1}\delta k)(2\delta k)^{-(n-1)/2+\varepsilon} \\
&\leqslant C_3\exp(-2\sqrt{1+\lambda_1}\delta k),
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega (b(x)-b^\infty)|\bar{u}_k|^{p+1} &\geqslant \int_{\omega\times\{|z-2ke_N|\leqslant 1\}} |b(x)-b^\infty||\bar{u}_k(x)|^{p+1} \\
&\geqslant C\int_{\omega\times\{|z|\leqslant 1\}} |\bar{u}(x)|^{p+1}\exp(-\delta(2k-1)) \\
&\geqslant C_4\exp(-2\delta k),
\end{aligned}
$$

where C_2, C_3, C_4 are some positive constants independent of k.

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Let k large enough, we have

$$\begin{aligned}\sup_{1/2\leqslant\alpha,\,\beta\leqslant 2} & I(\alpha u_0-\beta\bar{u}_k)\\ &\leqslant \sup_{\alpha\geqslant 0} I(\alpha u_0)+\sup_{\beta\geqslant 0} I^{\infty}(\beta\bar{u})\\ &\quad + C_1(C_2+C_3)\exp(-2\sqrt{1+\lambda_1}\delta k)-\frac{C_4}{2^{p+1}(p+1)}\exp(-2\delta k)\\ &< c_1+c_\infty.\end{aligned}$$

Thus $c_2 < c_1 + c_\infty$, which completes the proof of theorem 4.3. □

THEOREM 4.4. *Assume that $N \geqslant 3$, $b(x)$ satisfies condition (H1) and there exist positive constants C, δ, R_0 such that*

$$b(x) \geqslant b^\infty + C\exp(-\delta|x|) \quad \textit{for } |x| \geqslant R_0.$$

Then (1.1) has a solution that changes sign in $\boldsymbol{R}^N$ in addition to a positive solution.

Proof. Modifying the proof of theorem 4.3 and by proposition 3.7, we can prove theorem 4.4. We omit the details. □

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