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Frame and direction mappings for surfaces in $\mathbb{R}^3$

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We study frames in $\mathbb{R}^3$ and mapping from a surface M in $\mathbb{R}^3$ to the space of frames. We consider in detail mapping frames determined by a unit tangent principal or asymptotic direction field U and the normal field N. We obtain their generic local singularities as well as the generic singularities of the direction field itself. We show, for instance, that the cross-cap singularities of the principal frame map occur precisely at the intersection points of the parabolic and subparabilic curves of different colours. We study the images of the asymptotic and principal foliations on the unit sphere by their associated unit direction fields. We show that these curves are solutions of certain first order differential equations and point out a duality in the unit sphere between some of their configurations.

Keywords: Frames; principal directions and curves; asymptotic directions and curves; singularities; first order differential equations; divergent dagrams

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1. Introduction

Suppose given a geometrically interesting direction field on a surface M in 3-space, represented by a unit field $U(x)$; for example $U(x)$ might be (locally) one of the principal directions away from umbilics or the asymptotic directions at hyperbolic points. Then together with the normal vector N, we have a family of frames given by the triple $(U, N \wedge U,\ N)$; we can consider the corresponding map into the space of frames and the singularities of this mapping. It is natural that the singularities of this map will be related to the singularities of the individual components, the direction mappings $U,\ N \wedge U,\ N : M \to \mathbb{S}^2$, and we shall also consider these.

Moreover, we can consider these mappings along geometrically relevant curves, for example the parabolic curve, the flecnodal curve, the ridge curve. We can also consider the images of the local foliations given by (one part of) the principal or asymptotic curves in the frame space or the unit sphere $\mathbb{S}^2$.

In this paper, we will consider the above constructions and use an elementary result of Arnold (the triality theorem of [1]) to investigate the geometry of the above curves in $\mathbb{S}^2$. In the case of the asymptotic directions, we will replace M by the double cover of the hyperbolic region branched over the parabolic set obtained by considering the asymptotic directions in the sphere tangent bundle. Part of our aim is to link this approach with other results in the generic geometry of surfaces. We will see that the conditions emerging for various types of generic singularities of the frame and related maps link to singularities of the distance squared functions and folding maps in the case of the principal frame and contact with planes and lines for the asymptotic frame. The paper is organized as follows.

In § 2, we give some preliminaries about the singularities of mappings which will be used in the rest of the paper. In § 3, we study the set of frames $\mathbb{F}$ and give various representations of this set. We also give a simple proof of Arnold's triality theorem. In § 4, we consider the singularities of frame mappings $(U, V, N) : M \to \mathbb{F}$ as well as those of U and V. We give geometric conditions for the frame map to be singular and for the singularity to be a cross-cap. In § 5, we deal in detail of the cases when U is a unit principal or asymptotic direction field. In § 6, we obtain the generic configurations of the images of the principal (resp. asymptotic) foliation by the principal (resp. asymptotic) map to the unit sphere. We show that these are, in most cases, solutions of first order differential equations. We also consider the images of these foliations by the Gauss map and relate some of them by duality in the sphere.

2. Preliminaries

Let $\mathcal{E}_n$ be the local ring of germs of functions $\mathbb{R}^n, 0 \to \mathbb{R}$ and $\mathcal{M}_n$ its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{E}(n, p)$ the p-tuples of elements in $\mathcal{E}_n$. Let $\mathcal{A} = \mathcal{R} \times \mathcal{L} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ denote the group of right-left equivalence which acts smoothly on $\mathcal{M}_n.\mathcal{E}(n, p)$ by $(h, k) \cdot G = k \circ G \circ h^{-1}$.

The k-jet space of smooth map-germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ is by definition $J^k(n, p) = \mathcal{M}_n \cdot \mathcal{E}(n, p)/\mathcal{M}_n^{k+1} \cdot \mathcal{E}(n, p)$.

Let $\mathcal{A}_k$ be the subgroup of $\mathcal{A}$ whose elements have k-jets the germ of the identity. The group $\mathcal{A}_k$ is a normal subgroup of $\mathcal{A}$. Define $\mathcal{A}^{(k)} = \mathcal{A}/\mathcal{A}_k$. The elements of $\mathcal{A}^{(k)}$ are the k-jets of the elements of $\mathcal{A}$.

The action of $\mathcal{A}$ on $\mathcal{M}_n.\mathcal{E}(n, p)$ induces an action of $\mathcal{A}^{(k)}$ on $J^k(n, p)$ as follows. For $j^k f \in J^k(n, p)$ and $j^k h \in \mathcal{A}^{(k)}$, $j^k h.j^k f = j^k(h.f)$.

The tangent space to the $\mathcal{A}$-orbit of f at the germ f is given by

$$L\mathcal{A}{\cdot}f = \mathcal{M}_n{\cdot}\{f_{x_1}, \ldots, f_{x_n}\} + f^*(\mathcal{M}_p).\{e_1, \ldots, e_p\},$$

where f_{x_i} denotes partial derivatives with respect to x_i $(i = 1, \ldots, n)$, $e_1, \ldots, e_p$ denote the standard basis vectors of $\mathbb{R}^p$ considered as elements of $\mathcal{E}(n, p)$, and $f^*(\mathcal{M}_p)$ is the pull-back of the maximal ideal in $\mathcal{E}_p$. The extended tangent space to the $\mathcal{A}$-orbit of f at the germ f is given by $L_e\mathcal{A}{\cdot}f = \mathcal{E}_n{\cdot}\{f_{x_1}, \ldots, f_{x_n}\} + f^*(\mathcal{E}_p).\{e_1, \ldots, e_p\}$, and the codimension of the extended orbit is $d_e(f, \mathcal{A}) = \dim_{\mathbb{R}}(\mathcal{E}(n, p)/L_e\mathcal{A} \cdot f\,)$.

Table 1. $\mathcal{A}_e$-*codimension* $\leqslant 2$ *singularities of map-germs* $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$.

Name	Normal form	Algebraic conditions
Fold	(x, y^2)	$a_{22} \neq 0$
Cusp	$(x, xy + y^3)$	$a_{22} = 0,\ a_{21} \neq 0,\ a_{33} \neq 0$
Lips/beaks	$(x, y^3 \pm x^2 y)$	$a_{22} = 0,\ a_{21} = 0,\ a_{33} \neq 0,\ a_{32}^2 - 3a_{31}a_{33} \neq 0$
Goose	$(x, y^3 + x^3 y)$	$a_{22} = 0,\ a_{21} = 0,\ a_{32}^2 - 3a_{31}a_{33} = 0,\ a_{33} \neq 0,$ $27a_{41}a_{33}^3 - 18a_{42}a_{32}a_{33}^2 + 9a_{43}a_{32}^2 a_{33} - 4a_{44}a_{32}^3 \neq 0$
Swallowtail	$(x, xy + y^4)$	$a_{22} = 0,\ a_{33} = 0,\ a_{21} \neq 0,\ a_{44} \neq 0$
Butterfly	$(x, xy + y^5 \pm y^7)$	$a_{22} = 0,\ a_{33} = 0,\ a_{44} = 0,\ a_{21} \neq 0,\ a_{55} \neq 0,$ $(8a_{55}a_{77} - 5a_{66}^2)a_{21}^2 + 2a_{55}(a_{32}a_{66} - 20a_{43}a_{55})a_{21}$ $+ 35a_{32}^2 a_{55}^2 \neq 0$
Gulls	$(x, xy^2 + y^4 + y^5)$	$a_{22} = 0,\ a_{21} = 0,\ a_{33} = 0,\ a_{32} \neq 0,\ a_{44} \neq 0,$ $a_{55}a_{32}^2 - 2a_{43}a_{44}a_{32} + 4a_{31}a_{44}^2 \neq 0$

There are classifications of map-germs for various pairs (n, p). When $p = 1$, we have Arnold's extensive list of germs of functions (here we only need the group $\mathcal{R}$ of changes of coordinates in the source). We shall need the following representatives of $\mathcal{R}$-orbits:

$$A_k : \pm x^2 \pm y^{k+1}, k \geqslant 0, \quad D_k : x^2 y \pm y^{k-1}, k \geqslant 4.$$

For $(n, p) = (2, 2)$, that is, for map-germs from the plane to the plane, there are several classifications and we refer to Rieger [**15**] for that of singularities of $\mathcal{A}$-codimension $\leqslant 6$. In this paper, we need only the singularities of $\mathcal{A}_e$-codimension $\leqslant 2$. For map-germs $g(x, y) = (x, f(x, y))$ with

$$f(x, y) = a_{20}x^2 + a_{21}xy + a_{22}y^2 + \sum_{i=0}^{3} a_{3i}x^{3-i}y^i + \sum_{i=0}^{4} a_{4i}x^{4-i}y^i + O(5), \tag{2.1}$$

where $O(5)$ is a remainder of order 5, the conditions on the coefficients of f for the map-germ g to have a singularity at the origin of $\mathcal{A}_e$-codimension $\leqslant 2$ are as in table 1 (see for example [**11**]).

One can consider certain families of functions and mappings on a smooth surface M in $\mathbb{R}^3$ with the singularities of their members capturing some aspects of the extrinsic geometry of the surface. These families are as follows.

The family of height functions $H : M \times \mathbb{S}^2 \to \mathbb{R}$ on a smooth surface $M \subset \mathbb{R}^3$ is defined by

$$H(x, v) = H_v(x) = x \cdot v,$$

where '$\cdot$' denotes the scalar product in $\mathbb{R}^3$. For v fixed, the function H_v measures the contact of the surface M with parallel planes orthogonal to v. A transversality theorem asserts that for a generic surface M, for any $v \in \mathbb{S}^2$ and at any point x on M the function H_v can only have an $\mathcal{R}$-singularity at x of type A_1, A_2 or A_3. (Here, and in the rest of the paper, generic means for a residual subset of $C^\infty(U, \mathbb{R}^3)$ of local parametrizations of M. The set $C^\infty(U, \mathbb{R}^3)$ is endowed with the Whitney C^∞-topology.) A point x is a singularity of H_v if and only of $v = \pm N(x)$. The singularity is of type A_2 when x is a parabolic point and of type A_3 at special parabolic points

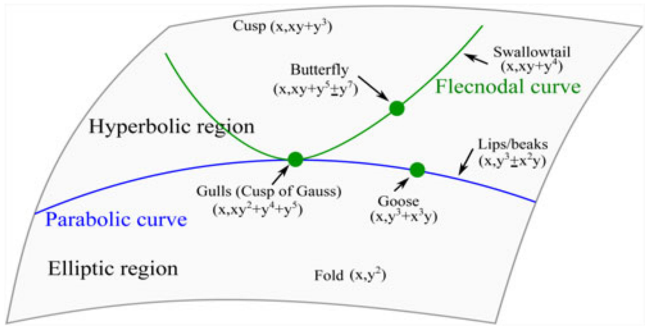

Figure 1. Generic local singularities of orthogonal projections of a surface in $\mathbb{R}^3$.

called cusps of Gauss (at such points the Gauss map N is $\mathcal{A}$-equivalent to the cusp singularity in table 1).

The family of distance squared functions $D : M \times \mathbb{R}^3 \to \mathbb{R}$ on M is defined by

$$D(x, a) = D_a(x) = (x - a) \cdot (x - a).$$

For a fixed, the function D_a measures the contact of M with spheres of centre a. Here, for a generic surface M, for any $a \in \mathbb{R}^3$ and at any point x on M the function D_a can only have an $\mathcal{R}$-singularity at x of type A_1, A_2, A_3, A_4 or D_4. (The D_4-singularities occur at umbilic points.)

The orthogonal projection P_v of M along the direction $v \in \mathbb{S}^2$ to the plane $T_v\mathbb{S}^2$ is given by $P_v(x) = x - (x \cdot v)v$. This can be represented locally by a map-germ from the plane to the plane. Varying v yields the family of orthogonal projection $P : M \times \mathbb{S}^2 \to T\mathbb{S}^2$ of M given by

$$P(x, v) = (v, P_v(x)).$$

The map P_v measures the contact of the surface with lines parallel to v. Again, a transversality theorem asserts that for a generic surface M, for any $v \in \mathbb{S}^2$ and at any point x on M the map-germ P_v at x has only one of local singularities in table 1; see figure 1. It is worth observing that the cusps of Gauss are the points where the orthogonal projection has a gulls singularity.

For a generic surface M, the orthogonal projection of M along an asymptotic direction can have a swallowtail singularity on a smooth curve on M. This curve is called the *flecnodal curve* and is the locus of the geodesic inflections of the asymptotic curves. Recall from [2] that for a generic surface the flecnodal curve meets (tangentially) the parabolic curve precisely at the cusps of the Gauss map N (or the gulls singularities of the orthogonal projection).

3. Frames and curves

In what follows, we will consider the space $\mathbb{F}$ of frames in 3-space. Our first result is just a set of basic facts concerning the space of frames.

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PROPOSITION 3.1.

(i) *The set $\mathbb{F}$ can be identified with the unit tangent bundle to the 2-sphere $\mathbb{S}^2$, or more directly as the set $\mathbb{F} = \{(a_1, a_2, a_3) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 : a_i \cdot a_j = \delta_{ij}\}$.*

(ii) *In turn, given a choice of standard frame, the set $\mathbb{F}$ can be identified with the special orthogonal group $SO(3)$, which is diffeomorphic to the real projective 3-space. In particular, the tangent space at each point can be identified with the Lie algebra of $SO(3)$, the space of skew-symmetric 3×3 matrices.*

(iii) *For each choice of standard frame, $\mathbb{F}$ has a natural contact structure.*

(iv) *$\mathbb{F}$ has a homogeneous geometry with the metric induced from the inclusion $\mathbb{F} \subset \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$, that is, given any two points of $\mathbb{F}$, there is an isometry of $\mathbb{F}$ interchanging the two.*

Proof.

(i) Is clear.

(ii) Given the standard orthonormal basis for $\mathbb{R}^3$, e_1, e_2, e_3 and a triple $(a_1, a_2, a_3) \in \mathbb{F}$, we can consider the orientation preserving transformation taking e_i to a_i that is, the matrix with columns a_1, a_2, a_3. Conversely given any element of $SO(3)$ represented as a matrix with respect to our standard basis, we consider the triple of vectors given by the columns. It is not hard to see that $\mathbb{F} = SO(3)$ is projective 3-space: orient $\mathbb{R}^3$ and to each point c in $\mathbb{R}^3$ assign the anticlockwise rotation about c through an angle $\pi||c||$ in a direction which looking out in the direction of c. Note that when $c = 0$ there is no problem since the angle is zero, that is, we have the identity. Note that antipodal points on the boundary sphere are identified since rotation about an axis through π clockwise is the same as rotation through π anticlockwise.

(iii) If $\mathbb{P}T^*\mathbb{S}^2$ is the projective cotangent sphere bundle to $\mathbb{S}^2$ then consider the map $\mathbb{F} \to PT^*\mathbb{S}^2$, given by $(a_1, a_2, a_3) \mapsto (a_1, \tau)$, where $\tau : T_{a_1}\mathbb{S}^2 \to \mathbb{R}$ is defined by $\tau(b) = a_2 \cdot b$; this is clearly a double cover. Indeed note that $b \in T_{a_1}\mathbb{S}^2$ if and only if $b \cdot a_1 = 0$ so each non-zero linear form on $T_{a_1}\mathbb{S}^2$ arises as such a τ, and this bundle map is an isomorphism. The projectivised cotangent bundle $P(T^*\mathbb{S}^2)$ has a canonical contact structure which lifts to $\mathbb{F}$.

(iv) We have the inclusions $\mathbb{F} \subset \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. We claim that given the induced metric $\mathbb{F}$ is a homogenous space; that is assuming the usual metric for $\mathbb{S}^2$, the product metric on $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$, and the induced metric on $\mathbb{F}$ then there is an isometry interchanging any two points. Indeed as we have seen there is a unique element of $SO(3)$ taking any triple $(a_1, a_2, a_3) \in \mathbb{F}$ to (e_1, e_2, e_3), and fixing $\mathbb{F}$, and this is an isometry of $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. □

REMARKS 3.2.

(i) Suppose given a smooth map $\phi : M \to \mathbb{F}$, then its derivative at a point x is a map $D\phi(x) : T_xM \to T_{\phi(x)}\mathbb{F} = T_{\phi(x)}SO(3)$. We then translate to the identity

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element in $SO(3)$ and consider the image of $D\phi(x)$ as a subset of the Lie algebra of 3×3 skew matrices $sk(3)$. Thus, we can think of the derivative as a skew matrix

$$\begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}.$$

Since $D\phi(x)$ is a linear map the ω_{ij} are 1-forms on M, the so-called connection forms. They measure the rate of turning of the three orthogonal vectors in the frame as we move in various directions on M.

(ii) Choosing a co-ordinate system on M, and three ordered orthonormal vectors $V_1(x)$, $V_2(x)$, $V_3(x)$ yielding our frame, then writing ω_{ij}^r for $\omega_{ij}(\partial/\partial x_r)$ we find that $\omega_{ij}^r = \partial V_i/\partial x_r \cdot V_j$, or alternatively $DV_i(e_r) \cdot V_j$, where the e_r are the images of the standard basis under the parametrization.

3.1. Curves in the unit 2-sphere

Let $\gamma : I \to \mathbb{S}^2$ be a unit speed curve, so $\gamma(t) \cdot \gamma(t) = 1$, $\gamma'(t) \cdot \gamma'(t) = 1$. Differentiating the first identity gives $\gamma(t) \cdot \gamma'(t) = 0$. Define the map $u_\gamma : I \to \mathbb{F}$ by

$$u_\gamma(t) = (\gamma(t), \gamma'(t) \wedge \gamma(t), \gamma'(t)).$$

As above, $\mathbb{F}$ has a natural contact structure with the projection $\mathbb{F} \to \mathbb{S}^2$ to the first component being the projection to the base of a Legendre fibration; the image of u_γ is a Legendrian curve. There is a result due to Arnold, labelled by him a triality theorem [**1**], which we need; its formulation requires us to orient $\mathbb{S}^2$. In the form we require it, the result is relatively easy and we give a quick proof here. We first need two definitions.

Definition 3.3. *If $\gamma : I \to \mathbb{S}^2$ is a smooth curve with a well-defined oriented tangent at each point (so e.g., if it has a regular parametrization) then:*

(i) *the dual of γ is the curve obtained from the original by moving a distance $\pi/2$ along the normals on the side determined by the orientation of γ and $\mathbb{S}^2$; in other words, we have the poles of great circles of the sphere tangent to the curve.*

(ii) *The derivative of γ is the curve obtained from the original by moving each point a distance $\pi/2$ in the positive direction along the great circle tangent to the curve at that point.*

Remarks 3.4. Arnold points out that:

(i) the definition of the dual works for co-oriented curves which are wavefronts having local singularities of the form $x^n = y^{n+1}$.

(ii) The dual is naturally oriented and is a wavefront equidistant (by $\pi/2$) from the original.

(iii) The second dual of a curve is antipodal to the original.

(iv) These definitions work provided that $\gamma'(t) \neq 0$ since we can re-parametrize to get a unit speed curve.

PROPOSITION 3.5 [1]. *Let* $\gamma : I \to \mathbb{S}^2$ *be a smooth unit speed curve and let* $\mathbb{F}$ *be the space of frames with the natural contact structure. There are three projections* $\pi_j : \mathbb{F} \to \mathbb{S}^2,\ \pi_1(a_1, a_2, a_3) = a_1,\ \pi_2(a_1, a_2, a_3) = a_2,\ \pi_3(a_1, a_2, a_3) = a_3$. *Then:*

(i) *the map* $u_\gamma : I \to \mathbb{F}$ *is a Legendrian map with Legendrian projection* π_1*, yielding the Legendrian image* γ*;*

(ii) *the map* $\pi_2 \circ u_\gamma : I \to \mathbb{S}^2$ *is (up to sign) a parametrization of the dual of* γ*;*

(iii) *the map* $\pi_3 \circ u_\gamma : I \to \mathbb{S}^2$ *is (up to sign) a parametrization of the derivative of* γ*;*

(iv) *the derivative of a wavefront coincides with the derivative of any of its parallels.*

Proof.

(i) Since $u_\gamma(t) = (\gamma(t), \gamma'(t) \wedge \gamma(t), \gamma'(t))$, we have $\pi_1 \circ u_{\gamma(t)} = \gamma(t)$ and the tangent great circle to γ at $\gamma(t)$ is orthogonal to $\gamma'(t) \wedge \gamma(t)$, the second component of u_γ. One now checks that v_γ is a Legendrian immersion with projection γ.

(ii) Clearly $\pi_2 \circ u_\gamma = \gamma'(t) \wedge \gamma(t)$ which is a pole of the great tangent circle.

(iii) Similarly $\pi_3 \circ u_{\gamma(t)} = \gamma(t) \wedge (\gamma'(t) \wedge \gamma(t)) = \gamma'(t)$ the derivative of γ.

(iv) An equidistant or parallel to γ is given by $\gamma_r(t) = \gamma(t) + r\gamma(t) \wedge \gamma'(t)$. Of course γ_r is no longer unit speed, but $\gamma_r'(t) = (\gamma' + r\gamma' \wedge \gamma' + r\gamma \wedge \gamma'')(t) = (\gamma' + r\gamma \wedge \gamma'')(t)$. However $\gamma,\ \gamma',\ \gamma''$ are orthogonal so $\gamma \wedge \gamma'' = \pm||\gamma''||\gamma'$ and γ_r' is a multiple of γ' as required. □

EXAMPLE 3.6. Suppose given a unit speed line of curvature on a surface $\alpha : I \to M$, composing with the normal map, we obtain a mapping $\gamma = N_\alpha : I \to \mathbb{S}^2,\ s \mapsto N \circ \alpha(s)$. Now since α is a line of curvature, we have $\gamma'(s) = (\kappa_1 P_1)(\alpha(s))$ where $\kappa_1,\ P_1$ are, respectively, the corresponding principal curvature and principal direction. The result above tells us that the curve $P_2(\alpha(s))$ is the dual of γ and $P_1(\alpha(s))$ the derivative map of γ (more on this in § 6.3).

4. Maps from a surface into the space of frames

Given a unit vector field U on M, we can consider the frame determined by the unit normal N, the vector field U and the third vector V making up the orthonormal triple; we choose $N \wedge U$. The frame $U,\ V,\ N$ gives a map from M to $\mathbb{F}$: we are interested in the singularities of this map. If the map is generic then we can expect the following stable singularities: transverse self-intersections, triple points

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and cross-caps. Given a geometrically relevant vector field U, for example, a field of principal directions or asymptotic directions, we consider the following questions:

(a) When does the frame mapping $M \to \mathbb{F}$ have a singularity?

(b) For a generic surface $M \subset \mathbb{R}^3$ is the frame map $M \to \mathbb{F}$ stable?

(c) If so what is the geometric interpretation of a cross-cap singularity? (The geometric interpretation of the multi-local singularities is immediate.)

(d) If not what singularities do we have?

(e) What are the images of the integral curves of U in $\mathbb{F}$ or $\mathbb{S}^2$ via the frame map or $U,\ V,\ N$?

We start then by considering arbitrary mappings $U,\ V : M \to \mathbb{S}^2$ with $U(x) \cdot V(x) \equiv 0$. Note that since $U \cdot U = V \cdot V = 1, U \cdot V = 0$ we have, using subscript i for partial differentiation with respect to x_i, $U \cdot U_i = V \cdot V_i = 0, U_i \cdot V + U \cdot V_i = 0$.

We first remark that given a smooth germ $f : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ and a submersion $g : \mathbb{R}^3, 0 \to \mathbb{R}^2$ then if f is singular $\ker Df(0) \subset \ker D(g \circ f)(0)$.

In particular, if $(U, V, U \wedge V) : M \to \mathbb{F}$ has a singular point at x then so do U and V, and if they both have rank 1 then $\ker DU(x) = \ker DV(x)$. The converse is also true for if $f : M \to \mathbb{F}$ is the frame map and $\pi_j : \mathbb{F} \to \mathbb{S}^2$ is projection to the j^{th} component of $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ then $\ker DU(x) = Df(x)^{-1}(\ker d\pi_1(x))$ and $\ker DV(x) = Df(x)^{-1}(\ker D\pi_2(x))$ but $\ker(D\pi_1(x)) \cap \ker(D\pi_2(x)) = \{0\}$. This establishes part (i) of the following proposition.

Proposition 4.1.

(i) *If the frame map* $(U, V, U \wedge V) : M \to \mathbb{F}$ *has a singular point at* x *then the maps* $U,\ V,\ U \wedge V : M \to \mathbb{S}^2$ *have singular points at* x.

(ii) *If* U *and* V *have singular points at* x *of rank* 1 *then either* $\ker DU(x) = \ker DV(x)$ *or* $\operatorname{im} DU(x)$ *and* $\operatorname{im} DV(x)$ *are parallel. In the first instance* (U, V) *is singular.*

(iii) *Suppose now that* $U,\ V$ *are fold maps at* x*. In the second instance in* (ii) *above the image of the derivative* U *and* V *at* x *are orthogonal to* U *and* V*.*

(iv) *The frame map has a singular point which is a cross-cap if* $\ker DU(x) = \ker DV(x)$ *and the singular sets* $\Sigma U,\ \Sigma V$ *are transverse at* x*.*

Proof. (ii), (iii) Suppose that U and V are singular, so we can write $U_2 = \lambda U_1,\ V_2 = \mu V_1$ (say). Then $U_2 \cdot V = \lambda U_1 \cdot V$ and $U \cdot V_2 = \mu U \cdot V_1$; but $U_j \cdot V = -U \cdot V_j$ so $\lambda = \mu$ or $U_j \cdot V = U \cdot V_j = 0,\ j = 1,\ 2$. In the first case clearly (U, V) is singular. In the second $U_1 \cdot V = V_1 \cdot U = U_1 \cdot U = V_1 \cdot V = 0$ so the one-dimensional images of the derivatives DU and DV are orthogonal to U and V.

(iv) Suppose that $U,\ V$ are singular at x and that $\ker DU(x) = \ker DV(x)$. We can take (x_1, x_2) as local parameters of M and suppose that the point of interest is the origin. We can rotate the coordinate axes if necessary and suppose that

$U_{x_2}(0,0) = V_{x_2}(0,0) = 0$, that is, the kernels of $DU(0,0)$ and $DV(0,0)$ are parallel to the x_2-axis. If we represent the frame as a map $f : \mathbb{M} \to \Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ given by $f(x) = (U(x), V(x))$, then by Whitney's criteria, f has a cross-cap singularity at the origin if and only if $(\partial^2 f/\partial x_1 \partial x_2)(0,0)$ and $(\partial^2 f/\partial x_2^2)(0,0)$ are linearly independent.

Using the fact that $(U, V) \in \Delta$, one can show that $(\partial^2 f/\partial x_1 \partial x_2)$ and $(\partial^2 f/\partial x_2^2)$ (evaluated at the origin) are linear combinations of the three independent vectors in the ordered set $B = \{(U \wedge V, 0), (0, U \wedge V), (V, -U)\}$ and have coordinates

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = (U_{x_1 x_2} \cdot U \wedge V,\, V_{x_1 x_2} \cdot U \wedge V,\, U_{x_1 x_2} \cdot V)_B$$

$$\frac{\partial^2 f}{\partial x_2^2} = (U_{x_2 x_2} \cdot U \wedge V,\, V_{x_2 x_2} \cdot U \wedge V,\, U_{x_2 x_2} \cdot V)_B$$

Observe that the third coordinate of $\partial^2 f/\partial x_1 \partial x_2$ (resp. $\partial^2 f/\partial x_2 \partial x_2$) can be replaced by $-V_{x_1 x_2} \cdot U$ (resp. $-V_{x_2 x_2} \cdot U$). We have, at any point x near the origin,

$$\begin{aligned} U_{x_1} &= (U_{x_1} \cdot V)V + (U_{x_1} \cdot U \wedge V)U \wedge V, \\ U_{x_2} &= (U_{x_2} \cdot V)V + (U_{x_2} \cdot U \wedge V)U \wedge V, \end{aligned}$$

$$\begin{aligned} V_{x_1} &= (V_{x_1} \cdot U)U + (V_{x_1} \cdot U \wedge V)U \wedge V, \\ V_{x_2} &= (V_{x_2} \cdot U)U + (V_{x_2} \cdot U \wedge V)U \wedge V, \end{aligned}$$

so the critical sets ΣU of U and ΣV of V have equations

$$\begin{aligned} \Sigma U &: \quad (U_{x_1} \cdot V)(U_{x_2} \cdot U \wedge V) - (U_{x_1} \cdot U \wedge V)(U_{x_2} \cdot V) = 0, \\ \Sigma V &: \quad (V_{x_1} \cdot U)(V_{x_2} \cdot U \wedge V) - (V_{x_1} \cdot U \wedge V)(V_{x_2} \cdot U) = 0. \end{aligned}$$

These sets are transverse at the origin if and only if

$$\begin{aligned} &(U_{x_1} \cdot V)(V_{x_1} \cdot U)\,[(U_{x_1 x_2} \cdot U \wedge V)(V_{x_2 x_2} \cdot U \wedge V) \\ &\quad -(U_{x_2 x_2} \cdot U \wedge V)(V_{x_1 x_2} \cdot U \wedge V)] \\ &- (U_{x_1} \cdot V)(V_{x_1} \cdot U \wedge V)\,[(U_{x_1 x_2} \cdot U \wedge V)(V_{x_2 x_2} \cdot U) \\ &\quad -(U_{x_2 x_2} \cdot U \wedge V)(V_{x_1 x_2} \cdot U)] \\ &+ (V_{x_1} \cdot V)(U_{x_1} \cdot U \wedge V)\,[(V_{x_1 x_2} \cdot U \wedge V)(U_{x_2 x_2} \cdot V) \\ &\quad -(V_{x_2 x_2} \cdot U \wedge V)(U_{x_1 x_2} \cdot V)] \neq 0 \end{aligned}$$

(where the left-hand side of the above inequality is evaluated at the origin). If transversality holds, then one of the expressions in the square brackets above is non-zero. The result follows from the fact that the expressions in the square brackets are precisely the 2×2-minors of the matrix formed by the vectors $(\partial^2 f/\partial x_1 \partial x_2)$ and $(\partial^2 f/\partial x_2^2)$ evaluated at the origin. □

Corollary 4.2.

(i) *If* $U,\ V : M \to \mathbb{S}^2$ *have a singular point at* $x \in M$ *and* $\ker DU(x) = \ker DV(x)$ *then* $W = U \wedge V$ *is singular and* $\ker D(U \wedge V)(x) = \ker DU(x) = \ker DV(x)$.

(ii) *If* $U,\ V : M \to \mathbb{S}^2$ *have a singular point at* $x \in M$ *and* $\operatorname{im}\ DU(x) = \operatorname{im}\ DV(x)$ *then generically* $W = U \wedge V$ *does not have a singularity; indeed it is singular if and only* U *or* V *has rank* 0.

Proof.

(i) Clearly, $W_j = (U \wedge V)_j = U_j \wedge V + U \wedge V_j$; as above if U is singular at x say then we may suppose that $U_2 = \lambda U_1$ and if V is singular there then we write $V_2 = \lambda V_1$ (same multiplier). So $W_2 = \lambda U_1 \wedge V + \lambda U \wedge V_1,\ W_1 = U_1 \wedge V + U \wedge V_1$ and W is singular, moreover, $W_2 = \lambda W_1$ so the kernels are identical.

(ii) Here we have $U_2 = \lambda U_1,\ \ V_2 = \mu V_1,\ \ \lambda \neq \mu$. So $W_1 = (U \wedge V)_1 = U_1 \wedge V + U \wedge V_1,\quad W_2 = (U \wedge V)_2 = U_2 \wedge V + U \wedge V_2 = \lambda U_1 \wedge V + \mu U \wedge V_1$. Clearly, these are dependent if and only if $\lambda = \mu$, ruled out, or one of $U_1 \wedge V$ or $U \wedge V_1 = 0$. But we already know that $U_1 \cdot V = U \cdot V_1 = 0$, so clearly the first of these conditions implies that $U_1 = 0$, the second $V_1 = 0$. □

Now replace $U,\ V,\ U \wedge V$ by $\mathcal{V}_1,\ \mathcal{V}_2,\ \mathcal{V}_3$ with connection forms as in remarks 3.2.

Corollary 4.3.

(i) *The frame map* $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3) : M \to \mathbb{F}$ *has a singular point if and only if the following matrix has rank* 1*:*

$$\begin{bmatrix} \omega_{12}^1 & \omega_{13}^1 & \omega_{23}^1 \\ \omega_{12}^2 & \omega_{13}^2 & \omega_{23}^2 \end{bmatrix}.$$

(ii) *We find that* $\ker D\mathcal{V}_1(x) \cap \ker D\mathcal{V}_2(x) \neq \{0\}$ *if and only if* $(\omega_{12}^1, \omega_{12}^2) \neq (0, 0)$.

(iii) *If we compose the frame* $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3) : M \to \mathbb{F}$ *with any of the three projections to* $\mathbb{S}^2$ *we obtain maps* $\mathcal{V}_i : M \to \mathbb{S}^2$*. The derivative of the map* $\mathcal{V}_1$ *(resp.* $\mathcal{V}_2,\ \mathcal{V}_3$*) at* x*, with respect the basis* $\{e_1, e_2\}$ *for* T_xM *and respectively,* $\{\mathcal{V}_2, \mathcal{V}_3\}, \{\mathcal{V}_1,\ \mathcal{V}_3\}, \{\mathcal{V}_1,\ \mathcal{V}_2\}$ *at the relevant* $T_{\mathcal{V}_i(x)}\mathbb{S}^2$*, is given, respectively, by*

$$\begin{bmatrix} \omega_{12}^1 & \omega_{13}^1 \\ \omega_{12}^2 & \omega_{13}^2 \end{bmatrix} \quad \left(resp. \begin{bmatrix} \omega_{21}^1 & \omega_{23}^1 \\ \omega_{21}^2 & \omega_{23}^2 \end{bmatrix}, \quad \begin{bmatrix} \omega_{31}^1 & \omega_{32}^1 \\ \omega_{31}^2 & \omega_{32}^2 \end{bmatrix} \right).$$

Proof. Clearly $D\mathcal{V}_i(e_r) = \sum (D\mathcal{V}_i(e_r) \cdot \mathcal{V}_j)\mathcal{V}_j = \sum \omega_{ij}^r \mathcal{V}_j$. □

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5. Geometric frames

In this section, we compute the connection forms for some key frames. We denote by S the shape operator of M. The following well-known result ([**14**] p. 230) will prove useful:

PROPOSITION 5.1. *Let* $\alpha : \mathbb{R}, 0 \to M$ *be a unit speed integral curve of* U*; for each* t*, we consider the frame determined by* $U(\alpha(t)),\ N(\alpha(t)) \wedge U(\alpha(t)),\ N(\alpha(t))$*. Then*

$$\begin{bmatrix} U' \\ V' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & g & \kappa \\ -g & 0 & \tau \\ -\kappa & -\tau & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ N \end{bmatrix}$$

where $\kappa = S(U) \cdot U$ *is the normal curvature of* M *in the* U *direction,* $\tau = S(U) \cdot V$ *and* g *is the geodesic curvature of* α*. The curve is (i) geodesic if and only if* $g \equiv 0$*, (ii) asymptotic if and only if* $\kappa \equiv 0$ *and (iii) principal if and only if* $\tau \equiv 0$*.*

PROPOSITION 5.2. *Consider a neighbourhood of a non-umbilic point on a smooth surface* $M \subset \mathbb{R}^3$*, with unit normal* N*, principal directions* $P_1,\ P_2$*. Choose a local parametrisation* X *of* M *with* $X_j = \lambda_j P_j,\ j = 1,\ 2;\ \lambda_1(x_1, 0) = 1,\ \lambda_2(0, x_2) = 1,$ *and the frame* $F = (P_1, P_2, N)$*.*

(i) *The derivative of* F *is determined by a pair of skew symmetric matrices* $\Omega_1,\ \Omega_2$*, with the connection* 1*-forms being the entries in* $\Omega_1 dx_1 + \Omega_2 dx_2$*, and these are:*

$$\begin{aligned}
\omega_{12} &= \frac{1}{\kappa_1 - \kappa_2}\left(\frac{\partial \kappa_1}{\partial x_2} dx_1 + \frac{\partial \kappa_2}{\partial x_1} dx_2\right), \\
\omega_{13} &= \kappa_1 dx_1, \\
\omega_{23} &= \kappa_2 dx_2.
\end{aligned}$$

(ii) *Suppose that a frame is determined by* $Q_1 = \cos\theta P_1 + \sin\theta P_2, Q_2 = -\sin\theta P_1 + \cos\theta P_2, N$ *where* θ *is a function of* x *and* y*. Then the corresponding entries are;*

$$\begin{aligned}
\omega_{12} &= \frac{1}{\kappa_1 - \kappa_2}\left(\left(\frac{\partial \theta}{\partial x_1} + \frac{\partial \kappa_1}{\partial x_2}\right) dx_1 + \left(\frac{\partial \theta}{\partial x_2} + \frac{\partial \kappa_2}{\partial x_1}\right) dx_2\right), \\
\omega_{13} &= -\kappa_1 \cos\theta dx_1 - \kappa_2 \sin\theta dx_2, \\
\omega_{23} &= \kappa_1 \sin\theta - \kappa_2 \cos\theta dx_2.
\end{aligned}$$

(iii) *It follows that the principal frame map* $(P_1, N \wedge P_1, N)$ *has a singularity when* $\kappa_1 = \partial\kappa_1/\partial x_2 = 0$ *or* $\kappa_2 = \partial\kappa_2/\partial\kappa_1 = 0$*. In case (ii), we find that the conditions are* $\kappa_1 = (\kappa_1 - \kappa_2)\partial\theta/\partial x_1 + \partial\kappa_1/\partial x_2 = 0$ *or* $\kappa_2 = (\kappa_1 - \kappa_2)\partial\theta/\partial x_2 + \partial\kappa_2/\partial x_1 = 0$*. In particular, we only ever have singular points on the parabolic set - clear since the Gauss map must have a singular point.*

Proof.

(i) This is a standard calculation, again see [**14**] p. 255, as usual using the orthogonality of the frames, and an application of the Codazzi equations, or equivalently using the equality of mixed partial derivatives of N and X.

(ii) This follows in the same way.

(iii) These are just a straightforward calculation. $\square$

5.1. Principal and normal frame

In this section, we shall consider the mapping into the space of frames locally defined by choosing one oriented principal direction P_1 and consider the triple $(P_1, N \wedge P_1, N)$; of course $N \wedge P_1$ is the other principal direction, which we denote by P_2.

Let $X : U \to \mathbb{R}^3$ be a local parametrization of an umbilic free surface patch M where the coordinate curves are the lines of principal curvature. Suppose that $\kappa_1 \neq 0$ in U. Then the focal sheet F_1 of the focal set is the image of the map $\phi(x) = X(x) + 1/\kappa_1(x)N(x)$. (If $\kappa_1 = 0$ then the corresponding focal points go to infinity.) We have

$$\frac{\partial \phi}{\partial x_1} = -\frac{1}{\kappa_1^2}\frac{\partial \kappa_1}{\partial x_1}N,$$
$$\frac{\partial \phi}{\partial x_2} = \frac{\kappa_2 - \kappa_1}{\kappa_1}P_2 - \frac{1}{\kappa_1^2}\frac{\partial \kappa_1}{\partial x_2}N.$$

As $\kappa_1 \neq \kappa_2$, the map ϕ has rank 2 unless $\partial\kappa_1/\partial x_1 = 0$. This is the condition for the contact between the surface and the corresponding sphere of curvature to be more degenerate than an A_2.

Remark 5.3. The condition $\partial\kappa_1/\partial x_1 = 0$ occurs generically on a smooth curve on M called the *ridge curve*. Then the focal set is singular and is diffeomorphic to a cuspidal edge or a swallowtail surface. Observe that the cusps of Gauss (or gulls singularities of the orthogonal projection) are the points of intersection of the parabolic and ridge curves associated to the same principal direction, that is, they are the points that satisfy

$$\kappa_1 = \frac{\partial \kappa_1}{\partial x_1} = 0 \quad \left(\text{resp. } \kappa_2 = \frac{\partial \kappa_2}{\partial x_2} = 0\right).$$

If $\partial\kappa_1/\partial x_1 \neq 0$, then F_1 is a smooth surface and the principal direction $P_1(x)$ is normal to F_1 at $\phi(x)$, so that P_1 is locally the Gauss map for F_1. (Similar construction works of course for the sheet F_2 of the focal set).

Proposition 5.4. *Away from umblic, parabolic and ridge points, the principal frame on M corresponds to a frame on F_1 with P_1 the normal, N, P_2 orthonormal tangent vectors at $\phi(x)$, and the principal curves corresponding to P_1 lift to geodesics on F_1.*

Proof. We choose the parametrization as above. The normal to F_1 is P_1, so that (P_1, N, P_2) is a geometric frame on F_1. Now consider the lift of the lines of curvature corresponding to P_1, $\beta(t) = X(t,c) + 1/\kappa_1(X(t,c))N(X(t,c))$. Differentiating with respect to t (just as above) we obtain a multiple of N. We reparametrize β by arclength s so that $\beta'(s) = N(s)$, with $N(s) = N(X(t(s), c))$. Then $\beta''(s) = N'(s)$ is parallel to P_1 which is normal to F_1, and this implies that β is a geodesic on F_1. □

The differential geometry of the focal set is studied in [**6**, **18**] using singularity theory. The contact of F_1 with a plane $\mathcal{P}$ is captured by the singularities of the folding map on M with respect to $\mathcal{P}$. The folding map is locally a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^3, 0$. It has an S_2-singularity at x if and only if $\phi(x)$ is an ordinary parabolic point on F_1, that is, the Gauss map P_1 has a fold singularity. (The S_k-singularities are those that are $\mathcal{A}$-equivalent to $(x, y^2, y^3 \pm x^{k+1}y)$, see [**12**].) The singularity of the folding map is of type S_3 if and only if $\phi(x)$ is a cusp of Gauss of F_1, that is, the Gauss map P_1 has a cusp singularity.

The parabolic set of the focal set is called *sub-parabolic curve* in [**6**, **18**]. We have the following geometric characterization of the sub-parabolic curve in terms of the principal curvatures.

Proposition 5.5 [**4**, **13**]. *Away from umbilic, parabolic and ridge points, and with a parametrization of the surface with coordinate curves the lines of principal curvature, a point $x \in M$ is on the sub-parabolic curve corresponding to the parabolic set of the focal F_1 (resp. F_2) if and only if*

$$\frac{\partial \kappa_2}{\partial x_1}(x) = 0 \left(\textit{resp. } \frac{\partial \kappa_1}{\partial x_2}(x) = 0 \right).$$

Remark 5.6. Suppose that x is not an extremum of κ_1 (resp. κ_2): generically, we will not have any points for which $\kappa_j = \partial \kappa_j / \partial x_1 = \partial \kappa_j / \partial x_2 = 0$. Then the condition

$$\kappa_1 = \frac{\partial \kappa_1}{\partial x_2} = 0 \left(\text{resp. } \kappa_2 = \frac{\partial \kappa_2}{\partial x_1} = 0 \right)$$

in Proposition 5.2(ii) for the principal frame to have a cross-cap singularity corresponds to point which is both parabolic with $\kappa_1 = 0$ (resp. $\kappa_2 = 0$) and sub-parabolic associated with a parabolic point on the focal sheet F_2 (resp. F_1). Observe that with the genericity condition above the focal sheet F_2 (resp. F_1) is a smooth surface.

We next consider the nature of the singular points of the principal frame map as well as that of the principal and normal maps. We consider only the principal map P_2, the results are similar for P_1. (We chose P_2 so that the generic conditions that appear in the proof can be matched with those in table 1).

THEOREM 5.7.

(i) *The principal and normal maps* P_1, P_2, $N : M \to \mathbb{S}^2$ *have singular points when:*

$$P_1 : \kappa_1 = 0 \text{ or } \partial\kappa_2/\partial x_1 = 0,$$

that is, at a parabolic point on M *or a parabolic point on* F_1

$$P_2 : \kappa_2 = 0 \text{ or } \partial\kappa_1/\partial x_2 = 0, \text{similar interpretation.}$$

$$N : \kappa_1\kappa_2 = 0, \text{that is, at a parabolic point.}$$

(ii) *The frame map* (P_1, P_2, N) *has a cross-cap when*

$$\kappa_1 = \frac{\partial\kappa_1}{\partial x_2} = 0,\ \frac{\partial\kappa_1}{\partial x_1} \neq 0,\ \frac{\partial^2\kappa_1}{\partial x_2^2} \neq 0,$$

or

$$\kappa_2 = \frac{\partial\kappa_2}{\partial x_1} = 0,\ \frac{\partial\kappa_2}{\partial x_2} \neq 0,\ \frac{\partial^2\kappa_2}{\partial x_1^2} \neq 0.$$

In other words, if we are at a parabolic point and sub-parabolic point corresponding to the parabolic point of the other sheet of the focal set. These points are generically distinct from the cusps of Gauss. We call them the cross-cap points of the principal frame map. *See figure 2.*

(iii) *The normal map* N *has a fold when the contact between the surface and its tangent plane is of type* A_2 *and a cusp when this contact is of type* A_3.

(iv) *If* $\kappa_2 \neq 0$ *and* $\partial\kappa_1/\partial x_2 = 0$ (*i.e., at a sub-parabolic point corresponding to a parabolic point on the focal sheet* F_2) *the principal direction map* P_2 *has a fold when the principal plane spanned by* P_1 *and* N *and the focal set* F_2 *at the focal point have* A_2 *contact, and generically, a cusp when they have* A_3 *contact.* (*This case includes parabolic points* $\kappa_1 = 0$.)

(v) *If* $\kappa_2 = 0$ *and* $\partial\kappa_1/\partial x_2 \neq 0$ *at* x, *generically, the principal direction map* P_2 *is a fold unless* x *is a goose singularity of the orthogonal projection of* M *along* P_2. *At a goose singularity of the projection, generically it is a cusp.*

(vi) *If* $\kappa_2 = \partial\kappa_1/\partial x_2 = 0$, *then the map* P_2, *generically, has a beaks singularity.*

Proof. The conditions in

(i) just follow from the computation of the connection forms for the principal frame.

(ii) We choose two of the vector fields making the frames, say P_1, N and apply the criterion in Proposition 4.1. We want $\ker DP_1(x) = \ker DN(x)$ and the critical sets, that is the parabolic and subparabolic curves, to be transverse, and these are the resulting conditions.

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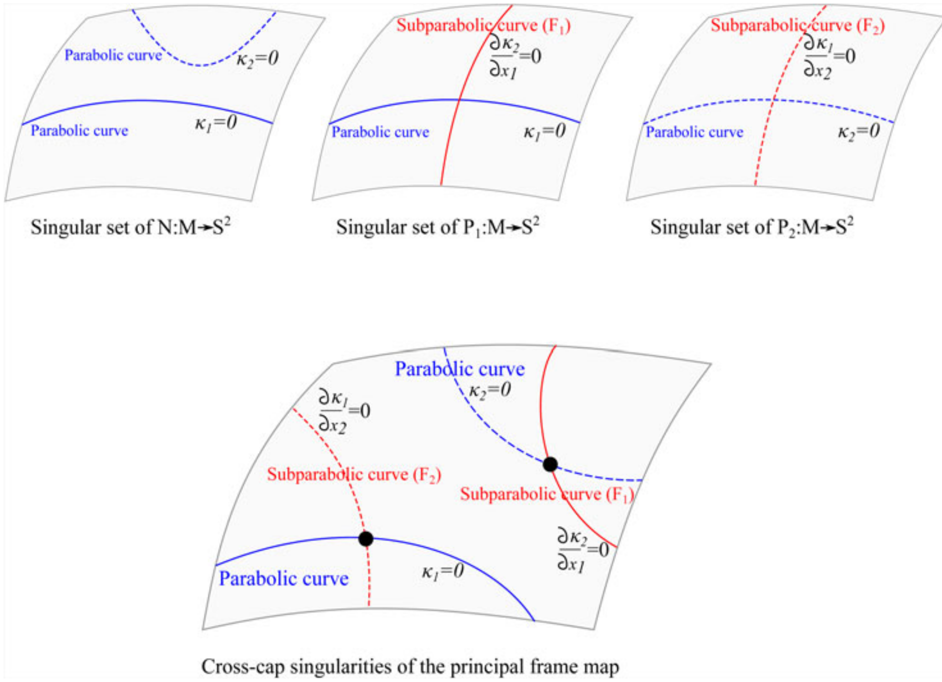

Figure 2. The singular sets of the maps N, P_1, P_2 and the cross-cap singularities of the principal frame map (thick dots).

(iii) This is well known. There are many interpretations/characterizations of these cusps of Gauss or godrons as they are also known; see for example [2].

(iv) At a regular point of F_2 the principal direction P_2 is normal, that is, P_2 is the Gauss map of F_2, and the result follows as in (ii). (The fact that generically there is a cusp when there is A_3 contact follows from [6]).

(v)-(vi) Here we need to dig a little deeper. We take M in Monge form $\phi(x, y) = (x, y, f(x, y))$ at the origin with f as in (2.1). We take the principal directions along the x- and y-axes at the origin so $a_{21} = 0$. As $\kappa_2(0, 0) = 0$, $a_{22} = 0$. The origin is not an umbilic point, so $a_{20} \neq 0$. In fact, we can choose $a_{20} > 0$ by reversing the direction of the z-axis if necessary.

As usual writing E, F, G, respectively l, m, n for the coefficients of the first, resp. second, fundamental form in the given co-ordinate system, the principal directions are solutions of the binary differential equation

$$(Em - lF)dx^2 + (En - lG)dxdy + (Fn - mG)dy^2 = 0,$$

with

$$E(x, y) = 1 + 4a_{20}x^2 + O(3),$$

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$$F(x,y) = O(3),$$
$$G(x,y) = 1 + O(3),$$
$$l(x,y) = 2a_{20} + 6a_{30}x + 2a_{31}y + 12a_{40}x^2 + 6a_{41}xy + 2a_{42}y^2 + O(3),$$
$$m(x,y) = 2a_{31}x + 2a_{32}y + 3a_{41}x^2 + 4a_{42}xy + 3a_{43}y^2 + O(3),$$
$$n(x,y) = 2a_{32}x + 6a_{33}y + 2a_{42}x^2 + 6a_{43}xy + 12a_{44}y^2 + O(3).$$

We seek the principal direction which is parallel to $\phi_y(0,0)$ at the origin, so we can set $q = dx/dy$ and with q a solution of

$$(Em - lF)q^2 + (En - lG)q + (Fn - mG) = 0. \tag{5.1}$$

Thus the unit principal direction on M which is parallel to $\phi_y(0,0)$ at the origin is given by

$$P_2 = \frac{1}{(Eq^2 + 2Fq + G)^{1/2}}(q\phi_x + \phi_y)$$

with q a solution of (5.1).

We consider an $\mathcal{A}$-equivalent map to P_2 by projecting $P_2 = (P_{21}, P_{22}, P_{23})$ to the tangent space of the sphere at $P_2(0,0) = (0,1,0)$. We write $\tilde{P}_2 = (P_{21}, P_{23})$. The calculations are carried out using Maple and the expression for the relevant jet of $\tilde{P}_2$ is too lengthy to reproduce here. We shall use the recognition criteria of singularities of map-germs from the plane to the plane (see e.g., [**16**]) to identify the singularities of $\tilde{P}_2$ (and hence of P_2). Let $g(x,y) = 0$ define the critical set of a map-germ $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of corank 1 and let η be a vector field such that $\eta h = 0$ on the critical set of h. Then, the singularity of h at the origin is a

$$\begin{array}{rcl} \text{fold} & \iff & g \text{ is regular and } \eta g \neq 0 \\ \text{cusp} & \iff & g \text{ is regular, } \eta g = 0 \text{ and } \eta\eta g \neq 0 \\ \text{beaks} & \iff & g \text{ has an } A_1^-\text{-singularity and } \eta\eta g \neq 0 \end{array}$$

The singular set Σ of $\tilde{P}_2$ is the zero set of the function $g(x,y) = \det D\tilde{P}_2(x,y)$. The kernel of $D\tilde{P}_2$ at points on Σ is parallel to $\eta = (-\partial P_{21}/\partial y, \partial P_{21}/\partial x)$. We consider η as a germ of a vector field in $\mathbb{R}^2$.

We have $\nabla g(0,0) = -((2a_{31})/(a_{20}))(a_{32}, 3a_{33})$, so generically, Σ is singular if and only if $a_{31} = 0$, equivalently, $\partial\kappa_1/\partial x_2(0,0) = 0$.

For (v) we have $a_{31} \neq 0$. Then the map $\tilde{P}_2$ has a fold singularity at the origin if and only if $\eta g(0,0) \neq 0$, which we found to be equivalent to $a_{32}^2 - 3a_{31}a_{33} \neq 0$.

If $a_{32}^2 - 3a_{31}a_{33} = 0$, then $\tilde{P}_2$ has a cusp singularity if and only if $\eta\eta g(0,0) \neq 0$, equivalently,

$$a_{32}(27a_{41}a_{33}^3 - 18a_{42}a_{32}a_{33}^2 + 9a_{43}a_{32}^2a_{33} - 4a_{44}a_{32}^3) \neq 0.$$

We observe that $a_{32} = 0$ if and only if $\partial\kappa_2/\partial x_1(0,0) = 0$. As we have already $\kappa_2 = \partial\kappa_1/\partial x_2 = 0$ at the origin, generically $a_{32} \neq 0$. Therefore, $\tilde{P}_2$ has a cusp singularity at precisely the goose singularities of the

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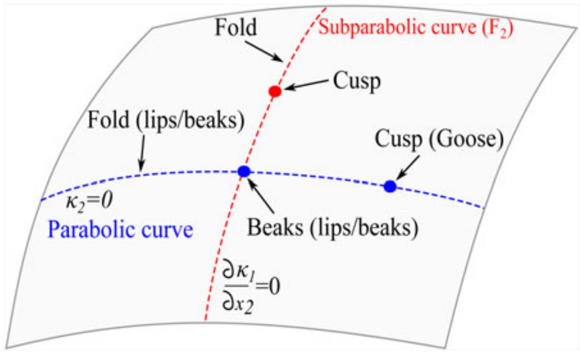

Figure 3. The generic singularities of the principal map $P_2 : M \to \mathbb{S}^2$. The singularities in bracket on the parabolic set are those of the orthogonal projection along the unique asymptotic direction.

orthogonal projection along P_2 (compare above conditions with those in table 1).

For (vi), we have $a_{31} = 0$. Then,

$$j^2 g(x, y) = \frac{2}{a_{20}^2}(a_{32}x + 3a_{33}y)(3a_{20}a_{41}x + (2a_{32}^2 - 3a_{30}a_{32} + 2a_{20}a_{42})y)$$

which generically has a Morse singularity of type A_1^-. We have $\eta\eta g(0, 0) = -12a_{32}^3 a_{41}/a_{20}^3$, and generically this is not zero. Therefore, the map-germ $\tilde{P}_2$ has a beaks singularity at the origin. □

5.2. Asymptotic and normal frame

We consider here the asymptotic and normal frame; we already know that any critical point of the frame map must be at a parabolic point of M, and this is exactly the curve distinguishing two and zero asymptotic directions. For completeness, we compute the connection forms in this case, but to carry out a detailed analysis, we require different tools.

PROPOSITION 5.8. *For the asymptotic frame obtained from a principal frame by rotation as in proposition 5.2, we find that*

$$\frac{\partial \theta}{\partial x_j} = \frac{1}{2\kappa_2(\kappa_2 - \kappa_1)\tan\theta}\left(\kappa_1 \frac{\partial \kappa_2}{\partial x_j} - \kappa_2 \frac{\partial \kappa_1}{\partial x_j}\right),$$

where $\tan^2\theta = -\kappa_1/\kappa_2$.

Proof. The sectional curvature in a direction inclined at an angle θ to the principal direction P_1 is $\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$ by Euler's Theorem. An asymptotic direction is one in which this sectional curvature is zero so assuming say $\kappa_2 \neq 0$, we can set $\tan^2\theta = -\kappa_1/\kappa_2$. Note that generically, there are no flat umbilics, that is points where all sectional curvatures vanish so this assumption is valid. If

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we write $r = -\kappa_1/\kappa_2$ say then $\partial r/\partial x_j = 2\tan\theta(1+\tan^2\theta)\partial\theta/\partial x_j$ so $\partial\theta/\partial x_j = \partial r/\partial x_j/2\tan\theta(1+\tan^2\theta)$ which reduces to the required result. $\square$

Proposition 5.9. *Let A be a smooth unit vector field determining asymptotic directions in a neighbourhood of a hyperbolic point $q \in M$. Then the map $A : M, q \to \mathbb{S}^2$ has a singular point if and only if the asymptotic curve determined by A has a geodesic inflection, and the kernel of the derivative of A at q contains $A(q)$, in other words, we are on the flecnodal curve of M.*

Proof. For any smooth unit speed curve $\gamma : I \to M$ through $q = \gamma(0)$, we have $A(\gamma(s)) \cdot A(\gamma(s)) \equiv 1$, $A(\gamma(s)) \cdot N(\gamma(s)) \equiv 0$. Differentiating and evaluating at 0 we find that $DA(q)(v) \cdot A(q) = 0$, $DA(q)(v) \cdot N(q) + A(q) \cdot S_q(v) = 0$ where $v = \gamma'(0)$. Let B be a vector field making A, B, N an orthonormal triple. If $v = A(q)$ then as $A(q) \cdot S_q(A(q)) = 0$ by definition of an asymptotic direction we find that $DA(q)(A(q))$ is orthogonal to both $A(q)$ and $N(q)$. So either $DA(q)(A(q))$ is zero or it is a non-zero multiple of $B(q)$. Suppose the latter and that we are at a singular point of A. Then the image of $DA(q)$ is generated by $B(q)$ and the second equation above reduces to $A(q) \cdot S_q(v) = 0$ for all $v \in T_qM$. It follows that S_q is singular, which contradicts our hypothesis that q is hyperbolic. So $DA(q)(A(q)) = \gamma''(0) = 0$, where γ is the asymptotic curve through q determined by A; in particular, this curve has a geodesic inflection. Conversely, if we have a geodesic inflection then $\gamma''(0) \cdot B(q) = 0$; but we already know that $\gamma''(0)$ is orthogonal to $A(q)$ and $N(q)$ so $\gamma''(0) = DA(q)(A(q)) = 0$ and A has a singular point. $\square$

At each hyperbolic point on M, there are two asymptotic directions, at each parabolic point one and none in the elliptic region. Therefore, near each hyperbolic point, we can find 2 sets of frames determined by an asymptotic direction A, a tangent direction orthogonal to A and the unit normal. We want to extend these families of frames across the parabolic set. For a generic surface M the parabolic set is smooth and we can consider the set of asymptotic directions in the projective tangent bundle $\tilde{M} \subset PTM$ to obtain a double cover of the non-elliptic points, ramified over the parabolic curve with each point of the double cover giving a frame, that is, we obtain a smooth mapping $\tilde{M} \to \mathbb{F}$. In particular, we have the maps $\tilde{A}, \tilde{N} : \tilde{M} \to \mathbb{S}^2$ given by $\tilde{A}(q) = A(\pi(\tilde{q}))$ and $\tilde{N}(q) = N(\pi(\tilde{q}))$, where $\pi : PTM \to M$ is the canonical projection.

Taking a local chart in PTM, the surface $\tilde{M}$ is given by

$$\tilde{M} = \{(x, y, p) : l(x, y)p^2 + 2m(x, y)p + n(x, y) = 0\}.$$

For a generic surface M, $\tilde{M}$ is a smooth 2-dimensional manifold, with the projection $\tilde{M} \to M$ having a fold singularity along the set consisting of parabolic points and their unique asymptotic direction.

To carry out the next set of calculations, we shall work with the representation of $\mathbb{F}$ as $\Delta = \{(a, b) \in \mathbb{S}^2 \times \mathbb{S}^2 : a \cdot b = 0\}$.

Lemma 5.10. *The tangent space to $\Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ at (a, b) is generated by*

$$\{(a \wedge b, 0), (0, a \wedge b), (b, -a)\}.$$

Proof. The set Δ is given by $a \cdot a = b \cdot b = 1,\ a \cdot b = 0$, so differentiating a path $(a(t), b(t))$ in $\mathbb{S}^2 \times \mathbb{S}^2$ we find that $(a'(0), b'(0)) = (u, v)$ is tangent at (a, b) if and only if $u \cdot a = v \cdot b = u \cdot b + v \cdot a = 0$. One checks that the three vectors above satisfy these conditions and are independent. $\square$

We consider now the frame $(\tilde{A}, \tilde{N}) : \tilde{M} \to \Delta$ at a point $\tilde{q} \in \tilde{M}$ corresponding to a parabolic point $q \in M$.

THEOREM 5.11. *Let $q \in M$ be a parabolic point but not a cusp of Gauss and let A_0 be the unique asymptotic direction at q, so $\tilde{q} = (q, [A_0])$.*

(i) *The germ $\tilde{A} : \tilde{M}, \tilde{q} \to \mathbb{S}^2$ has a fold singularity unless q is a goose singularity of the orthogonal projection along A_0. At the point corresponding to a goose singularity the map $\tilde{A}$ has a cusp singularity.*

(ii) *The germ $\tilde{N} : \tilde{M}, \tilde{q} \to \mathbb{S}^2$ is not finitely-$\mathcal{A}$-determined.*

(iii) *The frame germ $(\tilde{A}, \tilde{N}) : \tilde{M}, \tilde{q} \to \Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ is an immersion.*

Proof. We take M in Monge form $\phi(x, y) = (x, y, f(x, y))$ at the origin with f as in (2.1). We take the asymptotic direction at the origin, which we assume to be a parabolic point, along $(0, 1, 0)$, so $a_{21} = 0$ and $a_{22} = 0$.

The height function on M in the normal direction $(0, 0, 1)$, that is, the function f, has an A_2-singularity at the origin (the origin is not a cusp of Gauss). Therefore, $a_{20} \neq 0$ and $a_{33} \neq 0$.

The Gauss map is given by

$$N = \frac{1}{(f_x^2 + f_y^2 + 1)^{1/2}} (-f_x, -f_y, 1)$$

and choosing a chart in the target by projecting to the tangent plane to $\mathbb{S}^2$ at $(0, 0, 1)$, we can compute its initial terms

$$N = (-2a_{20}x - 3a_{30}x^2 - 2a_{31}xy - a_{32}y^2 + O(3), -a_{31}x^2 - 2a_{32}xy - 3a_{33}y^2 + O(3)).$$

(i) As usual writing E, F, G, respectively l, m, n for the coefficients of the first, resp. second, fundamental form in the given co-ordinate system, the asymptotic directions are solutions of the binary differential equation

$$l(x, y)dx^2 + 2m(x, y)dxdy + n(x, y)dy^2 = 0,$$

with

$$\begin{aligned} l(x, y) &= 2a_{20} + 6a_{30}x + 2a_{31}y + 12a_{40}x^2 + 6a_{41}xy + 2a_{42}y^2 + O(3), \\ m(x, y) &= 2a_{31}x + 2a_{32}y + 3a_{41}x^2 + 4a_{42}xy + 3a_{43}y^2 + O(3), \\ n(x, y) &= 2a_{32}x + 6a_{33}y + 2a_{42}x^2 + 6a_{43}xy + 12a_{44}y^2 + O(3). \end{aligned}$$

As $l(0, 0) \neq 0$, we can set $p = dx/dy$ and the equation of the asymptotic directions becomes

$$l(x, y)p^2 + 2m(x, y)p + n(x, y) = 0. \tag{5.2}$$

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Thus a unit asymptotic direction on M is given by

$$A = \frac{1}{(Ep^2 + 2Fp + G)^{1/2}}(p\phi_x + \phi_y)$$

with p a solution of (5.2).

As seen above the surface $\tilde{M} = \{(x, y, p) : l(x, y)p^2 + 2m(x, y)p + n(x, y) = 0\}$ is smooth and is the graph of a function $y = y(x, p)$. We find (using Maple) that

$$j^3y(x, p) = t_{10}x + t_{20}x^2 + t_{21}xp + t_{22}p^2 + t_{30}x^3 + t_{31}x^2p + t_{32}xp^2 + t_{33}p^3,$$

with

$$\begin{aligned}
t_{10} &= -\frac{1}{3a_{33}}a_{32},\\
t_{20} &= -\frac{1}{9a_{33}^3}(2a_{44}a_{32}^2 + 3a_{42}a_{33}^2 - 3a_{43}a_{32}a_{33}),\\
t_{21} &= -\frac{2}{9a_{33}^2}(3a_{31}a_{33} - a_{32}^2),\\
t_{22} &= -\frac{1}{3a_{33}}a_{20},\\
t_{30} &= \frac{1}{81a_{33}^5}(-24a_{44}^2a_{32}^3 - 36a_{44}a_{32}a_{42}a_{33}^2 + 54a_{44}a_{32}^2a_{43}a_{33} + 27a_{53}a_{32}a_{33}^3 +\\
&\quad 27a_{43}a_{33}^3a_{42} - 27a_{43}^2a_{33}^2a_{32} - 27a_{52}a_{33}^4 - 18a_{54}a_{32}^2a_{33}^2 + 10a_{55}a_{32}^3a_{33}),\\
t_{31} &= -\frac{1}{9a_{33}^4}(9a_{41}a_{33}^3 - 6a_{43}a_{33}^2a_{31} + 5a_{43}a_{32}^2a_{33} + 8a_{44}a_{32}a_{31}a_{33} - 4a_{44}a_{32}^3\\
&\quad - 6a_{42}a_{32}a_{33}^2),\\
t_{32} &= -\frac{1}{27a_{33}^3}(4a_{32}^3 - 15a_{31}a_{32}a_{33} - 9a_{43}a_{20}a_{33} + 12a_{44}a_{32}a_{20} + 27a_{30}a_{33}^2),\\
t_{33} &= \frac{2}{9a_{33}^2}a_{32}a_{20}.
\end{aligned}$$

As before when considering the 3-jet of the map $\tilde{A} : \tilde{M} \to \mathbb{S}^2$, with $\tilde{A}(x, p) = A(x, y(x, p))$, we choose a chart in the target by projecting to the tangent space, and the 3-jet is given by

$$\begin{aligned}
j^3(\tilde{A}_1, \tilde{A}_3) = \Bigg(p - \frac{1}{2}p^3, &\frac{3a_{33}a_{31} - a_{32}^2}{3a_{33}}x^2 + 2a_{20}xp\\
&+ \frac{27a_{41}a_{33}^3 - 18a_{42}a_{32}a_{33}^2 + 9a_{43}a_{32}^2a_{33} - 4a_{44}a_{32}^3}{27a_{33}^3}x^3\\
&+ \frac{27a_{30}a_{33}^2 - 6a_{31}a_{32}a_{33} + a_{32}^3}{9a_{33}^2}x^2p\Bigg).
\end{aligned}$$

Here we do not need to use the recognition criteria in the proof of Proposition 5.7 as elementary changes of coordinates in the source and target show that this 3-jet

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is $\mathcal{A}^{(3)}$-equivalent to

$$\left(p, \frac{3a_{33}a_{31} - a_{32}^2}{3a_{33}}x^2 + 2a_{20}xp + \frac{27a_{41}a_{33}^3 - 18a_{42}a_{32}a_{33}^2 + 9a_{43}a_{32}^2a_{33} - 4a_{44}a_{32}^3}{27a_{33}^3}x^3\right).$$

We can deduce immediately that $\tilde{A}$ has a singularity at the origin of type

$$\text{fold} \iff 3a_{31}a_{33} - a_{32}^2 \neq 0;$$

$$\text{cusp} \iff 3a_{31}a_{33} - a_{32}^2 = 0, 27a_{41}a_{33}^3 - 18a_{42}a_{32}a_{33}^2 + 9a_{43}a_{32}^2a_{33} - 4a_{44}a_{32}^3 \neq 0.$$

The asymptotic direction at the origin is along $(0, 1, 0)$ and the projection of M along this direction is the map-germ $P(x, y) = (x, f(x, y))$. The conditions for $\tilde{A}$ to be a fold (resp. cusp) are precisely those for P to have a lips/beaks (resp. goose) singularity, see table 1.

(ii) The Gauss map $N : M \to \mathbb{S}^2$ has a fold singularity at an ordinary point on the parabolic curve (i.e., away from the cusps of Gauss). The map $(x, p) \mapsto (x, y(x, p))$ is clearly a fold map $(t_{22} \neq 0)$. Its discriminant is the critical set of the Gauss map. It follows that the composite map $\tilde{N}(x, p) = N(x, y(x, p))$ is not finitely-$\mathcal{A}$-determined along its critical set. For if it was $\mathcal{A}$-finite then by Gaffney's geometric criterion for finite determinacy its complexification would be stable off a point, see [**17**]. But it is not hard to see that along the set of critical values of the composite the local multiplicity is at least 4, which means that the mapping fails to be stable.

(iii) We have $\tilde{A}_0 = \tilde{A}(0, 0) = (0, 1, 0)$ and $\tilde{N}_0 = \tilde{N}(0, 0) = (0, 0, 1)$. By lemma 5.10, the tangent space to $\Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ at $(\tilde{A}_0, \tilde{N}_0)$ is spanned by $(\tilde{A}_0 \wedge \tilde{N}_0; 0)$, $(0; \tilde{A}_0 \wedge \tilde{N}_0)$ and $(\tilde{N}_0, -\tilde{A}_0)$, that is by

$$((1, 0, 0); (0, 0, 0)),\ ((0, 0, 0); (1, 0, 0)),\ ((0, 0, 1); (0, -1, 0)).$$

Composing the map $(\tilde{A}, \tilde{N}) : \tilde{M} \to \Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ with the projection to $T_{(\tilde{A}_0, \tilde{N}_0)}\Delta$ yields an $\mathcal{A}$-equivalent map-germ. If we write $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ and $\tilde{N} = (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$ then

$$(\tilde{A}, \tilde{N}) \sim_{\mathcal{A}} (\tilde{A}_1, \tilde{N}_1, -\tilde{A}_3 + \tilde{N}_2).$$

The 1-jet of the $(\tilde{A}_1, \tilde{N}_1, -\tilde{A}_3 + \tilde{N}_2)$ is $(p, -2a_{20}x, 0)$, so $(\tilde{A}, \tilde{N})$ is an immersion. (We are assuming that $a_{20} \neq 0$, that is, $\kappa_1(0, 0) \neq 0$; generically this is the case.) □

THEOREM 5.12. *Let q be a cusp of Gauss of the surface M, with $\tilde{q}$ the corresponding point on $\tilde{M}$.*

(i) *Generically, the germ $\tilde{A} : \tilde{M}, \tilde{q} \to \mathbb{S}^2$ has a beaks singularity.*

(ii) *The germ $\tilde{N} : \tilde{M}, \tilde{q} \to \mathbb{S}^2$ is not finitely-$\mathcal{A}$-determined.*

(iii) *Generically, the frame map $(\tilde{A}, \tilde{N}) : \tilde{M}, \tilde{q} \to \Delta \subset \mathbb{S}^2 \times \mathbb{S}^2$ has an S_1-singularity.*

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Proof. We follow the same steps as in the proof of Theorem 5.11. Here, we take $a_{21}=a_{22}=a_{33}=0$ and $4a_{20}a_{44}-a_{32}^2\neq 0$ (the height function has an A_3-singularity). One easily checks that the set $\tilde{M}$ is a smooth surface if and only if $a_{32}\neq 0$, that is, if and only if the parabolic set on M is a smooth curve. When this is the case it can be parametrized as the graph of a smooth function $x=x(y,p)$ with

$$j^3x(y,p)=-\frac{6a_{44}}{a_{32}}y^2-2yp-\frac{a_{20}}{a_{32}}p^2+\frac{1}{a_{32}^2}(2(9a_{43}a_{44}-5a_{55}a_{32})y^3+ \\ 3(4a_{31}a_{44}+a_{43}a_{32})y^2p+3(a_{43}a_{20}+a_{31}a_{32})yp^2+2a_{31}a_{20}p^3).$$

(i) The map-germ $\tilde{A}=(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3)$ is $\mathcal{A}$-equivalent to $(\tilde{A}_1,\tilde{A}_3)$ whose 3-jet is

$$\left(p-\frac{1}{2}p^3,-8a_{44}y^3-\frac{3}{a_{32}}(4a_{20}a_{44}+a_{32}^2)py^2-6a_{20}p^2y-\frac{2a_{20}^2}{a_{32}}p^3\right).$$

Provided that $a_{44}a_{32}\neq 0$ this is $\mathcal{A}^{(3)}$-equivalent to

$$\left(p,-8a_{44}y^3+\frac{3(a_{32}^2-4a_{20}a_{44})^2}{8a_{44}a_{32}^2}yp^2\right),$$

which is a beaks singularity. The non-vanishing of $a_{32}a_{44}$ is a necessary condition for the projection along the asymptotic direction to have a gulls singularity, see table 1. (Generically, the projection has a gulls singularity at a cusp of Gauss.)

(ii) The statement follows, as in the proof of Theorem 5.11(ii).

(iii) Following the proof of Theorem 5.11(iii), $(\tilde{A},\tilde{N})$ is $\mathcal{A}$-equivalent to $(\tilde{A}_1,\tilde{N}_1,-\tilde{A}_3+\tilde{N}_2)$ which has a 3-jet $\mathcal{A}^{(3)}$-equivalent to

$$\left(p,\frac{12a_{20}a_{44}-a_{32}^2}{a_{32}}y^2,16a_{44}y^3-\frac{36a_{20}(a_{32}^2-4a_{20}a_{44})^2}{(12a_{20}a_{44}-a_{32}^2)^2}yp^2\right)$$

when $12a_{20}a_{44}-a_{32}^2\neq 0$. This is an S_1-singularity if furthermore $a_{44}\neq 0$. Both conditions are satisfied for generic surfaces at cusps of Gauss. The equality $12a_{20}a_{44}-a_{32}^2=0$ is the condition that the asymptotic curves have a folded saddle-node singularity (see [**5**]). □

REMARK 5.13. Theorem 5.12(iii) gives a geometric setting where the singularity S_1 (which is of $\mathcal{A}_e$-codimension 1) occurs stably.

6. Singular foliations of the sphere

There are geometric foliations on the surface M obtained by integrating the asymptotic (resp. principal) direction fields. We are interested in the image of these foliations in the sphere S^2 by the asymptotic (resp. principal) map. We also comment on the image of the foliations by the Gauss map. To do this, we consider the

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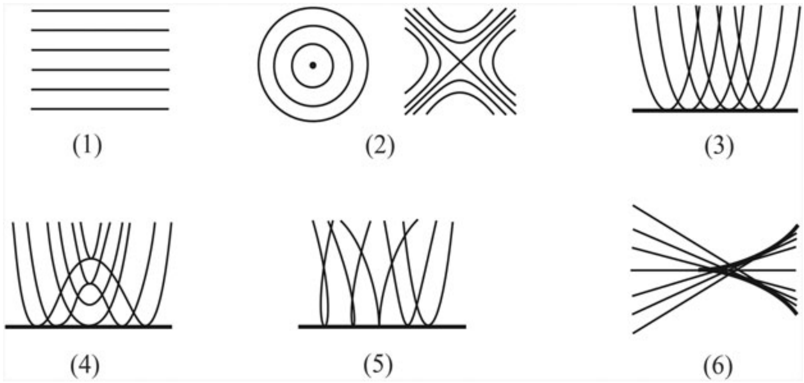

Figure 4. The generic types of divergent mapping diagrams.

foliation on the surface $\tilde{M}$ for the asymptotic curves and on M for the principal curves; when the foliation is regular, it is given locally as the level sets of a submersion μ. Let g denote the map $\tilde{A}$, $P_i, i = 1, 2$, or N and consider the divergent diagram (f, g):

$$\mathbb{R}, 0 \xleftarrow{\mu} \mathbb{R}^2, 0 \xrightarrow{g} \mathbb{R}^2, 0.$$

The above diagrams are studied by Dufour in [8, 9]. Two such germs (g_1, μ_1), (g_2, μ_2) of divergent diagrams are equivalent if the diagram

$$\begin{array}{ccccc} \mathbb{R}, 0 & \xleftarrow{\mu_1} & \mathbb{R}^2, 0 & \xrightarrow{g_1} & \mathbb{R}^2, 0 \\ \downarrow h & & \downarrow k & & \downarrow l \\ \mathbb{R}, 0 & \xleftarrow{\mu_2} & \mathbb{R}^2, 0 & \xrightarrow{g_2} & \mathbb{R}^2, 0 \end{array}$$

commutes for some germs of diffeomorphisms h, k, l.

Theorem 6.1 [8, 9]. *There are six generic types of divergent mapping diagrams and these are characterized as follows (figure 4):*

(1) *g is a diffeomorphism, μ is a submersion;*

(2) *g is a diffeomorphism, μ has a Morse singularity;*

(3) *g has a fold singularity, μ restricted to the singular set Σ_g of g is regular and $(g, \mu) : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ is regular;*

(4) *g has a fold singularity, $\mu|_{\Sigma_g}$ has a Morse singularity, and (g, μ) is regular;*

(5) *g has a fold singularity, (g, μ) is a cross-cap whose double points curve is transverse at 0 to the plane $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^3$;*

(6) *g has a cusp singularity and (g, μ) is regular.*

In [10], the authors studied divergent diagrams induced by germs of first order ordinary differential equations (or, briefly, equations) with independent first integral. An equation is defined to be the germ of the surface $N = F^{-1}(0)$, with

$F : PT^*\mathbb{R}^2, z \to \mathbb{R}, 0$ a germ of a smooth function. The projectivised cotangent bundle $PT^*\mathbb{R}^2$ of $\mathbb{R}^2$ is endowed with the canonical contact structure given by the 1-form $\alpha = dy - pdx$. The surface N is supposed to be smooth in [**10**], so is locally the image of a germ of an immersion $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$. The equation is then represented by the germ f.

Let $\pi : PT^*\mathbb{R}^2 \to \mathbb{R}^2$ be the natural projection. Two germs of immersions (equations) $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$ and $f' : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z'$ are said to be equivalent if there exist germs of diffeomorphisms $\psi : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ and $\phi : \mathbb{R}^2, \pi(z) \to \mathbb{R}^2, \pi(z')$ such that $\hat{\phi} \circ f = f' \circ \psi$, where $\hat{\phi} : PT^*\mathbb{R}^2, z \to PT^*\mathbb{R}^2, z'$ is the lift of ϕ.

Suppose that the equation f has a first integral, that is, there exists a germ of a submersion $\mu : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ such that $d\mu \wedge f^*\alpha = 0$. As the solutions of the equation in the plane are the images under $\pi \circ f$ of the level sets of μ, it is natural to consider the divergent mapping diagram $\mathbb{R}, 0 \xleftarrow{\mu} \mathbb{R}^2, 0 \xrightarrow{\pi \circ f} \mathbb{R}^2, 0$. Consider in general a divergent mapping diagram (g, μ)

$$\mathbb{R}, 0 \xleftarrow{\mu} \mathbb{R}^2, 0 \xrightarrow{g} \mathbb{R}^2, 0$$

where g is a smooth map germ and μ is a germ of a submersion. The diagram (g, μ) is called an *integral diagram* if there exists a germ of an immersion $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$ such that $d\mu \wedge f^*\alpha = 0$ and $g = \pi \circ f$ [**10**]. Then (g, μ) is said to be induced by f. Suppose given two germs of equations f and f' with first integrals and with the set of critical points of $\pi \circ f$ and $\pi \circ f'$ nowhere dense. Then f and f' are equivalent as equations if and only if the diagrams $(\pi \circ f, \mu)$ and $(\pi \circ f', \mu')$ are equivalent as mapping diagrams ([**10**, proposition 2.8]). (The 'if' part of [**10**, proposition 2.8] remains true if in the definition of f having a first integral one allows μ to have singularities. The 'only if' part holds when μ is a submersion.) A weaker equivalence relation between integral diagrams is introduced in [**10**]) and the following result proved there.

THEOREM 6.2 ([**10**, theorem B]). *An integral diagram of generic type is equivalent to one of the following integral diagrams* (g, μ)*:*

(1) *Non-singular:* $g = (u, v)$, $\mu = v$.

(2) *Regular fold:* $g = (u^2, v)$, $\mu = v - 1/3u^3$.

(3) *Clairaut fold:* $g = (u, v^2)$, $\mu = v - 1/2u$.

(4) *Regular cusp:* $g = (u^3 + uv, v)$, $\mu = 3/4u^4 + 1/2u^2v + \beta \circ g$, *where* $\beta(x, y)$ *is a germ of a smooth function with* $\beta(0) = 0$ *and* $\beta_y(0) = \pm 1$.

(5) *Clairaut cusp:* $g = (u, v^3 + uv)$, $\mu = v + \beta \circ g$, *where* $\beta(x, y)$ *is a germ of a smooth function with* $\beta(0) = 0$.

(6) *Mixed fold:* $g = (u, v^3 + uv^2)$, $\mu = 1/2v^2 + \beta \circ g$, *where* $\beta(x, y)$ *is a germ of a smooth function with* $\beta(0) = 0$ *and* $\beta_x(0) = 1$.

The configurations of the solutions of the associated equations are as shown in figure 5, (1)–(6).

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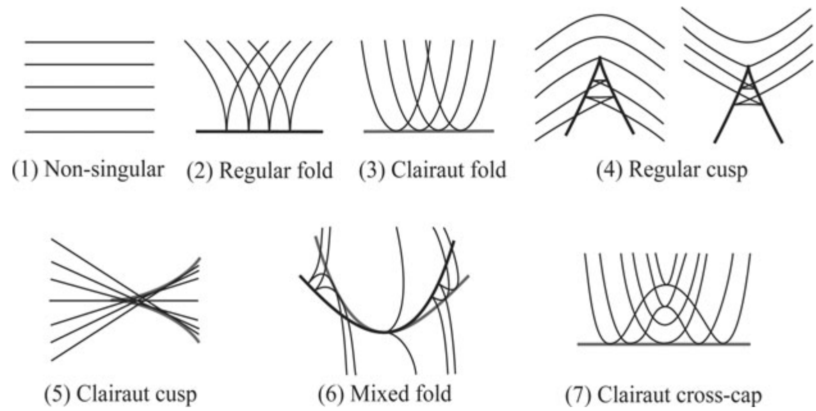

Figure 5. Configurations of generic integral diagrams (1)–(6) and of the Clairaut cross-cap (7).

The case when the surface N of the equation has a cross-cap singularity is studied in [**7**]. The generic model is the Clairaut cross-cap $g = (u, 1/4v^2)$, $\mu = v - 1/2u^2$ ([**7**, theorem 2.7]); see figure 5(7).

Remark 6.3. It is worth observing that there are pairs (g, μ) which are generic as mapping diagrams but not as integral diagrams and vice-versa (compare figures 4 and 5).

6.1. Images of the principal curves by the principal map

We start by considering the image of the principal curves in $\mathbb{S}^2$ by the principal map, say P_2. If P_2 is a regular map, then it will map the principal foliation $\mathcal{F}_2$ associated with P_2 (and $\mathcal{F}_1$ associated with P_1) to a regular foliation on the unit sphere. We shall suppose that the map P_2 is singular. Recall from proposition 5.7 that P_2 has critical points along the sub-parabolic and parabolic curves ($\kappa_2 = 0$ or $\partial\kappa_1/\partial x_2 = 0$). We take a coordinate system where the coordinate curves represent the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. The derivative map of P_2 at a point q with respect to the standard basis for T_qM and basis $\{P_1, N\}$ for the tangent space to $\mathbb{S}^2$ at $P_2(x)$ is

$$\begin{pmatrix} \dfrac{1}{\kappa_2 - \kappa_1}\dfrac{\partial\kappa_1}{\partial x_2} & \dfrac{1}{\kappa_2 - \kappa_1}\dfrac{\partial\kappa_2}{\partial x_1} \\ 0 & \kappa_2 \end{pmatrix}.$$

Theorem 6.4.

(1) *Away from the beaks singularities of the map P_2, the configurations of the images of the principal curves of the foliation $\mathcal{F}_2$ of a generic surface by the principal map P_2 are those of a generic divergent diagram and are as follows:*
 (i) *When $\kappa_2 \neq 0$ and $\partial\kappa_1/\partial x_2 = 0$:*
  - *At a fold singularity of P_2:*
   – *at most points on the sub-parabolic curve: figure 4(3).*

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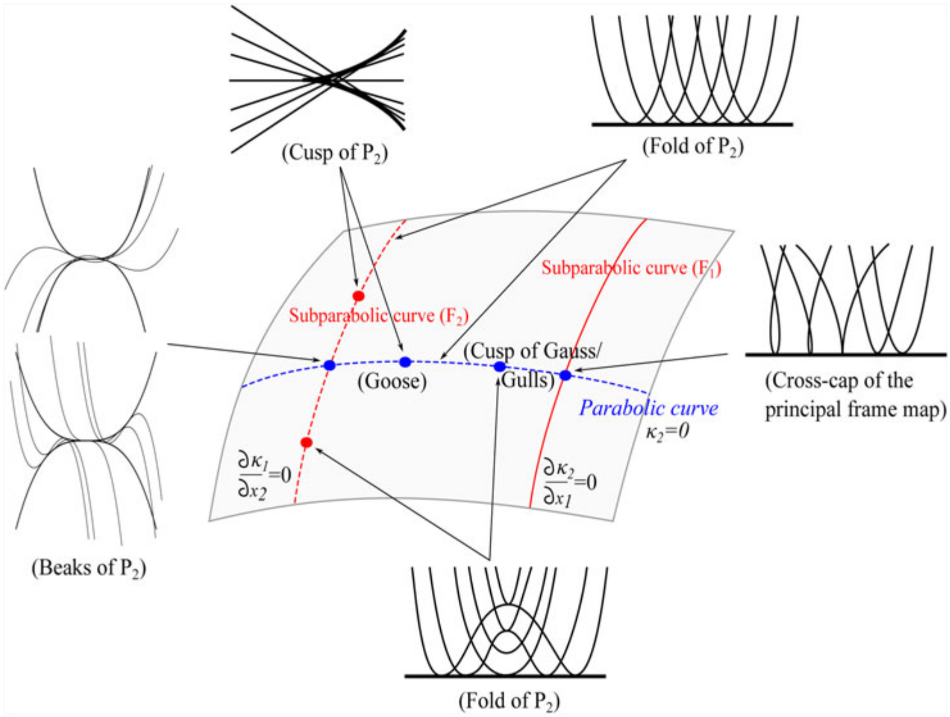

Figure 6. Images of the foliation $\mathcal{F}_2$ by the principal map P_2.

– *at isolated points on the sub-parabolic curve: figure 4(4).*

- *At a cusp singularity of P_2: figure 4(6).*

(ii) *When $\kappa_2 = 0$ and $\partial\kappa_1/\partial x_2 \neq 0$ at q:*
- *At a fold singularity of P_2:*
  – *at most points on the parabolic curve: figure 4(3).*

  – *at a cusp of Gauss: figure 4(4)*

  – *at a cross-cap point of the principal frame: figure 4(5).*

- *At a cusp singularity of P_2: figure 4(6).*

(2) *At a beaks singularity of P_2, that is, when $\kappa_2 = \partial\kappa_1/\partial x_2 = 0$ at q, the configurations are not one of a generic divergent diagrams. They are as in figure 7.*

See figure 6.

Proof. (1) We take the point q to be the origin.

(i) The sub-parabolic curve associated with the map P_2 is given by $\partial\kappa_1/\partial x_2 = 0$ and the kernel of DP_2 at each point on the sub-parabolic curve is parallel to $(1, 0)$. At a fold singularity of P_2, the sub-parabolic

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curve is transverse to $(1,0)$, that is, $\partial^2\kappa_1/\partial x_1\partial x_2 \neq 0$ at the point of interest. At generic points of the sub-parabolic curve, we also have $\partial^2\kappa_1/\partial x_2^2 \neq 0$, that is, the sub-parabolic curve is transverse to the foliation $\mathcal{F}_2$. In this case, we are in the case (3) of theorem 6.1 and the image of the foliation $\mathcal{F}_2$ by P_2 is as in figure 4(3).

At isolated points on the parabolic curve, which again we can translate to the origin, we can have $\partial^2\kappa_1/\partial x_2^2(0,0) = 0$. Then the contact of the sub-parabolic curve with the leaf $x_1 = 0$ of $\mathcal{F}_2$ is of order 2 (ordinary tangency) if and only if $\partial^3\kappa_1/\partial x_2^3(0,0) \neq 0$. This is, of course, satisfied for generic surfaces, so we are in the case (4) of theorem 6.1 and the image of the foliation $\mathcal{F}_2$ by P_2 is as in figure 4(4).

If the origin is a cusp singularity of the map P_2, the foliation $\mathcal{F}_2$ is transverse to the sub-parabolic curve. The map (P_2, x_1) is regular, so we are in the case (6) of theorem 6.1 and the image of the foliation $\mathcal{F}_2$ by the map P_2 is as in figure 4(6).

(ii) Here we are on that part of the parabolic curve $\kappa_2 = 0$ and the kernel of DP_2 at each point is parallel to $(\partial\kappa_2/\partial x_1, -\partial\kappa_1/\partial x_2)$. When P_2 has a fold singularity at q and $\partial\kappa_2/\partial x_1(q) \neq 0$ (i.e., q is not a sub-parabolic point associated with P_1), the line of principal curvature in $\mathcal{F}_2$ is transverse to the kernel of P_2. If the line of principal curvature is also transverse to the parabolic, that is, $\partial\kappa_2/\partial x_2(q) \neq 0$, then the configuration is as in figure 4(3). If it is tangent to the parabolic set ($\partial\kappa_2/\partial x_2(q) = 0$; occurs at a cusp of Gauss), we have generically the configuration in figure 4(4). When $\partial\kappa_2/\partial x_1(q) = 0$ (i.e. at a cross-cap point of the principal frame), we are generically in the case 5 of theorem 6.1, so the configuration is as in figure 4(5). Therefore, the image of a line of curvature in $\mathcal{F}_2$ is singular. If P_2 has a cusp singularity, then generically the line of curvature is transverse to the parabolic set (which is tangent to the kernel of DP_2) so the configuration is as in figure 4(6).

(2) When P_2 has a beaks singularity, the leaves of the foliation $\mathcal{F}_2$ are transverse to the parabolic and sub-parabolic curves as well as to the kernel of DP_2. We can make changes of coordinates in the source and target and take P_2 in the $\mathcal{A}$-normal form $g = (x, x^2y - y^3)$. We have a divergent diagram (g, μ) which is not one of the generic cases in theorem 6.1. Here, we have two possible configurations depending on the position of the fibre $\mu^{-1}(0)$. It can be mapped by g to the regions where each point has one pre-images by g (figure 7 left) or to the regions where each point has three pre-images (figure 7 right). We consider an example of μ for each case and draw the images of the fibres of μ by g. □

Theorem 6.5. *For a generic surface and away from parabolic points or at generic parabolic points, the images of the lines of principal curvature in* $\mathcal{F}_2$ *by the map* P_2 *are solutions of a first order differential equation.*

Proof. The lines of principal curvature are smooth curves as we are away from umbilic points, so locally they are the fibres of some submersion $\mu : M, q \to \mathbb{R}, 0$.

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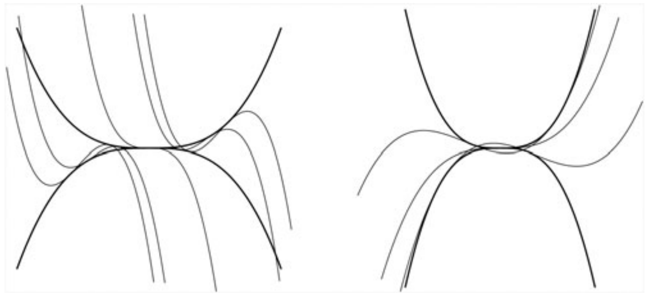

Figure 7. Image of the foliation $\mathcal{F}_2$ by P_2 at the intersection of the parabolic and sub-parabolic curves (thin curves; images of the parabolic and sub-parabolic curves in thick).

We need to show that the divergent diagram (P_2, μ) is an integral diagram. That is, we need to construct a germ of an immersion $f : M, q \to PT^*\mathbb{S}^2$ such that $\pi \circ f = P_2$ and $d\mu \wedge f^*\alpha = 0$, where $\pi : PT^*\mathbb{S}^2 \to \mathbb{S}^2$ is the canonical projection of the cotangent bundle and α is the canonical 1-form in $PT^*\mathbb{S}^2$. We take a local coordinate system in M where the coordinate curves are the lines of principal curvatures. Suppose that a line of curvature in $\mathcal{F}_2$ is parametrized by $(c_1, t + c_2)$. Then we require that its image by f to be a Legendrian curve, that is, we require that

$$\begin{aligned} f(c_1, t+c_2) &= (P_2(c_1, t+c_2), [(P_2(c_1, t+c_2))']) \\ &= (P_2(c_1, t+c_2)), \left[\frac{\partial P_2}{\partial x_2}(c_1, t+c_2)\right]\Big) \\ &= \left(P_2(c_1, t+c_2), \left[\left(\frac{1}{k_2 - k_1}\frac{\partial \kappa_2}{\partial x_1}P_1 + \kappa_2 N\right)(c_1, t+c_2)\right]\right). \end{aligned}$$

For this reason, we define $f : M, q \to PT^*\mathbb{S}^2$ by

$$\begin{aligned} f(x_1, x_2) &= \left(P_2(x_1, x_2), \left[\frac{\partial P_2}{\partial x_2}(x_1, x_2)\right]\right) \\ &= \left(P_2(x_1, x_2), \left[\left(\frac{1}{k_2 - k_1}\frac{\partial \kappa_2}{\partial x_1}P_1 + \kappa_2 N\right)(x_1, x_2)\right]\right). \end{aligned}$$

We need f to be a germ of an immersion (or of a cross-cap in the case of Clairaut cross-cap). At points where P_2 is regular f is an immersion, so we only need to analyse the cases when P_2 is singular. Following proposition 5.7, we have three cases to consider.

Case (i): $\partial\kappa_1/\partial x_2 = 0, \kappa_2 \neq 0$ at q. Here, $\partial P_2/\partial x_1(q) = 0$ and $\partial P_2/\partial x_2(q) \neq 0$, so we need $\partial^2 P_2/\partial x_1 \partial x_2(q) \neq 0$. Differentiating and doting with P_1 and N, we find that $\partial^2 P_2/\partial x_1 \partial x_2(q) = 0$ if and only if

$$\left(\frac{\partial}{\partial x_1}\left(\frac{1}{\kappa_2 - \kappa_1}\frac{\partial k_2}{\partial x_1}\right) - \kappa_1\kappa_2\right)(q) = 0 \quad \text{and}$$
$$\left(\frac{\partial k_2}{\partial x_1}\left(1 + \frac{1}{(\kappa_2 - \kappa_1)^2}\frac{\partial k_2}{\partial x_1}\right)\right)(q) = 0.$$

As we have already one condition $\partial\kappa_1/\partial x_2(q) = 0$, so for generic surfaces, the above two equalities do not hold simultaneously. Therefore, in this case, for generic surfaces, f is a germ of an immersion.

Case (ii): $\partial\kappa_1/\partial x_2 \neq 0, \kappa_2 = 0$ at q. Then $\partial P_2/\partial x_1(q)$ and $\partial P_2/\partial x_2(q)$ are parallel to $P_1(q)$, so we need $\partial^2 P_2/\partial x_1\partial x_2(q)$ and $\partial^2 P_2/\partial x_2^2(q)$ to be linearly independent. The two vectors can become dependent at isolated points on the parabolic set. These points are in general distinct from the cusp of Gauss (where the configuration in as in figure 4(4)), so we do not have a Clairaut cross-cap at such points (i.e. f is a cross-cap singularity; figure 5(7)). At such points (P_2, μ) is a generic divergent diagram but is not a generic integral diagram (see remark 6.3).

Case (iii): $\partial\kappa_1/\partial x_2 = \kappa_2 = 0$ at q. This is similar to case (i) as $\partial P_2/\partial x_1(q) = 0$ and $\partial P_2/\partial x_2(q) \neq 0$. Generically, $\partial^2 P_2/\partial x_1\partial x_2(q) \neq 0$, so (P_2, μ) is indeed an integral diagram but is not a generic one (the configurations in figure 7 are not part of figure 5). □

We consider now the image of the foliation $\mathcal{F}_1$ by P_2.

THEOREM 6.6. *Suppose that q is a singular point of the map P_2 but not of the map P_1. Then the images of the lines of principal curvature in $\mathcal{F}_1$ by P_2 are solutions of a first-order differential equation with an integral diagram of the following type.*

(i) *When $\kappa_2 \neq 0$ and $\partial\kappa_1/\partial x_2 = 0$ at q*
  - *At a fold singularity of P_2: Regular fold (figure 5(2)).*
  - *At a cusp singularity of P_2: Regular cusp (figure 5(4)).*

(ii) *When $\kappa_2 = 0$ and $\partial\kappa_1/\partial x_2 \neq 0$ at q:*
  - *At a fold singularity of P_2: Clairaut Fold (figure 5(3)).*
  - *At a cross-cap singularity of the principal frame: Clairaut cross-cap (figure 5(7)).*
  - *At a cusp singularity of P_2: Clairaut cusp (figure 5(5)).*

(ii) *When $\kappa_2 = \partial\kappa_1/\partial x_2 = 0$ at q: Mixed fold (figure 5(6)).*

See figure 8.

Proof. Suppose that a line of curvature in $\mathcal{F}_1$ is parametrized by $(t + c_1, c_2)$. Following the proof of Theorem 6.5 and using the notation of that proof, we need to construct an immersion $f : M \to PT^*\mathbb{S}^2$ which maps $(t + c_1, c_2)$ to a Legentrian curve, so

$$\begin{aligned} f(t + c_1, c_2) &= (P_2(t + c_1, c_2), [(P_2(t + c_1, c_2))']) \\ &= \left(P_2(t + c_1, c_2), \left[\frac{\partial P_2}{\partial x_1}(t + c_1, c_2)\right]\right) \end{aligned}$$

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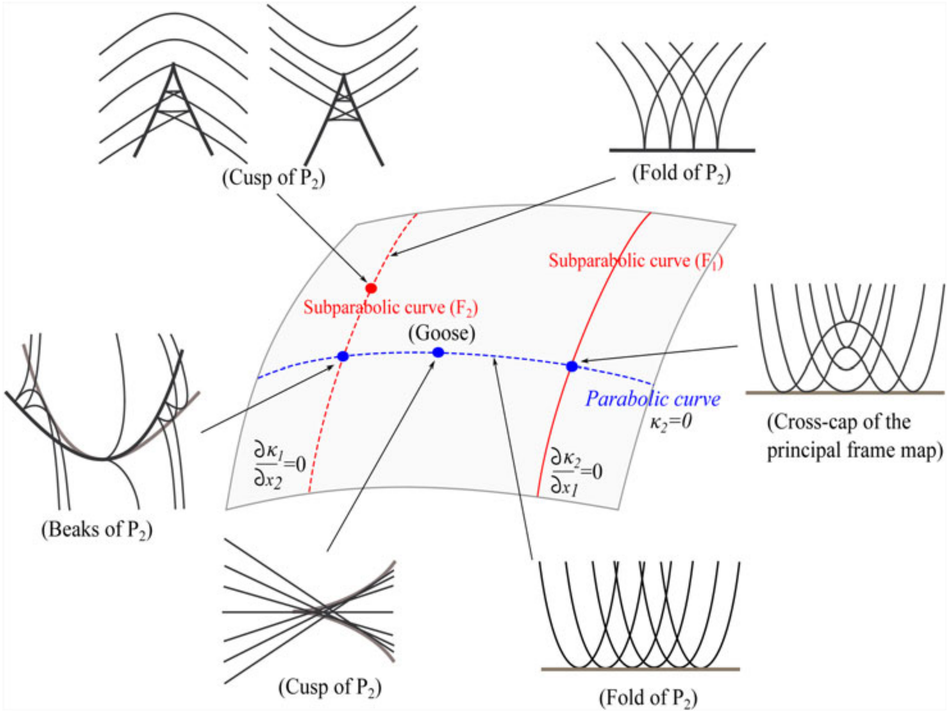

Figure 8. Images of the foliation $\mathcal{F}_1$ by the principal map P_2.

$$= \left(P_2(t+c_1, c_2), \left[\frac{\partial \kappa_1}{\partial x_2}(t+c_1, c_2)P_1(t+c_1, c_2)\right]\right)$$
$$= (P_2(t+c_1, c_2), [P_1(t+c_1, c_2)]) \text{ when } \frac{\partial \kappa_1}{\partial x_2} \neq 0.$$

For this reason, we define $f : M, q \to PT^*\mathbb{S}^2$ by

$$f(x_1, x_2) = (P_2(x_1, x_2), [P_1(x_1, x_2)]).$$

We have $d\mu \wedge f^*\alpha = 0$ by construction, where the fibres of μ are the leaves of $\mathcal{F}_1$. We only need to show that f is a germ of an immersion. This is straightforward using the fact that

$$DP_1 = \begin{pmatrix} \dfrac{1}{\kappa_2 - \kappa_1}\dfrac{\partial \kappa_1}{\partial x_2} & \kappa_1 \\ \dfrac{1}{\kappa_2 - \kappa_1}\dfrac{\partial \kappa_2}{\partial x_1} & 0 \end{pmatrix}, \quad DP_2 = \begin{pmatrix} \dfrac{1}{\kappa_2 - \kappa_1}\dfrac{\partial \kappa_1}{\partial x_2} & \dfrac{1}{\kappa_2 - \kappa_1}\dfrac{\partial \kappa_2}{\partial x_1} \\ 0 & \kappa_2 \end{pmatrix}$$

and the assumption that P_1 is not singular when P_2 is.

It is now a matter of identifying the type of the integral diagram from theorem 6.2. The arguments for the cases (i) and (ii) are similar to those in the proof of Proposition 6.4 and are omitted. For (iii) the map P_2 has a beaks singularity, the lines

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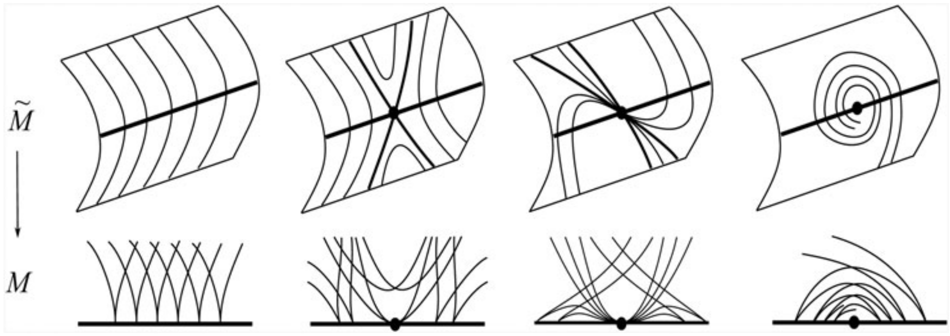

Figure 9. From left to right: family of cusps, folded saddle, folded node and folded focus.

of curvature in $\mathcal{F}_1$ are transverse to the two components of the critical set of P_2, are tangent to the kernel of DP_2 along one component of the critical set of P_2 and transverse to it along the other. This is case 6 in theorem 6.2. □

REMARK 6.7.

(1) At point when both P_2 and P_1 are singular, the map f is not an immersion.

(2) In the cases in theorem 6.6 (ii), the integral diagram (P_2, μ) is also a generic divergent diagram. For the cases (i) and (iii) of the same theorem, (P_2, μ) is not a generic divergent diagram but is a generic integral diagram.

6.2. Images of the lift of the asymptotic curves by the map $\tilde{A}$

Finally, we consider the asymptotic curves on the surface, which yield the standard family of cusps near an ordinary point of the parabolic set (figure 9, left) and generically there is one of the final three topological configurations in figure 9 at a cusp of Gauss. The asymptotic curves are lifted to a foliation on the surface $\tilde{M}$ which are integral curves of a vector field ξ. The vector field ξ is regular at a lift of an ordinary parabolic point and generically has a saddle, node or focus singularity at a cusp of Gauss with the singularities of the asymptotic curves labelled, accordingly, folded saddle, node or focus.

PROPOSITION 6.8.

(1) *Suppose that $q \in M$ is a parabolic point but not a cusp of Gauss, that is, the Gauss map has a fold at q, let $\tilde{q}$ denote the corresponding point of $\tilde{M}$. Then the images of the integral curves of ξ near $\tilde{q}$ by the map $\tilde{A} : \tilde{M} \to \mathbb{S}^2$ are those of a generic divergent diagram as well as of a generic integral diagram and are as follows:*

   (i) *If q is not a goose singularity of the orthogonal projection of M along the asymptotic direction: figure 4(3) or figure 5(3).*

   (ii) *If q is a goose singularity of the projection of M along the asymptotic direction: figure 4(6) or figure 5(5).*

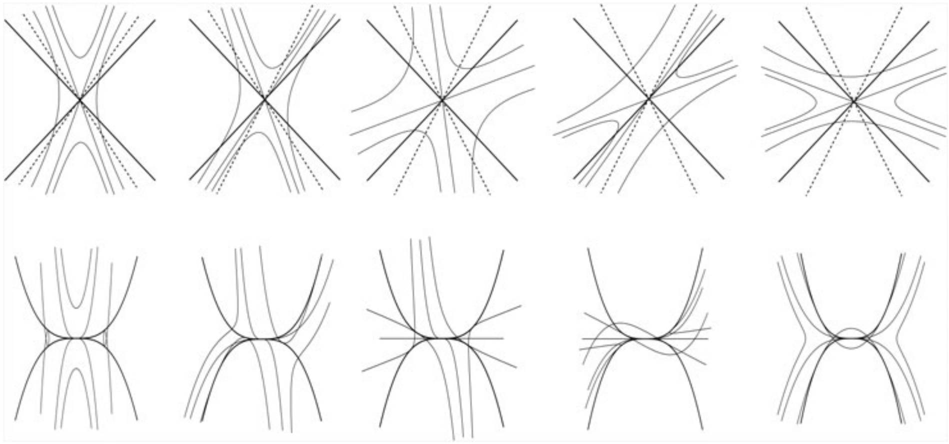

Figure 10. The Beaks-Saddle cases. Top figures: the configurations of the integral curves of ξ on $\tilde{M}$ in thin, the critical set of g in thick, and the set $S(g)$ in discontinuous. Bottom figures: the images of configurations of the top figures in $\mathbb{S}^2$ by the map $\tilde{A}$.

(2) *At a cusp of Gauss, the images of the integral curves of ξ by the map $\tilde{A}$ are neither those of a generic divergent diagram nor those of a generic integral diagram. They are generically as in figures 10, 11, 12 at respectively a saddle, node, focus singularity of ξ.*

Proof.

(1) The integral curves of ξ are smooth so are the level sets of a submersion g. We consider the divergent diagram $\mathbb{R}, 0 \xleftarrow{\mu} \tilde{M}, \tilde{q} \xrightarrow{\tilde{A}} \mathbb{S}^2, \tilde{A}(\tilde{q})$. The map $\tilde{A}$ is locally a fold in case (i) and a cusp in case (ii) (theorem 5.11). We need to prove that the germ $(\tilde{A}, \mu) : \tilde{M}, \tilde{q} \to \mathbb{S}^2 \times \mathbb{R}$ is regular (has maximal rank) at $\tilde{q}$ (see theorem 6.1). Following the proof of Theorem 5.11, an asymptotic curve $(x(t), y(t))$ at a parabolic point $q = (x(0), y(0))$ is lifted to the curve $\gamma(t) = (x(t), y(t), p(t))$ on $\tilde{M}$, with $\gamma(0) = \tilde{q}$ and $\gamma'(0)$ a multiple of $(0, 0, 1)$. Of course g has the integral curve γ as a fibre, so the condition for regularity is that $D\tilde{A}(\gamma(0))\gamma'(0) \neq 0$ in other words, $(\tilde{A} \circ \gamma)'(0) \neq 0$ (in fact it is clear from the picture that we need $\tilde{A} \circ \gamma$ to be an immersion at 0). This holds if and only if $\partial \tilde{A}/\partial p(\tilde{q}) \neq 0$, and this follows from the proofs of Theorems 5.11, and 5.12.

To show that $(\tilde{A}, \mu)$ is an integral diagram, we parametrize M by $(x, p) \mapsto (x, y(x, p), p)$ as in the proof Theorem 5.11 and consider the map germ $f : \mathbb{R}^2, 0 \to PT^*\mathbb{S}^2 \equiv PT^*\mathbb{R}^2$ given by $f(x, y) = (x, y(x, p), p)$ (taking an appropriate affine chart in $PT^*\mathbb{R}^2$). Clearly, f is a germ of an immersion, $(\mu, \pi \circ f) = (\tilde{A}, \mu)$ and $d\mu \wedge f^*\alpha = 0$ by construction.

(2) At a cusp of Gauss, $\tilde{A}$ has a beaks singularity (theorem 5.12), there are nearby flecnodal points where A is singular, and the lifted field has a saddle/node

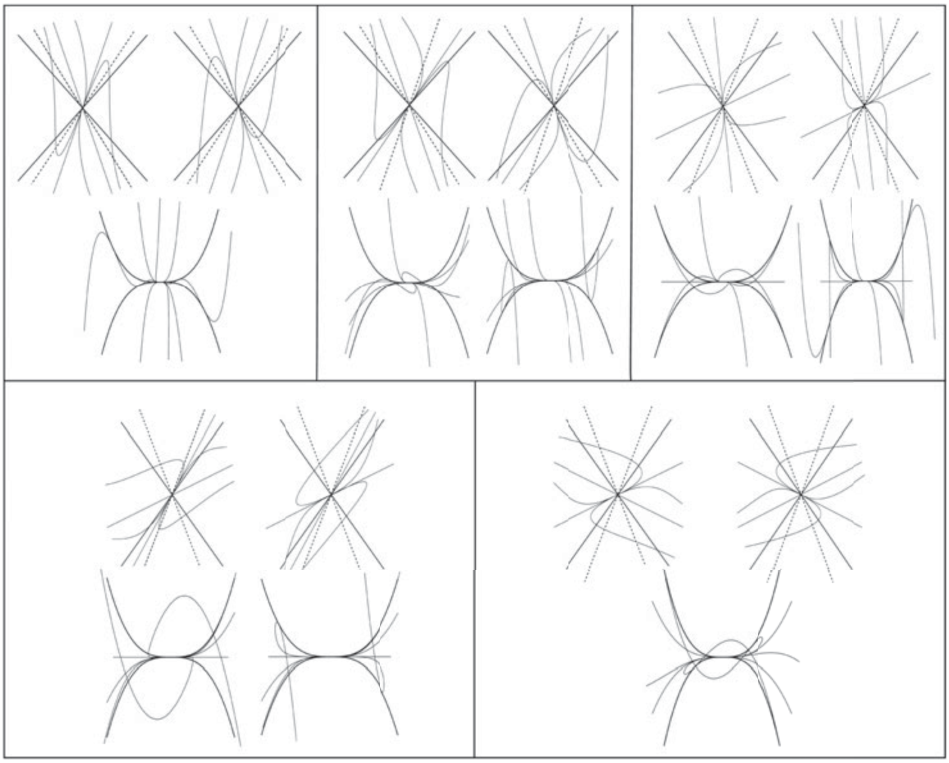

Figure 11. The Beaks-Node cases. Top figures: the configurations of the integral curves of ξ on $\tilde{M}$ in thin, the critical set of g in thick, and the set $S(g)$ in discontinuous. Bottom figures: the images of configurations of the top figures in $\mathbb{S}^2$ by the map $\tilde{A}$.

or focus singularity. Here, we certainly have a functional (even topological) modulus for the image of the asymptotic foliation in $\mathbb{S}^2$. However, one can still obtain some information on the configuration of the foliation in $\mathbb{S}^2$. We change coordinates and fix $\tilde{A}$ in normal form $g(u,v) = (u, u^2v - 1/3v^3)$. The critical set $\Sigma(g)$ of g is given by $(u-v)(u+v) = 0$, which is two transverse lines through the origin. The discriminant $\Delta(g)$ of g is the union of the curve $v = 2/3u^3$ and $v = -2/3u^3$. The discriminant locally separates the plane into regions with 1 or 3 pre-image points by g. The set $g^{-1}(\Delta)$ consists of $\Sigma(g)$ together with a set denoted by $S(g)$ which is the union of the two lines $v = 2u$ and $v = -2u$. The set $S(g)$ locally separates the source into regions which are mapped to those that have 1 or 3 pre-images by g.

In the plane (u,v), we have a vector field η with an elementary singularity at the origin and we seek to draw the images of the integral curves of η under the map g. When η has a saddle, the image foliation is determined by the relative position of the tangent lines to the separatrices with respect to the curves $\Sigma(g)$ and $S(g)$ (figure 10, top figures). When it has a node one has to

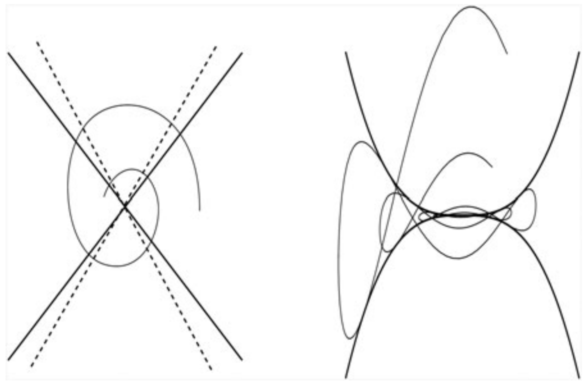

Figure 12. The Beaks-Focus case.

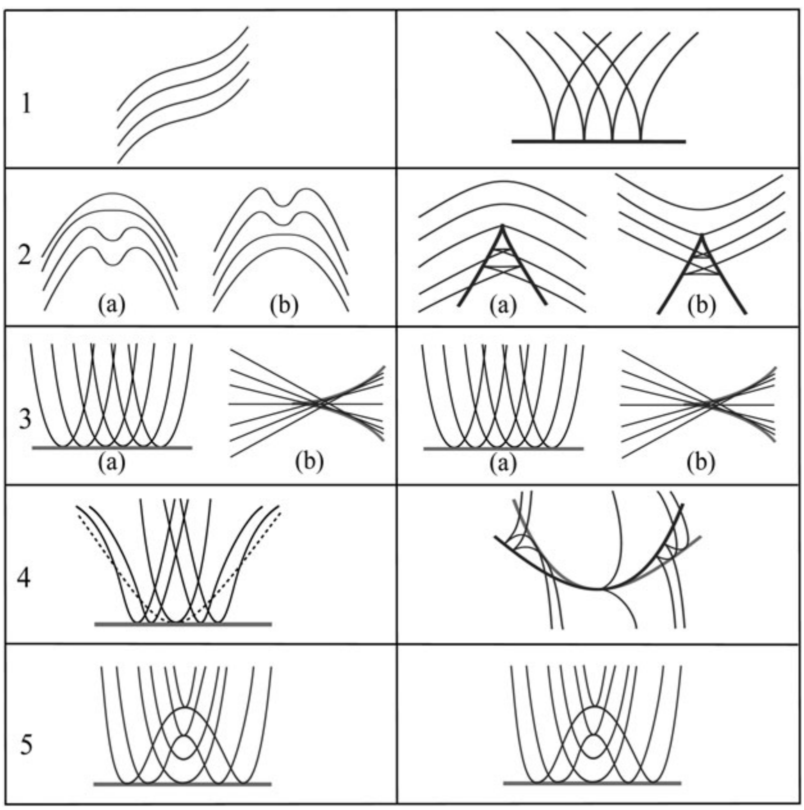

Figure 13. Foliations and their duals, left and right respectively, and vice-versa.

consider, in addition, the position of the line in the direction of the eigenvector corresponding to the largest modulus of the eigenvalues, figure 11, top figures in each case. There is only one possibility in the focus case. □

6.3. Images of foliations by the Gauss map and duality

We consider here only the images of the principal curves in $\mathbb{S}^2$ by the Gauss map. The asymptotic curves are singular along the parabolic set which is also the locus of points where the Gauss map is singular. This makes the singularities of their images by the Gauss map very degenerate (see theorems 5.11(ii) and 5.12(ii)).

Away from parabolic points, the Gauss map is a local diffeomorfism, so the images of the principal foliations (away from umbilic points) by the Gauss map are regular foliations on the sphere. Following example 3.6, if $\alpha(s)$ is a principal curve associated with P_1, then $P_2(\alpha(s))$ is the dual curve of $N(\alpha(s))$ in the sphere $\mathbb{S}^2$. When the lines of curvature in $\mathcal{F}_1$ have geodesic inflections (which occur along the sub-parabolic curve $\partial\kappa_1/\partial x_2 = 0$), their images in $N(\mathcal{F}_1)$ also have geodesic inflections in $\mathbb{S}^2$, and the dual foliation $P_2(\mathcal{F}_1)$ has the configuration in figure 5(2) (Regular fold) or figure 5(4) (Regular cusp) (see theorem 6.6(i)); see also figure 13 (1) and (2).

At a parabolic point $\kappa_1 = 0$, the kernel of DN is parallel to P_1. Away from cusps of Gauss, $N(\mathcal{F}_1)$ is a family of cusps (figure 5(2)) and $P_2(\mathcal{F}_1)$ is a regular foliation with geodesic inflections of its leaves along a curve transverse the foliation (figure 13 (1)). At a cusp of Gauss, $N(\alpha(s))$ has the configuration in figure 5(4) and $P_2(\mathcal{F}_1)$ is a regular foliation with geodesic inflections of the leaves along a curve tangent to the image of the principal curve at the cusp of Gauss (figure 13(2); see [**3**] for details).

At a parabolic point $\kappa_2 = 0$, we have the following cases:

Away from cusps of Gauss, cross-cap points of the principal map and cusp and beaks-singularities of P_2, $N(\alpha(s))$ has the configuration in Figure 5(3) (figure 13(3), (a) in first column) and its dual foliation $P_2(\mathcal{F}_1)$ has the same configuration (figure 13(3), (a) in second column).

At a cusp of Gauss, $N(\mathcal{F}_1)$ has the configuration in Figure 5(5) (figure 13(3), (b) in first column) and its dual foliation that of Figure 5(3) (figure 13(3), (a) in second column).

At a cusp of P_2 (goose singularity), $N(\mathcal{F}_1)$ has the configuration in figure 13(3), (a) in first column, and its dual foliation that in figure 13(3), (b) in second column.

At a beaks-singularity of P_2, $N(\mathcal{F}_1)$ still has the configuration in Figure 5(3) but its dual foliation has the Mixed fold configuration in figure 5(6). (This is because at the beaks-singularity of P_2, the leaves of $\mathcal{F}_1$ have geodesic inflections, see figure 13(4).)

At a cross-cap point of the principal map, the principal curve associated wit P_1 is tangent to the parabolic curve $\kappa_2 = 0$. Here, both $N(\mathcal{F}_1)$ and $P_2(\mathcal{F}_1)$ have the configuration in figure 5(7) (figure 13(5)).

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