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Explicit Formulas and Asymptotic Expansions for Certain Mean Square of Hurwitz Zeta-Functions: III

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Abstract. The main object of this paper is the mean square $I_h(s)$ of higher derivatives of Hurwitz zeta functions $\zeta(s, \alpha)$. We shall prove asymptotic formulas for $I_h(1/2 + it)$ as $t \to +\infty$ with the coefficients in closed expressions (Theorem 1). We also prove a certain explicit formula for $I_h(1/2 + it)$ (Theorem 2), in which the coefficients are, in a sense, not explicit. However, one merit of this formula is that it contains sufficient information for obtaining the complete asymptotic expansion for $I_h(1/2 + it)$ when h is small. Another merit is that Theorem 1 can be strengthened with the aid of Theorem 2 (see Theorem 3). The fundamental method for the proofs is Atkinson's dissection argument applied to the product $\zeta(u, \alpha)\zeta(v, \alpha)$ with the independent complex variables u and v.

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Key words. Riemann zeta function, Hurwitz zeta function, mean square, asymptotic expansion.

1. Introduction

The Hurwitz zeta function $\zeta(s, \alpha)$ is defined by the analytic continuation of the Dirichlet series

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s},$$

where $s = \sigma + it$ is a complex variable and $\alpha > 0$. Let $\zeta_1(s, \alpha) = \zeta(s, \alpha) - \alpha^{-s}$ and

$$I_h(s) = \int_0^1 |\zeta_1^{(h)}(s, \alpha)|^2 \mathrm{d}\alpha,$$

where $\zeta_1^{(h)}(s, \alpha)$ denotes the hth derivative of $\zeta_1(s, \alpha)$ with respect to s. The study of the case $h = 0$ is a classical problem, first investigated by Koksma and Lekkerkerker [KL] in 1952, and the recent developments were due to Andersson [A], Zhang [Z2], and the authors [KM1] [KM2] [KM3] [KM4]. In particular, the asymptotic series for $I_0(\frac{1}{2} + it)$ in the descending order of t was obtained in [KM3]. For the earlier history of this problem the readers are referred to the introductions of [KM3] [KM4].

It is the aim of the present paper to study the case $h \geqslant 1$. Research of this type of integrals was originated by Zhang [Z1] in the case of $h = 1$. He proved

$$I_1(\tfrac{1}{2}+it) = \tfrac{1}{3}\log^3(t/2\pi) + \gamma\log^2(t/2\pi) - 2B\log(t/2\pi) + A + \mathrm{O}\left(t^{-\frac{1}{6}}(\log t)^{\frac{10}{3}}\right) \tag{1.1}$$

for $t \geqslant 2$, where γ is Euler's constant. The constants A and B are given by certain integrals in Zhang's paper, but actually $A = 2\gamma_2$ and $B = -\gamma_1$, where the γ_j's are generalized Euler constants defined by

$$\zeta(1+s) = s^{-1} + \sum_{j=0}^{\infty}\gamma_j s^j, \tag{1.2}$$

(here $\zeta(s)$ is the Riemann zeta function and $\gamma_0 = \gamma$). Indeed Guo [G1] [G2] proved the following sharpening of (1.1):

$$I_1(\tfrac{1}{2}+it) = \tfrac{1}{3}\log^3(t/2\pi) + \gamma\log^2(t/2\pi) + 2\gamma_1\log(t/2\pi) + 2\gamma_2 + \mathrm{O}\left(t^{-1}(\log t)^2\right). \tag{1.3}$$

The basic tool of Zhang [Z1] is the approximate functional equation of $\zeta(s,\alpha)$, while Guo [G1] [G2] used the functional equation (formula (2.17.3) of Titchmarsh [T]) of $\zeta(s,\alpha)$.

In this paper, by a quite different method, we shall prove the following

THEOREM 1. *For any positive integer h and any $t \geqslant 2$, we have*

$$I_h(\tfrac{1}{2}+it) = \frac{1}{2h+1}\log^{2h+1}(t/2\pi) + \sum_{j=0}^{2h}\frac{(2h)!}{(2h-j)!}\gamma_j\log^{2h-j}(t/2\pi) - 2\mathrm{Re}\left\{\frac{h!\zeta^{(h)}(\frac{1}{2}+it)}{(\frac{1}{2}+it)^{h+1}}\right\} + \mathrm{O}\left(t^{-2}(\log t)^{2h}\right), \tag{1.4}$$

where the implied constant depends only on h.

Remarks. (1) In Theorem 3 below, we will give a refined version of Theorem 1.

(2) The case $h = 1$ of (1.4) gives a refinement of Guo's result (1.3).

(3) If $h \geqslant 2$, then the third explicit term on the right-hand side of (1.4) is absorbed into the error term.

(4) On the right-hand side of (1.4) there is no term of the order t^{-1}. Actually such terms appear in the course of the proof but they finally cancel.

(5) This theorem was announced in [KM4]. A weaker form in the case of $h = 1$ was already mentioned in [KM2].

(6) It is an interesting problem to derive the complete asymptotic series for $I_h(\frac{1}{2}+it)$ in the descending order of t. The existence of such series can be seen from Theorem 2 below, but it is desirable to obtain certain simple expression of the coefficients.

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Let $\Gamma(s)$ be the gamma-function. We denote by E the set of (u, v) at which some factor in (1.5) below has a singularity. The case $N = 1$ of the main theorem in [KM3] asserts that

$$\int_0^1 \zeta_1(u,\alpha)\zeta_1(v,\alpha)\mathrm{d}\alpha = \frac{1}{u+v-1} + R(u,v) - S_1(u,v) - S_1(v,u) - T_1(u,v) - T_1(v,u), \tag{1.5}$$

for $0 < \mathrm{Re}\, u < 2$, $0 < \mathrm{Re}\, v < 2$ and $(u, v) \notin E$, where

$$\begin{aligned} R(u,v) &= \Gamma(u+v-1)\zeta(u+v-1)\left\{\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right\} \\ &= 2(2\pi)^{u+v-2}\zeta(2-u-v)\Gamma(1-u)\Gamma(1-v)\cos\left(\frac{1}{2}\pi(u-v)\right), \end{aligned} \tag{1.6}$$

$$S_1(u,v) = \frac{\zeta(u)-1}{1-v}, \tag{1.7}$$

and

$$T_1(u,v) = \frac{u}{1-v}\sum_{l=1}^{\infty} l^{1-u-v}\int_l^{\infty}\beta^{u+v-2}(1+\beta)^{-u-1}\mathrm{d}\beta. \tag{1.8}$$

The formula (1.5) is the basis of all analysis in this paper. A different proof of (1.5) is given by Katsurada [K2] in a more generalized form. The advantage of (1.5) is that it includes independent variables u and v. This is the form appropriate to deduce the formula for the case of derivatives of any order, because we can differentiate it with respect to each variable. The same principle was first applied by Katsurada [K1] to the mean square of derivatives of Dirichlet L-functions.

When we consider the mean square on the critical line, we should take the limit

$$(u,v) \longrightarrow (\tfrac{1}{2}+it, \tfrac{1}{2}-it) \tag{1.9}$$

carefully, because $(\frac{1}{2}+it, \frac{1}{2}-it)$ belongs to the exceptional set E. In the case of $h = 0$, this process was done in Corollary 2 of [KM3]. One natural way of treating the case $h \geqslant 1$ is to differentiate both sides of (1.5) h-times with respect to u, and h-times with respect to v, and then take the limit (1.9). The result is Theorem 2 stated below, but it gives a quite complicated expression. This complexity mainly comes from the behaviour of $(\partial^{2h}/\partial u^h \partial v^h)R(u,v)$ near the point $(\frac{1}{2}+it, \frac{1}{2}-it)$. One idea to avoid this situation is replacing the Γ-factors in $R(u, v)$ by their Taylor expansions before carrying out the differentiation. This way leads to the proof of our Theorem 1, but considerable technical difficulties arise. For instance, it is by no means trivial that the terms of the order t^{-1} finally cancel.

On the other hand, it is to be noted that Theorem 2, though complicated, gives an explicit formula without O-terms. Let $A_k = A_k(t)$, $B_k = B_k(t)$, C_k be the Taylor

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expansion coefficients defined by

$$\frac{1}{\Gamma(\frac{1}{2}+it+z)} = \sum_{k=0}^{\infty} A_k z^k, \qquad \Gamma(\tfrac{1}{2}+it+z) = \sum_{k=0}^{\infty} B_k z^k,$$
$$\tfrac{1}{2}(2\pi)^z \operatorname{cosec}(\tfrac{1}{2}\pi(1+z)) = \sum_{k=0}^{\infty} C_k z^k,$$

and define

$$D_m(\mu, \nu) = \sum_{k=0}^{m} A_{k+\mu} B_{m-k+\nu} (-1)^{m-k+\nu} \frac{(k+\mu)!(m-k+\nu)!}{k!(m-k)!},$$
$$E_m = -C_{m+1} + \sum_{k=0}^{m} C_{m-k}(-1)^k \gamma_k$$

for nonnegative integers m, μ and ν.

THEOREM 2. *For any positive integer h and $t \geqslant 2$, we have*

$$I_h(\tfrac{1}{2}+it) = \frac{\partial^{2h}}{\partial u^h \partial v^h} \left\{ \frac{1}{u+v-1} + R(u, v) \right\}\Bigg|_{\substack{u=\frac{1}{2}+it \\ v=\frac{1}{2}-it}} - \\ - 2\mathrm{Re}\left\{ \frac{h!\zeta^{(h)}(\frac{1}{2}+it)}{(\frac{1}{2}+it)^{h+1}} \right\} - 2\mathrm{Re} \frac{\partial^{2h}}{\partial u^h \partial v^h} T_1(u, v)\Bigg|_{\substack{u=\frac{1}{2}+it \\ v=\frac{1}{2}-it}}, \tag{1.10}$$

and the first term on the right-hand side is equal to

$$2 \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu}(2h-\mu-\nu)! \times \\ \times \left\{ E_{2h-\mu-\nu} \mathrm{Re}\,(D_0(\mu, \nu)) + (-1)^{1-\mu-\nu} 2^{\mu+\nu-2h-2} \mathrm{Re}\left(D_{2h-\mu-\nu+1}(\mu, \nu)\right) \right\}. \tag{1.11}$$

Moreover, the third term on the right-hand side of (1.10) can be estimated as $\mathrm{O}(t^{-h-1})$.

This theorem gives a complete algorithm of writing down the explicit formula for $I_h(\frac{1}{2}+it)$ with fixed h. Let $\psi(s) = (\Gamma'/\Gamma)(s)$. The quantities $D_m(\mu, \nu)$ can be expressed in terms of ψ and its derivatives. However, the complexity of computations increases rapidly when h becomes large. Here we carry out the computations only for the cases $h = 1$ and $h = 2$, and we have the following corollary. For the convenience of computations we list the exact values of C_k $(0 \leqslant k \leqslant 5)$:

$$C_0 = \tfrac{1}{2}, \qquad C_1 = \tfrac{1}{2}\log 2\pi, \qquad C_2 = \tfrac{1}{16}\pi^2 + \tfrac{1}{4}(\log 2\pi)^2,$$
$$C_3 = \tfrac{1}{16}\pi^2 \log 2\pi + \tfrac{1}{12}(\log 2\pi)^3,$$
$$C_4 = \tfrac{5}{768}\pi^4 + \tfrac{1}{32}\pi^2(\log 2\pi)^2 + \tfrac{1}{48}(\log 2\pi)^4,$$
$$C_5 = \tfrac{5}{768}\pi^4 \log 2\pi + \tfrac{1}{96}\pi^2(\log 2\pi)^3 + \tfrac{1}{240}(\log 2\pi)^5.$$

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COROLLARY 2.1

$$
\begin{aligned}
I_1(\tfrac{1}{2}+it) = \mathrm{Re}\Big\{ & \tfrac{1}{3}\psi(\tfrac{1}{2}+it)^3 + 2E_0\psi(\tfrac{1}{2}+it)^2 - 4E_1\psi(\tfrac{1}{2}+it) - \\
& - \tfrac{1}{6}\psi''(\tfrac{1}{2}+it)\Big\} + 4E_2 - 2\mathrm{Re}\left\{\frac{\zeta'(\tfrac{1}{2}+it)}{(\tfrac{1}{2}+it)^2}\right\} - \\
& -2\mathrm{Re}\frac{\partial^2}{\partial u \partial v} T_1(u,v)\Big|_{\substack{u=\frac{1}{2}+it\\ v=\frac{1}{2}-it}},
\end{aligned} \tag{1.12}
$$

and

$$
\begin{aligned}
I_2(\tfrac{1}{2}+it) = \mathrm{Re}\Big\{ & \tfrac{1}{5}\psi(\tfrac{1}{2}+it)^5 + 2E_0\psi(\tfrac{1}{2}+it)^4 - 8E_1\psi(\tfrac{1}{2}+it)^3 + \\
& + 24E_2\psi(\tfrac{1}{2}+it)^2 - 48E_3\psi(\tfrac{1}{2}+it) - \psi(\tfrac{1}{2}+it)\psi'(\tfrac{1}{2}+it)^2 - \\
& - 2E_0\psi'(\tfrac{1}{2}+it)^2 + \tfrac{1}{30}\psi^{(4)}(\tfrac{1}{2}+it)\Big\} + 48E_4 - 4\mathrm{Re}\left\{\frac{\zeta''(\tfrac{1}{2}+it)}{(\tfrac{1}{2}+it)^3}\right\} - \\
& - 2\mathrm{Re}\frac{\partial^4}{\partial u^2 \partial v^2} T_1(u,v)\Big|_{\substack{u=\frac{1}{2}+it\\ v=\frac{1}{2}-it}}.
\end{aligned} \tag{1.13}
$$

This corollary implies, by using the well-known asymptotic formulas for ψ, ψ' etc. (see (3.11) and (3.12)), the cases $h = 1$ and $h = 2$ of Theorem 1.

Furthermore, from this corollary we can deduce the complete asymptotic series for $I_1(\frac{1}{2}+it)$ and $I_2(\frac{1}{2}+it)$ in the descending order of t. In fact, the asymptotic expansions for ψ, ψ', etc., are classically known, while the asymptotic expansion of

$$
\mathrm{Re}\frac{\partial^{2h}}{\partial u^h \partial v^h} T_1(u,v)\Big|_{\substack{u=\frac{1}{2}+it\\ v=\frac{1}{2}-it}}
$$

(for $h = 1, 2$) can be derived from (5.1) below (which is the case $N = 1$ of (2.2) in [KM3]).

We can show similar consequences for any fixed h, hence we may say that Theorem 2 includes more information than Theorem 1 for any fixed h. However, it seems to be not easy to deduce some simple expression of $I_h(\frac{1}{2}+it)$ for general h from Theorem 2. We content ourselves here with the following results, obtained from Corollary 2.1, which are not complete asymptotic series but sharper than (1.4) in the cases of $h = 1$ and 2.

COROLLARY 2.2.

$$
\begin{aligned}
I_1(\tfrac{1}{2}+it) &= \tfrac{1}{3}\log^3 t + 2E_0\log^2 t - (4E_1+\tfrac{1}{4}\pi^2)\log t + 4E_2 - \tfrac{1}{2}E_0\pi^2 - \\
&\quad - \left(\tfrac{1}{24}\log^2 t + \tfrac{1}{6}E_0\log t - \tfrac{1}{96}\pi^2 - \tfrac{1}{6}E_1 + \tfrac{1}{6}\right)\frac{1}{t^2} - \mathrm{Re}\left\{\frac{\zeta'(\frac{1}{2}+it)}{(\frac{1}{2}+it)^2}\right\} - \\
&\quad - 2\mathrm{Re}\frac{1}{(\frac{1}{2}+it)^2}\sum_{l=1}^{\infty} l^{-1}(l+1)^{-\frac{1}{2}-it}\log(l+1) - \\
&\quad - 2\mathrm{Re}\frac{1}{(\frac{1}{2}+it)^2}\sum_{l=1}^{\infty} l^{-2}(l+1)^{\frac{1}{2}-it}\log(l+1) + \mathrm{O}\left(\frac{\log^3 t}{t^3}\right),
\end{aligned}
\tag{1.14}
$$

and

$$
\begin{aligned}
I_2(\tfrac{1}{2}+it) &= \tfrac{1}{5}\log^5 t + 2E_0\log^4 t - \left(\tfrac{1}{2}\pi^2 - 8E_1\right)\log^3 t + (24E_2 - 3E_0\pi^2)\log^2 t + \\
&\quad + \left(\tfrac{1}{16}\pi^4 + 6E_1\pi^2 - 48E_3\right)\log t + \tfrac{1}{8}E_0\pi^4 - 6E_2\pi^2 + 48E_4 + \\
&\quad + \Bigg\{ -\tfrac{1}{24}\log^4 t - \tfrac{1}{3}E_0\log^3 t + \left(\tfrac{3}{16}\pi^2 + E_1\right)\log^2 t + \\
&\qquad + \left(\tfrac{1}{4}E_0\pi^2 - 2E_2 + 1\right)\log t - \left(\tfrac{1}{384}\pi^4 + \tfrac{1}{4}E_1\pi^2 - 2E_0 - 2E_3\right)\Bigg\}\frac{1}{t^2} - \\
&\qquad - 4\mathrm{Re}\,\frac{\zeta''(\frac{1}{2}+it)}{(\frac{1}{2}+it)^3} + \mathrm{O}\left(\frac{\log^4 t}{t^3}\right).
\end{aligned}
\tag{1.15}
$$

Remark. It is observed that the terms of the form $(\log t)^A/t$ do not appear in the formulas (1.14) and (1.15), while those of the form $(\log t)^B/t^2$ appear.

After our submission of the first version of the present paper, the referee kindly suggested the possibility of strengthening Theorem 1 with the aid of Theorem 2. We can in fact prove:

THEOREM 3. *For any postive integer h and any $t \geqslant 2$, we have*

$$
\begin{aligned}
I_h(\tfrac{1}{2}+it) &= \frac{1}{2h+1}\log^{2h+1}(t/2\pi) + \sum_{j=0}^{2h}\frac{(2h)!}{(2h-j)!}\gamma_j\log^{2h-j}(t/2\pi) + \\
&\quad + t^{-2}P_h(\log t, t^{-1}) - 2\mathrm{Re}\left\{\frac{h!\zeta^{(h)}(\frac{1}{2}+it)}{(\frac{1}{2}+it)^{h+1}}\right\} + \mathrm{O}\left(t^{-h-1}\right),
\end{aligned}
\tag{1.16}
$$

where $P_h(\log t, t^{-1})$ is a polynomial in $\log t$ and t^{-1}, and the implied constant depends only on h.

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The next four sections will be devoted to the proof of Theorem 1. Then in Section 6 we shall prove Theorem 2 with Corollary 2.1. Theorem 3 and Corollary 2.2 will be proved in Section 7. Explicit formulas similar to (1.10) can be shown on the line $\mathrm{Re}\, s = 1$, which will be presented in the last section.

The authors would like to thank the referee for many valuable comments and suggestions, especially for the recommendation of deducing Theorem 3 and Corollary 2.2 from Theorem 2.

2. The Beginning of the Proof of Theorem 1

The starting point of our proof is the fundamental formula (1.5). Putting $u = \frac{1}{2} + \xi + it$ and $v = \frac{1}{2} + \eta - it$, where $t \geqslant 2$ and ξ, η are small complex variables, we have

$$\begin{aligned} &\int_0^1 \zeta_1(\tfrac{1}{2} + \xi + it, \alpha)\zeta_1(\tfrac{1}{2} + \eta - it, \alpha)\mathrm{d}\alpha \\ &\quad = \frac{1}{\xi + \eta} + R(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) - \\ &\qquad - S_1(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) - S_1(\tfrac{1}{2} + \eta - it, \tfrac{1}{2} + \xi + it) - \\ &\qquad - T_1(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) - T_1(\tfrac{1}{2} + \eta - it, \tfrac{1}{2} + \xi + it). \end{aligned} \tag{2.1}$$

We have already mentioned in the introduction that the main difficulty lies in the treatment of

$$R(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) = \left(\frac{t}{2\pi}\right)^{-\xi-\eta} \zeta(1 - \xi - \eta)\Phi(\xi, \eta; t), \tag{2.2}$$

where, by (1.6),

$$\Phi(\xi, \eta; t) = \frac{1}{\pi} t^{\xi+\eta} \Gamma(\tfrac{1}{2} - \xi - it)\Gamma(\tfrac{1}{2} - \eta + it) \times \cosh(\pi t - \tfrac{1}{2}\pi i(\xi - \eta)). \tag{2.3}$$

The function Φ is clearly holomorphic with respect to ξ and η near the point $(\xi, \eta) = (0, 0)$, and the Taylor expansion

$$\Phi(\xi, \eta; t) = \sum_{m,n=0}^{\infty} b_{mn} \xi^m \eta^n \tag{2.4}$$

holds, where

$$b_{mn} = \frac{1}{m!n!} \frac{\partial^{m+n}\Phi}{\partial\xi^m \partial\eta^n}(0, 0; t).$$

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Also, by using (1.2), we get

$$\left(\frac{t}{2\pi}\right)^{-\xi-\eta}\zeta(1-\xi-\eta) = -\frac{1}{\xi+\eta} + \sum_{k=0}^{\infty} a_k(\xi+\eta)^k, \tag{2.5}$$

where

$$a_k = (-1)^k\left\{\frac{1}{(k+1)!}\log^{k+1}(t/2\pi) + \sum_{l=0}^{k}\frac{\gamma_l}{(k-l)!}\log^{k-l}(t/2\pi)\right\}. \tag{2.6}$$

From (2.2), (2.4) and (2.5), we obtain

$$R(\tfrac{1}{2}+\xi+it, \tfrac{1}{2}+\eta-it) = \left\{-\frac{1}{\xi+\eta} + A(\xi+\eta; t)\right\}\left\{b_{00} + B(\xi,\eta;t)\right\}, \tag{2.7}$$

where

$$A(z;t) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad B(\xi,\eta;t) = \sum_{m+n\geqslant 1} b_{mn}\xi^m\eta^n.$$

In order to study the right-hand side of (2.7) further, it is necessary to show several properties of b_{mn}, which will be given in the next section as a series of lemmas. As a preparation, we show here an alternative expression of Φ, by using the Taylor series

$$\log\Gamma(\tfrac{1}{2}-\xi-it) = \log\Gamma(\tfrac{1}{2}-it) + \sum_{j=1}^{J-1}\frac{(-1)^j}{j!}\psi^{(j-1)}(\tfrac{1}{2}-it)\xi^j + (-1)^J r_J(\xi;-t)\xi^J$$

and

$$\log\Gamma(\tfrac{1}{2}-\eta+it) = \log\Gamma(\tfrac{1}{2}+it) + \sum_{j=1}^{J-1}\frac{(-1)^j}{j!}\psi^{(j-1)}(\tfrac{1}{2}+it)\eta^j + (-1)^J r_J(\eta;t)\eta^J,$$

where J is a positive integer ($\geqslant 3$) and

$$r_J(z;x) = \int_0^1 \frac{(1-\tau)^{J-1}}{(J-1)!}\psi^{(J-1)}(\tfrac{1}{2}+ix-\tau z)\mathrm{d}\tau.$$

Here, and in what follows, x (resp. z) denotes a real (resp. complex) variable. Substituting the above series into (2.3), and noting

$$\Gamma(\tfrac{1}{2}+it)\Gamma(\tfrac{1}{2}-it) = \frac{\pi}{\cosh(\pi t)}, \tag{2.8}$$

we obtain

$$\Phi(\xi,\eta;t) = \frac{\cosh(\pi t - \frac{1}{2}\pi i(\xi-\eta))}{\cosh(\pi t)}\exp(\varphi(\xi;-t) + \varphi(\eta;t)), \tag{2.9}$$

where

$$\varphi(z;x) = \{\log|x| - \psi(\tfrac{1}{2}+ix)\}z + \sum_{j=2}^{J-1} \frac{(-1)^j}{j!}\psi^{(j-1)}(\tfrac{1}{2}+ix)z^j + \\ + (-1)^J r_J(z;x)z^J.$$

3. Properties of b_{mn}

LEMMA 1

(i) $b_{00} = 1$;
(ii) $b_{10} = b_{01} = \log t - \operatorname{Re}\psi(\frac{1}{2}+it)$;
(iii) $b_{20} = \frac{1}{2}\{-\frac{1}{4}\pi^2 + \theta(t) + \psi'(\frac{1}{2}-it)\}$,
$b_{11} = \frac{1}{4}\pi^2 + \theta(t) + \frac{1}{2}\{\psi(\frac{1}{2}+it) - \psi(\frac{1}{2}-it)\}^2$,
$b_{02} = \frac{1}{2}\{-\frac{1}{4}\pi^2 + \theta(t) + \psi'(\frac{1}{2}+it)\}$, *where*

$$\theta(t) = \{\log t - \psi(\tfrac{1}{2}+it)\}\{\log t - \psi(\tfrac{1}{2}-it)\}.$$

Proof. Since $\varphi(0;\pm t) = 0$, the first assertion is obvious. Next, the logarithmic differentiation of (2.8) gives

$$\psi(\tfrac{1}{2}+it) - \psi(\tfrac{1}{2}-it) = \frac{\pi i \sinh(\pi t)}{\cosh(\pi t)}, \tag{3.1}$$

hence,

$$\begin{aligned} b_{10} &= \frac{\partial\Phi}{\partial\xi}(0,0;t) \\ &= -\frac{1}{2}\pi i \cdot \frac{\sinh(\pi t)}{\cosh(\pi t)} + \frac{\partial\varphi}{\partial\xi}(0;-t) \\ &= -\frac{1}{2}\{\psi(\tfrac{1}{2}+it) - \psi(\tfrac{1}{2}-it)\} + \log t - \psi(\tfrac{1}{2}-it) \\ &= \log t - \operatorname{Re}\psi(\tfrac{1}{2}+it), \end{aligned}$$

and the case of b_{01} is similar. The proof of (iii) is also straightforward, so we omit the details. □

LEMMA 2. *We have*

$$\sum_{m+n=2} b_{mn}\xi^m\eta^n = (\xi+\eta)C(\xi,\eta;t),$$

where

$$C(\xi,\eta;t) = -\tfrac{1}{8}\pi^2(\xi+\eta) + \tfrac{1}{2}\theta(t)(\xi+\eta) + \tfrac{1}{2}\psi'(\tfrac{1}{2}-it)(\xi-\eta) + \frac{\pi^2\eta}{2\cosh^2(\pi t)}.$$

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Proof. Using (iii) of Lemma 1, we have

$$\sum_{m+n=2} b_{mn}\xi^m\eta^n = -\tfrac{1}{8}\pi^2(\xi^2 - 2\xi\eta + \eta^2) + \tfrac{1}{2}\theta(t)(\xi^2 + 2\xi\eta + \eta^2) + Y(\xi, \eta; t), \tag{3.2}$$

where

$$Y(\xi, \eta; t) = \tfrac{1}{2}\psi'(\tfrac{1}{2} - it)\xi^2 + \tfrac{1}{2}\{\psi(\tfrac{1}{2} + it) - \psi(\tfrac{1}{2} - it)\}^2\xi\eta + \tfrac{1}{2}\psi'(\tfrac{1}{2} + it)\eta^2. \tag{3.3}$$

Differentiating both sides of (3.1), we get

$$\psi'(\tfrac{1}{2} + it) + \psi'(\tfrac{1}{2} - it) = \frac{\pi^2}{\cosh^2(\pi t)}. \tag{3.4}$$

From (3.3), using (3.1) and (3.4), we get

$$Y(\xi, \eta; t) = \tfrac{1}{2}\psi'(\tfrac{1}{2} - it)(\xi^2 - \eta^2) + \frac{\pi^2}{2\cosh^2(\pi t)}(\xi + \eta)\eta - \tfrac{1}{2}\pi^2\xi\eta.$$

Substituting this into (3.2), and noting

$$-\tfrac{1}{8}\pi^2(\xi^2 - 2\xi\eta + \eta^2) - \tfrac{1}{2}\pi^2\xi\eta = -\tfrac{1}{8}\pi^2(\xi + \eta)^2,$$

we obtain the assertion of Lemma 2. □

LEMMA 3. *We have*

$$b_{mn} = \mathrm{O}(t^{-2}) \tag{3.5}$$

except for the cases $(m, n) = (0, 0)$ *or* $(2, 0)$ *or* $(0, 2)$*, while*

$$b_{20} = -\frac{1}{2it} + \mathrm{O}(t^{-2}) \tag{3.6}$$

and

$$b_{02} = \frac{1}{2it} + \mathrm{O}(t^{-2}). \tag{3.7}$$

Remark. The implied constants in (3.5) and in the following proof of Lemma 3 may depend on m and n.

Proof. We begin with (2.9), which can be rewritten as

$$\Phi(\xi, \eta; t) = \frac{e^{\pi t}}{2\cosh(\pi t)}\Lambda_{11}(\xi; t)\Lambda_{00}(\eta; t) + \frac{e^{-\pi t}}{2\cosh(\pi t)}\Lambda_{01}(\xi; t)\Lambda_{10}(\eta; t),$$

where

$$\Lambda_{pq}(z; t) = \exp\left\{(-1)^p\tfrac{1}{2}\pi iz + \varphi(z, (-1)^q t)\right\}.$$

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Hence,

$$\begin{aligned} b_{mn} = &\frac{1}{m!n!}\frac{e^{\pi t}}{2\cosh(\pi t)}\frac{\partial^m \Lambda_{11}}{\partial \xi^m}(0;\, t)\frac{\partial^n \Lambda_{00}}{\partial \eta^n}(0;\, t) + \\ &+ \frac{1}{m!n!}\frac{e^{-\pi t}}{2\cosh(\pi t)}\frac{\partial^m \Lambda_{01}}{\partial \xi^m}(0;\, t)\frac{\partial^n \Lambda_{10}}{\partial \eta^n}(0;\, t). \end{aligned} \tag{3.8}$$

In order to estimate the right-hand side of the above, we first study the behaviour of the function φ. Successive differentiations of the formula

$$\log \Gamma(s) = (s - \tfrac{1}{2})\log s - s + \tfrac{1}{2}\log 2\pi - \int_0^\infty \overline{B}_1(x)(x+s)^{-2}dx,$$

where $\overline{B}_1(x) = B_1(x - [x])$ is a periodic extension of the Bernoulli polynomial $B_1(x)$ (see Edwards [E], p. 109, 6.3, Formula (3)), gives

$$\psi(s) = \log s - \frac{1}{2s} + O(|s|^{-2}) \tag{3.9}$$

and

$$\psi^{(j)}(s) = (-1)^{j-1}(j-1)!s^{-j} + O(|s|^{-j-1}) \tag{3.10}$$

for any positive integer j. Hence we have

$$\psi(\tfrac{1}{2} \pm it) = \log t \pm \tfrac{1}{2}\pi i + O(t^{-2}), \tag{3.11}$$

$$\psi^{(j-1)}(\tfrac{1}{2} \pm it) = (-1)^j(j-2)!\big(\tfrac{1}{2} \pm it\big)^{-j+1} + O(t^{-j}) \tag{3.12}$$

(for $j \geqslant 2$) and

$$r_J(z; \pm t) = O(t^{-J+1})$$

uniformly for small z. Therefore we obtain

$$\varphi(z; (-1)^q t) = -(-1)^q \tfrac{1}{2}\pi i z + \frac{z^2}{1 + (-1)^q 2it} + O(t^{-2}) \tag{3.13}$$

uniformly for small z. Hence we have

$$\Lambda_{pq}(z;\, t) = \exp\{f_{pq}(z;\, t)\},$$

where

$$f_{pq}(z;\, t) = \{(-1)^p - (-1)^q\}\tfrac{1}{2}\pi i z + \frac{z^2}{1 + (-1)^q 2it} + O(t^{-2}) \tag{3.14}$$

uniformly for small z.

We now estimate the right-hand side of (3.8) by using

$$\frac{\partial^m \Lambda_{pq}}{\partial z^m}(0;\, t) = \frac{m!}{2\pi i}\int_{\mathfrak{K}} \frac{\Lambda_{pq}(z;\, t)}{z^{m+1}}dz, \tag{3.15}$$

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where $\mathfrak{R}$ denotes a small circle counterclockwise round the origin. The expression (3.15) is possible, since $\Lambda_{pq}(z; t)$ is holomorphic for small z. It is obvious from (3.14) that $f_{pq}(z; t)$ and $\Lambda_{pq}(z; t)$ are bounded for bounded z, hence

$$\frac{\partial^m \Lambda_{pq}}{\partial z^m}(0; t) = \mathrm{O}(1) \tag{3.16}$$

for any $m \geqslant 0$ and any pair (p, q). Next, we consider the cases $(p, q) = (0, 0)$ or $(1, 1)$ more closely. In these cases we have

$$f_{pq}(z; t) = \frac{z^2}{1 + (-1)^q 2it} + \mathrm{O}(t^{-2}),$$

hence

$$\begin{aligned}\int_{\mathfrak{R}} \frac{\Lambda_{pq}(z; t)}{z^{m+1}} \mathrm{d}z &= \int_{\mathfrak{R}} \frac{1 + f_{pq}(z; t) + \mathrm{O}(|f_{pq}(z; t)|^2)}{z^{m+1}} \mathrm{d}z \\ &= \int_{\mathfrak{R}} \frac{\mathrm{d}z}{z^{m+1}} + \int_{\mathfrak{R}} \frac{1}{1 + (-1)^q 2it} \cdot \frac{\mathrm{d}z}{z^{m-1}} + \int_{\mathfrak{R}} \frac{\mathrm{O}(t^{-2})}{z^{m+1}} \mathrm{d}z.\end{aligned} \tag{3.17}$$

If $m \geqslant 1$, then the first term vanishes. The second term also vanishes if $m \geqslant 3$ or $m = 1$, so in these cases we obtain

$$\frac{\partial^m \Lambda_{00}}{\partial z^m}(0; t) = \mathrm{O}(t^{-2}), \qquad \frac{\partial^m \Lambda_{11}}{\partial z^m}(0; t) = \mathrm{O}(t^{-2}). \tag{3.18}$$

If $m = 2$, then (3.17) implies

$$\frac{\partial^2 \Lambda_{00}}{\partial z^2}(0; t) = \mathrm{O}(t^{-1}), \qquad \frac{\partial^2 \Lambda_{11}}{\partial z^2}(0; t) = \mathrm{O}(t^{-1}). \tag{3.19}$$

Applying the estimates (3.16), (3.18) and (3.19) to (3.8), we obtain the conclusion (3.5) except for the cases $(m, n) = (0, 0)$, $(2, 0)$ or $(0, 2)$.

Lastly, in the cases of $(m, n) = (2, 0)$ or $(0, 2)$, the formula (3.6) or (3.7) can be deduced directly from (iii) of Lemma 1 with (3.11) and (3.12). The proof of Lemma 3 is now completed. □

4. Application of the Lemmas on b_{mn}

From (ii) of Lemma 1 we know $b_{10} = b_{01}$, which we denote by b_1. Then, noting Lemma 2, we have

$$B(\xi, \eta; t) = b_1(\xi + \eta) + (\xi + \eta)C(\xi, \eta; t) + D(\xi, \eta; t),$$

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where

$$D(\xi, \eta; t) = \sum_{m+n \geqslant 3} b_{mn} \xi^m \eta^n.$$

Hence, (2.7) with (i) of Lemma 1 implies

$$R(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) = -\frac{1}{\xi + \eta} - b_1 - C(\xi, \eta; t) - \frac{1}{\xi + \eta} D(\xi, \eta; t) + \\ + A(\xi + \eta; t) + A(\xi + \eta; t) B(\xi, \eta; t). \tag{4.1}$$

Substituting (4.1) into (2.1), we have

$$\int_0^1 \zeta_1(\tfrac{1}{2} + \xi + it, \alpha) \zeta_1(\tfrac{1}{2} + \eta - it, \alpha) \mathrm{d}\alpha = -b_1 - C(\xi, \eta; t) - \frac{1}{\xi + \eta} D(\xi, \eta; t) + \\ + A(\xi + \eta; t) + A(\xi + \eta; t) B(\xi, \eta; t) - \\ - S_1(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) - S_1(\tfrac{1}{2} + \eta - it, \tfrac{1}{2} + \xi + it) - \\ - T_1(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) - T_1(\tfrac{1}{2} + \eta - it, \tfrac{1}{2} + \xi + it). \tag{4.2}$$

This in particular implies that $(\xi + \eta)^{-1} D(\xi, \eta; t)$ is holomorphic when ξ and η are small, because all of the other members on (4.2) are holomorphic.

If $h \geqslant 1$, then

$$\frac{\partial^{2h}}{\partial \xi^h \partial \eta^h} \{-b_1 - C(\xi, \eta; t)\} = 0.$$

Hence, from (4.2) we have

$$\int_0^1 \frac{\partial^h}{\partial \xi^h} \zeta_1(\tfrac{1}{2} + \xi + it, \alpha) \frac{\partial^h}{\partial \eta^h} \zeta_1(\tfrac{1}{2} + \eta - it, \alpha) \mathrm{d}\alpha \\ = -\frac{\partial^{2h}}{\partial \xi^h \partial \eta^h} \left\{ \frac{1}{\xi + \eta} D(\xi, \eta; t) \right\} + \frac{\partial^{2h}}{\partial \xi^h \partial \eta^h} A(\xi + \eta; t) + \\ + \frac{\partial^{2h}}{\partial \xi^h \partial \eta^h} \{A(\xi + \eta; t) B(\xi, \eta; t)\} - \\ - \frac{\partial^{2h}}{\partial \xi^h \partial \eta^h} \{S_1(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) + S_1(\tfrac{1}{2} + \eta - it, \tfrac{1}{2} + \xi + it)\} - \\ - \frac{\partial^{2h}}{\partial \xi^h \partial \eta^h} \{T_1(\tfrac{1}{2} + \xi + it, \tfrac{1}{2} + \eta - it) + T_1(\tfrac{1}{2} + \eta - it, \tfrac{1}{2} + \xi + it)\} \\ = -X_1 + X_2 + X_3 - X_4 - X_5, \tag{4.3}$$

say. It is easy to see that

$$X_2 \longrightarrow (2h)! a_{2h} \tag{4.4}$$

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as $(\xi, \eta) \to (0, 0)$. Next, since

$$X_3 = \sum_{\kappa,\lambda=0}^{h} \binom{h}{\kappa}\binom{h}{\lambda} \frac{\partial^{\kappa+\lambda}}{\partial\xi^{\kappa}\partial\eta^{\lambda}} A(\xi+\eta;\, t) \frac{\partial^{2h-\kappa-\lambda}}{\partial\xi^{h-\kappa}\partial\eta^{h-\lambda}} B(\xi, \eta;\, t),$$

we find that

$$X_3 \longrightarrow \sum_{\substack{\kappa,\lambda=0 \\ (\kappa,\lambda)\neq(h,h)}}^{h} \binom{h}{\kappa}\binom{h}{\lambda} (\kappa+\lambda)!(h-\kappa)!(h-\lambda)! a_{\kappa+\lambda} b_{h-\kappa,h-\lambda} \tag{4.5}$$

as $(\xi, \eta) \to (0, 0)$. Except for the cases $(\kappa, \lambda) = (h, h-2)$ or $(h-2, h)$, we have

$$b_{h-\kappa,h-\lambda} = \mathrm{O}(t^{-2})$$

by Lemma 3. Hence, the right-hand side of (4.5) is equal to

$$2\binom{h}{h-2}(2h-2)! a_{2h-2}(b_{20}+b_{02}) + \mathrm{O}\Bigg(\sideset{}{^*}\sum_{\kappa,\lambda=0}^{h} |a_{\kappa+\lambda}| t^{-2}\Bigg),$$

where the symbol $\sum^*$ means that the cases $(\kappa, \lambda) = (h, h)$, $(h, h-2)$ and $(h-2, h)$ are excluded from the summation. From (3.6) and (3.7) we see that the terms of the order t^{-1} cancel, and so $b_{20} + b_{02} = \mathrm{O}(t^{-2})$. Also (2.6) implies that $a_k = \mathrm{O}((\log t)^{k+1})$. Therefore we find that

$$\lim_{(\xi,\eta)\to(0,0)} X_3 = \mathrm{O}(t^{-2}(\log t)^{2h}). \tag{4.6}$$

Next we consider X_1. We already remarked that $(\xi+\eta)^{-1} D(\xi, \eta;\, t)$ is holomorphic, hence it has the Taylor expansion $\sum_{\mu,\nu=0}^{\infty} c_{\mu,\nu}\xi^{\mu}\eta^{\nu}$. Therefore

$$\sum_{m+n\geqslant 3} b_{mn}\xi^m\eta^n = (\xi+\eta)\sum_{\mu,\nu=0}^{\infty} c_{\mu,\nu}\xi^{\mu}\eta^{\nu} = \sum_{\mu,\nu=0}^{\infty} (c_{\mu-1,\nu} + c_{\mu,\nu-1})\xi^{\mu}\eta^{\nu}, \tag{4.7}$$

with the notation $c_{\mu,\nu} = 0$ for $\mu < 0$ or $\nu < 0$. The relation (4.7) implies $c_{00} = c_{10} = c_{01} = 0$ and $c_{\mu-1,\nu} + c_{\mu,\nu-1} = b_{\mu\nu}$ if $\mu + \nu \geqslant 3$. Hence, by using Lemma 3 we see that $c_{m,0} = b_{m+1,0} = \mathrm{O}(t^{-2})$ for $m \geqslant 2$, and then using Lemma 3 repeatedly we find inductively

$$c_{m-r,r} = b_{m-r+1,r} - c_{m-r+1,r-1} = \mathrm{O}(t^{-2})$$

for $0 \leqslant r \leqslant m$. In particular we have $c_{hh} = \mathrm{O}(t^{-2})$, hence,

$$\lim_{(\xi,\eta)\to(0,0)} X_1 = (h!)^2 c_{hh} = \mathrm{O}(t^{-2}). \tag{4.8}$$

From (4.3), (4.4), (4.6) and (4.8) we obtain

$$\int_0^1 |\zeta_1^{(h)}(\tfrac{1}{2}+it,\alpha)|^2 d\alpha = (2h)!a_{2h} - \lim_{(\xi,\eta)\to(0,0)}(X_4+X_5) + \mathrm{O}(t^{-2}(\log t)^{2h}). \tag{4.9}$$

In the next section we will treat the remaining quantities X_4 and X_5.

5. Completion of the Proof of Theorem 1

Let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n (Pochhammer's symbol). For any positive integer K, we have

$$\begin{aligned} T_1(u,v) &= \sum_{k=1}^{K}(-1)^{k-1}\frac{(2-u-v)_{k-1}(u)_{1-k}}{1-v}\sum_{l=1}^{\infty} l^{-k}(l+1)^{-u-1+k} + \\ &\quad + (-1)^K\frac{(2-u-v)_K(u)_{1-K}}{1-v}\sum_{l=1}^{\infty} l^{1-u-v}\int_l^{\infty}\beta^{u+v-K-2}(1+\beta)^{-u-1+K}d\beta \\ &= \sum_{k=1}^{K} U_k(u,v) + V_K(u,v), \end{aligned} \tag{5.1}$$

say. This can be shown by integration by parts K-times from (1.8) (see (2.2) of [KM3]). Hence,

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} T_1(u,v) = \sum_{k=1}^{K}\frac{\partial^{2h}}{\partial u^h \partial v^h} U_k(u,v) + \frac{\partial^{2h}}{\partial u^h \partial v^h} V_K(u,v). \tag{5.2}$$

The right-hand side is estimated by the following two lemmas.

LEMMA 4. *For any* $|t| \geqslant 1$*,* $\sigma > 0$ *and any integer* $h \geqslant 1$*, we have*

$$\left.\frac{\partial^{2h}}{\partial u^h \partial v^h} U_k(u,v)\right|_{\substack{u=\sigma+it \\ v=\sigma-it}} = \begin{cases} \mathrm{O}(|t|^{-h-1}) & \text{if } 1 \leqslant k \leqslant h, \\ \mathrm{O}(|t|^{-k}) & \text{if } k \geqslant h+1, \end{cases} \tag{5.3}$$

where the implied constants depend only on σ*,* h *and* k*.*

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Proof. For any $k \geqslant 1$, we have

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} U_k(u,v) = (-1)^{k-1} \sideset{}{'}\sum_{\substack{e_1+e_2+e_3=h \\ f_1+f_2=h \\ e_1+f_1 \leqslant k-1}} \frac{(h!)^2}{e_1!e_2!e_3!f_1!} \times$$
$$\times \left\{\frac{\partial^{e_1+f_1}}{\partial u^{e_1}\partial v^{f_1}}(2-u-v)_{k-1}\right\}\left\{\frac{\mathrm{d}^{e_2}}{\mathrm{d}u^{e_2}}(u)_{1-k}\right\}(1-v)^{-f_2-1} \times$$
$$\times \sum_{l=1}^{\infty} l^{-k}(l+1)^{-u-1+k}(-\log(l+1))^{e_3}$$

for $\mathrm{Re}\, u > 0$, where $\sum'$ means that the additional condition $e_2 = 0$ is required if $k = 1$. If $u = \sigma + it$ with $|t| \geqslant 1$, then

$$\frac{\mathrm{d}^{e_2}}{\mathrm{d}u^{e_2}}(u)_{1-k} = \frac{\mathrm{d}^{e_2}}{\mathrm{d}u^{e_2}}\left(\frac{1}{(u-1)(u-2)\cdots(u-k+1)}\right) = \mathrm{O}(|t|^{-k+1-e_2}),$$

hence,

$$\left.\frac{\partial^{2h}}{\partial u^h \partial v^h} U_k(u,v)\right|_{\substack{u=\sigma+it \\ v=\sigma-it}} \ll \sideset{}{'}\sum_{\substack{e_1+e_2+e_3=h \\ f_1+f_2=h \\ e_1+f_1 \leqslant k-1}} |t|^{-k-e_2-f_2}. \tag{5.4}$$

Since $f_1 \leqslant k-1-e_1 \leqslant k-1$ we have $f_2 \geqslant \max\{0, h-k+1\}$. Hence,

$$k + e_2 + f_2 \geqslant k + \max\{0, h-k+1\},$$

therefore (5.4) implies the results of the lemma. □

Similar estimates hold for V_K, that is

LEMMA 5. *For any* $|t| \geqslant 1$, $0 < \sigma < 2$ *and any integer* $h \geqslant 1$, *we have*

$$\left.\frac{\partial^{2h}}{\partial u^h \partial v^h} V_K(u,v)\right|_{\substack{u=\sigma+it \\ v=\sigma-it}} = \begin{cases} \mathrm{O}(|t|^{-h-1}) & \text{if } 1 \leqslant K \leqslant h, \\ \mathrm{O}(|t|^{-K-1}) & \text{if } K \geqslant h+1, \end{cases} \tag{5.5}$$

where the implied constants depend only on σ, h *and* K.

Proof. First consider the case $K \geqslant h+1$. Replacing K in (5.2) by $K+1$ and comparing it with the original (5.2), we have

$$\begin{aligned} &\left.\frac{\partial^{2h}}{\partial u^h \partial v^h} V_K(u,v)\right|_{\substack{u=\sigma+it \\ v=\sigma-it}} \\ &\quad = \left.\frac{\partial^{2h}}{\partial u^h \partial v^h} U_{K+1}(u,v)\right|_{\substack{u=\sigma+it \\ v=\sigma-it}} + \left.\frac{\partial^{2h}}{\partial u^h \partial v^h} V_{K+1}(u,v)\right|_{\substack{u=\sigma+it \\ v=\sigma-it}}, \end{aligned} \tag{5.6}$$

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and the first term on the right-hand side is $O(|t|^{-K-1})$ by Lemma 4. As for the second term, we have

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} V_{K+1}(u,v) = (-1)^{K+1} \sum_{\substack{e_1+\cdots+e_5=h \\ f_1+\cdots+f_4=h \\ e_1+f_1 \leqslant K+1}} \frac{(h!)^2}{e_1!e_2!e_3!e_4!e_5!f_1!f_3!f_4!} \times$$
$$\times \left\{\frac{\partial^{e_1+f_1}}{\partial u^{e_1}\partial v^{f_1}}(2-u-v)_{K+1}\right\}\left\{\frac{\mathrm{d}^{e_2}}{\mathrm{d}u^{e_2}}(u)_{-K}\right\}(1-v)^{-f_2-1} \times$$
$$\times \sum_{l=1}^{\infty} l^{1-u-v}(-\log l)^{e_3+f_3}$$
$$\times \int_l^{\infty} \beta^{u+v-K-3}(\log\beta)^{e_4+f_4}(1+\beta)^{-u+K}(-\log(1+\beta))^{e_5}\,\mathrm{d}\beta$$

by termwise differentiation. This is valid if $\operatorname{Re} u > 0$ and $\operatorname{Re} v < 2$, because the integral on the right-hand side is absolutely convergent and

$$O\{(1+l)^{\operatorname{Re} v-2}(\log(1+l))^{e_4+f_4+e_5}\}$$

for $\operatorname{Re} v < 2$. We have

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} V_{K+1}(u,v)\bigg|_{\substack{u=\sigma+it \\ v=\sigma-it}} \ll \sum_{\substack{e_1+\cdots+e_5=h \\ f_1+\cdots+f_4=h \\ e_1+f_1 \leqslant K+1}} |t|^{-K-e_2-f_2-1} \ll |t|^{-K-1}.$$

Substituting this into (5.6), we obtain the second estimate of (5.5). Next, in case $K \leqslant h$, we use

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} V_K(u,v)\bigg|_{\substack{u=\sigma+it \\ v=\sigma-it}} = \sum_{k=K+1}^{h+1} \frac{\partial^{2h}}{\partial u^h \partial v^h} U_k(u,v)\bigg|_{\substack{u=\sigma+it \\ v=\sigma-it}} + \frac{\partial^{2h}}{\partial u^h \partial v^h} V_{h+1}(u,v)\bigg|_{\substack{u=\sigma+it \\ v=\sigma-it}},$$

which can be deduced from (5.2) similarly to (5.6). The right-hand side of the above is $O(|t|^{-h-1})$ because of Lemma 4 and the second half of (5.5). This completes the proof of the lemma. □

Applying Lemmas 4 and 5 to (5.2), we obtain

LEMMA 6. *For any* $|t| \geqslant 1$, $0 < \sigma < 2$ *and any integer* $h \geqslant 1$, *we have*

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} T_1(u,v)\bigg|_{\substack{u=\sigma+it \\ v=\sigma-it}} = O(|t|^{-h-1}), \tag{5.7}$$

where the implied constant depends only on σ *and* h.

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Remark. Although the parameter K does not appear in the conclusion (5.7), it is necessary for the proof.

Lemma 6 in particular implies that

$$\lim_{(\xi,\eta)\to(0,0)} X_5 = 2\mathrm{Re}\frac{\partial^{2h}}{\partial u^h \partial v^h} T_1(u, v)\bigg|_{\substack{u=\frac{1}{2}+it \\ v=\frac{1}{2}-it}} = \mathrm{O}(t^{-h-1}). \tag{5.8}$$

Finally, since

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} S_1(u, v) = \frac{h!}{(1-v)^{h+1}}\zeta^{(h)}(u) \tag{5.9}$$

for $h \geqslant 1$, we have

$$\lim_{(\xi,\eta)\to(0,0)} X_4 = 2\mathrm{Re}\left\{\frac{h!}{(\frac{1}{2}+it)^{h+1}}\zeta^{(h)}(\tfrac{1}{2}+it)\right\}. \tag{5.10}$$

Substituting (5.8) and (5.10) into (4.9), and recalling the definition (2.6) of a_{2h}, we arrive at the assertion of Theorem 1.

We note that the last assertion of Theorem 2 is already proved by the estimate (5.8).

6. Proof of Theorem 2 and Corollary 2.1

In this section we use the notations A_k, B_k, C_k, $D_m(\mu, v)$ and E_m which were defined in Section 1.

We first prove Theorem 2. Let

$$F(z) = \Gamma(z-1)\zeta(z-1)$$

and

$$G(u, v) = \frac{\Gamma(1-v)}{\Gamma(u)}.$$

Then from (1.6) we have

$$R(u, v) = F(u+v)\{G(u, v) + G(v, u)\},$$

hence the repeated differentiations give

$$\begin{aligned}\frac{\partial^{2h}}{\partial u^h \partial v^h}&\left\{\frac{1}{u+v-1}+R(u,v)\right\}\Bigg|_{\substack{u=\frac{1}{2}+it+z\\ v=\frac{1}{2}-it+z}}\\ &=\frac{(2h)!}{(2z)^{2h+1}}+\sum_{\mu,\nu=0}^{h}\binom{h}{\mu}\binom{h}{\nu}F^{(2h-\mu-\nu)}(1+2z)\times\\ &\quad\times\frac{\partial^{\mu+\nu}}{\partial u^\mu \partial v^\nu}\{G(u,v)+G(v,u)\}\Bigg|_{\substack{u=\frac{1}{2}+it+z\\ v=\frac{1}{2}-it+z}}\end{aligned} \tag{6.1}$$

for any small complex number z.

From the functional equation of $\zeta(s)$, we have

$$F(1+2z)=\tfrac{1}{2}(2\pi)^{2z}\operatorname{cosec}\left(\tfrac{1}{2}\pi(1+2z)\right)\zeta(1-2z)=-\frac{1}{4z}+\sum_{l=0}^{\infty}E_l(2z)^l,$$

and, hence,

$$\begin{aligned}&F^{(2h-\mu-\nu)}(1+2z)\\ &\quad=-\tfrac{1}{2}(-1)^{2h-\mu-\nu}(2h-\mu-\nu)!(2z)^{-(2h-\mu-\nu+1)}+\\ &\quad\quad+\sum_{l=0}^{\infty}E_{l+2h-\mu-\nu}\frac{(l+2h-\mu-\nu)!}{l!}(2z)^l.\end{aligned} \tag{6.2}$$

Next noting

$$\frac{\partial^\mu}{\partial u^\mu}\frac{1}{\Gamma(u)}\bigg|_{u=\frac{1}{2}+it+z}=\frac{\partial^\mu}{\partial z^\mu}\frac{1}{\Gamma(\frac{1}{2}+it+z)}=\sum_{k=0}^{\infty}A_{k+\mu}\frac{(k+\mu)!}{k!}z^k$$

and

$$\frac{\partial^\nu}{\partial v^\nu}\Gamma(1-v)\bigg|_{v=\frac{1}{2}-it+z}=\frac{\partial^\nu}{\partial z^\nu}\Gamma(\tfrac{1}{2}+it-z)=\sum_{l=0}^{\infty}(-1)^{l+\nu}B_{l+\nu}\frac{(l+\nu)!}{l!}z^l,$$

we find that the Taylor series expansion

$$\frac{\partial^{\mu+\nu}}{\partial u^\mu \partial v^\nu}\{G(u,v)+G(v,u)\}\bigg|_{\substack{u=\frac{1}{2}+it+z\\ v=\frac{1}{2}-it+z}}=\sum_{m=0}^{\infty}\left\{D_m(\mu,\nu)+\overline{D_m(\nu,\mu)}\right\}z^m \tag{6.3}$$

follows, where $\overline{D_m(\nu,\mu)}$ is the complex conjugate of $D_m(\nu,\mu)$.

Substituting (6.2) and (6.3) into the second term on the right-hand side of (6.1), we obtain

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} R(u, v)\Big|_{\substack{u=\frac{1}{2}+it+z \\ v=\frac{1}{2}-it+z}}$$
$$= \sum_{\mu, v=0}^{h} \binom{h}{\mu}\binom{h}{v}\Big\{ -\tfrac{1}{2}(-1)^{2h-\mu-v}(2h-\mu-v)!(2z)^{-(2h-\mu-v+1)} +$$
$$+ \sum_{l=0}^{\infty} E_{l+2h-\mu-v}\frac{(l+2h-\mu-v)!}{l!}(2z)^l\Big\} \times \sum_{m=0}^{\infty}\{D_m(\mu, v) + \overline{D_m(v, \mu)}\}z^m. \tag{6.4}$$

We rewrite the right-hand side of (6.4) in the ascending order of powers of z. Then the lowest order term is

$$-\tfrac{1}{2}(2h)!\{D_0(0,0) + \overline{D_0(0,0)}\}(2z)^{-2h-1},$$

and it cancels with the first term on the right-hand side of (6.1). The other negative order terms should be identically zero; this is because the left-hand side of (6.1) is holomorphic at $z = 0$, which can be seen from (1.5). Therefore, letting $z \to 0$ in (6.1), we see that the right-hand side tends to the constant term on the right-hand side of (6.4), which is equal to (1.11). This proves Theorem 2. □

We next proceed to the proof of Corollary 2.1. Let $P_k = P_k(t)$ and $Q_k = Q_k(t)$ $(k = 0, 1, 2, \ldots)$ be the functions defined by

$$\left(\frac{1}{\Gamma}\right)^{(k)}(\tfrac{1}{2}+it) = \frac{P_k}{\Gamma(\frac{1}{2}+it)}, \qquad \Gamma^{(k)}(\tfrac{1}{2}+it) = Q_k\Gamma(\tfrac{1}{2}+it).$$

Then noting $k!A_k = (1/\Gamma)^{(k)}(\frac{1}{2}+it)$ and $k!B_k = \Gamma^{(k)}(\frac{1}{2}+it)$, we find

$$D_m(\mu, v) = \sum_{k=0}^{m} P_{k+\mu}Q_{m-k+v}\frac{(-1)^{m-k+v}}{k!(m-k)!}. \tag{6.5}$$

Next for any analytic function $f = f(s)$ we define inductively the sequence of functions $F_k = F_k(f(s), f'(s), \ldots, f^{(k-1)}(s))$ $(k = 0, 1, 2, \ldots)$ by

$$F_k = \frac{\mathrm{d}}{\mathrm{d}s}F_{k-1} + fF_{k-1} \quad (k \geqslant 1); \qquad F_0 = 1. \tag{6.6}$$

For convenience of our later computations we list here the exact form of F_k $(0 \leqslant k \leqslant 5)$:

$$F_1 = f, \quad F_2 = f' + f^2, \quad F_3 = f'' + 3ff' + f^3,$$
$$F_4 = f^{(3)} + 6f'f^2 + 3(f')^2 + 4ff'' + f^4,$$
$$F_5 = f^{(4)} + 10f^3f' + 15f(f')^2 + 10f^2f'' + 10f'f'' + 5ff^{(3)} + f^5.$$

It is easy to see that F_k is a polynomial in $f, f', \ldots, f^{(k-1)}$ with coefficients in integers. By induction on k we can prove

LEMMA 7. *For any $k \geqslant 0$ we have*

$$P_k(t) = F_k\big(-\psi(\tfrac{1}{2}+it), -\psi'(\tfrac{1}{2}+it), \ldots, -\psi^{(k-1)}(\tfrac{1}{2}+it)\big)$$

and

$$Q_k(t) = F_k\big(\psi(\tfrac{1}{2}+it), \psi'(\tfrac{1}{2}+it), \ldots, \psi^{(k-1)}(\tfrac{1}{2}+it)\big).$$

The equality (6.5) therefore shows that $D_m(\mu, \nu)$ is a polynomial in $\psi^{(j)}(\frac{1}{2}+it)$, and its exact form for given m, μ, and ν can be calculated by Lemma 7. Substituting the resulting expressions for $D_m(\mu, \nu)$ $(0 \leqslant m \leqslant 5; 0 \leqslant \mu, \nu \leqslant 2)$ into (1.11), we obtain the assertions of Corollary 2.1. All the calculations are rather lengthy and tiresome, but straightforward. □

7. Proof of Theorem 3 and Corollary 2.2

To prove Theorem 3 we first show the following refinements of (3.11) and (3.12).

LEMMA 8. *For any positive integers h and k, and any $t \geqslant 2$, we have*

$$\psi(\tfrac{1}{2}+it) = \log t + \tfrac{1}{2}\pi i + \sum_{l=1}^{h} c_l(0)t^{-l} + \mathrm{O}(t^{-h-1})$$

and

$$\psi^{(k)}(\tfrac{1}{2}+it) = \sum_{l=k}^{h} c_l(k)t^{-l} + \mathrm{O}(t^{-h-1}),$$

where the $c_l(k)$'s are some constants depending only on k and l.

Proof. Stirling's formula with the exact error term asserts that, for any integer $\nu \geqslant 1$,

$$\begin{aligned}\log\Gamma(s) = \big(s-\tfrac{1}{2}\big)\log s - s + \tfrac{1}{2}\log 2\pi +\\ + \sum_{j=1}^{\nu}\frac{B_{2j}}{2j(2j-1)s^{2j-1}} - \int_0^\infty \frac{\overline{B}_{2\nu+1}(x)}{(2\nu+1)(s+x)^{2\nu+1}}\,\mathrm{d}x\end{aligned} \tag{7.1}$$

as $s \to \infty$ in $|\arg s| < \pi$, where B_j is the jth Bernoulli number and $\overline{B}_j(x) = B_j(x-[x])$ is a periodic extension of the jth Bernoulli polynomial $B_j(x)$ (cf. [E], p. 109, 6.3, Formula (3)). Repeated differentiations of both sides of (7.1) give

$$\begin{aligned}\psi^{(k)}(s) &= \frac{(-1)^{k-1}(k-1)!}{s^k} + \frac{(-1)^{k-1}k!}{2s^{k+1}} + \\ &\quad + \sum_{j=1}^{v}(-1)^{k-1}B_{2j}(2j+1)(2j+2)\cdots(2j+k-1)s^{-2j-k} + \\ &\quad + (-1)^k(2v+2)(2v+3)\cdots(2v+k-1)\int_0^\infty \frac{\overline{B}_{2v+1}(x)}{(s+x)^{2v+k+2}}\,\mathrm{d}x\end{aligned} \tag{7.2}$$

for any $k \geqslant 1$. We set $s = \frac{1}{2} + it$ with $t \geqslant 2$. Since $\overline{B}_{2v+1}(x)$ is bounded for $x \geqslant 0$, the order of the last term on the right-hand side of (7.2) is $\mathrm{O}(t^{-2v-k})$, which is absorbed in $\mathrm{O}(t^{-h-1})$ if we choose v sufficiently large. Substituting

$$s^{-m} = \left(\tfrac{1}{2} + it\right)^{-m} = (it)^{-m}\sum_{n=0}^{\infty}\binom{-m}{n}(2it)^{-n} \quad (m = 1, 2, \ldots)$$

into each term on the right-hand side of (7.2), and then collecting the terms of the respective orders t^{-l} with $k \leqslant l \leqslant h$ and $l > h$ in the resulting expression, we obtain the second assertion of the lemma. The derivation of the first assertion is similar. □

Here we mention how to prove Corollary 2.2. From the above proof of Lemma 8 we can see that $c_1(0) = 0$, $c_2(0) = -1/24$,

$$c_k(k) = \frac{(-1)^{k-1}(k-1)!}{i^k} \quad (k \geqslant 1)$$

and $c_{k+1}(k) = 0$ $(k \geqslant 1)$. Hence,

$$\begin{aligned}\psi\left(\tfrac{1}{2} + it\right) &= \log t + \tfrac{1}{2}\pi i - \frac{1}{24t^2} + \mathrm{O}(t^{-3}),\\ \psi'\left(\tfrac{1}{2} + it\right) &= -\frac{i}{t} + \mathrm{O}(t^{-3}),\\ \psi''\left(\tfrac{1}{2} + it\right) &= t^{-2} + \mathrm{O}(t^{-3})\end{aligned}$$

and $\psi^{(4)}(\frac{1}{2} + it) = \mathrm{O}(t^{-4})$. Also by Lemmas 4 and 5 we have

$$\frac{\partial^2}{\partial u \partial v}T_1 = \frac{\partial^2}{\partial u \partial v}U_1 + \frac{\partial^2}{\partial u \partial v}U_2 + \mathrm{O}(t^{-3})$$

and $(\partial^4/\partial u^2 \partial v^2)T_1 = \mathrm{O}(t^{-3})$. Using these facts, we can deduce Corollary 2.2 from Corollary 2.1 by straightforward calculations, the details being omitted. □

Now we prove Theorem 3. Let L be the first term on the right-hand side of (1.10) for brevity. In view of Theorem 2, (6.5) and Lemma 7, we see that L is expressed as a linear combination of the real parts of polynomials of $\psi^{(j)}(\frac{1}{2} + it)$

($0 \leqslant j \leqslant 2h+1$), and, hence, from Lemma 8,

$$L = Q(\log t, t^{-1}) + \mathrm{O}(t^{-h-1}). \tag{7.3}$$

Here $Q(\log t, t^{-1})$ is a polynomial in $\log t$ and t^{-1}, whose degree with respect to t^{-1} is at most $h+1$. Since the third term on the right-hand side of (1.10) is of the order $\mathrm{O}(t^{-h-1})$ (see (5.8)), Theorem 2 and (7.3) show

$$I_h(\tfrac{1}{2}+it) = Q(\log t, t^{-1}) - 2\mathrm{Re}\left\{\frac{h!\zeta^{(h)}(\frac{1}{2}+it)}{(\frac{1}{2}+it)^{h+1}}\right\} + \mathrm{O}(t^{-h-1}). \tag{7.4}$$

We write $Q(\log t, t^{-1}) = \sum_{j=0}^{h+1} t^{-j} Q_j(\log t)$, where $Q_j(\log t)$ ($0 \leqslant j \leqslant h+1$) are polynomials in $\log t$. Then comparing (1.4) and (7.4), we find that

$$Q_0(\log t) = \frac{1}{2h+1}\log^{2h+1}(t/2\pi) + \sum_{j=0}^{2h} \frac{(2h)!}{(2h-j)!}\gamma_j \log^{2h-j}(t/2\pi),$$
$$Q_1(\log t) = 0$$

and the error term $\mathrm{O}(t^{-2}(\log t)^{2h})$ on the right-hand side of (1.4) is of the form

$$\sum_{j=2}^{h+1} t^{-j} Q_j(\log t) + \mathrm{O}(t^{-h-1}) = t^{-2} P_h(\log t, t^{-1}) + \mathrm{O}(t^{-h-1}),$$

say, where $P_h(\log t, t^{-1})$ is a polynomial in $\log t$ and t^{-1}. This completes the proof of Theorem 3. □

8. Explicit Formulas on the Line $\mathrm{Re}\, s = 1$

The exceptional set E includes other important cases such as $(u, v) = (1+it, 1-it)$ ($t \geqslant 2$) or $(u, v) = (m, m)$ (m is an integer, $m \neq 1$). In this section we consider the asymptotic behaviour of $I_h(1+it)$. Note that several explicit formulas for $I_0(m)$ were obtained in [KM4].

Let $t \geqslant 1$, and $\widetilde{A}_k = \widetilde{A}_k(t)$, $\widetilde{B}_k = \widetilde{B}_k(t)$, $\widetilde{C}_k$ be the Taylor expansion coefficients defined by

$$\frac{1}{\Gamma(1+it+z)} = \sum_{k=0}^{\infty} \widetilde{A}_k z^k, \qquad \Gamma(it+z) = \sum_{k=0}^{\infty} \widetilde{B}_k z^k, \qquad \Gamma(1+z) = \sum_{k=0}^{\infty} \widetilde{C}_k z^k,$$

and set

$$\widetilde{D}_m(\mu, \nu) = \sum_{k=0}^{m} \widetilde{A}_{k+\mu} \widetilde{B}_{m-k+\nu} (-1)^{m-k+\nu} \frac{(k+\mu)!(m-k+\nu)!}{k!(m-k)!},$$
$$\widetilde{E}_m = \widetilde{C}_{m+1} + \sum_{k=0}^{m} \widetilde{C}_{m-k}\gamma_k.$$

Then the following explicit formula holds on the line $\mathrm{Re}\, s = 1$.

THEOREM 4. *For any positive integer h and any $t \geqslant 2$, we have*

$$\begin{aligned} I_h(1+it) &= (2h)! + \frac{\partial^{2h}}{\partial u^h \partial v^h} R(u,v)\Big|_{\substack{u=1+it\\ v=1-it}} - \\ &\quad - 2\mathrm{Re}\left\{\frac{h!\zeta^{(h)}(1+it)}{(it)^{h+1}}\right\} - 2\mathrm{Re}\frac{\partial^{2h}}{\partial u^h \partial v^h} T_1(u,v)\Big|_{\substack{u=1+it\\ v=1-it}}, \end{aligned} \tag{8.1}$$

and the second term on the right-hand side is equal to

$$\begin{aligned} &2\sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu}(2h-\mu-\nu)!\times \\ &\times\left\{\widetilde{E}_{2h-\mu-\nu}\mathrm{Re}\,(\widetilde{D}_0(\mu,\nu)) + (-1)^{2h-\mu-\nu}2^{\mu+\nu-2h-1}\mathrm{Re}\,(\widetilde{D}_{2h-\mu-\nu+1}(\mu,\nu))\right\}. \end{aligned} \tag{8.2}$$

Moreover, the third term on the right-hand side of (8.1) can be estimated as $\mathrm{O}(t^{-h-1})$.

From this theorem we have

COROLLARY 4.1

$$\begin{aligned} I_1(1+it) &= 2 + \mathrm{Re}\left[\frac{1}{it}\left\{\tfrac{1}{3}\psi''(1+it) - \frac{1}{it}\psi'(1+it) - \tfrac{2}{3}\psi(1+it)^3 + \right.\right. \\ &\qquad \left.\left. + \frac{1}{it}\psi(1+it)^2 - 2\widetilde{E}_1\left(2\psi(1+it) - \frac{1}{it}\right) - \frac{2}{(it)^3}\right\}\right] + \\ &\qquad + \frac{1}{(it)^4} - 2\mathrm{Re}\,\frac{\zeta'(1+it)}{(it)^2} - 2\mathrm{Re}\frac{\partial^2}{\partial u \partial v} T_1(u,v)\Big|_{\substack{u=1+it\\ v=1-it}}, \end{aligned} \tag{8.3}$$

and

$$\begin{aligned} I_2(1+it) &= 24 + \mathrm{Re}\Bigg[4\widetilde{E}_1\left\{-\frac{1}{it}\psi(1+it)^3 + \frac{3}{(it)^2}\psi(1+it)^2 - \right. \\ &\quad \left. - \frac{2}{(it)^3}\psi(1+it) - \frac{1}{(it)^2}\psi'(1+it)\right\} + \\ &\quad + 8\widetilde{E}_2\left\{\frac{3}{it}\psi(1+it)^2 - \frac{3}{(it)^2}\psi(1+it)\right\} + \\ &\quad + 24\widetilde{E}_3\left\{-\frac{4}{it}\psi(1+it) + \frac{1}{(it)^2}\right\} - \frac{1}{15it}\psi^{(4)}(1+it) - \frac{4}{3(it)^3}\psi''(1+it) + \end{aligned}$$

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$$+\frac{2}{it}\psi(1+it)\psi'(1+it)^2-\frac{1}{(it)^2}\psi'(1+it)^2-\frac{2}{(it)^2}\psi(1+it)^2\psi'(1+it)+$$
$$+\frac{4}{(it)^3}\psi(1+it)\psi'(1+it)-\frac{2}{5it}\psi(1+it)^5+\frac{1}{(it)^2}\psi(1+it)^4-$$
$$-\frac{4}{3(it)^3}\psi(1+it)^3+\frac{1}{3(it)^2}\psi^{(3)}(1+it)\Bigg]+\frac{4}{(it)^6}-$$
$$-4\mathrm{Re}\frac{\zeta''(1+it)}{(it)^3}-2\mathrm{Re}\frac{\partial^4}{\partial u^2\partial v^2}T_1(u,v)\Big|_{\substack{u=1+it\\ v=1-it}}. \tag{8.4}$$

Using the asymptotic formulas (3.9) and (3.10), we can rewrite (8.3) and (8.4) as

$$\begin{aligned} I_1(1+it) &= 2-\pi\frac{\log^2 t}{t}+\left(\frac{1}{12}\pi^3-2\pi\widetilde{E}_1\right)\frac{1}{t}+\mathrm{O}\left(\frac{\log^2 t}{t^3}\right)- \\ &\quad -2\mathrm{Re}\frac{\zeta'(1+it)}{(it)^2}-2\mathrm{Re}\frac{\partial^2}{\partial u\partial v}T_1(u,v)\Big|_{\substack{u=1+it\\ v=1-it}} \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} I_2(1+it) &= 24-\pi\frac{\log^4 t}{t}+\left(\tfrac{1}{2}\pi-12\pi\widetilde{E}_1\right)\frac{\log^2 t}{t}+24\pi\widetilde{E}_2\frac{\log t}{t}+ \\ &\quad +\left(\pi^3\widetilde{E}_1-24\pi\widetilde{E}_3-\frac{1}{80}\pi^5\right)\frac{1}{t}+\mathrm{O}\left(\frac{\log^4 t}{t^3}\right)- \\ &\quad -4\mathrm{Re}\frac{\zeta''(1+it)}{(it)^3}-2\mathrm{Re}\frac{\partial^4}{\partial u^2\partial v^2}T_1(u,v)\Big|_{\substack{u=1+it\\ v=1-it}}. \end{aligned} \tag{8.6}$$

Now we prove Theorem 4 and its corollary. The frame of the proof is similar to that of Theorem 2, so we omit the details.

We can show that

$$\begin{aligned} F^{(2h-\mu-\nu)}(2+2z) &= (-1)^{2h-\mu-\nu}(2h-\mu-\nu)!(2z)^{-(2h-\mu-\nu+1)}+ \\ &\quad +\sum_{l=0}^{\infty}\widetilde{E}_{l+2h-\mu-\nu}\frac{(l+2h-\mu-\nu)!}{l!}(2z)^l \end{aligned}$$

and

$$\frac{\partial^{\mu+\nu}}{\partial u^\mu\partial v^\nu}\{G(u,v)+G(v,u)\}\Big|_{\substack{u=1+it+z\\ v=1-it+z}}=\sum_{m=0}^{\infty}\{\widetilde{D}_m(\mu,\nu)+\overline{\widetilde{D}_m(\nu,\mu)}\}z^m.$$

Therefore

$$\frac{\partial^{2h}}{\partial u^h \partial v^h} R(u, v)\bigg|_{\substack{u=1+it+z \\ v=1-it+z}}$$
$$= \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu}\bigg\{(-1)^{2h-\mu-\nu}(2h-\mu-\nu)!(2z)^{-(2h-\mu-\nu+1)} +$$
$$+ \sum_{l=0}^{\infty} \widetilde{E}_{l+2h-\mu-\nu} \frac{(l+2h-\mu-\nu)!}{l!}(2z)^l\bigg\} \times \sum_{m=0}^{\infty}\Big\{\widetilde{D}_m(\mu, \nu) + \overline{\widetilde{D}_m(\nu, \mu)}\Big\} z^m. \tag{8.7}$$

Since $R(1+it+z, 1-it+z)$ is holomorphic at $z=0$, when $z \to 0$ the right-hand side of (8.7) tends to its constant term, which is equal to (8.2). This proves Theorem 4, because the last assertion follows from Lemma 6.

Next let $\widetilde{P}_k = \widetilde{P}_k(t)$ and $\widetilde{Q}_k = \widetilde{Q}_k(t)$ $(k = 0, 1, 2, \ldots)$ be the functions defined by

$$\left(\frac{1}{\Gamma}\right)^{(k)}(1+it) = \frac{\widetilde{P}_k}{\Gamma(1+it)}, \qquad \Gamma^{(k)}(it) = \widetilde{Q}_k\Gamma(it).$$

Then noting $k!\widetilde{A}_k = (1/\Gamma)^{(k)}(1+it)$ and $k!\widetilde{B}_k = \Gamma^{(k)}(it)$, we find

$$\widetilde{D}_m(\mu, \nu) = \frac{1}{it}\sum_{k=0}^{m} \widetilde{P}_{k+\mu}\widetilde{Q}_{m-k+\nu}\frac{(-1)^{m-k+\nu}}{k!(m-k)!}. \tag{8.8}$$

Also we can prove

LEMMA 9. *For any $k \geqslant 0$ we have*

$$\widetilde{P}_k(t) = F_k\big(-\psi(1+it), -\psi'(1+it), \ldots, -\psi^{(k-1)}(1+it)\big)$$

and

$$\widetilde{Q}_k(t) = F_k\big(\psi(it), \psi'(it), \ldots, \psi^{(k-1)}(it)\big).$$

From (8.8) and Lemma 9, together with the identity

$$\psi^{(j)}(it) = \psi^{(j)}(1+it) - \frac{(-1)^j j!}{(it)^{j+1}},$$

we can show the corollary.

We conclude this section with some observations and discussions.

An explicit formula for $I_0(1+it)$ was given in Corollary 3 of [KM3]. Taking $N=1$ in that formula and using (3.9), we can show

$$8I_0(1+it) = 1 - \frac{\pi}{t} + O\left(\frac{1}{t^3}\right) - 2\mathrm{Re}\,\frac{\zeta(1+it)}{it} - 2\mathrm{Re}\,\{T_1(1+it, 1-it)\}. \tag{8.9}$$

From (8.5), (8.6) and this formula, we can observe that the terms of the form $At^{-2}(\log t)^B$ do not appear. In fact, in the course of the proofs, such kind of terms appear but cancel each other. This is a phenomenon similar to the cancellation of the terms Ct^{-1} in the case of $I_h(\frac{1}{2}+it)$ (see Remark 4 for Theorem 1).

The above discussion suggests that, as an analogue of Theorem 1, the following asymptotic formula would hold for any $h \geqslant 0$:

$$
\begin{aligned}
I_h(1+it) = (2h)! + \frac{1}{t}P_h(\log t) + \mathrm{O}\left(\frac{(\log t)^d}{t^3}\right) \\
- 2\mathrm{Re}\left\{\frac{h!\zeta^{(h)}(1+it)}{(it)^{h+1}}\right\} - 2\mathrm{Re}\frac{\partial^{2h}}{\partial u^h \partial v^h} T_1(u,v)\Big|_{\substack{u=1+it\\ v=1-it}},
\end{aligned}
\tag{8.10}
$$

where P_h is a polynomial of degree $d = d(h)$. Perhaps $d = 2h$.

Furthermore, similary to the case of $I_h(\frac{1}{2}+it)$ (see Remark 6 for Theorem 1), it is an interesting problem to derive the complete asymptotic series for $I_h(1+it)$ in the descending order of t with the coefficients of certain simple expressions.

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