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THE NON-AXIOMATIZABILITY OF O-MINIMALITY

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Abstract. Fix a language $\mathcal{L}$ extending the language of ordered fields by at least one new predicate or function symbol. Call an $\mathcal{L}$-structure $\mathcal{R}$ *pseudo-o-minimal* if it is (elementarily equivalent to) an ultraproduct of o-minimal structures. We show that for any recursive list of $\mathcal{L}$-sentences Λ, there is a real closed field $\mathcal{R}$ satisfying Λ which is not pseudo-o-minimal. This shows that the theory $\mathrm{T}^{\text{o-min}}$ consisting of those $\mathcal{L}$-sentences true in *all* o-minimal $\mathcal{L}$-structures, also called the *theory of o-minimality (for $\mathcal{L}$)*, is not recursively axiomatizable. And, in particular, there are locally o-minimal, definably complete expansions of real closed fields which are not pseudo-o-minimal.

§1. Introduction. In [1], Ax proved that the *theory of finite fields*, consisting of all those sentences in the language of fields which are true in all finite fields, is recursively axiomatizable. To accomplish this, Ax showed first that the infinite fields which are elementarily equivalent to ultraproducts of finite fields, called *pseudofinite fields*, are precisely those fields which are perfect, pseudoalgebraically closed and have an algebraic extension of each degree. And second, he showed that these properties are all first order definable by recursive axiom schemas (in the language of fields). The *theory of finite linear orders*, the set of sentences true in all finite linear orders in the language $\{<\}$, provides another example of the same phenomenon: *pseudofinite linear orders* are finitely axiomatized by the statement that the order is discrete and has a first and last element (see [13]). Thus, a linear ordering is elementarily equivalent to an ultraproduct of finite linear orderings if and only if it is finite or has order type $\omega + L \cdot \eta + \omega^*$ for some linear order L, where η denotes the order type of $\mathbb{Z}$.

Both of these examples fall into the following general framework: fix a language $\mathcal{L}$, let $\mathbb{K}$ be a class of $\mathcal{L}$-structures and let $\mathrm{Th}(\mathbb{K})$ be the theory consisting of those $\mathcal{L}$-sentences which are true in all $\mathcal{M} \in \mathbb{K}$. Then the class of models of $\mathrm{Th}(\mathbb{K})$ is exactly the class $\overline{\mathbb{K}}$ obtained by closing $\mathbb{K}$ under isomorphism, elementary equivalence and ultraproducts [5, Corollary 8.5.13]. In other words, $\mathcal{M}$ is a model of $\mathrm{Th}(\mathbb{K})$ if and only if $\mathcal{M}$ is elementarily equivalent to an ultraproduct of members of $\mathbb{K}$. Now, the axiomatizability results above say that with $\mathbb{K}$ the class of finite fields, or the class of finite linear orders, the theory $\mathrm{Th}(\mathbb{K})$ is recursively axiomatizable or finitely axiomatizable, respectively.

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O-minimality is a condition on ordered structures $(\mathcal{R}, <, \dots)$ which states that every one-variable definable set is a finite union of points and intervals. See [15] for a thorough introduction to o-minimality. For many languages $\mathcal{L}$, this property is not *first order*, by which we mean that there is no schema $(\varphi_i)_{i \in I}$ of first-order $\mathcal{L}$-sentences such that an $\mathcal{L}$-structure $\mathcal{R}$ is o-minimal if and only if $\mathcal{R} \models \varphi_i$ for all $i \in I$. By Łos' Theorem, this is equivalent to saying that an ultraproduct of o-minimal $\mathcal{L}$-structures need not be o-minimal. For $\mathcal{L}_{OF}$ the language of ordered fields, an $\mathcal{L}_{OF}$ structure is o-minimal if and only if it is real closed; i.e., an ultraproduct of o-minimal pure ordered fields is o-minimal. But for most languages, not only might an ultraproduct of o-minimal structures not be o-minimal, it might not be NIP (see e.g., [4, Example 6.19]).

Notice that a property P being first order in the sense above is a condition that is separate from the condition of P being preserved under elementary equivalence; i.e., that an $\mathcal{L}$-structure $\mathcal{M}$ has property P if and only if $\mathcal{N} \equiv \mathcal{M}$ has property P. These two conditions might either obtain or fail to obtain together in some cases, but in the case of o-minimality, they are not the same: it was shown early on in the study of o-minimality (see [7]) that for any two elementarily equivalent structures, either both or neither is o-minimal.

Whenever a property is not first order we can ask whether, nonetheless, it is recursively axiomatizable. For *finite fields* and *finite linear orders*, this indeed the case. A collection of consequences of the property of being a finite field or of being a finite linear order were found, and then it was shown that these axiomatized the property. In the case of o-minimality, given a fixed language $\mathcal{L}$, we can ask the same question: is the common theory of o-minimal $\mathcal{L}$-structures recursively axiomatizable? Any attempt to construct such an axiomatization will necessarily result in a collection of first-order consequences of o-minimality; i.e., first-order properties that hold in all o-minimal $\mathcal{L}$-structures. Any such consequence we come across can thus be regarded as a *weakening* of o-minimality—the class of ordered $\mathcal{L}$-structures satisfying this consequence will necessarily be larger than the class of o-minimal structures.

In fact, many weakenings of o-minimality, such as *weak o-minimality* ([8]), *quasi-o-minimality* ([2]), *d-minimality* ([10]), *o-minimal open core* ([3]) etc., have been studied in the literature (for even more, see [4,9,12,16]). These weakenings generalize the *tameness* of o-minimal structures, and occur naturally. For instance, Van den Dries showed in [14] that the real field with a predicate for the subgroup $2^{\mathbb{Z}}$ is *d-minimal* but not o-minimal. For us, our interest falls only on those weakenings which can be shown to be first order, since they will be building blocks for a potential axiomatization. Two that are of particular interest to us here are the following conditions on an ordered structure $(\mathcal{R}, <, \dots)$:

Definition 1.1 (Local O-minimality (LOM)). *For every definable subset $A \subseteq R^1$, and for every point $a \in R$, there is an interval I containing a such that $A \cap I$ is a finite union of points and intervals.*[1]

[1] If we strengthen this by allowing the point to possibly be $\pm\infty$, we get a property that Schoutens calls *type completeness (TC)* in [11]. Local o-minimality, though prima facie weaker, is only actually weaker if the structure in question does not have a multiplicative group structure. Since we will be primarily interested in expansions of ordered fields, this distinction is immaterial for us.

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Definition 1.2 (Definable Completeness (DC)). *Every definable subset $A \subseteq R^1$ which is bounded above has a supremum.*

Ordered fields satisfying both of these properties are necessarily real closed (this follows easily from [9, Proposition 2.5]) and have particularly nice definable sets: for instance, every definable $A \subseteq R^1$ has a discrete boundary which first, has no accumulation points in the topology on R, and second, is either finite, or has order type $\omega + L \cdot \eta + \omega^*$ for some linear order L. This should be seen as a sort of pseudo finite analogue of the definition of o-minimality.

Since LOM and DC are expressible by first-order axiom schemas,[2] and true in *every* o-minimal structure, they are a starting point to look for a recursive axiomatization of o-minimality. In [11], Schoutens hypothesizes that LOM and DC together (from now on, just LOM+DC) possibly with a first-order variant of the pigeonhole principle for discrete definable sets, might be enough to axiomatize $\mathrm{T}^{\text{o-min}}$. And in [4], Fornasiero investigated LOM+DC fields. They showed that such structures have many important tameness properties in analogy with o-minimal structures, helping build the case that o-minimality might be recursively axiomatizable.

The main theorem of this paper will show on the contrary that LOM+DC, or even LOM+DC strengthened by any recursive list of axioms, does not axiomatize $\mathrm{T}^{\text{o-min}}$:

Theorem 1.3. *Let $\mathcal{L}$ be a language extending the language of ordered fields by at least one new function or predicate symbol, and let Λ be a consistent, recursive list of $\mathcal{L}$-sentences extending RCF. Then there is an $\mathcal{L}$-structure which satisfies Λ but is not elementarily equivalent to an ultraproduct of o-minimal $\mathcal{L}$-structures.*

The rest of the paper has two parts: in the first, we set up some terminology and definitions; and in the second we prove Theorem 1.3.

§2. Preliminaries.

Let $\mathcal{L}$ extend the language of (pure) ordered fields $\mathcal{L}_{OF}$ by a new unary predicate symbol N, and possibly other new symbols. Let $\mathcal{R}$ be an $\mathcal{L}$-structure which is an expansion of a real closed field.

As we mentioned in the introduction, any ordered field considered as an $\mathcal{L}_{OF}$-structure with the usual interpretations of the symbols is o-minimal if and only if it is real closed. Since being a real closed field is first order, ultraproducts of real closed fields are real closed fields, and hence o-minimal. Thus, in $\mathcal{L}_{OF}$, pseudo-o-minimal structures are o-minimal. This is why we insist that $\mathcal{L}$ be an extension of $\mathcal{L}_{OF}$ containing at least one new function or predicate symbol.

For every one-variable $\mathcal{L}$-formula $\varphi(z)$, we fix a new variable x not occuring in $\varphi(z)$ and define:

Definition 2.1. *$\varphi^{\leq x}(z)$ is the formula $\varphi(z)$, but whenever '$N(t)$' appears for some $\mathcal{L}$-term t, it is replaced by '$N(t) \wedge t \leq x$'.*

For a subset $X \subseteq R^n$, we define $X^{\leq r} := \{x \in X \mid x_1, \dots x_n \leq r\}$, and make similar definitions for $X^{\geq r}$, $X^{<r}$ and $X^{>r}$.

[2] Both of these claims are easy lemmas. The case of definable completeness is a straightforward translation, and for local o-minimality, the key observation is that for a small enough interval I, there are only *eight* possibilities for $A \cap I$.

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If $(\mathcal{M}_i)_{i\in I}$ is an I-indexed set of $\mathcal{L}$-structures, and $\mathcal{U}$ is a (nonprincipal) ultrafilter on I, we write $\prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ for the corresponding ultraproduct. And if P is a property of $\mathcal{L}$-structures, then we say $\mathcal{U}$-*most* index models $\mathcal{M}_i$ have property P if

$$\{i \in I \mid \mathcal{M}_i \textit{ has property } \mathtt{P}\} \in \mathcal{U}.$$

Let PA be the *relational* theory of Peano Arithmetic (see [6] for an introduction to the model theory of PA). That is, we consider PA in a *relational* language with symbols for only the ordering and the graphs of addition and multiplication. Note that this is essentially equivalent to the usual theory PA in its usual language with function symbols for addition, multiplication, and successor: a (nonfinite) model of relational PA, can be definitionally expanded to a model of ordinary PA, and vice versa. The relevant difference of us is that a substructure of a model of relational PA can be finite.

Though the word *o-minimalistic* is used in [11] to describe those structures elementarily equivalent to ultraproducts of o-minimal structures, the terminology *pseudo-o-minimal* is more in line with examples like pseudofinite fields and pseudofinite orderings.

§3. Proof of Theorem 1.3. Fix from now on a consistent, recursive list Λ of $\mathcal{L}$-sentences extending RCF, and suppose for contradiction that it is an axiomatization of $\mathrm{T}^{\text{o-min}}$.

Extend $\mathcal{L}$ by two ternary predicate symbols α and μ, and let T be the $\mathcal{L}(\alpha,\mu)$-theory consisting of the following informally stated, but nonetheless first-order axiom schemas (where $\mathcal{R} = (R,+,\times,<,0,1,N,\alpha,\mu)$ is a model of the axioms):

1. $(R,+,\times,<,0,1) \models RCF$.
2. $\mathcal{N} := (N,\alpha,\mu,<,0,1) \models PA$.
3. α and μ are the graphs of $+$ and $\times$ intersected with N^3.
4. For each $\psi \in \Lambda$, $(\overline{R}, N, \alpha, \mu) \models \forall x\ \psi^{\leq x}$.

The fourth schema ensures that when the model $\mathcal{N}$ of PA defined in $\mathcal{R}$ is restricted to any initial segment $\mathcal{N}^{\leq x} := (N^{\leq x}, \alpha^{\leq x}, \mu^{\leq x}, <)$, the axioms in Λ are forced to hold in $\mathcal{R}$ with $\mathcal{N}$ replaced by $\mathcal{N}^{\leq x}$. That is, the structure $(R,+,\times,0,1,N^{\leq x},\alpha^{\leq x},\mu^{\leq x})$ must model Λ.

In order that we can work with T, we first need to know that it is consistent. Indeed, we will show that not only does T have a model, but we proceed to show that there is a model of T with a *reduct* that satisfies Λ but which could not possibly be an ultraproduct of o-minimal structures.

Claim. T *is consistent*.

Observe that the real field, $\overline{\mathbb{R}} = (\mathbb{R},+,\times,<,0,1)$ with added predicates for $\mathbb{N}$ and the graphs of addition and multiplication on $\mathbb{N}$ is a model of T. For this we only need to check the fourth schema. But, we immediately see that for any $r \in \mathbb{R}$, $\mathbb{N}^{\leq r}$ is a *finite* initial segment of $\mathbb{N}$, so this subset, and the induced partial graphs of multiplication and addition on this set were definable inside $\overline{\mathbb{R}}$ already. Thus, $(\overline{\mathbb{R}}, \mathbb{N}^{\leq r}, \alpha^{\leq r}, \mu^{\leq r})$ is just a definitional expansion of $\overline{\mathbb{R}}$. Now, since $\overline{\mathbb{R}}$ is o-minimal, $(\overline{\mathbb{R}}, \mathbb{N}^{\leq r}, \alpha^{\leq r}, \mu^{\leq r})$ thus satisfies all the first-order consequences of o-minimality; and in particular, Λ. Thus, $(\overline{\mathbb{R}}, \mathbb{N}) \models$ T. ■

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Since every model of T interprets a model of PA (in fact, defines one), Gödel's Second Incompleteness Theorem applies, allowing us to conclude that $T + \neg\mathrm{Con}(T)$ is also consistent.[3] So from now on, let $(\mathcal{R}, \mathcal{N}) \models T + \neg\mathrm{Con}(T)$.

Let $\alpha \in \mathcal{N}$ be a Gödel code for a proof of $\neg\mathrm{Con}(T)$, and let $\alpha < r \in \mathcal{R}$ be sufficiently large (we ask that the codes for any symbols occuring in the proof coded by α be contained in $\mathcal{N}^{\leq r}$ etc.). Since $(\mathcal{R}, \mathcal{N})$ is a model of T, $(\mathcal{R}, \mathcal{N}^{\leq r})$ satisfies Λ. And since $\mathcal{N}^{\leq r}$ is an initial segment of a model $\mathcal{N}$ of PA with bounded portions of addition and multiplication, it is a Δ_0-elementary substructure of $\mathcal{N}$ (by [6, Theorem 2.7]). Noting that α being a code for a proof of $0 = 1$ in T is a Δ_0-property of $\alpha \in \mathcal{N}^{\leq r}$, and that r was chosen sufficiently large, we have that $\mathcal{N}^{\leq r} \models \neg\mathrm{Con}(T)$.

Claim. *$(\mathcal{R}, \mathcal{N}^{\leq r})$ is not pseudo-o-minimal.*

Suppose for contradiction that $(\mathcal{R}, \mathcal{N}^{\leq r})$ was elementarily equivalent to an ultraproduct of o-minimal structures:

$$(\mathcal{R}, \mathcal{N}^{\leq r}) \equiv (\mathcal{S}, \mathcal{M}) = \prod_{i \in I} (\mathcal{S}_i, \mathcal{M}_i)/\mathcal{U}$$

with $\mathcal{U}$ a nonprincipal ultrafilter on I, and $\mathcal{U}$-most $(\mathcal{S}_i, \mathcal{M}_i)$ o-minimal.

Since $\mathcal{N}^{\leq r}$ is an initial segment of a model of PA, so is $\mathcal{M}$. And, in particular, we have $(\mathcal{S}, \mathcal{M}) \models \neg\mathrm{Con}(T)$ by elementary equivalence.

But $\mathcal{M}$ is discrete in $\mathcal{S}$, so $\mathcal{U}$-most of the sets $\mathcal{M}_i$ must also be discrete in the $\mathcal{S}_i$ by Łos' Theorem. And since $\mathcal{U}$-most index models $(\mathcal{S}_i, \mathcal{M}_i)$ are o-minimal, $\mathcal{U}$-most $\mathcal{M}_i$ must then be finite, being discrete and definable. But any finite $\mathcal{M}_i$ is (isomorphic to) a finite initial segment of $\mathbb{N}$. That is, $\mathcal{U}$-most $\mathcal{M}_i$ are isomorphic to a structure $\mathbb{N}^{\leq n(i)}$ consisting, for some $n(i) \in \mathbb{N}$, of the first $n(i)$ elements of $\mathbb{N}$, together with the graphs of addition, multiplication, and ordering restricted to this set.

Finally, $\mathcal{M} \models \neg\mathrm{Con}(T)$, so there is $\alpha \in \mathcal{M}$ such that α is a code for a proof of $0 = 1$ in T. But then for an index i such that $(\mathcal{S}_i, \mathcal{M}_i)$ is o-minimal, and $\mathcal{M}_i$ is isomorphic to some $\mathbb{N}^{\leq n(i)}$ as above, the i-th coordinate of α, i.e., the element $\alpha_i \in \mathcal{M}_i$, must be a code for a proof of $0 = 1$ in T as well.

But this implies that there is a standard code (i.e., in $\mathbb{N}$) for the proof of $\neg\mathrm{Con}(T)$. From the existence of a standard code for a proof we could recover an *actual* proof of $\neg\mathrm{Con}(T)$. Hence, T would actually be inconsistent, a contradiction. □

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[3]The meaning of "Con(T)" relies, of course, on a construction of Gödel coding done inside the model of PA definable in a given model of T. We will not reproduce the details of such a construction here.

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