NATURAL HEDGING IN LONG-TERM CARE INSURANCE

BY

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ABSTRACT

We investigate the application of natural hedging strategies for long-term care (LTC) insurers by diversifying both longevity and disability risks affecting LTC annuities. We propose two approaches to natural hedging: one built on a multivariate duration, the other on the Conditional Value-at-Risk minimization of the unexpected loss. Both the approaches are extended to the LTC insurance using a multiple state framework. In order to represent the future evolution of mortality and disability transition probabilities, we use the stochastic model of Cairns *et al.* (2009) with cohort effect under parameter uncertainty through a semi-parametric bootstrap procedure. We calculate the optimal level of a product mix and measure the effectiveness provided by the interaction of LTC stand alone, deferred annuity and whole-life insurance. We compare the results obtained by the two approaches and find that a natural hedging strategy for LTC insurers is attainable with a product mix of LTC and annuities, but including low proportion of LTC.

KEYWORDS

Long-term care insurance, natural hedging, disability risk, multivariate duration, conditional VaR minimization.

1. INTRODUCTION

Long term care (LTC) insurance has reached a global relevance due to the increased number of elderly in the world, which generate a higher demand for LTC services. In fact, most of the LTC recipients are over 65 years of age and around 60% are women because of their higher life expectancy combined with a higher prevalence of disability in old age (OECD, 2013). Consequently, public expenditure on LTC continues to grow significantly, weighing on government budgets and debt levels in developed countries. In Italy, public expenditure on LTC as a share of GDP was estimated at 1.9% in 2013 (of which about two-thirds paid to people aged 65 and over) and could achieve 3.3% of GDP by 2060 (RGS, 2014).

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In Italy, public LTC benefits and services are provided by Institutions of different nature and structure, i.e., the Municipalities, Local Health Authorities, Nursing homes and the National Institute of Social Security (Istituto Nazionale Previdenza Sociale, INPS). Thus, the funding and the management of LTC services are spread over local, regional and central state authorities, with different principles and eligibility criteria fixed in the institutional models of each region (Tediosi and Gabriele, 2010). Where the public provision of care services is partial, fragmented (as in Italy) or not available, care is mostly financed out-of-pocket. In this context, private LTC insurance can play an important role, providing solutions integrated with the governments and health care institutions services.

However, insurance companies are conservative by nature and this is particularly true for LTC insurance underwriting, for the presence of risk elements such as the uncertainty with disability rates and the uncertainty in mortality improvement that can affect their underwriting profit. Developing effective risk management strategies can help insurance companies to underwrite LTC policies, resulting in lower premiums and leading to an expansion of the market.

From a technical point of view, disability benefits are affected by biometric risks (longevity and disability risks). While longevity risk has been widely and accurately defined in the insurance literature, the same cannot be said for disability risk. In the Solvency II Directive (2009/138/EC), the latter is described as the risk of adverse changes in the value of insurance liabilities, resulting from changes in the level, trend or volatility of disability rates. Both of these risks are systematic as they arise from the uncertainty of future development of mortality and disability rates and expose LTC insurers to potential unexpected losses.

Insurers can use different approaches to protect themselves from the effect of biometric risks: they can internally reduce the risk exposure using stochastic mortality forecasting models and natural hedging or they can decide to partially transfer the risk through traditional reinsurance or via mortality-linked securities traded on the financial market. However, mortality-linked securitization covers only longevity risk, while there are no specific tools for disability risk on the market.

Our paper focuses on the application of natural hedging for LTC insurers by examining the effects of portfolio diversification, which is essential for the purposes of both risk management and solvency capital requirements for longevity and disability risks under Solvency II. Specifically, our paper addresses the changes in the level of longevity and disability rates, while it does not consider changes in trend or volatility.

Natural hedging of LTC insurance is obtained by diversifying both longevity and disability risks through a suitable mix of insurance benefits within a policy or a portfolio. The main advantages of a natural hedging approach are it does not require the insurer to find counterparties, there are no transaction costs and it is an internal tool for insurers to diversify biometric risks.

In the context of life insurance, natural hedging can be defined as "an interaction of life insurance and annuities in response to a change in mortality to stabilize the cash flow for insurers" (Wang *et al.*, 2009); in a more general case, natural hedging consists in adapting the insurance portfolio in order to minimize the overall exposure to a well-defined risk (or risks). The literature on this topic shows that natural hedging can considerably reduce the sensitivity of an insurance portfolio against longevity risk. See, for example, Cox and Lin (2007) that introduced natural hedging for the mortality risk management, Tsai *et al.* (2010) that propose a Conditional Value-at-Risk (CVaR) minimization approach to obtain an optimal product mix for insurance companies who want to hedge against systematic mortality risk; and Wang *et al.* (2009) that propose an immunization model, including a stochastic mortality dynamic, to calculate the optimal life insurance-annuity product mix ratio to hedge against longevity risk. In addition, Zhu and Bauer (2014) pointed out that higher order variations in death rates may affect the performance of natural hedging and consider a non-parametric mortality model to better capture the shifts in mortality rates at different ages. Recently, Li and Haberman (2015) consider multi-population mortality models such as the correlated Poisson Lee–Carter model and Poisson common factor model to provide an assessment of the effectiveness of natural hedging between annuity and life products.

In the context of LTC insurance, very few papers have addressed the interaction of life insurance and LTC insurance. For example, Rickayzen (2007) investigates a special type of annuity, the "disability-linked annuity" compared with the corresponding traditional whole life annuity and concludes that longevity and morbidity risks included in the product work in opposite directions and this fact "should make the overall risk more controllable". Maegebier and Gatzert (2014) analyze the diversification benefits within an insurance portfolio due to the different types of biometric risks, measuring the effectiveness of natural hedging between annuity, disability and term life insurance and taking into account assets and liabilities.

We extend the existing literature on diversification of longevity risk and disability risk, adopting two different approaches to natural hedging. The first one uses the multivariate duration to study the sensitivity of the portfolio value to the change in the transition probabilities. The idea of a multivariate duration has been proposed in Reitano (1991) to study the interest rate sensitivity of the price of a portfolio of assets and liabilities, when the yield curve shifts are multivariate. In our paper, this concept is widened to the case of a LTC portfolio where the liabilities are sensitive to the changes of mortality and disability transition probabilities. Therefore, we define a multivariate duration based on the transition probabilities underlying the multiple state model that describe the insurance benefits and we propose an immunization model in a stochastic environment that combines disability and life benefits in an optimal proportion. However, the multivariate duration matching approach is based on the restrictive assumption that the future transition probabilities changes are produced by parallel shifts and it does not allow to consider parameter risk, which is considered fundamental for addressing biometric risk. To overcome this limitation, we consider a second approach based on the CVaR minimization criterion (see

Tsai *et al.* 2010 that use this criterion to find the optimal product mix in life insurance companies). We apply the CVaR minimization approach to the natural hedging of LTC insurance involving both mortality and disability risk. This latter approach allows us to base the analysis on comprehensive simulation of future transition probabilities. We then provide a numerical application of both these approaches to natural hedging and make a comparison of the results.

Even if natural hedging could be useful for insurers, it should be noticed that it is sometimes impossible to realized or inconvenient. Cox and Lin (2007) already observed that for corporate pension plan or annuity writer it may not be legal to issue life insurance and that for an insurer specialized in annuities entering in the insurance business is not practical. Wang *et al.* (2009) underline the difficulties to implement natural hedging strategies: first, for the differences in duration between annuities sold to older policyholders and life insurance sold to younger policyholders; second, because the natural hedging requires a change in business composition that may induce the insurers to reduce or increase the price of their annuity or life insurance products in order to change their attractiveness, producing adverse effects. Similarly, to other papers on this topic (see, e.g., Zhu and Bauer 2014) we assume that there are not demand-side effects when we increase the supply of insurance products, nonetheless such limitation could engender doubt on natural hedging effectiveness.

This paper is organized as follows. Section 2 introduces the stochastic model proposed for LTC insurance: the multiple state model and the LTC expected liabilities. Section 3 discusses the natural hedging strategy for LTC insurers based on a multivariate duration analysis and on the CVaR minimization approach. Section 4 shows the results of a numerical application to the Italian dataset. Finally, Section 5 reports conclusions.

2. A STOCHASTIC MODEL FOR LTC INSURANCE

2.1. The multiple state model

We consider an LTC insurance contract that pays an annuity benefit if the policyholder is disabled as long as he remains disabled. In the literature, a three-state model is generally applied to LTC insurance to reproduce the following states of a policyholder: active, disabled and dead. The disabled state may be split in more than one state according to different disability degrees, but requires a wide range of statistical data, often not available. The database we refer to allows to consider only one level of disability.

Let $\{S(\tau); \tau = 0, 1, 2, \ldots, T\}$ be a Markovian process describing the development of a single policy in discrete time, where the random variable $S(\tau)$ represents the state of the process at time τ and $[0, T]$ be a fixed finite time horizon. The LTC insurance is modeled by a multiple state model with a finite state space S=$\{1 = \text{healthy}, 2 = \text{disabled}, 3 = \text{dead}\}$ and a set of transitions according to Figure 1 (see Haberman and Pitacco (1999) for a review of multiple state

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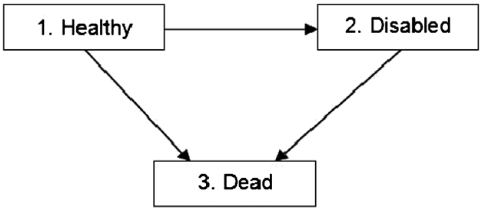

FIGURE 1: Set of states and set of transitions for LTC insurance.

models in LTC insurance). We assume that at initial time the policyholder is healthy and we disregard the possibility of recovery from the disabled state due to the usually chronic character of disability for elderly. This assumption is consistent with the empirical evidence emerging from our dataset, where LTC benefits are paid only to individuals who have a severe disability. During the observation period, the annual average rate of decrements other than death is 3.4%. Note that decrements other than death include both recoveries and undue benefits, while the INPS database do not allow to deduce the proportion between these two causes of decrement.

Let $x (x \geq 0)$ be the entry age at time t; the transition probabilities of a policyholder being in state j at age $x+\tau$, given that the policyholder is in state i at age x are defined as follows:

$$ {}_{\tau}p_{x,t}^{ij} = \mathbb{P}\left\{S(t+\tau) = j \,\middle|\, S(t) = i\right\} \qquad i, j \in S, \;\; i \neq j, \tag{2.1}$$

while the probability of a policyholder being in state i at age x to remain in the same state up to age $x+\tau$ is

$$ {}_{\tau}p_{x,t}^{ii} = \mathbb{P}\{S(t+z) = i \;\; \text{for all} \;\; z \in [0, \tau], S(t) = i\}. \tag{2.2}$$

To help understanding of the definitions presented below, the following notation is used:

- $D_{x,t}^{ij}$: observed number of transitions from state i to j $(i \neq j)$ in 1 year for policyholders aged x in year t in the reference population.
- $l_{x,t}^{i}$: number alive in state i at age x in year t in the reference population.
- $E_{x,t}^{i}$: number of person years lived in state i in 1 year for policyholders aged x in year t in the reference population.

According to this notation, the annual probabilities $p_{x,t}^{ij}$ for $i \neq j$ and $p_{x,t}^{ii}$ are, respectively, estimated by

$$ p_{x,t}^{ij} = \frac{D_{x,t}^{ij}}{l_{x,t}^{i}}, \tag{2.3}$$

$$ p_{x,t}^{ii} = \frac{l_{x+1,t+1}^{i}}{l_{x,t}^{i}}. \tag{2.4}$$

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2.2. LTC expected liabilities

We consider an insurance company offering LTC stand alone (ltc) policies paying an annuity at rate b_t^2 in year t if the insured is disabled (state 2). We denote by n_t^i the number of annuitants in state i at time t and refer to an insurance portfolio consisting of n_0^1 annuitants (all in state 1) aged x at time $t = 0$. Since we consider one cohort only, age can be omitted.

The expected liabilities at time t for a policyholder in state 1, V_t^1, and state 2, V_t^2, are given by

$$V_t^1 = b_t^2 \cdot a_{x+t}^{12}, \tag{2.5}$$

$$V_t^2 = b_t^2 \cdot a_{x+t}^{22}, \tag{2.6}$$

where

$$a_{x+t}^{ij} = \sum_{h=1}^{\omega-x} {}_h p_x^{ij} \cdot d(0,h) \quad \forall i, j \in 1, 2, \tag{2.7}$$

where $d(0, t)$ is the discount factor (for simplicity, the interest rate is assumed to be deterministic) and ω the maximum attainable age. We consider single premium coverages. Single premium is calculated according to the equivalence principle and is denoted by $\pi_0 = V^1$, where V^1 is the present value (at time 0) for the healthy beneficiaries of all cash flows of the contract until the maximum attainable age. The portfolio premium income at initial time is therefore $\Pi_0 = n_0^1 \cdot \pi_0$. While the portfolio expected liabilities at time t are calculated as follows:

$$V_t = \sum_{i \in 1,2} n_t^i \cdot V_t^i. \tag{2.8}$$

3. Natural hedging strategy for LTC insurers

Natural hedging occurs when the liabilities of different insurance products move in opposite directions in response to a change in the underlying key variables. It is a diversification strategy that allows to stabilize aggregate liability cash flows offsetting risks involved in different lines of business (LOB). In this paper, we consider the risk situation of an insurance company offering LTC and life policies. For this insurer, natural hedging employs the interaction of life (e.g., whole life or annuity) and LTC insurance in response to a change in the transition probabilities set to stabilize the cash flows. These insurance products do not necessarily move in opposite directions, but rather in different directions that anyway could allow for a diversification of risks. Therefore, the insurance company may change the portfolio composition to realize a potential hedge arising from the mix of policies liabilities.

In the following, we analyze a product mix including whole life, annuity and LTC insurance and study the effectiveness of natural hedging between these

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types of benefits. First, natural hedging is realized according to the multivariate duration matching approach that is simple to implement and based on the assumption that the future transition probabilities changes are produced by parallel shifts and does not take into account parameter risk. To overcome this limitation, we then consider the approach based on the CVaR minimization criterion that allow to include a comprehensive simulation of future transition probabilities and parameter uncertainty in the model.

Considering an insurance portfolio composed of different products or LoB, the insurer's expected total liability of the product mix at initial time is the sum of the expected liabilities of each LoB in the portfolio:

$$V_{\text{mix}}(\hat{\boldsymbol{p}}) = \sum_{\text{LoB}} V_{\text{LoB}}(\hat{\boldsymbol{p}}), \tag{3.1}$$

where $\hat{\boldsymbol{p}} = (\hat{p}^{13}, \hat{p}^{23}, \hat{p}^{12})$ is the expected transition probabilities vector at initial time defined by the multiple state model in Figure 1 and $V_{\text{LoB}}(\hat{\boldsymbol{p}})$ is then a multivariate function of $\hat{\boldsymbol{p}} = E(\boldsymbol{p})$ (note that for simplicity, age has been omitted).

In order to measure the hedge effectiveness, we define the present value of the unexpected cash flows from each LoB and from the product mix, respectively, as

$$X_{\text{LoB}} = V_{\text{LoB}}(\boldsymbol{p}) - V_{\text{LoB}}(\hat{\boldsymbol{p}}), \tag{3.2}$$

$$X_{\text{mix}} = V_{\text{mix}}(\boldsymbol{p}) - V_{\text{mix}}(\hat{\boldsymbol{p}}), \tag{3.3}$$

where $V_{\text{LoB}}(\boldsymbol{p})$ and $V_{\text{mix}}(\boldsymbol{p})$ represents the realized total liability of each LoB and of the product mix, respectively, for example, after experiencing a shock on the transition probabilities.

The corresponding proportion of present value of the unexpected cash flows are defined as

$$x_{\text{LoB}} = \frac{V_{\text{LoB}}(\boldsymbol{p}) - V_{\text{LoB}}(\hat{\boldsymbol{p}})}{V_{\text{LoB}}(\hat{\boldsymbol{p}})}, \tag{3.4}$$

$$x_{\text{mix}} = \sum_{\text{LoB}} \omega_{\text{LoB}} \cdot x_{\text{LoB}}, \tag{3.5}$$

where $\omega_{\text{LoB}} = \frac{V_{\text{LoB}}(\hat{\boldsymbol{p}})}{V_{\text{mix}}(\hat{\boldsymbol{p}})}$ is the liability proportion of each LoB on total portfolio liability.

3.1. Multivariate duration (MD) approach

As described in the introduction, Wang *et al.* (2009) proposed an immunization strategy based on a one-variable duration analysis as they study the mortality rate sensitivity of the value of a life insurance portfolio considered as a function of mortality rate. They analyze the situation of an insurer selling both life insurance and annuities and measure the effect of mortality changes on liabilities by defining an "effective mortality duration" under the assumption of a constant force of mortality, μ, as follows: $D^{V}_{e\mu} = \frac{V^{+} - V^{-}}{2V\Delta\mu}$, where V is the expected total

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liability from the portfolio, $\Delta\mu$ is the change in the mortality curve, V^{+} and V^{-} are the insurance liabilities value at higher mortality $\mu+\Delta\mu$ and lower mortality $\mu-\Delta\mu$, respectively. Then, they find the optimal proportion of life insurance, ω^{*}_{life}, that realize natural hedging between life insurance and annuities, through the effective mortality duration. The idea of mortality duration was introduced by Coughlan (2007) through the "q-duration" denoting the sensitivity of the value of insurance liabilities for a change in future mortality rates. It was then applied by, e.g., Li and Hardy (2011) and Plat (2011).

In the context of LTC insurance, where the model framework comprises multiple states, the portfolio value depends on a set of transition probabilities and a single variable duration is not sufficient to catch the portfolio liability value changes. Therefore, a multivariate duration analysis is necessary to represent the portfolio sensitivity to changes in a set of transition probabilities. A multivariate approach has been proposed in Reitano (1991) to study the interest rate sensitivity of the price of a portfolio of assets and liabilities, where the yield curve shifts are multivariate, i.e., when the yield curve moves in different directions for each maturity. Moving from these considerations, we extend the idea of a multivariate duration to the case of an LTC insurance portfolio and define a multivariate duration depending on mortality and disability transition probabilities. Then, we propose a natural hedging strategy based on it.

In the following, we adopt a notation in line with the one used by Reitano (1991). We denote by Δp^{s} the shift on the transition rate $\hat{p}^{s}$, with $s \in (1 \to 3, 2 \to 3, 1 \to 2)$, therefore $\Delta \boldsymbol{p} = (\Delta p^{13}, \Delta p^{23}, \Delta p^{12})$ is the vector of the shifts on the transition probabilities affecting the expected total liability of the insurer.

$V(\hat{\boldsymbol{p}}+\Delta\boldsymbol{p})$ viewed as a function of $\Delta\boldsymbol{p}$, reflects the sensitivity of the insurers' total liability to shifts of the transition probabilities. Let $V^{s}(\hat{\boldsymbol{p}})$ denote the partial derivative of $V(\hat{\boldsymbol{p}})$ respect to the transition probability $\hat{p}^{s}$ and $V'(\hat{\boldsymbol{p}})$ the vector of partial derivatives. We define the total (modified) duration vector, $\mathbf{D}(\hat{\boldsymbol{p}})$, for $\mathbf{V}(\hat{\boldsymbol{p}}) \neq 0$ as follows:

$$\mathbf{D}(\hat{\boldsymbol{p}}) = -\frac{V'(\hat{\boldsymbol{p}})}{V(\hat{\boldsymbol{p}})}. \tag{3.6}$$

The total duration vector is composed of the partial durations, $D^{s}(\hat{\boldsymbol{p}}) = -\frac{V^{s}(\hat{\boldsymbol{p}})}{V(\hat{\boldsymbol{p}})}$:

$$\mathbf{D}(\hat{\boldsymbol{p}}) = \left(D^{13}(\hat{\boldsymbol{p}}), D^{23}(\hat{\boldsymbol{p}}), D^{12}(\hat{\boldsymbol{p}})\right). \tag{3.7}$$

Using the first-order Taylor series approximation, the insurer's total liability variation can be approximated through the total duration as follows:

$$\frac{V(\hat{\boldsymbol{p}}+\Delta\boldsymbol{p}) - V(\hat{\boldsymbol{p}})}{V(\hat{\boldsymbol{p}})} \approx -\mathbf{D}(\hat{\boldsymbol{p}}) \cdot \Delta\boldsymbol{p}. \tag{3.8}$$

As a consequence, it can be expressed as a function of the partial durations as

$$\frac{V(\hat{\boldsymbol{p}}+\Delta\boldsymbol{p}) - V(\hat{\boldsymbol{p}})}{V(\hat{\boldsymbol{p}})} \approx -\sum_{s} D^{s}(\hat{\boldsymbol{p}}) \cdot \Delta\boldsymbol{p}^{s}. \tag{3.9}$$

In a multiple state framework, the ability of an immunization model to predict the sensitivity of the total liability of an insurance portfolio depends on the validity of the underlying transition probabilities assumption. First, we assume parallel shifts on transition probabilities, where shifts are the same for each age of the transition probabilities curves, but not necessary the same for each transition probability. Besides, if we assume that each transition probability changes of the same size, Δ, Equation (3.9) becomes as follows:

$$\frac{V(\hat{\boldsymbol{p}} + \Delta \boldsymbol{p}) - V(\hat{\boldsymbol{p}})}{V(\hat{\boldsymbol{p}})} \approx -\Delta \sum_s D^s(\hat{\boldsymbol{p}}). \tag{3.10}$$

Similarly to interest rate effective duration, we can define the total effective duration, that can better capture the transition probabilities dynamics respect to the total (modified) duration:

$$\mathbf{D}_e(\hat{\boldsymbol{p}}) = -\frac{V(\hat{\boldsymbol{p}} + \Delta \boldsymbol{p}) - V(\hat{\boldsymbol{p}} - \Delta \boldsymbol{p})}{2V(\hat{\boldsymbol{p}})\Delta \boldsymbol{p}}, \tag{3.11}$$

where $V(\hat{\boldsymbol{p}} + \Delta \boldsymbol{p})$ and $V(\hat{\boldsymbol{p}} - \Delta \boldsymbol{p})$ denote the insurance liabilities value with a positive and a negative shift, respectively.

The total effective duration vector is a vector consisting of the partial effective durations, $D_e^s(\hat{\boldsymbol{p}}) = -\frac{V(\hat{\boldsymbol{p}}+\Delta p^s) - V(\hat{\boldsymbol{p}}-\Delta p^s)}{2V(\hat{\boldsymbol{p}})\Delta p^s}$.

To neutralize the effects of the transition probabilities changes on the insurance portfolio liability, we define a natural hedging strategy based on the effective duration. Considering the Taylor first-order approximation, the difference $V(\hat{\boldsymbol{p}} + \Delta \boldsymbol{p}) - V(\hat{\boldsymbol{p}})$ becomes zero when

$$\sum_s D^s(\hat{\boldsymbol{p}}) \cdot \Delta \boldsymbol{p}^s = 0. \tag{3.12}$$

However, shifts may have different sign and size, therefore a sufficient condition for the sum in Equation (3.12) to be zero is that each partial duration is zero:

$$D^s(\hat{\boldsymbol{p}}) = 0 \qquad \forall s \in (1 \to 3, 2 \to 3, 1 \to 2). \tag{3.13}$$

However, if we assume that each transition probability changes of the same size, the change in the expected total liability becomes zero when the sum of the partial durations in Equation (3.10) is zero:

$$\sum_s D^s(\hat{\boldsymbol{p}}) = 0. \tag{3.14}$$

This approach allows to reduce the biometric risks (longevity and disability risk) affecting an insurance portfolio through the construction of a natural hedging based on the total duration. Clearly, it does not realize a Redington immunization, as it does not consider the constraint on the convexity.

The partial durations of the insurance portfolio are defined as follows:

$$D^{s}_{\text{mix}}(\hat{\boldsymbol{p}}) = \sum_{\text{LoB}} \omega_{\text{LoB}} \cdot D^{s}_{\text{LoB}}(\hat{\boldsymbol{p}}), \tag{3.15}$$

where $D^{s}_{\text{LoB}}(\hat{\boldsymbol{p}})$ is the partial duration of the LoB respect to the transition probability s. A budget constraint is added to ensure that the portfolio weights' sum is equal to 1, $\sum_{\text{LoB}} \omega_{\text{LoB}} = 1$.

If each transition probability changes of the same size, from Equation (3.15), the proportion of the liability respecting Equation (3.14), $\omega^{\text{MD}}_{\text{LoB}}$, under the budget constraint, is the solution of the following system of linear equations:

$$\begin{aligned} \sum_{s} \sum_{\text{LoB}} \omega_{\text{LoB}} \cdot D^{s}_{\text{LoB}}(\hat{\boldsymbol{p}}) &= 0 \\ \sum_{\text{LoB}} \omega_{\text{LoB}} &= 1. \end{aligned} \tag{3.16}$$

If the system has a solution and the equations are independent, the solution is unique if and only if variables (LoB) are two. If we consider more than two insurance products other constrains should be added to the system to have a unique solution.

Otherwise, if each transition probability changes of different sizes, the proportion of the liability respecting Equation (3.13), $\omega^{\text{MD}}_{\text{LoB}}$, under the budget constraint, is the solution of the following system of linear equations:

$$\begin{aligned} \sum_{\text{LoB}} \omega_{\text{LoB}} \cdot D^{13}_{\text{LoB}}(\hat{\boldsymbol{p}}) &= 0, \\ \sum_{\text{LoB}} \omega_{\text{LoB}} \cdot D^{23}_{\text{LoB}}(\hat{\boldsymbol{p}}) &= 0, \\ \sum_{\text{LoB}} \omega_{\text{LoB}} \cdot D^{12}_{\text{LoB}}(\hat{\boldsymbol{p}}) &= 0, \\ \sum_{\text{LoB}} \omega_{\text{LoB}} &= 1. \end{aligned} \tag{3.17}$$

If the equations of the above system are independent, we need at least four variables (LoB) to have a solution. As we consider less than four insurance products, the system should be reduced by removing one or more equations to find a solution.

As long as the insurer continues to hold a liability proportion of each LoB in the portfolio mix, $\omega^{\text{MD}}_{\text{LoB}}$, he obtains a natural hedge of longevity and disability risk.

3.2. Conditional VaR minimization approach

As explained in the introduction, the natural hedging of longevity risk based on the multivariate duration has some limitations. For this reason, Tsai *et al.* (2010) proposed a criterion based on a risk measure minimization in the context of the natural hedging of longevity risk. In particular, they choose to minimize the CVaR of the portfolio loss for the well-known better properties of the CVaR than VaR. We apply the CVaR minimization approach to the natural hedging of LTC insurance involving two biometric risks: longevity and disability.

We find the liability proportion of each LoB on total portfolio liability, $\omega_{\text{LoB}}^{\text{CVaR}}$, minimizing the CVaR of x_{mix} under the budget constraint, by solving the following optimization problem:

$$\begin{aligned} \min_{\omega_{\text{LoB}}} \; & E[x_{\text{mix}} \mid x_{\text{mix}} \geq x_{\text{mix}}(\epsilon)] \\ & \sum_{\text{LoB}} \omega_{\text{LoB}} = 1, \end{aligned} \tag{3.18}$$

where $E[x_{\text{mix}} \mid x_{\text{mix}} \geq x_{\text{mix}}(\epsilon)]$ is the conditional expectation of the proportion of the present value of the unexpected cash flows from the product mix exceeding the threshold $x_{\text{mix}}(\epsilon)$ with a ϵ confidence level. Following Tsai *et al.* (2010), to calculate x_{mix}, we firstly apply the forecasting model to simulate the distribution of $V_{\text{LoB}}(\boldsymbol{p})$, then obtaining the distribution of x_{LoB} and x_{mix} by Equation (3.5). Several algorithms, written in various computer program languages, are available for solving the optimization problem in Equation (3.18). Specifically, we use the R package "DEoptim" (Mullen *et al.*, 2011).

4. Numerical application

This section concerns the application of our hedging model, introduced in the previous section, to Italian disability data. In Italy, LTC insurance offered by insurance companies is not very widespread and data on disability claims are still poor and inadequate for pricing. However, the Italian Government has been paying since 1980 a disability benefit, called "indennita' di accompagnamento", to individuals residing in Italy who have suffered a disability which leaves them non-self-sufficient or unable to work. It is a universal cash benefit unconnected to a means' test and not subject to age limitations. Therefore, in the numerical application, we use data on people qualified to the "indennita' di accompagnamento", collected by the Italian National Institute of Social Security (INPS).

4.1. Dataset description and analysis

The INPS database provides standardized information on this disability benefit only from 2002. To be eligible to receive the "indennita' di accompagnamento", beneficiaries must be assessed of being 100% invalid and non-self-sufficient according to the activities of daily living (ADL) criterion, evaluating the individual's ability to perform daily activities essential for independent living. Moreover, beneficiaries must not reside in institutions with costs charged to the public administration. Since gender differences in disability among elderly are remarkable, the model is tested on both the genders. Specifically, our analysis focuses on Italian males and females using data from 2002 to 2012 for ages 40 to 89. The dataset is structured as follows:

- Deaths among disabled.
- Number of disabled.
- Inceptions on disabled state.

As the disability benefit paid by INPS is universal and the reference population well represented by the Italian general population, the number of healthy people can be calculated as the difference between the Italian general population, taken from the Italian National Institute of Statistics (ISTAT), and Italian disabled population given by the INPS dataset. The reference sample has the following size: the number of people becoming disabled from 2002 to 2012 for ages 40 to 89 is 1.1 million for males and 1.7 million for females. While the corresponding number of exposures to risk is from 13.4 million in 2002 to 15.2 million in 2012 for males, and from and 15.3 million in 2002 to 16.9 million in 2012 for females.

4.2. The forecasting of transition probabilities

Recently, Levantesi and Menzietti (2012) proposed to use stochastic models often applied in mortality projections (such as the Lee–Carter model (Lee and Carter, 1992) and the Cairns, Blake and Dowd model (Cairns *et al.*, 2006)) to forecast the transition probabilities defining a multiple state model for LTC insurance. Following this idea, in this paper, we compare the fitting results of the seven stochastic models considered by Cairns *et al.* (2009) using the INPS database. The goodness-of-fit is measured by the Bayes Information Criterion (BIC). We find that an extension of the CBD model including a cohort component (M8 model) fits Italian data very well. It has the highest BIC value for all the transition probabilities for females. As regards the males, it is the first one for $p^{13}_{x,t}$, the second one (but very close to the first one) for $p^{23}_{x,t}$ and the third one for $p^{12}_{x,t}$. The BIC values for all models fitted are displayed in Table 1 for males and Table 2 for females.

The transition probabilities $p^{ij}_{x,t}$ according to M8 are described by the following equation:

$$\operatorname{logit}(p^{ij}_{x,t}) = \ln\left(\frac{p^{ij}_{x,t}}{1-p^{ij}_{x,t}}\right) = {}^{ij}k^{(1)}_t + {}^{ij}k^{(2)}_t(x-\bar{x}) + {}^{ij}\gamma^{(3)}_c(x_c - x)\,, \qquad (4.1)$$

where x is the age, $\bar{x}$ the mean age in the sample age range (in our analysis $\bar{x} = 64.5$), t the time, $c = t - x$ the cohort and x_c a constant parameter that does

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TABLE 1

BIC STATISTIC AND RANK FOR EACH MODEL, AGES 40–89 AND YEARS 2002–2012, ITALIAN MALES.

	$p^{13}_{x,t}$	$p^{23}_{x,t}$	$p^{12}_{x,t}$
Model	BIC (Rank)	BIC (Rank)	BIC (Rank)
Lee–Carter Model (M1)	−3098 (4)	−2861 (5)	−3282 (6)
Renshaw–Haberman Model (M2)	−3217 (6)	−2917 (6)	−3040 (5)
Currie Age–Period–Cohort Model (M3)	−3041 (3)	−2604 (4)	−2794 (4)
Original Cairns–Blake–Dowd (CBD) Model (M5)	−5080 (7)	−4040 (7)	−5248 (7)
CBD Model with a Cohort Effect (M6)	−3123 (5)	−2586 (3)	−2708 (2)
Quadratic CBD Model with Cohort Effect (M7)	−2959 (2)	−2442 (1)	−2703 (1)
CBD Model with an Age-Dependent Cohort Effect (M8)	−2943 (1)	−2445 (2)	−2791 (3)

TABLE 2

BIC STATISTIC AND RANK FOR EACH MODEL, AGES 40–89 AND YEARS 2002–2012, ITALIAN FEMALES.

	$p^{13}_{x,t}$	$p^{23}_{x,t}$	$p^{12}_{x,t}$
Model	BIC (Rank)	BIC (Rank)	BIC (Rank)
Lee–Carter Model (M1)	−2996 (5)	−2625 (4)	−3938 (5)
Renshaw–Haberman Model (M2)	−3102 (6)	−2790 (5)	−3118 (3)
Currie Age–Period–Cohort Model (M3)	−2934 (4)	−2505 (3)	−3074 (2)
Original Cairns–Blake–Dowd (CBD) Model (M5)	−11071 (7)	−8766 (7)	−22133 (7)
CBD Model with a Cohort Effect (M6)	−2860 (3)	−2910 (6)	−4214 (6)
Quadratic CBD Model with Cohort Effect (M7)	−2730 (2)	−2442 (2)	−3306 (4)
CBD Model with an Age-Dependent Cohort Effect (M8)	−2720 (1)	−2413 (1)	−2957 (1)

not vary with age or time (in our analysis $x_c = 65$). To avoid any identifiability problem, we introduce the constraint $\sum_c^{ij} \gamma_c^{(3)} = 0$.

Model parameters are estimated separately for each transition probability as a trivariate CBD model. The maximum-likelihood estimates of the CBD model parameters are shown in Appendix A.1.

To forecast transition probabilities, we model the parameters ${}^{ij}k_t^{(1)}$, ${}^{ij}k_t^{(2)}$ and ${}^{ij}\gamma_c^{(3)}$ through a multivariate ARIMA time series model. The multivariate ARIMA model can be expressed as $K_{s+1} = K_s + \phi\,(K_{s-2} - K_{s-1}) + \delta + C\,Z_{s+1}$, where K_s is a 9×1 vector of parameters at the step s, s is the time or the cohort depending on the parameter; ϕ is a 9×1 vector of parameters of the autoregressive part of the model; δ is a 9×1 vector of the drifts of the model; C is a 9×9 constant upper triangular matrix so that CC' is the covariance matrix and Z is a 9×1 vector of standard normal random variables.

The choice of ARIMA process for the nine parameters has been made mainly according to Information Criteria as Akaike Information Criterion

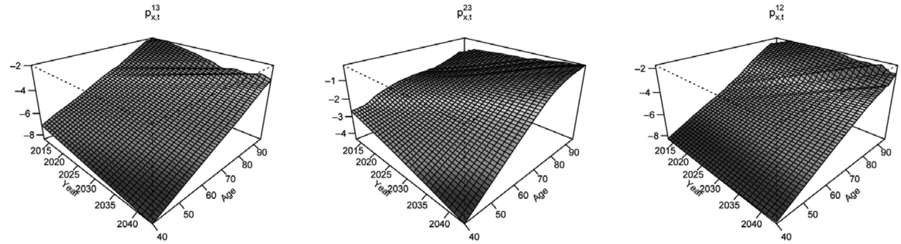

FIGURE 2: Forecasted central transition probabilities on log scale, years 2013–2043. Males.

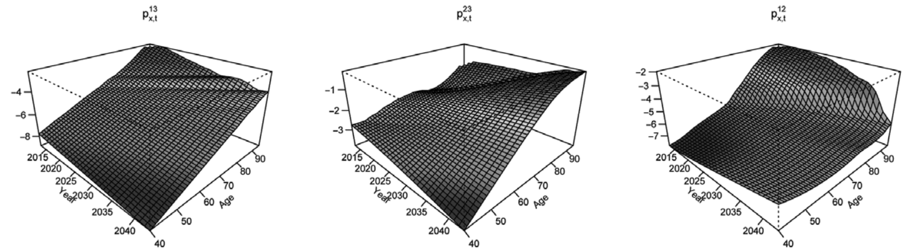

FIGURE 3: Forecasted central transition probabilities on log scale, years 2013–2043. Females.

(AIC), Schwartz Information Criterion (SIC) and Hannan and Quinn (HQ) and to the analysis of residuals. The selected ARIMA models and the corresponding parameters (σ^2: variance of the error terms; δ: drift; ϕ: autoregressive parameter) are reported in Appendix A.1.

As usual in the models with cohort effect, we assume that the cohort effect has dynamics that are independent of the period effect (see, e.g., Cairns *et al.* 2011), therefore ${}^{ij}\gamma_c^{(3)}$ is not correlated with ${}^{ij}k_t^{(1)}$ and ${}^{ij}k_t^{(2)}$. For the fitted correlation matrices ρ, see Appendix A.1.

Parameter uncertainty is incorporated in the forecasting of the CBD model through a semi-parametric bootstrap procedure as proposed by Brouhns *et al.* (2005). We generate B bootstrap samples of the number of transitions $D_{x,t}^{ij\,(b)}$, $b = 1, 2, \ldots, B$ by sampling from the Poisson distribution with mean $\widehat{D}_{x,t}^{ij}$ (under the assumptions that the number of transitions follows a Poisson distribution) that is fitted to real data by the CBD model.

The resulting forecasted transition probabilities on log scale are shown in Figures 2 and 3 for males and females, respectively.

4.3. Natural hedging strategy under MD approach

In the numerical application, we study the interaction of the following LoB: LTC stand alone (ltc), deferred annuity (annuity) and whole life (life). In Table 3, we describe the policy conditions and the interest rate assumption for the three LoB. The risk-free term structure used in the calculations is provided

TABLE 3

INSURANCE PRODUCTS CHARACTERISTICS.

Insurance Product	LTC Annuity	Life Annuity	Whole Life
Gender	Male; Female	Male; Female	Male; Female
Term Structure	CEIOPS 2007	CEIOPS 2007	CEIOPS 2007
Payout Benefit	€2 (per year)	€1 (per year)	€10
Premium Payment	Single Premium	Single Premium	Single Premium
Deferred Period	–	–	–
V_{LoB} (Entry Age 40)	0.4082; 0.1480	3.7342; 4.5493	1.2211; 0.3892
V_{LoB} (Entry Age 50)	0.5192; 0.1659	5.8286; 6.9607	1.8078; 0.6847
V_{LoB} (Entry Age 65)	0.7805; 0.2545	12.3531; 14.3038	3.3025; 1.4191

TABLE 4

EXPECTED LIABILITY VARIATION, ENTRY AGE: 65.

	Males			Females		
V^+	LTC	Annuity	Life	LTC	Annuity	Life
$V^+_{\hat{p}^{13}}$	−1.3023%	−1.0095%	2.2782%	−0.8225%	−1.1406%	8.6331%
$V^+_{\hat{p}^{23}}$	−0.2332%	−0.0074%	0.0156%	−0.3055%	−0.0027%	0.0137%
$V^+_{\hat{p}^{12}}$	8.8717%	−0.6880%	1.6902%	38.1023%	−0.7943%	6.8554%

by CEIOPS (CEIOPS, 2010) in the fifth quantitative impact study. Considering that most of the LTC recipients are over 65 years of age, we analyzed single premium coverages written for a male/female policyholder aged 65 (in 2013) as a benchmark case.

To ensure perfect comparability between insurance contracts, life insurance is modeled using the same multiple state model of LTC insurance. Therefore, the generic probability of death $q_{x,t}$ required for the computation of V_{life} and V_{annuity} is calculated as $q_{x,t} = \frac{l^1_{x,t}}{l_{x,t}} p^{13}_{x,t} + \frac{l^2_{x,t}}{l_{x,t}} p^{23}_{x,t}$.

In Table 4, we show the percentage variation of the expected liability, V_{LoB}, of the insurance products in case of a 0.001 positive shift of transition probabilities.

From this table, we observe that the changes in the death probability of healthy people, $\hat{p}^{13}$, have a positive impact on life insurance and a smaller negative impact on LTC and life annuity for both males and females; the changes in the death probability of disabled, $\hat{p}^{23}$, have a very small impact on the portfolio liabilities for both genders and the changes in the transition probabilities from healthy to disabled state, $\hat{p}^{12}$, have a strong positive impact on LTC (especially for females) and a positive impact on life insurance, while have a negative effect on life annuity for both genders. From these results, it can be guessed that a mixed portfolio could compensate positive and negative effects of different

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TABLE 5

TOTAL EFFECTIVE DURATIONS, ENTRY AGE: 65.

	Males			Females		
Duration	LTC	Annuity	Life	LTC	Annuity	Life
$D_e^{13}(\hat{\boldsymbol{p}})$	13.1393	10.1696	−22.9923	8.2795	11.5005	−87.2707
$D_e^{23}(\hat{\boldsymbol{p}})$	2.3372	0.0738	−0.1562	3.0640	0.0273	−0.1373
$D_e^{12}(\hat{\boldsymbol{p}})$	−89.3275	6.9324	−17.0646	−383.888	8.0117	−69.3304
$\sum_s D_e^s(\hat{\boldsymbol{p}})$	−73.8510	17.1758	−40.2131	−372.5445	19.5395	−156.7384

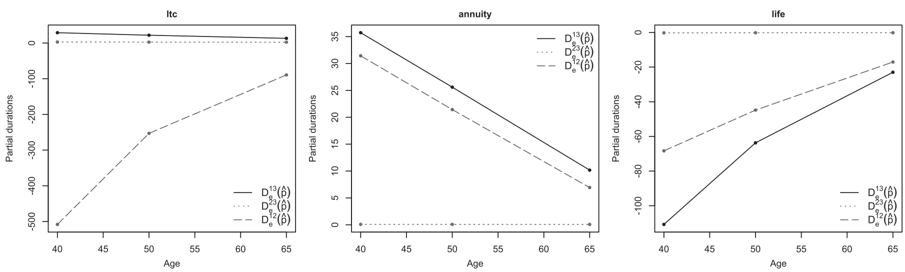

FIGURE 4: Partial durations by entry age 40, 50 and 65, males.

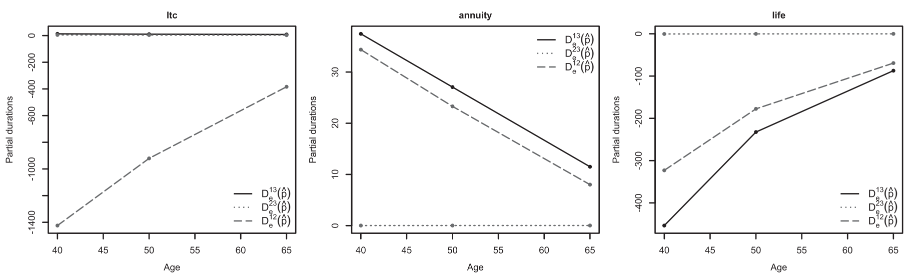

FIGURE 5: Partial durations by entry age 40, 50 and 65, females.

shifts on transition probabilities. In Table 5, we show the total effective duration vector, $\mathbf{D}_e(\hat{\boldsymbol{p}}) = \left(D_e^{13}(\hat{\boldsymbol{p}}), D_e^{23}(\hat{\boldsymbol{p}}), D_e^{12}(\hat{\boldsymbol{p}})\right)$, for the analyzed insurance products.

Moreover, we perform a sensitivity analysis of partial durations by entry age and gender, results are plotted in Figures 4 and 5 for males and females, respectively. We observe that $D_e^{23}(\hat{\boldsymbol{p}})$ continues to be negligible at all the ages. Results on $D_e^{12}(\hat{\boldsymbol{p}})$ show that the insurance products are less affected by shifts of $\hat{p}^{12}$ when the entry age increases. For life annuity and life insurance, results on $D_e^{13}(\hat{\boldsymbol{p}})$ lead to similar conclusion of $D_e^{12}(\hat{\boldsymbol{p}})$, while values of $D_e^{13}(\hat{\boldsymbol{p}})$ are minimal for LTC.

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TABLE 6

MIX OF TWO *LoB*: OPTIMAL PROPORTION OF THE *LoB* IN THE PRODUCT MIX. MD APPROACH.

	Mix 1 $\omega_{\text{ltc}}^{\text{MD}}$		Mix 2 $\omega_{\text{ltc}}^{\text{MD}}$		Mix 3 $\omega_{\text{life}}^{\text{MD}}$	
Age	Males	Females	Males	Females	Males	Females
40	12.37%	4.86%	−60.40%	−123.40%	27.27%	8.46%
50	17.09%	5.26%	−90.68%	−82.65%	30.24%	10.94%
65	18.87%	4.98%	−119.55%	−72.63%	29.93%	11.08%

The natural hedging strategy is obtained by a product mix that is a combination of two LoB: ltc and life, ltc and annuity or a combination of three LoB: ltc, life and annuity, as follows:

- Deferred annuity and LTC stand alone (mix 1).
- Whole life and LTC stand alone (mix 2).
- Whole life and annuity (mix 3).
- Deferred annuity, LTC stand alone and whole life (mix 4).

The third product mix has been considered for comparative purposes with the model of Wang *et al.* (2009) based on the mortality duration. To make results comparable, we set the liability of each product mix to the same amount of 1,000 monetary units.

4.3.1. *MD approach with shifts of the same size.* When each transition probability changes of the same size, Δ, the proportions of insurance liability optimal according to the multivariate duration approach, $\omega_{\mathbf{LoB}}^{\text{MD}}$, are obtained from the solution of the system of linear equations 3.16. As previously stated, this system has a unique solution when the LoB are two. The proportions of insurance liability, $\omega_{\mathbf{LoB}}^{\text{MD}}$, are shown in Table 6.

The optimal proportion for natural hedging of LTC in the mix 1 is in the range of 12%–19% for males and increases as the entry age becomes higher, while it is about 5% for females for all the entry ages. In terms of number of policies, e.g., at age 65, considering the expected liabilities of each LoB, the LTC proportion of 18.87% (4.98%) is equivalent to hold 241.7 (195.8) LTC policies and 65.7 (66.4) annuities for males (females). In the second mix, that combines LTC and life insurance, the optimal proportion of LTC is negative, which implies a proportion of life insurance that exceeds 100% as these two LoB do not really offset. Looking at the optimal portfolio composition in terms of number of policies at age 65, the product mix should be composed of −1531.6 (−2853.8) LTC policies and 664.8 (1216.4) life insurance policies for males (females). Finally, the optimal proportion of life insurance in the mix 3 is in the range of 27%–30% for males and 8%–11% for females. In terms of number of policies at age 65, the insurer has to hold 90.6 (78.1) life insurance policies and 56.7 (62.2) annuities for males (females).

TABLE 7

MIX OF TWO LOB: PRESENT VALUE OF THE UNEXPECTED CASH FLOWS. MD APPROACH. SHIFTS OF THE SAME SIZE. MALES.

Age	Shift	X_{mix1}	X_{mix2}	X_{mix3}	X_{ltc}	X_{annuity}	X_{life}
40	−0.001	0.67	−0.68	0.44	−487.93	69.66	−184.16
	0.001	0.64	−0.60	0.43	464.98	−64.92	174.72
50	−0.001	0.22	−0.68	0.08	−233.32	48.36	−111.31
	0.001	0.21	−0.63	0.08	223.98	−45.91	106.18
65	−0.001	−0.03	−0.13	−0.05	−75.10	17.42	−40.95
	0.001	−0.03	−0.12	−0.05	72.63	−16.93	39.49

TABLE 8

MIX OF TWO LOB: PRESENT VALUE OF THE UNEXPECTED CASH FLOWS. MD APPROACH. SHIFTS OF THE SAME SIZE. FEMALES.

Age	Shift	X_{mix1}	X_{mix2}	X_{mix3}	X_{ltc}	X_{annuity}	X_{life}
40	−0.001	0.86	−13.83	0.31	−1000.00	74.60	−803.14
	0.001	0.81	−12.74	0.30	1371.69	−69.20	751.99
50	−0.001	0.29	−5.31	−0.04	−926.76	51.81	−422.28
	0.001	0.28	−4.98	−0.04	886.93	−48.99	398.62
65	−0.001	−0.04	−0.90	−0.10	−379.44	19.86	−160.16
	0.001	−0.04	−0.86	−0.09	365.79	−19.23	153.40

To measure the hedge effectiveness, we make a sensitivity test setting the shift for all probabilities to a negative value of −0.001 and a positive value of 0.001 and calculate the present value of the unexpected cash flows X according to Equations (3.2) and (3.3), where $V(\boldsymbol{p})$ is the realized total liability after the shift on the expected transition probabilities vector (see Tables 7 and 8).

Our numerical analysis based on the multivariate duration approach under the assumption of parallel shifts of the same size shows that mix 1 is able to strongly reduce the sensitivity of the portfolio to longevity and disability risks respect to the individual LoB, especially with respect to the LTC insurance that has the highest risk level. Similar results can be achieved from mix 3, where, however, there are no LTC benefits. Numerical results from mix 2 show that the combination of life insurance and LTC are not so good as the other mixes, especially for females. We recall that mix 2 requires negative proportions of LTC insurance. In product mix 1, the pooling of longevity risk in the life annuity with the disability risk in the LTC insurance reduces the aggregate risk for the insurer since the two risks do not work in the same direction. Actually, as stated by Rickayzen (2007) for the disability-linked annuities, "the longer an individual stays healthy and receives the standard life annuity, the lower the present value of the LTC annuity enhancement; whereas, the earlier the individual becomes severely

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TABLE 9

MIX OF THREE LOB: OPTIMAL PROPORTION OF EACH LOB IN THE PRODUCT MIX 4. MD APPROACH.

	Males			Females		
Age	$\omega_{\text{ltc}}^{\text{MD}}$	$\omega_{\text{annuity}}^{\text{MD}}$	$\omega_{\text{life}}^{\text{MD}}$	$\omega_{\text{ltc}}^{\text{MD}}$	$\omega_{\text{annuity}}^{\text{MD}}$	$\omega_{\text{life}}^{MD}$
40	1.33%	74.35%	24.33%	0.49%	91.90%	7.60%
50	0.90%	70.47%	28.63%	0.25%	89.34%	10.41%
65	−0.43%	69.81%	30.63%	−0.26%	88.60%	11.65%

disabled [...], the shorter the overall term of the annuity since the individual's life expectancy is likely to be compromised by the illness". Our results are consistent with this statement and the multivariate duration approach allows to obtain the proportions that achieve a significative risk compensation under the assumption of parallel and same size shifts.

4.3.2. *MD approach with shifts of different sizes.* When transition probabilities change of different sizes, the proportions of insurance liability optimal according to the multivariate duration approach, $\omega_{\mathbf{LoB}}^{\text{MD}}$, are obtained from the solution of the system of linear Equations (3.17). As mentioned in Section 3, if the linear system equations are independent, we need at least four LoB to have a solution. As the changes in the mortality of disabled, $\hat{p}^{23}$, have a very small impact on the portfolio liabilities, we remove the second equation from the linear system so that three LoB are sufficient to achieve a solution. Therefore, we set to 0 only two partial durations, $D_{\text{mix}}^{13}(\hat{\boldsymbol{p}})$ and $D_{\text{mix}}^{12}(\hat{\boldsymbol{p}})$.

In the following, we analyze an insurance portfolio consisting of three LoB: deferred annuity, LTC stand alone and whole life (mix 4). The optimal proportions for natural hedging for the product mix 4 are shown in Table 9. The numerical analysis shows that the optimal proportion of LTC is close to 0, as the best hedging is obtained by combining life and annuity for both the genders. The contribution of LTC insurance is then residual in the product mix 4. However, from the point of view of insurers selling LTC benefits, the natural hedging strategy can be realized by setting a constraint on the minimum share of LTC in the system of linear equations 3.17.

To evaluate the hedge effectiveness, we carry out the following sensitivity tests:

- Sensitivity test 1: we suppose that each transition probability can shift of ± 0.0005 and ± 0.001 producing 4^3 different cases. Results are shown in Table 10.
- Sensitivity test 2: to realize a sensitivity test close to the real shifts experienced by the transition probabilities over time, we use historical shifts detected by each transition probability on the period 2002–2012. Note that in this case, shifts are not only of different sizes but also non-parallel. Results are shown in Table 11.

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TABLE 10

SENSITIVITY TEST 1: STATISTICAL PARAMETERS OF THE PRESENT VALUE OF THE UNEXPECTED CASH FLOWS. MD APPROACH. SHIFTS OF DIFFERENT SIZES.

		Males				Females			
Age		X_{mix4}	X_{ltc}	$X_{annuity}$	X_{life}	X_{mix4}	X_{ltc}	$X_{annuity}$	X_{life}
40	Min	0.19	−544.10	−181.90	−299.40	0.33	−1000.00	−193.00	−864.40
	Mean	6.77	290.80	−15.87	60.45	4.43	888.70	−31.23	378.00
	Max	23.05	1593.00	136.70	498.40	11.00	4275.00	88.85	2118.00
	Var	39.15	615799.10	7717.03	52699.04	9.21	4488505.00	6693.88	772190.70
50	Min	−0.10	−505.50	−130.90	−288.60	−0.19	−1000.00	−139.20	−739.10
	Mean	1.18	73.37	−2.65	8.36	0.86	431.10	−13.42	113.00
	Max	4.48	836.70	128.70	304.40	3.24	2796.00	92.59	1131.00
	Var	1.56	219847.10	4953.03	26681.64	1.27	2291899.00	4315.17	287043.60
65	Min	−0.48	−311.70	−49.39	−127.50	−1.56	−992.70	−55.89	−476.40
	Mean	−0.12	−2.44	0.64	−1.87	−0.40	32.38	−0.03	−2.54
	Max	0.02	317.20	53.82	114.30	0.00	1197.00	58.03	441.20
	Var	0.02	41481.43	771.45	4181.73	0.19	647052.70	959.74	60834.64

TABLE 11

SENSITIVITY TEST 2: STATISTICAL PARAMETERS OF THE PRESENT VALUE OF THE UNEXPECTED CASH FLOWS. MD APPROACH. SHIFTS OF DIFFERENT SIZES.

		Males				Females			
Age		X_{mix4}	X_{ltc}	$X_{annuity}$	X_{life}	X_{mix4}	X_{ltc}	$X_{annuity}$	X_{life}
40	Min	−0.65	−57.05	−1.24	−88.23	−18.66	−317.46	−14.21	−532.35
	Mean	2.16	−34.98	10.64	−61.29	−5.13	−205.22	11.12	−426.36
	Max	11.37	153.98	41.41	16.23	14.58	385.55	32.18	352.17
	Var	10.84	5078.04	143.16	1019.29	96.95	61766.57	245.52	80174.39
50	Min	−6.26	−68.68	−2.29	−95.96	−24.39	−383.22	−13.06	−422.65
	Mean	−0.54	−52.34	10.25	−66.69	−11.01	−280.78	9.86	−347.10
	Max	5.95	172.81	40.95	25.86	18.25	467.87	30.43	282.82
	Var	12.98	6876.43	144.37	1338.65	157.54	95126.45	211.80	51443.20
65	Min	−8.85	−96.92	−1.16	−89.20	−23.82	−628.73	−11.22	−351.89
	Mean	−5.80	−75.40	7.58	−63.63	−14.11	−482.98	5.87	−301.34
	Max	7.02	195.66	33.46	25.45	17.70	714.45	28.66	242.00
	Var	23.61	10525.55	100.09	1260.86	159.72	214946.01	153.93	37320.85

As regards the sensitivity test 1 (see Table 10), results indicate a very high variability of LTC insurance, high variability of the whole life and a lower one of the annuity. Note that the product mix 4 almost completely cancels the variability of the individual LoB, as evidenced by the narrow range between

TABLE 12

SENSITIVITY TEST 2: RELATIVE RISK REDUCTION OF THE PRODUCT MIX RESPECT TO EACH LOB. MD APPROACH.

	Males			Females		
Age	$R_{\text{mix,ltc}}$	$R_{\text{mix,annuity}}$	$R_{\text{mix,life}}$	$R_{\text{mix,ltc}}$	$R_{\text{mix,annuity}}$	$R_{\text{mix,life}}$
40	99.79%	92.43%	98.94%	99.84%	60.51%	99.88%
50	99.81%	91.01%	99.03%	99.83%	25.62%	99.69%
65	99.78%	76.41%	98.13%	99.93%	−3.76%	99.57%

minimum and maximum values of the distribution of X and the corresponding variance.

Concerning the second test (see Table 11), results confirm the variability characteristics of the individual LoB (highest for LTC, medium for whole life and lowest for annuity). The values for mix 4 show a strong volatility reduction for males, while the same is not true for females.

To better understand the level of variance reduction, we calculate an indicator of relative risk reduction, R, given by

$$R_{\text{mix,LoB}} = 1 - \frac{\sigma^2(X_{\text{mix}})}{\sigma^2(X_{\text{LoB}})}. \tag{4.2}$$

Natural hedge is said to be effective if the distribution of X_{mix} is significantly less variable than the distribution of each product mix component X_{LoB}. Therefore, a higher value of the relative risk reduction R indicates a better hedge effectiveness. The R values obtained by the ratio between the variance of the distribution of X from the product mix 4 and from the single product are shown in Table 12. The analysis demonstrates that natural hedging between ltc, annuity and life is very good for males with values of relative risk reduction higher than 99% respect to ltc, higher than 98% respect to life and higher than 75% respect to annuity. Concerning females, the values of R are similar for ltc and life, while values for annuity are not satisfactory, in particular at age 65.

It appears clearly from these results that the MD approach is not always effective to construct natural hedging when the shift of the transition probabilities are non-parallel, such as those experienced in the observation period 2002–2012.

4.4. Natural hedging strategy under CVaR minimization (CVaRM) approach

Similarly to the application of MD approach, in this case, we also consider product mixes of two and three LoB and evaluate the hedging effectiveness according to the variability level of the distribution of the present value of the unexpected cash flows (X) from the single LoB respect to the product mix.

To investigate the natural hedging under the CVaRM approach, we simulate 10,000 realizations of the future transition probabilities and calculate the

TABLE 13

MIX OF TWO LOB: OPTIMAL PROPORTION OF EACH LOB. CVARM APPROACH.

		Mix 1 $\omega_{\text{ltc}}^{\text{CVaRM}}$		Mix 2 $\omega_{\text{ltc}}^{\text{CVaRM}}$		Mix 3 $\omega_{\text{life}}^{\text{CVaRM}}$	
Age	ϵ	Males	Females	Males	Females	Males	Females
40	95.0%	14.78%	2.58%	−43.18%	−49.68%	28.72%	8.37%
	99.0%	13.63%	2.16%	−39.76%	−75.93%	29.22%	7.65%
	99.5%	13.19%	2.04%	−38.66%	−79.99%	29.85%	7.35%
50	95.0%	11.14%	2.68%	−41.97%	−41.59%	25.57%	9.08%
	99.0%	9.99%	2.43%	−36.35%	−63.23%	25.94%	8.45%
	99.5%	9.60%	2.28%	−37.37%	−74.11%	25.90%	8.22%
65	95.0%	3.32%	1.22%	−19.98%	−35.92%	19.81%	6.81%
	99.0%	2.61%	1.05%	−17.76%	−45.62%	19.61%	6.37%
	99.5%	2.32%	0.88%	−17.26%	−48.92%	19.76%	6.34%

simulated distribution of X for the all product mixes previously analyzed. We find the liability proportion of each LoB on total portfolio liability that minimize the CVaR of x_{mix} by solving the optimization problem in Equation (3.18). The values of liability proportions $\omega_{\text{LoB}}^{\text{CVaRM}}$ are calculated according to different confidence levels: $\epsilon = 99.5\%$, 99% and 95%.

The values of the optimal proportion under the CVaRM approach are calculated using forecasted transition probabilities incorporating parameter uncertainty through a semi-parametric bootstrap procedure as described in Section 4.2. First, we draw one set of CBD model's parameters for determining the portfolio optimal proportions then we evaluate the performance of the portfolio through another set of parameters. Therefore, we assess the performance of the natural hedging by out-of-sample values of the loss distribution.

For the product mixes that are a combination of two LoB (mix 1, mix 2 and mix 3), the values of $\omega_{\text{LoB}}^{\text{CVaRM}}$ are shown in Table 13, while the corresponding CVaRM values in Table 14 for males and Table 15 for females. The values of liability proportions obtained through the CVaRM approach are very different from those calculated according to the MD approach. For males, the differences increase with increasing age due to the behavior of ω_{ltc} that works in the opposite direction under the CVaRM approach.

The CVaR values obtained from mix 1 and mix 3 are lower than those calculated on the single LoB, especially for mix 3 which however does not include LTC insurance. Finally, the product mix 2 produces CVaR values better than the ltc and life, but worse than the annuity. It is thus confirmed (as in the MD approach) that mix "life + ltc" is not effective in terms of natural hedging. Looking at the number of policies and considering, e.g., product mix 1 with entry age 65 and 99.5% confidence level, the LTC proportion of 2.32% (0.88%) is equivalent to hold 29.7 (34.8) LTC policies and 79.1 (69.3) annuities for males (females).

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TABLE 14

MIX OF TWO LOB: OUT-OF-SAMPLE CVAR VALUES. MALES.

		CVaR$_\epsilon$					
Age	ϵ	x_{ltc}	x_{annuity}	x_{life}	x_{mix1}	x_{mix2}	x_{mix3}
40	95.0%	1637.29	168.77	564.18	67.99	352.68	35.40
	99.0%	2428.57	191.74	695.26	85.72	418.74	43.86
	99.5%	2747.93	199.28	743.14	91.76	443.81	46.77
50	95.0%	1305.43	114.44	439.48	61.39	283.63	24.73
	99.0%	1899.66	134.29	543.47	79.17	344.28	31.56
	99.5%	2146.47	141.48	579.80	85.68	365.80	34.42
65	95.0%	1013.48	51.04	241.54	46.16	191.30	13.59
	99.0%	1431.04	63.31	298.60	58.15	239.00	17.86
	99.5%	1605.64	68.39	319.63	62.96	257.58	19.06

TABLE 15

MIX OF TWO LOB: OUT-OF-SAMPLE CVAR VALUES. FEMALES.

		CVaR$_\epsilon$					
Age	ϵ	x_{ltc}	x_{annuity}	x_{life}	x_{mix1}	x_{mix2}	x_{mix3}
40	95.0%	2014.32	39.96	1177.06	30.40	1060.94	9.85
	99.0%	3212.41	46.00	2104.92	35.62	2027.97	14.16
	99.5%	3806.16	48.04	2529.91	37.56	2459.84	16.20
50	95.0%	1521.83	31.05	700.96	24.16	545.05	8.32
	99.0%	2453.66	36.67	1265.19	29.46	951.15	12.33
	99.5%	2880.97	38.68	1539.34	31.36	1141.12	14.10
65	95.0%	1919.73	22.98	662.62	19.69	395.92	6.47
	99.0%	2948.12	27.49	1106.57	24.58	565.37	8.92
	99.5%	3303.37	29.02	1255.54	26.24	622.91	10.13

For the product mixes that are a combination of three LoB (mix 4), the proportions of insurance liability are shown in Table 16, while the corresponding CVaR values in Table 17 for males and Table 18 for females. These results are then compared with those from the MD approach.

In the mix of three products, the proportions of insurance liability obtained by the CVaRM approach are much more similar to the MD approach respect to the mixes of two products. However, the differences are stronger at age 65. For example, at age 65, the value of CVaR$_{99.5\%}$ under the MD approach is 80% higher compared to the CVaR value under the CVaRM approach for males, while for females the value under the MD approach is 6.5 times higher than that under the CVaRM approach. Under the CVaRM approach, the CVaR of the

TABLE 16

MIX OF THREE LOB: OPTIMAL PROPORTION OF EACH LOB IN THE PRODUCT MIX 4. CVARM APPROACH.

		Males			Females		
Age	ϵ	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$
40	95.0%	3.80%	73.02%	23.17%	0.58%	91.44%	7.98%
	99.0%	3.36%	72.39%	24.25%	0.53%	92.15%	7.33%
	99.5%	3.25%	72.26%	24.49%	0.57%	92.47%	6.96%
50	95.0%	0.66%	74.69%	24.66%	0.46%	91.02%	8.52%
	99.0%	0.22%	73.99%	25.78%	0.43%	91.64%	7.93%
	99.5%	0.19%	74.00%	25.81%	0.45%	91.97%	7.58%
65	95.0%	−1.46%	79.33%	22.13%	−0.40%	92.73%	7.67%
	99.0%	−1.57%	79.37%	22.20%	−0.45%	93.07%	7.37%
	99.5%	−1.75%	79.10%	22.65%	−0.44%	93.20%	7.24%

TABLE 17

MIX OF THREE LOB: OUT-OF-SAMPLE CVAR VALUES. MALES.

		CVaR_ϵ				
Age	ϵ	x_{ltc}	x_{annuity}	x_{life}	$x_{\text{mix}}^{\text{CVaRM}}$	$x_{\text{mix}}^{\text{MD}}$
40	95.0%	1637.29	168.77	564.18	31.25	34.81
	99.0%	2428.57	191.74	695.26	39.97	43.44
	99.5%	2747.93	199.28	743.14	42.96	46.79
50	95.0%	1305.43	114.44	439.48	24.53	27.61
	99.0%	1899.66	134.29	543.47	31.49	34.06
	99.5%	2146.47	141.48	579.80	34.38	36.34
65	95.0%	1013.48	51.04	241.54	11.15	25.10
	99.0%	1431.04	63.31	298.60	14.59	28.28
	99.5%	1605.64	68.39	319.63	16.05	29.26

product mix is consistently less than the CVaR of the single LoB, even for the annuity. Therefore, the natural hedging turns out to be always effective, even if the optimal hedging is obtained with very low LTC proportions (becoming negative at entry age 65).

To provide a graphical representation of the level of risk reduction achieved by both the approaches, we show in Figures 6–8 for the entry age 40, 50 and 65, respectively, the distribution of the present value of unexpected cash flows from the single LoB and the product mix 4 under MD and CVaRM approach with a 99.5% confidence level. The figures confirm the strong risk reduction obtained by the product mix 4 under both MD and CVaRM approaches.

TABLE 18

MIX OF THREE LOB: OUT-OF-SAMPLE CVAR VALUES. FEMALES.

		CVaR_ϵ				
Age	ϵ	x_{ltc}	x_{annuity}	x_{life}	$x_{\mathrm{mix}}^{\mathrm{CVaRM}}$	$x_{\mathrm{mix}}^{\mathrm{MD}}$
40	95.0%	2014.32	39.96	1177.06	8.00	8.19
	99.0%	3212.41	46.00	2104.92	12.49	13.08
	99.5%	3806.16	48.04	2529.91	14.48	17.04
50	95.0%	1521.83	31.05	700.96	7.78	13.46
	99.0%	2453.66	36.67	1265.19	11.91	29.68
	99.5%	2880.97	38.68	1539.34	13.31	37.72
65	95.0%	1919.73	22.98	662.62	5.69	29.60
	99.0%	2948.12	27.49	1106.57	7.66	49.12
	99.5%	3303.37	29.02	1255.54	8.45	55.06

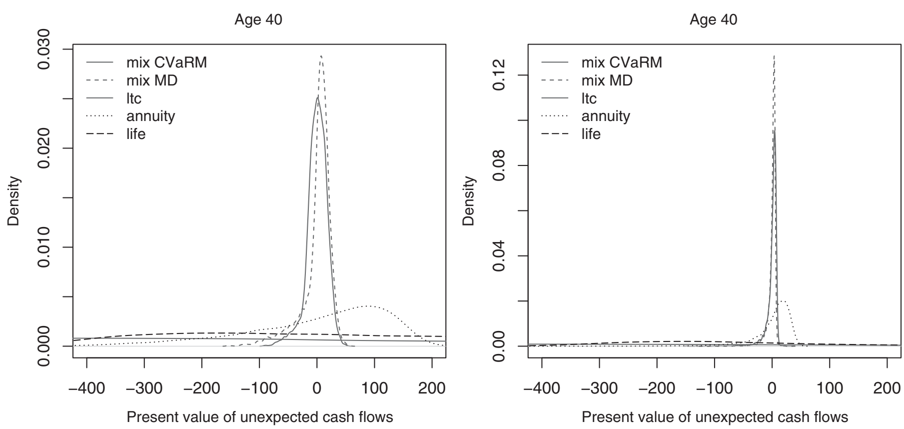

FIGURE 6: Out-of-sample values of the loss distribution, mix 4. MD and CVaRM approach with a 99.5% confidence level. Age 40. Left: males, right: females.

In order to measure how robust the choice of the portfolio optimal proportions is under CVaRM approach, we simulate the values of the loss distribution and then we calculate the optimal proportion of each LoB in the portfolio using different samples of CBD model's parameters in the bootstrap procedure. A comparison between in-sample and out-of-sample values is shown in Table 19 for males and Table 20 for females according to five samples and to a 99.5% confidence level. The portfolio optimal proportions are calculated according to the sample 1, thus the in-sample and the out-of-sample values of CVaR of the loss distribution of the product mix are equal in this sample. Sample 2 is the sample chosen for generating out-of-sample values of the distribution of X shown in Tables 17 and 18. Looking at these results, the performance of the model is still

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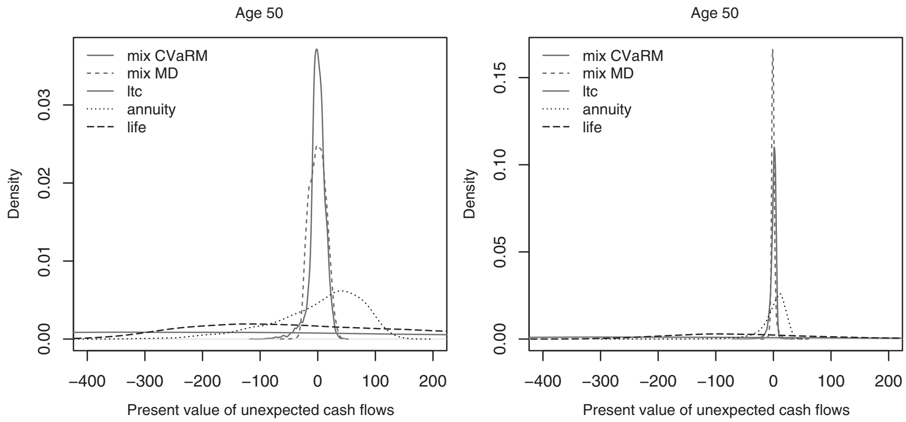

FIGURE 7: Out-of-sample values of the loss distribution, mix 4. MD and CVaRM approach with a 99.5% confidence level. Age 50. Left: males, right: females.

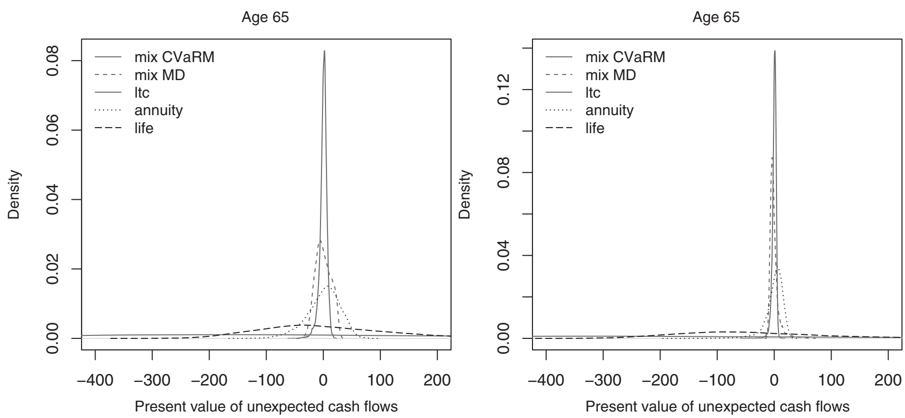

FIGURE 8: Out-of-sample values of the loss distribution, mix 4. MD and CVaRM approach with a 99.5% confidence level. Age 65. Left: males, right: females.

very good when changing the sample, and the values of the optimal proportions $\omega_{\mathrm{LoB}}^{\mathrm{CVaRM}}$ are not very sensitive to the sample's change.

4.5. Testing the effectiveness of natural hedging for a portfolio mix with different age profiles

When dealing with natural hedging between life insurance and annuities, it is important to bear in mind that the buyers of these two products typically have different initial age, e.g., 40 for life insurance and 65 for annuities. The same consideration holds for LTC insurance that is usually sold to different cohorts than annuities. In the following, we will study the effectiveness of natural hedging based on the CVaRM approach considering insurance products

TABLE 19

MIX OF THREE LOB: OPTIMAL PROPORTION OF EACH LOB IN THE PRODUCT MIX 4 AND CVAR VALUES. CVARM APPROACH WITH $\epsilon = 99.5\%$. MALES.

Age	Sample	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$	Out-of-Sample $\text{CVaR}_\epsilon(x_{\text{mix}}^{\text{CVaRM}})$	In-Sample $\text{CVaR}_\epsilon(x_{\text{mix}}^{\text{CVaRM}})$
40	1	3.25%	72.26%	24.49%	42.02	42.02
	2	3.12%	70.77%	26.10%	42.96	42.60
	3	3.15%	71.59%	25.26%	41.39	41.35
	4	3.25%	71.87%	24.88%	41.92	41.88
	5	3.59%	72.06%	24.35%	41.79	41.68
50	1	0.19%	74.00%	25.81%	33.05	33.05
	2	0.66%	74.89%	24.45%	34.38	34.30
	3	0.70%	74.51%	24.79%	34.52	34.30
	4	−0.18%	73.89%	26.30%	35.20	35.11
	5	0.61%	75.10%	24.29%	34.13	33.92
65	1	−1.75%	79.10%	22.65%	16.18	16.18
	2	−1.79%	79.19%	22.61%	16.05	16.04
	3	−1.67%	79.81%	21.86%	16.32	16.24
	4	−1.63%	79.83%	21.80%	16.61	16.49
	5	−1.58%	79.56%	22.02%	16.18	16.14

TABLE 20

MIX OF THREE LOB: OPTIMAL PROPORTION OF EACH LOB IN THE PRODUCT MIX 4 AND CVAR VALUES. CVARM APPROACH WITH $\epsilon = 99.5\%$. FEMALES.

Age	Sample	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$	Out-of-Sample $\text{CVaR}_\epsilon(x_{\text{mix}}^{\text{CVaRM}})$	In-Sample $\text{CVaR}_\epsilon(x_{\text{mix}}^{\text{CVaRM}})$
40	1	0.57%	92.47%	6.96%	12.18	12.18
	2	0.69%	92.80%	6.52%	14.48	13.96
	3	0.73%	92.91%	6.35%	15.09	14.06
	4	0.60%	92.76%	6.64%	13.30	13.08
	5	0.81%	93.17%	6.02%	17.32	14.71
50	1	0.45%	91.97%	7.58%	10.59	10.59
	2	−0.56%	93.14%	7.43%	13.31	12.28
	3	0.63%	92.69%	6.67%	14.54	13.08
	4	0.59%	92.55%	6.86%	13.46	12.58
	5	0.53%	92.85%	6.62%	17.02	14.04
65	1	−0.44%	93.20%	7.24%	7.56	7.56
	2	0.35%	92.45%	7.20%	8.45	8.34
	3	−0.52%	93.22%	7.30%	8.10	8.03
	4	−0.50%	93.37%	7.13%	8.83	8.60
	5	−0.45%	93.55%	6.90%	10.60	9.70

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TABLE 21

BIC STATISTIC FOR M9, AGES 40–89 AND YEARS 2002–2012, ITALIAN POPULATION BY GENDER.

	$p^{13}_{x,t}$	$p^{23}_{x,t}$	$p^{12}_{x,t}$
Model (Gender)	BIC	BIC	BIC
M9 (Males)	−2945	−2457	−2728
M9 (Females)	−2735	−2410	−2960

with different age profiles. We consider a mix where life insurance and LTC coverage are purchased at age 40, while annuities at age 65. The M8 used in the forecasting of transition probabilities is characterized by low-dimensional stochastic factors and not fully captures the different transition probabilities evolution of each age group in the population. This could affect the natural hedging effectiveness when a portfolio mix with different age profiles is considered. In order to overcome this limitation, but taking advantage of the operational simplicity of the forecasting model, we propose an extension of the M8 that includes a different factor for the older cohorts (suitable for annuities) that does not affect the younger policyholders (suitable for life/LTC insurance). The proposed model, denoted as M9, has a set of four stochastic factors for each transition ij and the transition probabilities $p^{ij}_{x,t}$ are described by the following equation:

$$\mathrm{logit}(p^{ij}_{x,t}) = \ln\left(\frac{p^{ij}_{x,t}}{1-p^{ij}_{x,t}}\right) = {}^{ij}k^{(1)}_t + {}^{ij}k^{(2)}_t(x-\bar{x}) + {}^{ij}k^{(3)}_t(x-\bar{x})^+ + {}^{ij}\gamma^{(4)}_c(x_c-x)\,, \tag{4.3}$$

where x is the age, $\bar{x}$ the mean age in the sample age range (in our analysis $\bar{x} = 64.5$), $(x-\bar{x})^+ = \max(x-\bar{x}, 0)$, t the time, $c = t-x$ the cohort and x_c a constant parameter that does not vary with age or time (in our analysis, $x_c = 65$). To avoid any identifiability problem, we introduce the constraint $\sum_c^{ij} \gamma^{(4)}_c = 0$.

The set of factors ${}^{ij}k^{(3)}_t$ are added to capture the different dynamics of transition probabilities at lower rather than upper ages. The other parameters have the same meaning of the M8 as described in Equation (4.1). Similarly to M8, parameters are estimated separately for each transition probability.

The BIC values displayed in Table 21 are similar to those obtained from M8.

The maximum-likelihood estimates of the M9 parameters are shown in Appendix A.2.

To forecast transition probabilities, the parameters ${}^{ij}k^{(1)}_t$, ${}^{ij}k^{(2)}_t$, ${}^{ij}k^{(3)}_t$ and ${}^{ij}\gamma^{(4)}_c$ are modeled through a multivariate ARIMA model $K_{s+1} = K_s + \phi\,(K_{s-2} - K_{s-1}) + \delta + C\,Z_{s+1}$. Where K_s is a 12×1 vector of parameters at the step s, s is the time or the cohort depending on the parameter; ϕ is a 12×1 vector of parameters of the autoregressive part of the model; δ is a 12×1 vector of the drifts of the model; C is a 12×12 constant upper triangular matrix so that CC'

TABLE 22

MIX OF THREE LOB: OPTIMAL PROPORTION OF EACH LOB IN THE PRODUCT MIX 4. AGE AT ISSUE: ltc AND life: 40; annuity: 65. CVARM APPROACH. M8.

	Males			Females		
ϵ	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$
95.0%	1.05%	94.84%	4.10%	−0.04%	98.10%	1.94%
99.0%	0.11%	94.18%	5.71%	−0.10%	98.34%	1.76%
99.5%	0.27%	94.15%	5.58%	−0.09%	98.51%	1.58%

TABLE 23

MIX OF THREE LOB: OPTIMAL PROPORTION OF EACH LOB IN THE PRODUCT MIX 4. AGE AT ISSUE: ltc AND life: 40; annuity: 65. CVARM APPROACH. M9.

	Males			Females		
ϵ	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$	$\omega_{\text{ltc}}^{\text{CVaRM}}$	$\omega_{\text{annuity}}^{\text{CVaRM}}$	$\omega_{\text{life}}^{\text{CVaRM}}$
95.0%	−0.39%	95.52%	4.87%	1.39%	96.96%	1.64%
99.0%	−0.30%	95.91%	4.39%	1.20%	97.46%	1.34%
99.5%	−0.69%	95.69%	5.00%	1.17%	97.65%	1.18%

is the covariance matrix; and Z is a 12×1 vector of standard normal random variables.

As for M8, the choice of ARIMA process for the parameters has been made mainly according to Information Criteria and to the analysis of residuals. The selected ARIMA models and the corresponding parameters are reported in Appendix A.2. We assume that the cohort effect has dynamics that are independent of the period effect; therefore, ${}^{ij}\gamma_c^{(4)}$ is not correlated with ${}^{ij}k_t^{(1)}$, ${}^{ij}k_t^{(2)}$ and ${}^{ij}k_t^{(3)}$. For the fitted correlation matrices ρ, see Appendix A.2.

Also in this case, parameter uncertainty is incorporated in the model's forecasting through a semi-parametric bootstrap procedure (see Section 4.2 for further details).

In order to evaluate the natural hedging effectiveness, we only consider the product mix 4 that is a combination of three LoB (LTC, life insurance and annuity), where the entry age is 40 for life and LTC insurance and 65 for annuity. We find the liability optimal proportions by solving the optimization problem in Equation (3.18) with different confidence levels under the two forecasting models, M8 (Table 22) and M9 (Table 23).

From the previous tables, we observe that the optimal portfolio is almost exclusively composed of annuities. We can argue that when considering products with different entry ages, the natural hedging is not really effective and the best choice for the insurer is to sell the less risky product. In M9, the optimal proportions are similar to those obtained by M8. Consequently, the CVaR of the

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TABLE 24

MIX OF THREE LOB: CVAR VALUES. AGE AT ISSUE: ltc AND life: 40; annuity: 65. M8.

	s_ϵ (Males)				CVaR$_\epsilon$ (Females)			
ϵ	x_{ltc}	x_{annuity}	x_{life}	$x_{\text{mix}}^{\text{CVaRM}}$	x_{ltc}	x_{annuity}	x_{life}	$x_{\text{mix}}^{\text{CVaRM}}$
95.0%	1637.29	51.04	564.18	41.19	2014.32	22.98	1177.06	19.19
99.0%	2428.57	63.31	695.26	52.29	3212.41	27.49	2104.92	23.86
99.5%	2747.93	68.39	743.14	56.59	3806.16	29.02	2529.91	25.07

TABLE 25

MIX OF THREE LOB: CVAR VALUES. AGE AT ISSUE: ltc AND life: 40; annuity: 65. M9.

	CVaR$_\epsilon$ (Males)				CVaR$_\epsilon$ (Females)			
ϵ	x_{ltc}	x_{annuity}	x_{life}	$x_{\text{mix}}^{\text{CVaRM}}$	x_{ltc}	x_{annuity}	x_{life}	$x_{\text{mix}}^{\text{CVaRM}}$
95.0%	1407.24	41.38	540.07	35.83	2319.14	47.09	1634.06	35.67
99.0%	2024.75	52.12	677.03	46.05	4001.39	55.21	2336.37	44.51
99.5%	2257.58	55.89	721.58	49.50	4747.18	58.20	2547.01	48.38

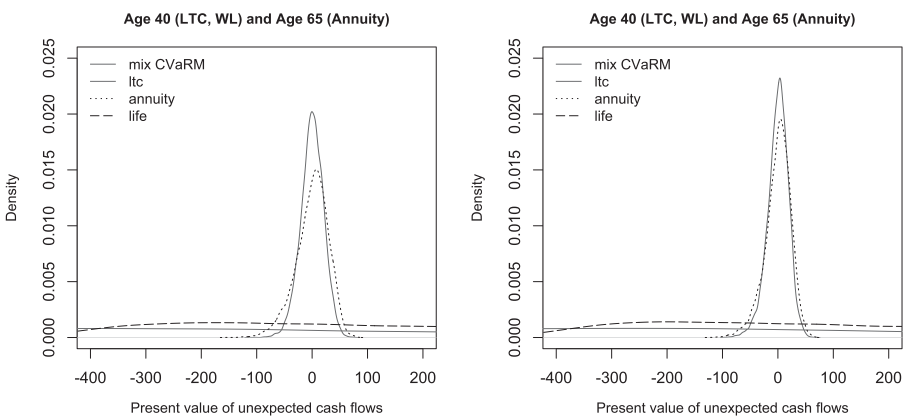

FIGURE 9: Out-of-sample values of the loss distribution, mix 4. CVaRM approach. Males. Left: M8, right: M9.

product mix 4 is very close to the CVaR of the annuity for both M8 (Table 24) and M9 (Table 25).

The level of risk reduction achieved by the mix of LTC, annuity and life insurance respect to the single LoB can be deduced from the distribution of the present value of unexpected cash flows shown in Figure 9 (males) and Figure 10 (females), where the entry age is 40 for life and LTC insurance and 65 for annuity. The left plot is relative to M8, while the right plot to M9.

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TABLE 26

MIX 1: OPTIMAL PROPORTION OF EACH LOB. CVARM APPROACH.

	M8		M9	
	$\omega_{\mathrm{ltc}}^{\mathrm{CVaRM}}$		$\omega_{\mathrm{ltc}}^{\mathrm{CVaRM}}$	
ϵ	Males	Females	Males	Females
95.0%	3.02%	0.79%	2.11%	2.61%
99.0%	2.62%	0.58%	1.86%	2.16%
99.5%	2.53%	0.47%	1.84%	1.90%

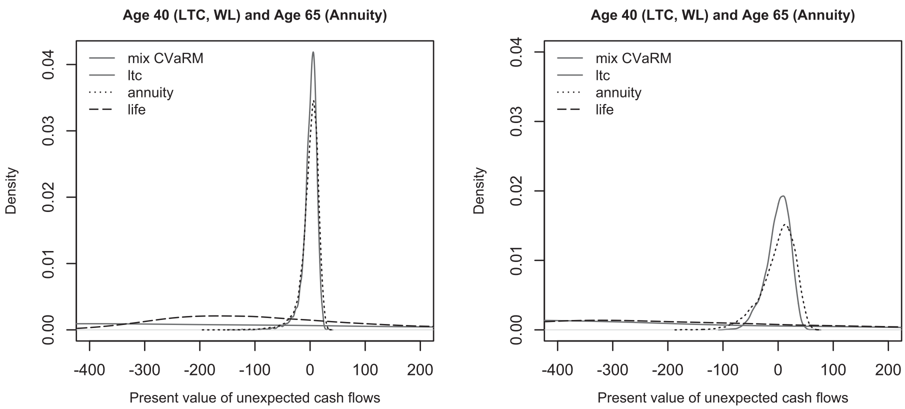

FIGURE 10: Out-of-sample values of the loss distribution, mix 4. CVaRM approach. Females. Left: M8, right: M9.

On the other hand, given the difficulties of building a portfolio with three different LoB for an insurer specialized in annuities, we analyze the product mix 1 to be considered more practical for this type of insurers. The optimal proportions of insurance liability for product mix 1 are shown in Table 26.

The LTC proportion is very low under both M8 and M9, suggesting that the portfolio is composed almost exclusively of annuitants. However, considering the expected liability of each LoB, the LTC proportion of 2.53% (0.47%) at age 65 for males (females) under M8 is equivalent to hold a portfolio of 61.9 (31.9) LTC policies and 78.9 (69.6) annuities. Therefore, the number of LTC male (female) policyholders should be about 78% (46%) with respect to the number of annuitants. Under M9, the LTC proportion of 1.84% (1.90%) at age 65 for males (females) is equivalent to hold a portfolio of 38.0 (112.1) LTC policies and 79.1 (69.3) annuities. In this latter case, the number of LTC male (female) policyholders should be about 48% (162%) with respect to the number of annuitants.

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5. Conclusions

As stated by Levantesi and Menzietti (2012), LTC stand alone contracts are more risky than life annuities and enhanced pension due to the higher volatility of the disability component that has a much greater importance in the LTC contracts. In this paper, the high riskiness of LTC policies is recognized by the higher values of the partial durations and CVaR, as well as by the longer tails of the simulated distributions of the present value of unexpected cash flows related to LTC benefits. Therefore, it seems crucial for insurance companies to identify adequate tools to manage longevity and disability risk in this kind of insurance.

We propose a risk minimization strategy for LTC contracts based on two different approaches built in a multiple state framework. The first one is based on the multivariate duration under restrictive assumptions on the transition probabilities shifts, while the second one on the minimization of the CVaR that is valid under a more general representation of the transition probabilities evolution.

Under the MD approach (that is shift-based), our analysis shows that where the shifts are of the same size, good levels of natural hedging can be achieved by combining LTC insurance and life annuity. This is the consequence of an increase in the transition probability from healthy to disabled leading to higher LTC benefits, but at the same time producing a reduction in life expectancy and a decrease in the present value of the life annuity. In practice, longevity and disability risks contained inside this product mix act in different directions reducing the aggregate risk.

On the other hand, under the MD approach, in case of shifts of different size, the natural hedging obtained by combining LTC, life annuity and whole life appears to be effective respect to LTC stand alone and whole life, but less effective respect to the life annuity.

The CVaRM approach produces very different optimal proportions respect to the MD approach, when combining two LoB. If we consider a product mix consisting of three LoB (LTC stand alone, whole life and deferred annuity), the differences between the two approaches are less pronounced, but increase with increasing entry age.

Our analysis demonstrates that LTC insurers can nevertheless reduce their portfolio riskiness through a product mix but including very low proportion of LTC. Disability risk is indeed more difficult to hedge through a combination of LoB than longevity risk and this is true under both the approaches here considered. In order to investigate the validity of these conclusions, we consider a more realistic product mix with different entry ages consisting of annuity, LTC and life insurance. Therefore, we introduce a forecasting model of transition probabilities including a stochastic factor that does not affect the younger ages, in order to obtain transition probabilities evolution varying with ages. Our analysis introduces further perplexity about the effectiveness of natural hedging for insurers having to deal with disability risk: the optimal product mix almost exclusively consists of annuities and the CVaR values of the product mix are then close to those of the annuity.

On the other hand, if we consider the product mix 1 (annuities and LTC insurance), which is a representative portfolio composition, e.g., for pension funds, the optimal proportions' results indicate a predominant weight of the annuity in terms of expected liability, but with the number of annuitants on the whole similar to that of the LTC policyholders. Therefore, we come to conclusion that a natural hedging strategy for insurers specialized in LTC and annuities and dealing with biometric risks, is feasible; although the level of risk reduction is not high compared to a portfolio only including annuities, the hedging should be considered effective compared to a portfolio consisting of only LTC policies.

Some aspects of our analysis require further research. First, our results relies to a specific data set and to conventional stochastic models to represent the evolution of the transition probabilities. The choice of conventional stochastic models belonging to the CBD family, and built in a multivariate framework, has been made for their parsimony, tractability and forecasting power. However, a topic for further research could be the development of non-parametric models in a multiple state framework to test the effectiveness of natural hedging as well as the use of different data set. Moreover, further studies can be addressed on alternative tools of risk management that might be more effective than natural hedging. For example, it would be interesting to explore the possibility to construct a swap written on the survival of disabled people and to measure their hedge effectiveness.

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APPENDIX A. PARAMETERS ESTIMATION OF THE CBD MODELS

A.1. Model M8

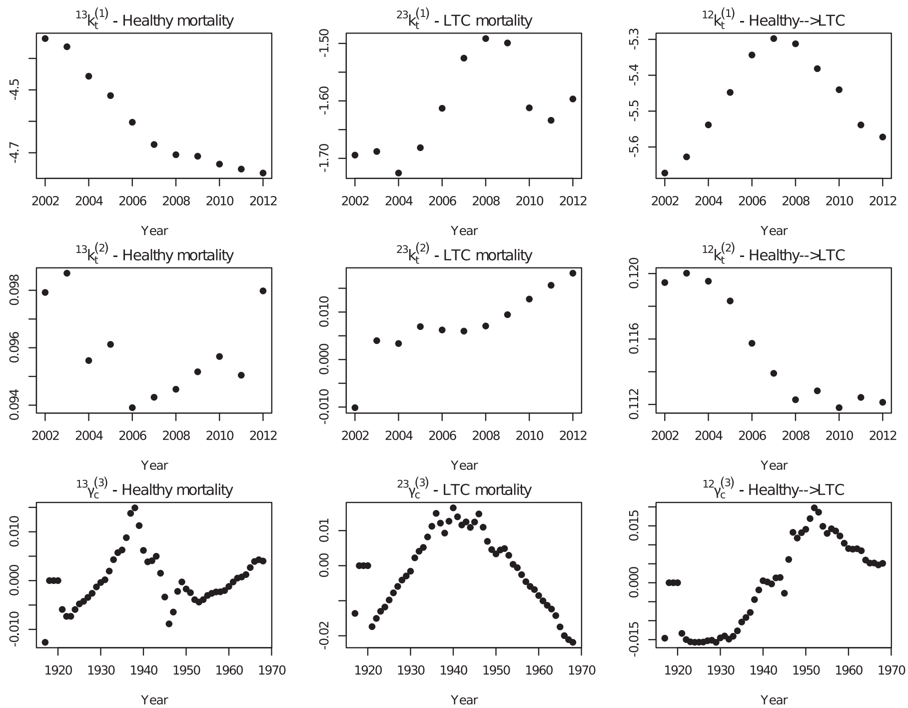

FIGURE A1: Estimated parameters of M8, ages 40–89 and years 2002–2012, males. Left: $p^{13}_{x,t}$, middle: $p^{23}_{x,t}$, right: $p^{12}_{x,t}$

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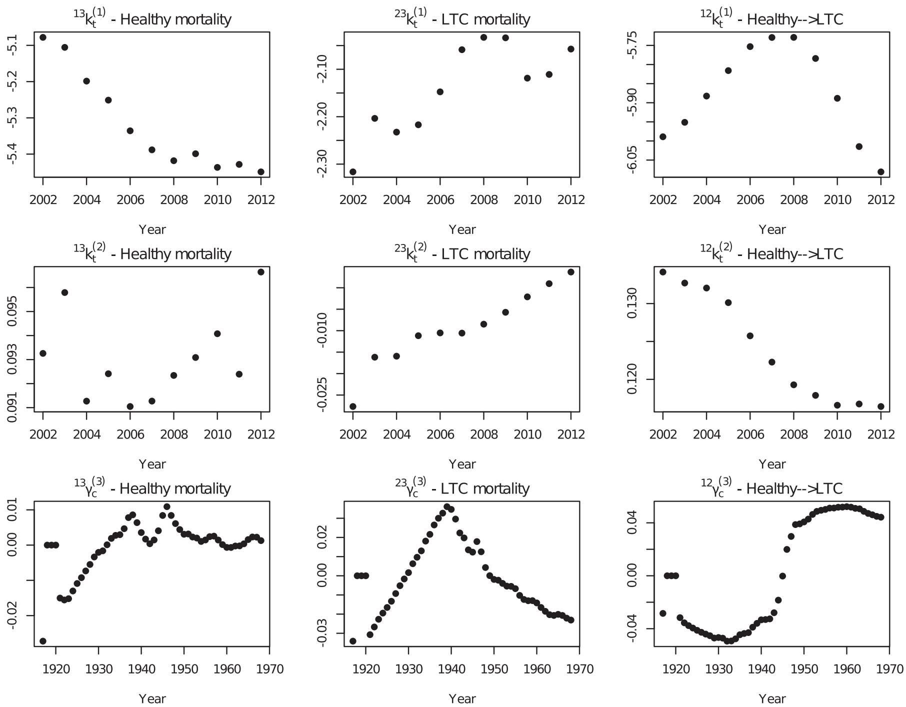

FIGURE A2: Estimated parameters of M8, ages 40–89 and years 2002–2012, females. Left: $p_{x,t}^{13}$, middle: $p_{x,t}^{23}$, right: $p_{x,t}^{12}$.

TABLE A1

FITTED PARAMETERS OF THE ARIMA MODELS. M8.

	Males				Females			
Parameter	ARIMA	σ^2	δ	ϕ	ARIMA	σ^2	δ	ϕ
${}^{13}k_t^{(1)}$	(1,1,0)	0.000794	−0.042768	0.467670	(0,1,0)	0.001278	−0.037101	–
${}^{13}k_t^{(2)}$	(1,0,0)	0.000002	0.095892	0.485742	(0,0,0)	0.000003	0.093053	–
${}^{13}\gamma_c^{(3)}$	(1,0,0)	0.000008	−0.000016	0.890373	(1,0,0)	0.000021	−0.000042	0.862711
${}^{23}k_t^{(1)}$	(1,0,0)	0.002858	−0.447964	0.722562	(0,1,0)	0.003406	0.025897	–
${}^{23}k_t^{(2)}$	(0,1,0)	0.000018	0.012261	–	(0,1,0)	0.000011	0.019070	–
${}^{23}\gamma_c^{(3)}$	(1,0,0)	0.000014	−0.000643	0.953314	(1,0,0)	0.000051	−0.000198	0.935790
${}^{12}k_t^{(1)}$	(1,1,0)	0.001880	0.021158	0.769094	(1,1,0)	0.001355	−0.010993	0.835668
${}^{12}k_t^{(2)}$	(1,0,0)	0.000001	0.169896	0.931216	(1,1,0)	0.000001	−0.011685	0.577086
${}^{12}\gamma_c^{(3)}$	(1,0,0)	0.000012	−0.000032	0.950738	(1,1,0)	0.000047	0.001426	0.469796

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TABLE A2

THE FITTED CORRELATION MATRIX ρ, MALES. M8.

1	0.585969	0	0.234900	0.408428	0	−0.631857	0.289442	0
0.585969	1	0	−0.044542	0.634420	0	0.153251	0.388055	0
0	0	1	0	0	0.354541	0	0	0.420511
0.234900	−0.044542	0	1	−0.280318	0	−0.284169	−0.426169	0
0.408428	0.634420	0	−0.280318	1	0	0.115349	0.561262	0
0	0	0.354541	0	0	1	0	0	0.665747
−0.631857	0.153251	0	−0.284169	0.115349	0	1	−0.028082	0
0.289442	0.388055	0	−0.426169	0.561262	0	−0.028082	1	0
0	0	0.420511	0	0	0.665747	0	0	1

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TABLE A3

THE FITTED CORRELATION MATRIX ρ, FEMALES. M8.

1	0.484591	0	−0.041576	0.324403	0	−0.525643	0.602555	0
0.484591	1	0	0.145906	0.687334	0	0.250951	0.278517	0
0	0	1	0	0	0.798916	0	0	0.731983
−0.041576	0.145906	0	1	0.291364	0	0.429681	−0.372514	0
0.324403	0.687334	0	0.291364	1	0	0.173947	0.119925	0
0	0	0.798916	0	0	1	0	0	0.733570
−0.525643	0.250951	0	0.429681	0.173947	0	1	−0.253267	0
0.602555	0.278517	0	−0.372514	0.119925	0	−0.253267	1	0
0	0	0.731983	0	0	0.733570	0	0	1

A.2. Model M9

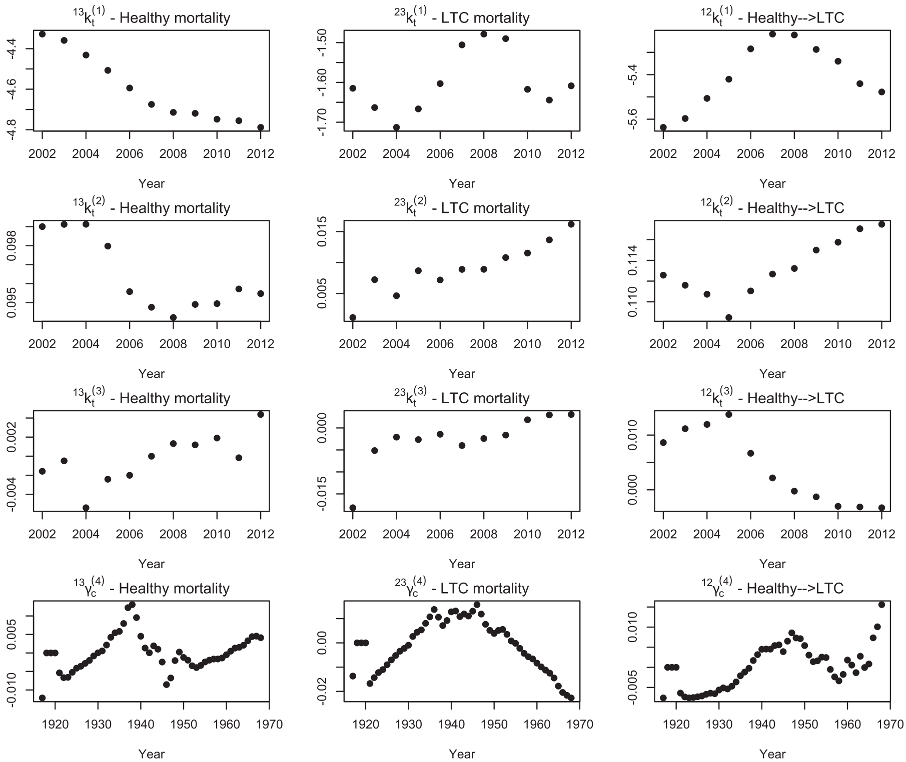

FIGURE A3: Estimated parameters of M9, ages 40–89 and years 2002–2012, males. Left: $p_{x,t}^{13}$, middle: $p_{x,t}^{23}$, right: $p_{x,t}^{12}$.

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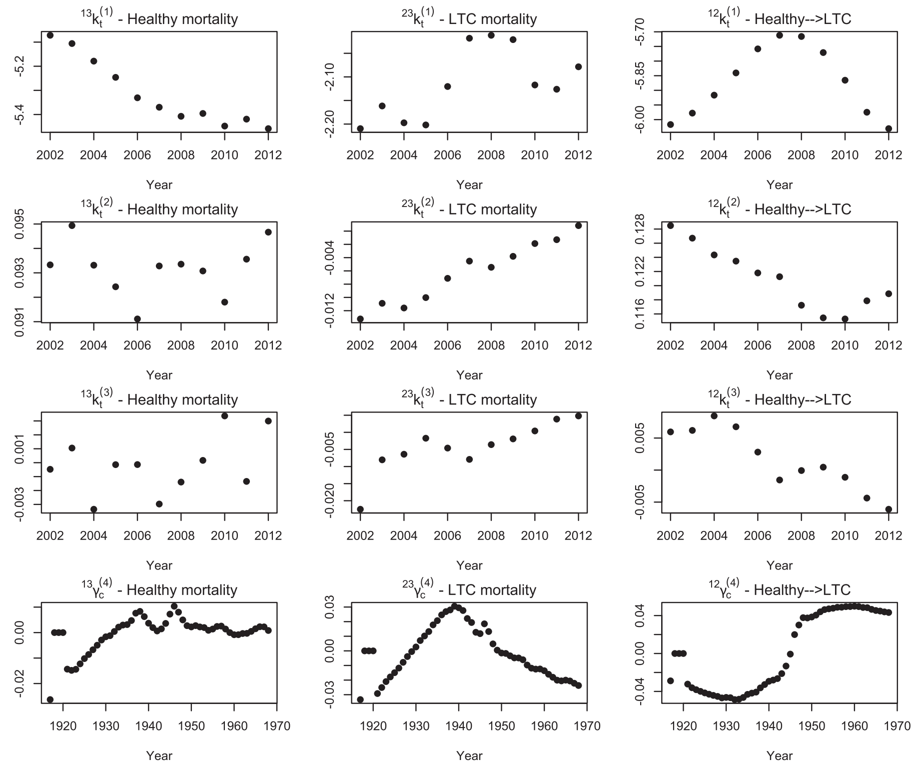

FIGURE A4: Estimated parameters of M9, ages 40–89 and years 2002–2012, females. Left: $p_{x,t}^{13}$, middle: $p_{x,t}^{23}$, right: $p_{x,t}^{12}$.

TABLE A4

FITTED PARAMETERS OF THE ARIMA MODELS. M9.

	Males				Females			
Parameter	ARIMA	σ^2	δ	ϕ	ARIMA	σ^2	δ	ϕ
${}^{13}k_t^{(1)}$	(1,1,0)	0.000591	−0.046061	0.567125	(1,1,0)	0.001261	−0.038679	0.046790
${}^{13}k_t^{(2)}$	(1,0,0)	0.000001	−0.000005	0.617498	(1,0,0)	0.002928	0.093169	0.174231
${}^{13}k_t^{(3)}$	(1,0,0)	0.000006	−0.001880	0.727019	(0,1,0)	0.001269	0.000347	–
${}^{13}\gamma_c^{(4)}$	(1,0,0)	0.000007	0.132043	0.877771	(1,0,0)	0.000019	0.075159	0.857923
${}^{23}k_t^{(1)}$	(1,0,0)	0.003318	0.096441	0.869012	(1,0,0)	0.000001	−0.000134	0.682577
${}^{23}k_t^{(2)}$	(1,1,0)	0.000002	−0.318211	−0.923826	(0,1,0)	0.000002	−0.671305	–
${}^{23}k_t^{(3)}$	(1,0,0)	0.000019	−0.000301	0.857469	(0,1,0)	0.000004	0.001407	–
${}^{23}\gamma_c^{(4)}$	(1,0,0)	0.000014	0.010350	0.955437	(1,0,0)	0.000047	−0.018765	0.935625
${}^{12}k_t^{(1)}$	(1,1,0)	0.002554	−0.000299	0.454729	(1,1,0)	0.000011	0.016809	0.837573
${}^{12}k_t^{(2)}$	(1,0,0)	0.000002	0.015220	0.809532	(1,0,0)	0.000024	−0.000944	0.911731
${}^{12}k_t^{(3)}$	(1,0,0)	0.000009	0.037086	0.883038	(0,1,0)	0.000005	−0.000227	–
${}^{12}\gamma_c^{(4)}$	(1,0,0)	0.000005	0.000197	0.954700	(1,1,0)	0.000047	0.001413	0.418683

TABLE A5

THE FITTED CORRELATION MATRIX ρ, MALES. M9.

1	0.419779	0.334020	0	0.179146	−0.041772	0.112324	0	−0.848814	0.351991	0.005262	0
0.419779	1	−0.158944	0	−0.561929	0.395559	0.318140	0	−0.350001	−0.153184	0.584233	0
0.334020	−0.158944	1	0	0.433352	0.105553	−0.080144	0	−0.278235	0.311475	−0.237048	0
0	0	0	1	0	0	0	0.312528	0	0	0	0.390282
0.179146	−0.561929	0.433352	0	1	−0.623783	−0.517089	0	−0.315739	0.384321	−0.587268	0
−0.041772	0.395559	0.105553	0	−0.623783	1	0.516998	0	0.432181	−0.149640	0.458033	0
0.112324	0.318140	−0.080144	0	−0.517089	0.516998	1	0	0.162790	−0.200142	0.361882	0
0	0	0	0.312528	0	0	0	1	0	0	0	0.408692
−0.848814	−0.350001	−0.278235	0	−0.315739	0.432181	0.162790	0	1	−0.381819	0.142494	0
0.351991	−0.153184	0.311475	0	0.384321	−0.149640	−0.200142	0	−0.381819	1	−0.835778	0
0.005262	0.584233	−0.237048	0	−0.587268	0.458033	0.361882	0	0.142494	−0.835778	1	0
0	0	0	0.390282	0	0	0	0.408692	0	0	0	1

TABLE A6

THE FITTED CORRELATION MATRIX ρ, FEMALES. M9.

1	0.497906	−0.230210	0	0.107373	−0.115650	0.184585	0	−0.568221	0.293340	−0.031216	0
0.497906	1	−0.046071	0	0.291867	−0.081447	0.535378	0	0.194905	0.233263	0.214784	0
−0.230210	−0.046071	1	0	−0.034635	0.338595	0.239645	0	0.064611	−0.164836	0.109473	0
0	0	0	1	0	0	0	0.802980	0	0	0	0.744265
0.107373	0.291867	−0.034635	0	1	0.374493	−0.303746	0	0.000938	−0.069284	−0.395483	0
−0.115650	−0.081447	0.338595	0	0.374493	1	−0.133120	0	0.185421	0.329399	−0.682235	0
0.184585	0.535378	0.239645	0	−0.303746	−0.133120	1	0	0.152179	−0.058808	0.482099	0
0	0	0	0.802980	0	0	0	1	0	0	0	0.774737
−0.568221	0.194905	0.064611	0	0.000938	0.185421	0.152179	0	1	0.173800	−0.005432	0
0.293340	0.233263	−0.164836	0	−0.069284	0.329399	−0.058808	0	0.173800	1	−0.551021	0
−0.031216	0.214784	0.109473	0	−0.395483	−0.682235	0.482099	0	−0.005432	−0.551021	1	0
0	0	0	0.744265	0	0	0	0.774737	0	0	0	1