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A BIJECTION OF INVARIANT MEANS ON AN AMENABLE GROUP WITH THOSE ON A LATTICE SUBGROUP

JOHN HOPFENSPERGER

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Abstract

Suppose G is an amenable locally compact group with lattice subgroup Γ. Grosvenor ['A relation between invariant means on Lie groups and invariant means on their discrete subgroups', *Trans. Amer. Math. Soc.* **288**(2) (1985), 813–825] showed that there is a natural affine injection $\iota : \mathrm{LIM}(\Gamma) \to \mathrm{TLIM}(G)$ and that ι is a surjection essentially in the case $G = \mathbb{R}^d$, $\Gamma = \mathbb{Z}^d$. In the present paper it is shown that ι is a surjection if and only if G/Γ is compact.

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1. Introduction

Let H be a closed subgroup of the locally compact group G. The subgroup H is called *cofinite* if G/H carries a G-invariant probability measure and *cocompact* if G/H is compact. After the recent paper of Bader *et al.* [1], G is said to have property (M) if every cofinite subgroup is cocompact. The letter M is for Mostow, who proved that solvable Lie groups have property (M).

A discrete subgroup Γ is called a lattice if it is cofinite. Cocompact lattices are better known as *uniform* lattices. Nonuniform lattices are not too hard to come by, but the most famous examples are nonamenable. For example, $\mathrm{SL}_2(\mathbb{Z})$ is a nonuniform lattice in $\mathrm{SL}_2(\mathbb{R})$, which is not amenable because $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ generate a free subgroup. In fact, [1, Theorem 1.7] states that every finite direct product of amenable linear groups has property (M). Hence, every lattice in such a group is uniform.

When G is amenable, a paper of Grosvenor [3] constructs a natural affine injection ι from the invariant means on a lattice subgroup Γ into the topological invariant means on the whole group G. Grosvenor proved that ι is surjective when G is an abelian Lie group with finitely many connected components. In light of the results of [1], it is natural to suppose ι might be surjective when Γ is uniform. This is precisely what the present paper proves.

2. Definitions

Let G be a locally compact group. Define left-translation by $l_x f(y) = f(xy)$ for $f \in L_\infty(G)$ and $x, y \in G$. The set of left-uniformly continuous functions on G is $\mathrm{LUC}(G) = \{f \in L_\infty(G) : \lim_{x\to e} \|f - l_x f\|_\infty = 0\}$. Let S be either $L_\infty(G)$ or $\mathrm{LUC}(G)$. The set of means on S is $\mathrm{M}(S) = \{\mu \in S^* : \mu \geq 0 \text{ and } \mu(\mathbb{1}_G) = 1\}$, endowed with the w^*-topology to make it compact. The set of left-invariant means on S is $\mathrm{LIM}(S) = \{\mu \in \mathrm{M}(S) : (\forall x \in G)\ (\forall f \in S)\ \mu(f - l_x f) = 0\}$. $\mathrm{LIM}(G)$ is shorthand for $\mathrm{LIM}(L_\infty(G))$. We say G is amenable if $\mathrm{LIM}(G)$ is nonempty. If G is amenable, so are all its closed subgroups.

Let $P_1(G) = \{p \in L_1(G) : p \geq 0 \text{ and } \|p\|_1 = 1\}$. The set of topological left-invariant means on S is $\mathrm{TLIM}(S) = \{\mu \in \mathrm{M}(S) : (\forall p \in P_1)\ (\forall f \in S)\ \mu(f - p * f) = 0\}$. Pick any $p \in P_1(G)$ and $\mu \in \mathrm{LIM}(\mathrm{LUC}(G))$. There is exactly one way to extend μ to $\widetilde{\mu} \in \mathrm{TLIM}(L_\infty(G))$, given by $\widetilde{\mu}(f) = \mu(p * f)$ for $f \in L_\infty(G)$. This works because every left-invariant mean on $\mathrm{LUC}(G)$ is topological left-invariant (see [2, Lemma 2.2.2]). The map $\mu \mapsto \widetilde{\mu}$ is an isometry from $\mathrm{LIM}(\mathrm{LUC}(G))$ onto $\mathrm{TLIM}(L_\infty(G))$. Thus it is common to associate the two sets of means and write $\mathrm{TLIM}(G)$ for both.

Let $\mathscr{C}(G)$ be the compact symmetric neighbourhoods of e in G. For $F \in \mathscr{C}(G)$, define the mean μ_F on $L_\infty(G)$ by $\mu_F(f) = (1/|F|) \int_F f(x)\,dx$, where $|F|$ is the Haar measure of F. For $E, F \in \mathscr{C}(G)$,

$$\|\mu_E - \mu_F\| = \left\| \frac{\mathbb{1}_E}{|E|} - \frac{\mathbb{1}_F}{|F|} \right\|_1 = \frac{|E \setminus F|}{|E|} + \frac{|F \setminus E|}{|F|} + \left| \frac{1}{|E|} - \frac{1}{|F|} \right|. \tag{2.1}$$

The pair (K, ε) always signifies $K \in \mathscr{C}(G)$ and $\varepsilon > 0$. We say $F \in \mathscr{C}(G)$ is (K, ε)-invariant if $|KF \setminus F| < \varepsilon |F|$. A net $\{F_\alpha\}$ in $\mathscr{C}(G)$ is called a Følner net if it is eventually (K, ε)-invariant for any (K, ε). See [5, Theorem 4.13] for a proof that every amenable group admits a Følner net satisfying this definition.

The following well-known result is due to Weil [7, Section 9]. Suppose $H < G$ is any closed subgroup. The coset space G/H admits a G-invariant Radon measure λ if and only if $\Delta_H = \Delta_G|_H$. In this case, λ is determined up to normalisation by

$$\int_G f(g)\,dg = \int_{G/H} \int_H f(xh)\,dh\,d\lambda(xH), \quad f \in C_0(G). \tag{2.2}$$

If (2.2) holds, λ is said to satisfy the *standard normalisation*.

Henceforth, let $\Gamma < G$ be a lattice subgroup and let $\pi : G \to G/\Gamma$ be the quotient map. By definition of a lattice, Γ is discrete and the coset space G/Γ admits a G-invariant Radon probability measure λ. Assume that $d\gamma$ is the counting measure on Γ and that λ satisfies the standard normalisation. It is a well-known corollary of Weil's theorem that G must be unimodular to admit a lattice (see [6, Remark I.1.9]).

LEMMA 2.1 [4, Theorem 5.9]. *Suppose G is unimodular and amenable. For each $\mu \in \mathrm{TLIM}(G)$, there exists a Følner net $\{F_\alpha\}$ such that $\lim_\alpha \mu_{F_\alpha} = \mu$.*

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3. The nonuniform case

Given $m \in \mathrm{LIM}(\Gamma)$, define $P_m : \mathrm{LUC}(G) \to C(G/\Gamma)$ by

$$P_m f(x\Gamma) = \langle m, (l_x f)|_\Gamma \rangle, \quad f \in \mathrm{LUC}(G). \tag{3.1}$$

Then $P_m f$ is well defined because m is left-invariant and it is continuous because $f \in \mathrm{LUC}(G)$. It satisfies $P_m(l_y f) = l_y(P_m(f))$ because $l_x l_y = l_{yx}$, and it is onto because

$$P_m(h \circ \pi) = h, \quad h \in C(G/\Gamma). \tag{3.2}$$

Define $\iota : \mathrm{LIM}(\Gamma) \to \mathrm{TLIM}(G)$ by

$$\iota m(f) = \int_{G/\Gamma} P_m f \, d\lambda, \quad f \in \mathrm{LUC}(G). \tag{3.3}$$

The map ιm is linear, positive, unital and left-invariant because m and λ are. By [3, Theorem 3.2], ι is injective, but the following proof is somewhat simpler.

LEMMA 3.1. *The map ι is injective.*

PROOF. Since Γ is discrete, let $U \subset G$ be a sufficiently small neighbourhood of e such that $U^{-1}U \cap \Gamma = \{e\}$. By Urysohn's lemma, there exists a continuous $h \geq 0$ with support contained in U, such that $\|h\|_1 = 1$. Define an embedding $\ell_\infty(\Gamma) \hookrightarrow \mathrm{LUC}(G)$ by

$$\phi \mapsto \Phi, \quad \text{where } \Phi(x) = \begin{Bmatrix} 0 & \text{for } x \notin U\Gamma \\ h(x\gamma^{-1})\phi(\gamma) & \text{for } x \in U\gamma \end{Bmatrix}.$$

For $t \in U$, $\|l_t\Phi - \Phi\| \leq \|l_t h - h\| \cdot \|\phi\|$, which shows $\Phi \in \mathrm{LUC}(G)$. A direct computation shows $\iota m(\Phi) = m(\phi)$. □

THEOREM 3.2. *If G is an amenable group with nonuniform lattice subgroup Γ, then ι is not onto.*

PROOF. Pick compact sets $K, U \subset G/\Gamma$ with $\lambda(K) > 0$ and $K \subset U^\circ$. Let $\{F_\alpha\} \subset \mathscr{C}(G)$ be a Følner net for G. For each α, pick $y_\alpha \in G/\Gamma$ outside the compact set $F_\alpha U$. Thus $F_\alpha y_\alpha \cap U = \varnothing$.

Let $\delta(y) \in C(G/\Gamma)^*$ denote the point mass at y. Define μ_α by the weak integral

$$\mu_\alpha = \frac{1}{|F_\alpha|} \int_{F_\alpha} \delta(ty_\alpha) \, dt. \tag{3.4}$$

Thus, $\{\mu_\alpha\}$ is a net of means on $C(G/\Gamma)$ converging to G-invariance. Fix an accumulation point λ_2 of this net, so λ_2 is a G-invariant mean on $C(G/\Gamma)$. Choose $h \in C(G/\Gamma)$ with $\mathbb{1}_K \leq h \leq \mathbb{1}_U$. By construction, $\lambda_2(h) = 0$.

Pick any $m \in \mathrm{LIM}(\Gamma)$. With P_m as in (3.1), define $\nu \in \mathrm{TLIM}(G)$ by

$$\nu(f) = \langle \lambda_2, P_m f \rangle, \quad f \in \mathrm{LUC}(G). \tag{3.5}$$

By (3.2), $\nu(h \circ \pi) = 0$, whereas $\iota\mu(h \circ \pi) = \lambda(h) > 0$ for each $\mu \in \mathrm{LIM}(\Gamma)$. □

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Although amenable nonuniform lattices are hard to come by, they do exist. Suppose $\{q_n\}_{n\geq 0}$ is a sequence of prime powers with $\sum_n 1/q_n < \infty$. Let F_{q_n} be the finite field of order q_n, with multiplicative group $F_{q_n}^\times$. Form the groups $\Lambda = \bigoplus_n F_{q_n}$, $S = \prod_n F_{q_n}^\times$ and $G = \Lambda \rtimes S$. G is abelian-by-abelian, hence amenable. From [1, Theorem 1.11], G contains uncountably many pairwise noncommensurable nonuniform lattices.

4. The uniform case

Henceforth, let Γ be a uniform lattice.

LEMMA 4.1. *G/Γ has a transversal T that is a precompact neighbourhood of e.*

PROOF. Pick any compact neighbourhood $U \subset G$ about e which is small enough so that $U^{-1}U \cap \Gamma = \{e\}$. Thus $\pi|_{tU}$ is injective for any $t \in G$. Note that G is covered by $\{tU : t \in G\}$, hence G/Γ is covered by $\{\pi(tU) : t \in G\}$. Let $\{\pi(t_1U), \ldots, \pi(t_nU)\}$ be a finite subcover. Without loss of generality, suppose $t_1 = e$. Let $T_1 = t_1U$. Inductively let $T_{k+1} = T_k \cup [t_{k+1}U \setminus T_k\Gamma]$. The desired transversal is T_n. □

Suppose $S \subseteq T$. We have assumed λ satisfies the standard normalisation, hence

$$|S| = \int_G \mathbb{1}_S = \int_{G/\Gamma}\int_\Gamma \mathbb{1}_S(t\gamma)\,d\gamma\,d\lambda(t\Gamma) = \int_{G/\Gamma} \mathbb{1}_{\pi(S)}\,d\lambda = \lambda(\pi(S)). \tag{4.1}$$

In particular, $|T| = 1$. Let $\#(D)$ denote the cardinality of $D \subset \Gamma$. Since G is unimodular,

$$\#(D) = \sum_{\gamma\in D} |T| = \sum_{\gamma\in D} |T\gamma| = |TD|. \tag{4.2}$$

Equation (4.1) also implies

$$\int_{G/\Gamma} F\,d\lambda = \int_T F(\pi(t))\,dt, \quad F \in L_\infty(G/\Gamma). \tag{4.3}$$

When $m \in \mathrm{LIM}(\Gamma)$, we can substitute the function $P_m f$ in (4.3) to conclude that the following definition of $\iota : \ell_\infty^*(\Gamma) \to \mathrm{LUC}(G)^*$ extends the definition given in (3.3):

$$\iota m(f) = \int_T \langle m, (l_t f)|_\Gamma\rangle\,dt, \quad f \in \mathrm{LUC}(G). \tag{4.4}$$

For $D \subset \Gamma$, we observe that

$$\iota\mu_D(f) = \frac{1}{\#(D)}\sum_{\gamma\in D}\int_T f(t\gamma)\,dt = \frac{1}{|TD|}\int_{TD} f(t)\,dt = \mu_{TD}(f). \tag{4.5}$$

LEMMA 4.2. *The map $\iota : \ell_\infty^*(\Gamma) \to \mathrm{LUC}(G)^*$ as defined in (4.4) is w^*-to-w^* continuous.*

PROOF. Suppose $\lim_\alpha m_\alpha = 0 \in \ell_\infty^*(\Gamma)$. Pick $f \in \mathrm{LUC}(G)$ and $\varepsilon > 0$. Partition T into precompact sets $\{T_k\}_{k=1}^n$ such that $\sup_{t,s\in T_k}|f(t) - f(s)| < \varepsilon$. For each k, pick $t_k \in T_k$.

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Thus

$$\left|\iota m_\alpha(f) - \sum_{k=1}^{n} |T_k|\,\langle m_\alpha, (l_{t_k} f)|_\Gamma\rangle\right| < \varepsilon. \tag{4.6}$$

This implies $\lim_\alpha |\iota m_\alpha(f)| \le \varepsilon$. Since ε was arbitrary, $\lim_\alpha m_\alpha(f) = 0$. □

LEMMA 4.3. *Suppose $\{F_\alpha\}$ is a Følner net for G, and let $D_\alpha = \{\gamma \in \Gamma : F_\alpha \cap T\gamma \neq \varnothing\}$. Then $\{D_\alpha\}$ is a Følner net for Γ and $\lim_\alpha \|\mu_{F_\alpha} - \mu_{TD_\alpha}\| = 0$.*

PROOF. Pick $K \in \mathscr{C}(\Gamma)$ and $\varepsilon > 0$. If α is large enough that F_α is (TKT^{-1}, ε)-invariant, then

$$\#(KD_\alpha \setminus D_\alpha) = |TKD_\alpha \setminus TD_\alpha| \le |TKT^{-1}F_\alpha \setminus F_\alpha| < \varepsilon|F_\alpha| \le \varepsilon\#(D_\alpha). \tag{4.7}$$

In other words, D_α is (K, ε)-invariant. This shows $\{D_\alpha\}$ is a Følner net for Γ.

If α is large enough that F_α is (TT^{-1}, ε)-invariant, then

$$|TD_\alpha \setminus F_\alpha| \le |TT^{-1}F_\alpha \setminus F_\alpha| < \varepsilon|F_\alpha|. \tag{4.8}$$

Since $F_\alpha \subseteq TD_\alpha$, applying (2.1) gives

$$\|\mu_{TD_\alpha} - \mu_{F_\alpha}\| = \frac{|TD_\alpha \setminus F_\alpha|}{|TD_\alpha|} + \left(\frac{1}{|F_\alpha|} - \frac{1}{|TD_\alpha|}\right) < \varepsilon + \left(1 - \frac{1}{1+\varepsilon}\right) < 2\varepsilon. \tag{4.9}$$

This proves $\lim_\alpha \|\mu_{TD_\alpha} - \mu_{F_\alpha}\| = 0$. □

THEOREM 4.4. *If G is an amenable group with uniform lattice subgroup Γ, then the map $\iota : \mathrm{LIM}(\Gamma) \to \mathrm{TLIM}(G)$ is surjective.*

PROOF. Pick $\nu \in \mathrm{TLIM}(G)$. By Lemma 2.1, G admits a Følner net $\{F_\alpha\}$ such that $\nu = \lim_\alpha \mu_{F_\alpha}$. Define $\{D_\alpha\}$ as in Lemma 4.3. Since the unit ball of $\ell_\infty^*(\Gamma)$ is compact, $\{\mu_{D_\alpha}\}$ admits a convergent subnet $\{\mu_{D_\beta}\}$. Evidently $m = \lim_\beta \mu_{D_\beta} \in \mathrm{LIM}(\Gamma)$. Applying (4.5) and Lemmas 4.2 and 4.3,

$$\iota m = \lim_\beta \iota\mu_{D_\beta} = \lim_\beta \mu_{TD_\beta} = \lim_\beta \mu_{F_\beta} = \nu. \tag{4.10}$$

□

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JOHN HOPFENSPERGER, Department of Mathematics,
University at Buffalo, Buffalo, NY 14260-2900, USA
e-mail: johnhopf@buffalo.edu

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