J. Appl. Prob. **56,** 153–173 (2019)
*doi:*10.1017/jpr.2019.11

PRESERVATION OF THE MEAN RESIDUAL LIFE ORDER FOR COHERENT AND MIXED SYSTEMS

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Abstract

The signature of a coherent system has been studied extensively in the recent literature. Signatures are particularly useful in the comparison of coherent or mixed systems under a variety of stochastic orderings. Also, certain signature-based closure and preservation theorems have been established. For example, it is now well known that certain stochastic orderings are preserved from signatures to system lifetimes when components have independent and identical distributions. This applies to the likelihood ratio order, the hazard rate order, and the stochastic order. The point of departure of the present paper is the question of whether or not a similar preservation result will hold for the *mean residual life* order. A counterexample is provided which shows that the answer is negative. Classes of distributions for the component lifetimes for which the latter implication holds are then derived. Connections to the theory of order statistics are also considered.

Keywords: Coherent system; mixed system; signature; stochastic ordering; mean residual life order; decreasing failure rate; order statistics; spacings

2010 Mathematics Subject Classification: Primary 60E15
Secondary 62N05

1. Introduction

Consider a coherent system with n binary components as studied, e.g. in the monograph by Barlow and Proschan [3]. Suppose that the component lifetimes $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) with cumulative distribution function F. Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be their ordered values, and let T be the lifetime of the system. Samaniego [24] introduced the *signature*, $\boldsymbol{s} = (s_1, \dots, s_n)$, of the system which, when F is continuous, is given as

$$s_i = \mathbb{P}(T = X_{i:n}) \quad \text{for } i = 1, \dots, n. \tag{1}$$

A key property of system signatures is that $\boldsymbol{s}$ depends only on the system structure and does not depend on the distribution F of component lifetimes. Moreover (see [25, Theorem 3.1]), the survival function of the lifetime T of the system can be represented as a function of $\boldsymbol{s}$ and

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F as follows:

$$\mathbb{P}(T>t)=\sum_{i=1}^{n} s_i\mathbb{P}(X_{i:n}>t)=\sum_{i=1}^{n} s_i\sum_{k=0}^{i-1}\binom{n}{k}(F(t))^k(\bar{F}(t))^{n-k}. \tag{2}$$

By changing the order of summation and changing the summation variable k to $j=n-k$ in (2), we may write

$$\mathbb{P}(T>t)=\sum_{j=1}^{n}\left(\sum_{i=n-j+1}^{n} s_i\right)\binom{n}{j}\bar{F}(t)^jF(t)^{n-j}=\sum_{j=1}^{n} a_j\binom{n}{j}\bar{F}(t)^jF(t)^{n-j}, \tag{3}$$

where

$$a_j=\sum_{i=n-j+1}^{n} s_i \quad \text{for } j=1,\dots,n. \tag{4}$$

As shown in [5],

$$a_j=\frac{\#\text{ path sets of size } j}{\binom{n}{j}}.$$

It follows that the a_j can be interpreted as the probability that the system functions when j components function. Hence, in particular, we have $a_n=1$. Coolen and Coolen-Maturi [7] introduced the term *survival signature* for the vector $\boldsymbol{a}=(a_1,\dots,a_n)$. Note that there is a 1–1 correspondence between the signature vector and the vector defining the survival signature. As will become clear, representation (3) for the survival function of a system lifetime will be fundamental in the approach of the present paper. Note that (2) and (3) are valid for both discrete and continuous distributions F. In the discrete case we will no longer have (1), but, as noted by Kochar *et al.* [15], we may write

$$s_k=\frac{\#\text{ of orderings of components for which the } k\text{th failure causes system failure}}{n!}.$$

The considerations so far are restricted to *coherent* systems. It is, however, useful to extend the class of systems to include so-called *mixed* systems; see [25, pp. 28–31]. In the following, we shall refer to a system with n components as an n-system. A mixed n-system is a stochastic mixture of a number of coherent n-systems. It is easily verified that results (2) and (3) continue to hold for mixed systems; see [25, p. 30]. Note that any probability vector $\boldsymbol{s}=(s_1,\dots,s_n)$ can serve as the signature of a mixed system. One possible representation of such a mixed system is the one which gives weight s_i to an i-out-of-n system for $i=1,\dots,n$.

Samaniego [25, Chapters 4 and 5] demonstrated the utility of signatures in various reliability contexts. For example, signatures have been shown to be useful in establishing certain closure and preservation theorems in reliability, and they can play a useful role in the comparison of coherent or mixed systems. One example of the former is the IFR closure problem that was first considered by Samaniego [24]. Samaniego [25] also presented a collection of preservation theorems, showing that certain types of orderings of signatures imply like orderings of the corresponding system lifetime distributions. Since the calculation of the lifetime distributions of complex systems is often challenging, the utility of comparing the much simpler indices for system designs in the form of signature vectors should be evident. To be more specific, Samaniego [25, Chapter 4] presented preservation results of this kind for stochastic comparisons with respect to likelihood ratio, hazard rate, and stochastic ordering. For example, the preservation result for stochastic ordering states that if a system has a signature $\boldsymbol{s}$ which is stochastically smaller than the signature $\boldsymbol{t}$ of another system, the former system will have a stochastically smaller lifetime, whatever the distribution F of the component lifetimes. Samaniego [25] showed that similar results hold for the other two orderings mentioned above.

The point of departure of the present paper is the question of whether or not a similar preservation result will hold for the *mean residual life order*, i.e. whether system lifetimes will be ordered with respect to the mean residual life order if this ordering holds for the system signatures. As we shall see, the answer to this question is negative. A simple counterexample, essentially involving easy hand calculations, is given in Section 2. The example shows that there are indeed systems with signatures which are ordered according to the mean residual life order, but where the system lifetimes are not similarly ordered for some specific component distribution. Sections 3 and 4 are devoted to the description of nested classes of component distributions, depending on the system size n, for which the preservation of the mean residual life order takes place. For example, one finds that any decreasing failure rate distribution is contained in each of these classes. Further, it is shown that these classes can be characterized by properties of spacings of order statistics.

The importance of the mean residual life order as a way of comparing components' or systems' performance has been highlighted in the recent reliability literature. It is well known that the hazard rate order implies the mean residual life order, but as illustrated by Belzunce *et al.* [4], in practical applications it is often seen that the hazard rate order does not hold, while the mean residual life order does obtain. They demonstrated this by considering Weibull distributions with different shape parameters, as well as by an empirical study of daily return data from two Spanish companies, an electric utility company and a banking company. In a recent paper, Navarro and Gomis [21] obtained comparison results for the performance of coherent systems with respect to the mean residual life order, while Mirjalili *et al.* [18] considered the mean residual life of a coherent system with a cold standby component. Ghitany *et al.* [11] treated an application to finance which shows how the mean residual life function is used in risk measurements appropriate for the evaluation of market risk or credit risk of a portfolio.

This paper is organized as follows. In Section 2 we define the different orders to be considered in the paper, with emphasis on the mean residual life order. In addition, the definitions of the orders in terms of signature vectors are given. Finally, a necessary and sufficient condition for the mean residual life ordering of system lifetimes is given, together with an example where two systems have signatures that are ordered with respect to the mean residual life order, while the system lifetimes are not. In Section 3 we study the classes of component lifetime distributions F for which the mean residual life ordering is preserved. In particular, the sufficiency of their defining property is proved. In Section 4 we examine the case where the distributions F are absolutely continuous and in Section 5 we consider the connection between our results on the mean residual life ordering of system lifetimes and the theory of order statistics. In Sections 6 and 7 we provide additional examples and some concluding remarks.

2. Ordering of lifetimes and signatures: a counterexample

The likelihood ratio order, the hazard rate order, and the stochastic order are the most studied orderings among random variables. Of these, the two last mentioned will be the most relevant in the present paper. Their definitions, given below, apply both for discrete and continuous pairs of random variables (X, Y).

Note that we will use *increasing* to mean *nondecreasing* and *decreasing* to mean *nonincreasing*. In the paper, we will also let $a/0 = \infty$ for $a > 0$, while $0/0$ is indeterminate, but, when occurring, will correspond to cases without relevance.

Definition 1. Let X and Y be nonnegative random variables with corresponding survival functions $\bar{F}$ and $\bar{G}$. Then X is smaller than Y in the *stochastic order*, denoted by $X \leq_{\mathrm{st}} Y$, if and only if $\bar{F}(x) \leq \bar{G}(x)$ for all x; in the *hazard rate order*, denoted by $X \leq_{\mathrm{hr}} Y$, if and only if $\bar{G}(x)/\bar{F}(x)$ is increasing in x.

In the present paper, our main concern is with a different order, *the mean residual life order*; see [27, Chapter 2.A]. Recall that if X is a positive random variable with a survival function $\bar{F}$ and a finite mean, then the mean residual life function of X at $t \geq 0$ is defined as

$$m(t) = \mathbb{E}(X - t \mid X > t) \quad \text{for } t \geq 0. \tag{5}$$

The definition of the mean residual life ordering of two random variables is given below.

Definition 2. Let X and Y be nonnegative random variables with corresponding survival functions $\bar{F}$ and $\bar{G}$, and corresponding mean residual life functions $m(t)$ and $l(t)$. Then X is smaller than Y in the *mean residual life order*, denoted by $X \leq_{\mathrm{mrl}} Y$, if and only if

$$m(t) \leq l(t) \quad \text{for all } t > 0,$$

or, equivalently, $X \leq_{\mathrm{mrl}} Y$ if and only if

$$\frac{\int_t^\infty \bar{G}(u)\,\mathrm{d}u}{\int_t^\infty \bar{F}(u)\,\mathrm{d}u} \quad \text{is increasing in } t > 0.$$

It is well known ([27, Chapters 1, 2]) that

$$X \leq_{\mathrm{hr}} Y \implies X \leq_{\mathrm{st}} Y \quad \text{and} \quad X \leq_{\mathrm{hr}} Y \implies X \leq_{\mathrm{mrl}} Y. \tag{6}$$

However, neither of the orders '$\leq_{\mathrm{st}}$' and '$\leq_{\mathrm{mrl}}$' implies the other.

Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be the signature vectors of two systems. Following common notation, we shall let $\boldsymbol{s} \leq_{\mathrm{order}} \boldsymbol{t}$ in a specific order (st, hr, or mrl) mean that the corresponding discrete random variables are ordered in this way. Although Definitions 1 and 2 cover both discrete and continuous distributions, we find it convenient to have separate definitions for orderings of signature vectors, which will be given in terms of properties of the corresponding survival signatures.

Definition 3. Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be signature vectors for two n-systems with corresponding survival signatures given respectively by $\boldsymbol{a}$ and $\boldsymbol{b}$ as defined in (4). Then $\boldsymbol{s}$ is smaller than $\boldsymbol{t}$ in the

- *stochastic order*, $\boldsymbol{s} \leq_{\mathrm{st}} \boldsymbol{t}$, if and only if $b_i/a_i \geq 1$ for all i,
- *hazard rate order*, $\boldsymbol{s} \leq_{\mathrm{hr}} \boldsymbol{t}$, if and only if b_i/a_i is decreasing in i,
- *mean residual life order*, $\boldsymbol{s} \leq_{\mathrm{mrl}} \boldsymbol{t}$, if and only if $\big(\sum_{i=1}^k b_i\big)/\big(\sum_{i=1}^k a_i\big)$ is decreasing in k.

Theorems 4.3 and 4.4 of [25] state that if $\boldsymbol{s}$ and $\boldsymbol{t}$ are the signatures of two mixed n-systems having components with i.i.d. lifetimes and common distribution F, and if S and T are the respective system lifetimes, then we have

$$\boldsymbol{s} \leq_{\mathrm{st}} \boldsymbol{t} \implies S \leq_{\mathrm{st}} T, \tag{7}$$

$$\boldsymbol{s} \leq_{\mathrm{hr}} \boldsymbol{t} \implies S \leq_{\mathrm{hr}} T. \tag{8}$$

https://doi.org/10.1017/jpr.2019.11 Published online by Cambridge University Press

The present paper is concerned with the question of whether, or possibly under what conditions, the following implication can be added to those in (7) and (8) above:

$$\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t} \quad \Longrightarrow \quad S \leq_{\text{mrl}} T. \tag{9}$$

Throughout the paper, the component lifetime distributions F are assumed to have support in $[0, \infty)$ and to have finite expectation. Letting F denote cumulative distribution functions, we let $\bar{F} = 1 - F$ be the corresponding survival function. We have also found it convenient to allow discrete distributions F to have a positive point mass at 0. In the absolutely continuous case, the corresponding probability density function will be denoted by f, and will be assumed to have a support set which is a closed subinterval of $[0, \infty)$. This ensures that the distribution function F is strictly increasing in this subinterval. For later reference, the resulting subclass of absolutely continuous distributions will be denoted by $\mathcal{C}$.

By combining Definition 2 and (3) for the survival function of a system lifetime, we obtain the following result giving a necessary and sufficient condition for $S \leq_{\text{mrl}} T$.

Proposition 1. *Let* $\boldsymbol{s}$ *and* $\boldsymbol{t}$ *be the signatures of two mixed n-systems, and assume that* $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$. *Suppose that the systems have components with i.i.d. lifetimes with common distribution F. Let S and T be the respective lifetimes of the systems. Then* $S \leq_{\text{mrl}} T$ *if and only if*

$$\frac{\sum_{i=1}^{n} b_i \binom{n}{i} \int_t^\infty \bar{F}(u)^i F(u)^{n-i}\, \mathrm{d}u}{\sum_{i=1}^{n} a_i \binom{n}{i} \int_t^\infty \bar{F}(u)^i F(u)^{n-i}\, \mathrm{d}u} \quad \textit{is increasing in } t \geq 0, \tag{10}$$

where $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are the respective survival signatures corresponding to* $\boldsymbol{s}$ *and* $\boldsymbol{t}$*, as defined in (4).*

Suppose that $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ and that also $\boldsymbol{s} \leq_{\text{hr}} \boldsymbol{t}$. By (8), we then have $S \leq_{\text{hr}} T$, which, by (6), implies that $S \leq_{\text{mrl}} T$. Thus, the conclusion in (9) holds for any component lifetime distribution F if $\boldsymbol{s}, \boldsymbol{t}$ satisfy both $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ and $\boldsymbol{s} \leq_{\text{hr}} \boldsymbol{t}$. The remaining case of interest is therefore when $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ and $\boldsymbol{s} \not\leq_{\text{hr}} \boldsymbol{t}$. Since the orders st, hr, and mrl are equivalent for signatures of 2-systems (see Definition 3), we need only consider $n \geq 3$.

The following simple example shows that (9) does not hold in general, in the sense that, for some n, there are signatures $\boldsymbol{s}$ and $\boldsymbol{t}$ with $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ for which (10) does not hold for all distributions F. In view of this somewhat surprising and disappointing result, we turn our attention to the characterization of classes of component lifetime distributions F for which (10) holds, for given n, for any pair of systems with signatures satisfying $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$.

Example 1. (*A counterexample.*) Let $n = 3$, and consider signature vectors

$$\boldsymbol{s} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right), \qquad \boldsymbol{t} = \left(\frac{3}{8}, 0, \frac{5}{8}\right), \tag{11}$$

with corresponding survival signatures

$$\boldsymbol{a} = \left(\frac{1}{2}, \frac{3}{4}, 1\right), \qquad \boldsymbol{b} = \left(\frac{5}{8}, \frac{5}{8}, 1\right).$$

From Definition 3, we note that $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$, while neither of the orderings hr and st hold between $\boldsymbol{s}$ and $\boldsymbol{t}$.

Let the component lifetime distribution F be a discrete distribution giving mass p at time 0 and $q = 1 - p$ at time 1, where $0 < p < 1$. Let S and T be the lifetimes of the systems with signature $\boldsymbol{s}$ and $\boldsymbol{t}$, respectively. Then also S and T have only two possible values, 0 and 1.

For this simple F, we obtain, directly from the definition of mean residual life (5),

$$S \leq_{\text{mrl}} T \quad \text{if and only if} \quad \mathbb{P}(S=1) \leq \mathbb{P}(T=1). \tag{12}$$

It also follows from (3) that, for n-systems with the component lifetime distribution F above,

$$\mathbb{P}(S=1) = \sum_{j=1}^{n} a_j \binom{n}{j} q^j p^{n-j} = p^n \sum_{j=1}^{n-1} a_j \binom{n}{j} \left(\frac{q}{p}\right)^j + q^n.$$

A corresponding result holds for $\mathbb{P}(T=1)$ if the a_j are replaced by b_j. Hence by (12) we have $S \leq_{\text{mrl}} T$ if and only if

$$\sum_{j=1}^{n-1} a_j \binom{n}{j} \left(\frac{q}{p}\right)^j \leq \sum_{j=1}^{n-1} b_j \binom{n}{j} \left(\frac{q}{p}\right)^j. \tag{13}$$

Now let $n=3$ and let the survival signatures be given as in (11). Calculating each side of (13), with $c = q/p\,(>0)$, we see that $S \leq_{\text{mrl}} T$ if and only if

$$\frac{1}{2} \cdot 3 \cdot c + \frac{3}{4} \cdot 3 \cdot c^2 \leq \frac{5}{8} \cdot 3 \cdot c + \frac{5}{8} \cdot 3 \cdot c^2,$$

which after simplification is equivalent to $c^2 \leq c$ or $c \leq 1$. Hence, $S \leq_{\text{mrl}} T$ does *not* hold if $c > 1$, i.e. if $p < \frac{1}{2}$. (In this case, we have $T \leq_{\text{mrl}} S$.)

3. The main result

Theorem 1 defines, for given n, a class $\mathcal{F}_n$ of distributions F such that (9) holds for all n-systems whenever F is in $\mathcal{F}_n$. The theorem is the main result of the present paper and will be proved through a series of lemmas. Although the case $n=2$ has already been settled in Section 2, where it is concluded that in this case (9) holds for any distribution F, we shall find it convenient to include $n=2$ in the definition and treatment of the classes $\mathcal{F}_n$.

Theorem 1. *For $n \geq 2$, let*

$$\mathcal{F}_n = \left\{F\colon \binom{n}{i} \int_0^\infty \bar{F}(u)^i F(u)^{n-i}\, \mathrm{d}u \text{ is decreasing in } i = 1, 2, \ldots, n\right\}. \tag{14}$$

Then, for any two n-systems with signatures $\mathbf{s}$ and $\mathbf{t}$ satisfying $\mathbf{s} \leq_{\text{mrl}} \mathbf{t}$, and with i.i.d. component lifetimes with common distribution $F \in \mathcal{F}_n$, the corresponding system lifetimes are ordered as $S \leq_{\text{mrl}} T$.

The classes $\mathcal{F}_n$ are strictly nested, $\mathcal{F}_n \subset \mathcal{F}_{n-1}$ for all $n \geq 3$, and have a nonempty intersection.

Lemma 1. *For any distribution F and $n \geq 2$, we have $F \in \mathcal{F}_n$ if and only if*

$$\binom{n}{i} \int_t^\infty \bar{F}(u)^i F(u)^{n-i}\, \mathrm{d}u \quad \text{is decreasing in } i = 1, \ldots, n \text{ for all } t \geq 0. \tag{15}$$

Proof. Define $g_i(t)$ as the difference between (15) for $i+1$ and i, i.e. let

$$g_i(t) = \binom{n}{i+1} \int_t^\infty \bar{F}(u)^{i+1} F(u)^{n-i-1}\, \mathrm{d}u - \binom{n}{i} \int_t^\infty \bar{F}(u)^i F(u)^{n-i}\, \mathrm{d}u.$$

Thus, (15) is equivalent to $g_i(t) \leq 0$ for $i = 1, \ldots, n-1$ and all $t \geq 0$. It is straightforward to obtain

$$g_i(t) = \frac{n!}{i!\,(n-i-1)!}\left\{\int_t^{\infty} \bar{F}(u)^i F(u)^{n-i-1}\left[\frac{\bar{F}(u)}{i+1} - \frac{F(u)}{n-i}\right] \mathrm{d}u\right\}. \tag{16}$$

Observe first that $\lim_{t\to\infty} g_i(t) = 0$ for all i. This follows by considering (15) and showing that

$$\lim_{t\to\infty} \int_t^{\infty} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u = 0 \quad \text{for all } i = 1, 2, \ldots, n.$$

This is a consequence of the fact that

$$\int_t^{\infty} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u \leq \int_t^{\infty} \bar{F}(u) \,\mathrm{d}u \quad \text{for } i = 1, \ldots, n,$$

where the right-hand side tends to 0 because F has a finite expectation.

Now consider the derivative $g_i'(t)$ for $t \geq 0$. This is found by putting $t = u$ in the integrand of (16) and changing the sign. Using the fact that $\bar{F} = 1 - F$ in the expression in square brackets in (16), it follows that $g_i'(t) > 0$ when $F(t) > (n-i)/(n+1)$ and $g_i'(t) < 0$ when $F(t) < (n-i)/(n+1)$. Since $\lim_{t\to\infty} g_i(t) = 0$, it is apparent that $g_i(t) \leq 0$ for all $t \geq 0$ if and only if $g_i(0) \leq 0$. But this is equivalent to the statement of the lemma, and thus the proof is complete. □

Lemma 2. *For any distribution F and for $n \geq 2$, it follows that*

$$\frac{\int_t^{\infty} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u}{\int_s^{\infty} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u} \tag{17}$$

is decreasing in $i = 1, 2 \ldots, n$ for any fixed s and t such that $0 \leq s < t$.

Proof. Considering the difference between (17) for $i+1$ and i we find that the difference is negative if and only if

$$\left(\int_t^{\infty} \bar{F}(u)^{i+1} F(u)^{n-i-1} \,\mathrm{d}u\right)\left(\int_s^{\infty} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u\right) \leq \left(\int_t^{\infty} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u\right)\left(\int_s^{\infty} \bar{F}(u)^{i+1} F(u)^{n-i-1} \,\mathrm{d}u\right).$$

Utilizing the fact that $\int_s^{\infty} = \int_s^t + \int_t^{\infty}$, and cancelling terms, we see that the above is equivalent to

$$\left(\int_t^{\infty} \bar{F}(u)^{i+1} F(u)^{n-i-1} \,\mathrm{d}u\right)\left(\int_s^{t} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u\right) \leq \left(\int_t^{\infty} \bar{F}(u)^i F(u)^{n-i} \,\mathrm{d}u\right)\left(\int_s^{t} \bar{F}(u)^{i+1} F(u)^{n-i-1} \,\mathrm{d}u\right). \tag{18}$$

Fix $t > 0$ and define $h_i(s)$ for $0 \leq s \leq t$ to be the difference obtained by subtracting the right-hand side from the left-hand side of (18). We need to show that $h_i(s) \leq 0$ for $0 \leq s \leq t$. Clearly, $h_i(t) = 0$. Furthermore, by differentiation of $h_i(s)$ with respect to s, for all $0 \leq s \leq t$,

$$h_i'(s) = \bar{F}(s)^i F(s)^{n-i-1} \int_t^{\infty} \bar{F}(u)^i F(u)^{n-i-1} [F(u)\bar{F}(s) - \bar{F}(u)F(s)] \,\mathrm{d}u. \tag{19}$$

The expression in square brackets in (19) equals $F(u) - F(s)$, which is nonnegative for all $u \geq t$ since $s < t$. But then $h_i'(s) \geq 0$ for $0 \leq s \leq t$. Since $h_i(t) = 0$, this implies that $h_i(s) \leq 0$ for all $0 \leq s \leq t$, and the lemma follows. □

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In the proof of Theorem 1, we will make use of a result given in [6]. For distributions F and G on $\mathbb{R}$, Caperaa [6] defined the order $F >_{(+)} G$ (F is uniformly stochastically larger than G) if and only if $(1-G)/(1-F)$ is decreasing. Caperaa showed the following result as a corollary of his main result.

Lemma 3. (Caperaa [6].) *Let F and G be two distributions on $\mathbb{R}$. Then $F >_{(+)} G$ is a necessary and sufficient condition for*

$$\frac{\int_{-\infty}^{\infty} \alpha(x)\,\mathrm{d}F(x)}{\int_{-\infty}^{\infty} \beta(x)\,\mathrm{d}F(x)} \geq \frac{\int_{-\infty}^{\infty} \alpha(x)\,\mathrm{d}G(x)}{\int_{-\infty}^{\infty} \beta(x)\,\mathrm{d}G(x)} \tag{20}$$

to hold for all functions α and β, integrable with respect to F and G, such that β is nonnegative, and α/β and β are nondecreasing.

For distributions on $[0, \infty)$, the ordering $(+)$ is the same as the hazard rate ordering hr (see Definition 1). Note that Joag-Dev *et al.* [13] cited Caperaa's theorem and sketched a proof. We will use the following version of Lemma 3 which applies to discrete positive distributions.

Lemma 4. *Let $n \geq 2$ be given, and let K and L be two probability distributions on $\{1, 2, \dots, n\}$ satisfying $K \leq_{\mathrm{hr}} L$. Let $k(i)$ and $l(i)$ denote the point mass functions of the distributions K and L, respectively. Furthermore, let $\alpha(i)$ and $\beta(i)$ for $i = 1, \dots, n$ be numbers such that $\beta(i)$ is positive, and $\alpha(i)/\beta(i)$ and $\beta(i)$ are increasing in $i = 1, 2, \dots, n$. Then*

$$\frac{\sum_{i=1}^{n} \alpha(i)\, l(i)}{\sum_{i=1}^{n} \beta(i)\, l(i)} \geq \frac{\sum_{i=1}^{n} \alpha(i)\, k(i)}{\sum_{i=1}^{n} \beta(i)\, k(i)}. \tag{21}$$

Now, let $\boldsymbol{a}$ be the survival signature for a mixed n-system. Let $\tilde{a}_i = a_i / \sum_{i=1}^{n} a_i$ for $i = 1, \dots, n$. Thus we have normalized the a_i so that the $\tilde{a}_i$ sum to 1. Now define the cumulative distribution function K on $\{0, 1, \dots, n\}$ by $K(0) = 0$ and

$$K(i) = \tilde{a}_n + \tilde{a}_{n-1} + \cdots + \tilde{a}_{n-i+1}, \qquad i = 1, \dots, n, \tag{22}$$

so that $K(n) = 1$. Let the survival function corresponding to K be $\bar{K} = 1 - K$. Thus, $\bar{K}(n) = 0$ and

$$\bar{K}(i) = \tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_{n-i}, \qquad i = 0, 1, \dots, n-1. \tag{23}$$

For another survival signature $\boldsymbol{b}$, let $\tilde{b}_i = b_i / \sum_{i=1}^{n} b_i$ for $i = 1, \dots, n$, and define the cumulative distribution function L and survival function $\bar{L}$ similarly, with $\tilde{b}_i$ replacing $\tilde{a}_i$ for $i = 1, \dots, n$ in (22) and (23). Thus,

$$L(i) = \tilde{b}_n + \tilde{b}_{n-1} + \cdots + \tilde{b}_{n-i+1}, \qquad i = 1, \dots, n, \tag{24}$$

We now prove the following lemma.

Lemma 5. *Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are the respective survival signatures of systems with signatures $\boldsymbol{s}$ and $\boldsymbol{t}$, and let the distributions K and L be defined by (22) and (24). Then $\boldsymbol{s} \leq_{\mathrm{mrl}} \boldsymbol{t}$ if and only if $K \leq_{\mathrm{hr}} L$.*

Proof. As found in [27, Chapter 1.B], $K \leq_{\mathrm{hr}} L$ if and only if $\bar{L}(i)/\bar{K}(i)$ is increasing in i, i.e. if and only if

$$\frac{\tilde{b}_1 + \tilde{b}_2 + \cdots + \tilde{b}_n}{\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n} \leq \frac{\tilde{b}_1 + \tilde{b}_2 + \cdots + \tilde{b}_{n-1}}{\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_{n-1}} \leq \cdots \leq \frac{\tilde{b}_1}{\tilde{a}_1}.$$

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The latter inequalities will of course hold if the $\tilde{a}_i$ are replaced by a_i and the $\tilde{b}_i$ are replaced by b_i. Then this is exactly the definition of $\boldsymbol{s} \le_{\text{mrl}} \boldsymbol{t}$ as given in Definition 3. This proves the lemma. □

Proof of Theorem 1. Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be signatures of n-systems satisfying $\boldsymbol{s} \le_{\text{mrl}} \boldsymbol{t}$. We will show that (10) holds when it is assumed that $F \in \mathcal{F}_n$ (see (14)). Given the survival signatures $\boldsymbol{a}$ and $\boldsymbol{b}$ corresponding to $\boldsymbol{s}$ and $\boldsymbol{t}$, let the cumulative distribution functions K and L be defined as before Lemma 5. The corresponding point masses are

$$k(i) = a_{n-i+1}, \qquad l(i) = b_{n-i+1}, \quad i = 1, \dots, n.$$

For $0 \le s < t$ and $i = 1, \dots, n$, let

$$\alpha(i) = \binom{n}{n-i+1} \int_t^\infty \bar{F}(u)^{n-i+1} F(u)^{i-1} \, \mathrm{d}u, \tag{25}$$

$$\beta(i) = \binom{n}{n-i+1} \int_s^\infty \bar{F}(u)^{n-i+1} F(u)^{i-1} \, \mathrm{d}u. \tag{26}$$

Then, by Lemma 1, $\beta(i)$ is increasing in i, and, by Lemma 2, $\alpha(i)/\beta(i)$ is increasing in i. (It should here be noted that we have reversed the order of the terms in (15) and (17) in defining $\alpha(\cdot)$ and $\beta(\cdot)$.) Since $K \le_{\text{hr}} L$ by Lemma 5, it follows that the conditions of Lemma 4 are satisfied. Substitution in (21) and reversing the order of the terms in the sums gives

$$\frac{\sum_{i=1}^n b_i \binom{n}{i} \int_t^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u}{\sum_{i=1}^n b_i \binom{n}{i} \int_s^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u} \ge \frac{\sum_{i=1}^n a_i \binom{n}{i} \int_t^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u}{\sum_{i=1}^n a_i \binom{n}{i} \int_s^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u}.$$

This implies (10) since s and t were arbitrarily chosen, subject only to the restriction $0 \le s < t$. The first part of Theorem 1 thus follows.

The inclusion statement at the end of Theorem 1 is proved as follows. Let

$$c_{n,i} = \binom{n}{i} \int_0^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u \tag{27}$$

for $i = 1, \dots, n$. Then suppose that $F \in \mathcal{F}_n$. By using the identity $1 = \bar{F}(u) + F(u)$ and multiplying the integrand of $c_{n-1,i}$ by this sum, it is seen that, for $i = 1, 2, \dots, n-1$,

$$\begin{aligned} c_{n-1,i} &= \binom{n-1}{i} \int_0^\infty \bar{F}(u)^{i+1} F(u)^{n-i-1} \, \mathrm{d}u + \binom{n-1}{i} \int_0^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u \\ &= \frac{i+1}{n} c_{n,i+1} + \frac{n-i}{n} c_{n,i}. \end{aligned} \tag{28}$$

In order to prove that $F \in \mathcal{F}_{n-1}$, we need to show that $c_{n-1,i+1} - c_{n-1,i} \le 0$ for $i = 1, \dots, n-2$. From (28), we get

$$\begin{aligned} c_{n-1,i+1} - c_{n-1,i} &= \frac{i+2}{n} c_{n,i+2} + \frac{n-i-1}{n} c_{n,i+1} - \frac{i+1}{n} c_{n,i+1} - \frac{n-i}{n} c_{n,i} \\ &= \frac{i+2}{n} (c_{n,i+2} - c_{n,i+1}) + \frac{n-i}{n} (c_{n,i+1} - c_{n,i}), \end{aligned}$$

which is less than or equal to 0 by the assumption that $F \in \mathcal{F}_n$. Hence, $F \in \mathcal{F}_{n-1}$ as well, so $\mathcal{F}_n \subset \mathcal{F}_{n-1}$.

To see that these inclusions are strict, we consider Example 1. For the discrete distribution F of that example, we have, by (27), $c_{n,i} = \binom{n}{i} q^i p^{n-i}$. From this,

$$c_{n,i+1} - c_{n,i} = \binom{n}{i} (i+1)^{-1} q^i p^{n-i-1} [n - i - (n+1)p], \tag{29}$$

which implies that $c_{n,i+1} - c_{n,i} \leq 0$ for all $i = 1, \ldots, n-1$, and hence that $F \in \mathcal{F}_n$, if and only if $p \geq (n-1)/(n+1)$. Hence, if p is chosen so that

$$\frac{n-2}{n} \leq p < \frac{n-1}{n+1}, \tag{30}$$

we have $F \in \mathcal{F}_{n-1}$, while $F \notin \mathcal{F}_n$. Thus, $\mathcal{F}_n \subset \mathcal{F}_{n-1}$.

In order to prove that the intersection of the classes $\mathcal{F}_n$ is nonempty, we show that the standard exponential distribution, for which $F(t) = 1 - \mathrm{e}^{-t}$, is contained in $\mathcal{F}_n$ for all n. Now, for $i = 1, \ldots, n$,

$$\binom{n}{i} \int_0^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u = \binom{n}{i} \int_0^\infty \mathrm{e}^{-iu} (1 - \mathrm{e}^{-u})^{n-i} \, \mathrm{d}u = \frac{1}{i},$$

where we substituted $z = \mathrm{e}^{-u}$ in the second integral and used the well-known formula for the beta integral. The result is clearly decreasing in i, so $F \in \mathcal{F}_n$ by (14). The proof of Theorem 1 is hence complete. (Note that we show more generally, in Theorem 3, that $F \in \mathcal{F}_n$ for all n whenever F has a decreasing density.) □

4. Absolutely continuous *F*

Let $\mathcal{C}$ denote the set of absolutely continuous F as defined in Section 2. The following proposition is a corollary to Proposition 1 for the case when the component lifetime distribution F is in $\mathcal{C}$.

Proposition 2. *Let* $\boldsymbol{s}$ *and* $\boldsymbol{t}$ *be the signatures of two mixed n-systems, for which* $\boldsymbol{s} \leq_{\mathrm{mrl}} \boldsymbol{t}$. *Suppose that the systems have components with i.i.d. lifetimes with common absolutely continuous distribution* $F \in \mathcal{C}$ *and density* f. *Let* S *and* T *be the respective lifetimes of the systems. Then* $S \leq_{\mathrm{mrl}} T$ *if and only if*

$$\frac{\sum_{i=1}^n b_i \binom{n}{i} \int_0^u (z^i (1-z)^{n-i} / f(\bar{F}^{-1}(z))) \, \mathrm{d}z}{\sum_{i=1}^n a_i \binom{n}{i} \int_0^u (z^i (1-z)^{n-i} / f(\bar{F}^{-1}(z))) \, \mathrm{d}z} \quad \text{is decreasing in } u \text{ for } 0 < u < 1, \tag{31}$$

where $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are the respective survival signatures corresponding to* $\boldsymbol{s}$ *and* $\boldsymbol{t}$.

Proof. The result follows by substituting $z = \bar{F}(u)$ in the integral in (10), which gives

$$\int_t^\infty \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u = \int_0^{\bar{F}(t)} \frac{z^i (1-z)^{n-i}}{f(\bar{F}^{-1}(z))} \, \mathrm{d}z.$$

□

We now state the version of Theorem 1 that is valid for the absolutely continuous case.

Theorem 2. *For* $n \geq 2$, *let*

$$\tilde{\mathcal{F}}_n = \left\{ F \in \mathcal{C} \colon \binom{n}{i} \int_0^1 \frac{z^i (1-z)^{n-i}}{f(\bar{F}^{-1}(z))} \, \mathrm{d}z \text{ is decreasing in } i = 1, 2, \ldots, n \right\}. \tag{32}$$

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Then, for any two n-systems with signatures $\mathbf{s}$ *and* $\mathbf{t}$ *satisfying* $\mathbf{s} \le_{\text{mrl}} \mathbf{t}$, *and with i.i.d. component lifetimes with common distribution* $F \in \tilde{\mathcal{F}}_n$, *the corresponding system lifetimes satisfy* $S \le_{\text{mrl}} T$.

The classes $\tilde{\mathcal{F}}_n$ *are strictly nested,* $\tilde{\mathcal{F}}_n \subset \tilde{\mathcal{F}}_{n-1}$ *for all* $n \ge 3$, *and have a nonempty intersection.*

Proof. The conclusion of the mean residual life ordering of S and T follows by substituting $z = \bar{F}(u)$ in (14), as in the proof of Proposition 2.

The proof of inclusions $\tilde{\mathcal{F}}_n \subseteq \tilde{\mathcal{F}}_{n-1}$ is identical to that of the corresponding property in Theorem 1, now restricting to absolutely continuous F.

To prove that the inclusions are also strict in the absolutely continuous case, we consider the corresponding part of the proof of Theorem 1. Thus, let F be the discrete distribution that assigns probability p to $t = 0$ and $q = 1 - p$ to $t = 1$, where $0 < p < 1$. Now, for any $\varepsilon > 0$, we can find an absolutely continuous distribution $F_\varepsilon \in \mathcal{C}$, with support in [0, 1], such that

$$\lim_{\varepsilon \to 0} F_\varepsilon(t) = F(t) \quad \text{for all } 0 < t < 1.$$

One possible choice is to let

$$F_\varepsilon(t) = \begin{cases} \left(\dfrac{p}{\varepsilon}\right) t & \text{for } 0 \le t \le \varepsilon, \\ \max\left\{p + \varepsilon(t - \varepsilon),\, 1 + \dfrac{t-1}{\varepsilon}\right\} & \text{for } \varepsilon \le t \le 1. \end{cases}$$

Following (27), define

$$c^{\varepsilon}_{n,i} = \binom{n}{i} \int_0^1 \bar{F}_\varepsilon(u)^i F_\varepsilon(u)^{n-i}\, \mathrm{d}u.$$

It follows by the bounded convergence theorem that

$$\lim_{\varepsilon \to 0} c^{\varepsilon}_{n,i} = c_{n,i} = \binom{n}{i} q^i p^{n-i}$$

for each fixed n and i. Hence, by fixing n and choosing a p with strict inequality to the left in (30), we see from (29) that by choosing $\varepsilon > 0$ small enough we have

$$c^{\varepsilon}_{n-1,i+1} - c^{\varepsilon}_{n-1,i} < 0 \quad \text{for } i = 1, \ldots, n-2,$$

while

$$c^{\varepsilon}_{n,2} - c^{\varepsilon}_{n,1} > 0.$$

For such an ε, $F_\varepsilon \in \tilde{\mathcal{F}}_{n-1}$, but $F_\varepsilon \notin \tilde{\mathcal{F}}_n$, so $\mathcal{F}_n \subset \mathcal{F}_{n-1}$.

That the intersection of the $\tilde{\mathcal{F}}_n$ is nonempty follows from the proof of Theorem 1, where it was shown that the standard exponential distribution is contained in all the $\mathcal{F}_n$, or from the more general result Theorem 3 to be given below. This completes the proof of Theorem 2. □

The proof of Theorem 3 below uses the following lemma.

Lemma 6. *Let* $g(x)$ *be a function defined on* [0, 1] *such that, for some* $0 < c < 1$, $g(x) \le 0$ *for* $x \in [0, c)$ *and* $g(x) \ge 0$ *for* $x \in [c, 1]$, *and such that* $\int_0^1 g(x)\, \mathrm{d}x = 0$. *If* $h(x)$ *is a nonnegative decreasing function defined on* (0, 1], *then* $\int_0^1 g(x)h(x)\, \mathrm{d}x \le 0$.

Proof. Note first that, since $g(x) \leq 0$ on $[0, c)$, and $h(x)$ is decreasing, we have $g(x)h(x) \leq g(x)h(c)$ when $x \in (0, c)$. Furthermore, since $g(x) \geq 0$ on $[c, 1]$, and $h(x)$ is decreasing, we have $g(x)h(x) \leq g(x)h(c)$ also when $x \in [c, 1]$. But then

$$\int_0^1 g(x)h(x)\,\mathrm{d}x \leq h(c)\int_0^1 g(x) = 0.$$

□

Theorem 3. *If $F \in \mathcal{C}$ has density $f(t)$ which is decreasing in t, a property that is implied by the condition that F is a decreasing failure rate (DFR) distribution, then $F \in \tilde{\mathcal{F}}_n$ for all $n \geq 2$.*

Proof. Let $F \in \mathcal{C}$ have a decreasing density f. For given n, let d_i be the difference between the integral expression in (32) for $i+1$ and i. Then a straightforward calculation gives

$$\begin{aligned} d_i &= \frac{n!}{i!\,(n-i-1)!}\int_0^1 \frac{z^i(1-z)^{n-i-1}}{f(\bar{F}^{-1}(z))}\,\frac{(n+1)z-(i+1)}{(i+1)(n-i)}\,\mathrm{d}z \\ &= \frac{n!}{i!\,(n-i-1)!}\int_0^1 v_i(z)w(z)\,\mathrm{d}z \end{aligned} \tag{33}$$

for $i = 1, 2, \ldots, n-1$, where

$$v_i(z) = \frac{z^i(1-z)^{n-i-1}[(n+1)z-(i+1)]}{(i+1)n-i}, \qquad w(z) = \frac{1}{f(\bar{F}^{-1}(z))}.$$

Using the beta integral, we get

$$\binom{n}{i}\int_0^1 z^i(1-z)^{n-i}\,\mathrm{d}z = \frac{1}{n+1},$$

which does not depend on i. Hence, it is seen from (32) that d_i, by its definition as a difference, would equal 0 if $w(z) \equiv 1$ in (33). Consequently, $\int_0^1 v_i(z)\,\mathrm{d}z = 0$, and it is furthermore seen that $v_i(z) < 0$ if and only if $z < (i+1)/(n+1)$. Since $f(t)$ is decreasing in t, $f(\bar{F}^{-1}(z))$ is increasing in z. Thus, $w(z)$ is a decreasing function of z. Lemma 6 with $g = v_i$ and $h = w$ hence implies that d_i is nonpositive for all i. This proves that $F \in \tilde{\mathcal{F}}_n$ by (32).

Although it is well known that a DFR distribution has a decreasing density, we give the following simple argument for the sake of completeness. Let F be DFR with density $f(t)$ and hazard rate $\lambda(t)$. Then $f(t) = \lambda(t)\bar{F}(t)$, which is decreasing in t since both λ and $\bar{F}$ are decreasing. This completes the proof of Theorem 3. □

5. Connections to results on order statistics

Representation (3) for the survival function of a system lifetime may alternatively be given in terms of order statistics as follows. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the component lifetime distribution F and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics. Note that in (3), $\binom{n}{i}\bar{F}(t)^i F(t)^{n-i}$ can be interpreted as the probability that exactly i components are working at time t. If this is expressed by the order statistics of the component lifetimes, we have the identity

$$\binom{n}{i}\bar{F}(t)^i F(t)^{n-i} = \mathbb{P}(X_{(n-i):n} \leq t < X_{(n-i+1):n}) \tag{34}$$

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for any fixed n and $i = 1, \ldots, n$. From this, we obtain a characterization of the class $\mathcal{F}_n$ in terms of order statistics as given in the following proposition. Let

$$D_{i,n} = X_{i:n} - X_{(i-1):n} \quad \text{for } i = 1, \ldots, n$$

define the *sample spacings* between the order statistics, with $X_{0:n} \equiv 0$.

Proposition 3. *Let $n \geq 2$, and let $\mathcal{F}_n$ be the class of distributions defined in Theorem 1. Then $F \in \mathcal{F}_n$ if and only if $\mathbb{E}[D_{i,n}]$ is increasing in $i = 1, 2, \ldots, n$.*

Proof. Using (34), we have

$$\begin{aligned}\int_0^\infty \binom{n}{i} \bar{F}(u)^i F(u)^{n-i} \, \mathrm{d}u &= \int_0^\infty \mathbb{P}(X_{(n-i):n} \leq u < X_{(n-i+1):n}) \, \mathrm{d}u \\ &= \int_0^\infty \mathbb{P}(X_{(n-i+1):n} > u) \, \mathrm{d}u - \int_0^\infty \mathbb{P}(X_{(n-i):n} > u) \, \mathrm{d}u \\ &= \mathbb{E}[X_{(n-i+1):n} - X_{(n-i):n}] \\ &= \mathbb{E}[D_{n-i+1,n}]. \qquad (35)\end{aligned}$$

It is interesting to note that David [8, p. 50] attributed this identity to a classic paper by Francis Galton [10]. Now, by Theorem 1, $F \in \mathcal{F}_n$ if and only if the term on the left-hand side of (35) is decreasing in i. This is as claimed in the proposition. □

A careful examination of the arguments above shows that the proposition is also valid for discrete distributions F. For example, if t is a value to which F assigns a positive probability, then the event {exactly i components are functioning at time t} means that $\{X_{(n-i):n} \leq t < X_{(n-i+1):n}\}$ (cf. (34)). Expressed differently, this is the event where exactly i components are 'at risk' immediately after time t. If, on the other hand, we restrict attention to absolutely continuous F, then it is clear that Proposition 3 holds if $\mathcal{F}_n$, as defined in (14), is replaced by $\tilde{\mathcal{F}}_n$, defined in (32).

There is an extensive literature on the properties of sample spacings and their usage, particularly in goodness-of-fit testing and in reliability applications (see, e.g. [19], [28], and the references therein). Barlow and Proschan [2] proved that if F is DFR then the corresponding successive *normalized* spacings $(n - i + 1)D_{i,n}$ are stochastically ordered. Kochar and Kirmani [14] strengthened this result by proving a similar result for the hazard rate order. Later, Misra and van der Meulen [19] showed that a corresponding result holds for the *non*normalized spacings $\{D_{i:n}, i = 1, \ldots, n\}$. The following result is a special case of their Theorem 4.2.

Proposition 4. (Misra and van der Meulen [19].) *Suppose that F is absolutely continuous and DFR. Then, for $n \geq 2$,*

$$D_{i,n} \leq_{\text{hr}} D_{i+1,n} \quad \textit{for } i = 1, 2, \ldots, n-1.$$

Thus, under the condition of Proposition 4, the inequality $\mathbb{E}[D_{i,n}] \leq \mathbb{E}[D_{i+1,n}]$ holds for $i = 1, \ldots, n-1$. By Proposition 3, this implies that any DFR distribution F is in $\mathcal{F}_n$ for all n. This is of course in accordance with Theorem 2. Under a weaker condition than that in Proposition 4, Theorem 2 also implies the following result.

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Proposition 5. *Suppose that F is absolutely continuous with a decreasing density f. Then, for given $n \geq 2$,*

$$\mathbb{E}[D_{i,n}] \leq \mathbb{E}[D_{i+1,n}] \quad \text{for } i = 1, 2, \ldots, n-1.$$

6. Examples

Example 2. (*The power function distribution.*) This distribution has cumulative distribution function given by $F(t) = (t/\theta)^\alpha$ for $0 \leq t \leq \theta$, where $\alpha, \theta > 0$. In the following, we consider the case $\theta = 1$ in illustrating some aspects of the theoretical results obtained in preceding sections.

Let $F_\alpha(t) = t^\alpha$ for $0 \leq t \leq 1$, where $\alpha > 0$. The density of F_α is $f_\alpha(t) = \alpha t^{\alpha-1}$, $0 \leq t \leq 1$, which is decreasing if and only if $\alpha \leq 1$. It follows that $F_\alpha \in \tilde{F}_n$ for all n if $\alpha \leq 1$.

Since the property of decreasing density is only a sufficient condition for $F \in \tilde{\mathcal{F}}_n$, one may want to check the condition in (32) directly. Note then that $\bar{F}^{-1}(z) = (1-z)^{1/\alpha}$, so that $f(\bar{F}^{-1}(z)) = \alpha(1-z)^{1-1/\alpha}$. The integral in (32) thus becomes

$$y_i = \alpha^{-1}\binom{n}{i}\int_0^1 z^i(1-z)^{n-i-1+1/\alpha}\,\mathrm{d}z = \alpha^{-1}\binom{n}{i}\frac{\Gamma(i+1)\Gamma(n-i+1/\alpha)}{\Gamma(n+1+1/\alpha)},$$

where we used the beta integral. Noting that $\Gamma(i+1) = i!$, and using the identity $\Gamma(k+1) = k\Gamma(k)$, we obtain

$$y_{i+1} - y_i = \frac{\alpha-1}{\alpha^2}\,\frac{n!}{(n-i)!}\,\frac{\Gamma(n-i-1+1/\alpha)}{\Gamma(n+1+1/\alpha)}.$$

This difference is nonpositive if and only if $\alpha \leq 1$, and, hence, for each n, we have $F_\alpha \in \tilde{\mathcal{F}}_n$ if and only if $\alpha \leq 1$.

Recall now that $F \in \tilde{\mathcal{F}}_n$ is only a sufficient condition for $S \leq_{\text{mrl}} T$. Thus, the case $\alpha > 1$ is still undecided. A computer study has, however, indicated that even for $\alpha > 1$ very close to 1 there are $\boldsymbol{s}$ and $\boldsymbol{t}$ with $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ for which $S \not\leq_{\text{mrl}} T$.

For illustration, we use the necessary and sufficient condition for $S \leq_{\text{mrl}} T$ given in Proposition 2. When adapted to the power function distribution, we find that, for given n, $\boldsymbol{s}$, and $\boldsymbol{t}$ with $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$, we have $S \leq_{\text{mrl}} T$ if and only if

$$\frac{\sum_{i=1}^n b_i\binom{n}{i}B(u; i+1, n-i+1/\alpha)}{\sum_{i=1}^n a_i\binom{n}{i}B(u; i+1, n-i+1/\alpha)} \tag{36}$$

is decreasing in u for $0 < u \leq 1$, where

$$B(u; c, d) = \int_0^u z^{c-1}(1-z)^{d-1}\,\mathrm{d}z \quad \text{for } c, d > 0$$

is the incomplete beta function.

In Figure 1 the ratio in (36) is plotted as a function of u for two examples of $\boldsymbol{s}$ and $\boldsymbol{t}$ with $n = 3$, where $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$. The curves for $\alpha \leq 1$ are, as guaranteed by Theorem 2, seen to be monotonically decreasing. This is not the case, however, for the plotted curves for $\alpha > 1$. Thus, by Proposition 2, we do not have $S \leq_{\text{mrl}} T$ in the latter cases.

Example 3. (*The Weibull distribution.*) The Weibull distribution with scale parameter 1 and shape parameter $\alpha > 0$ has density function $f_\alpha(t) = \alpha t^{\alpha-1}\mathrm{e}^{-t^\alpha}$, which is decreasing if and only if $\alpha \leq 1$. Hence, $F \in \tilde{\mathcal{F}}_n$ for all n if $\alpha \leq 1$.

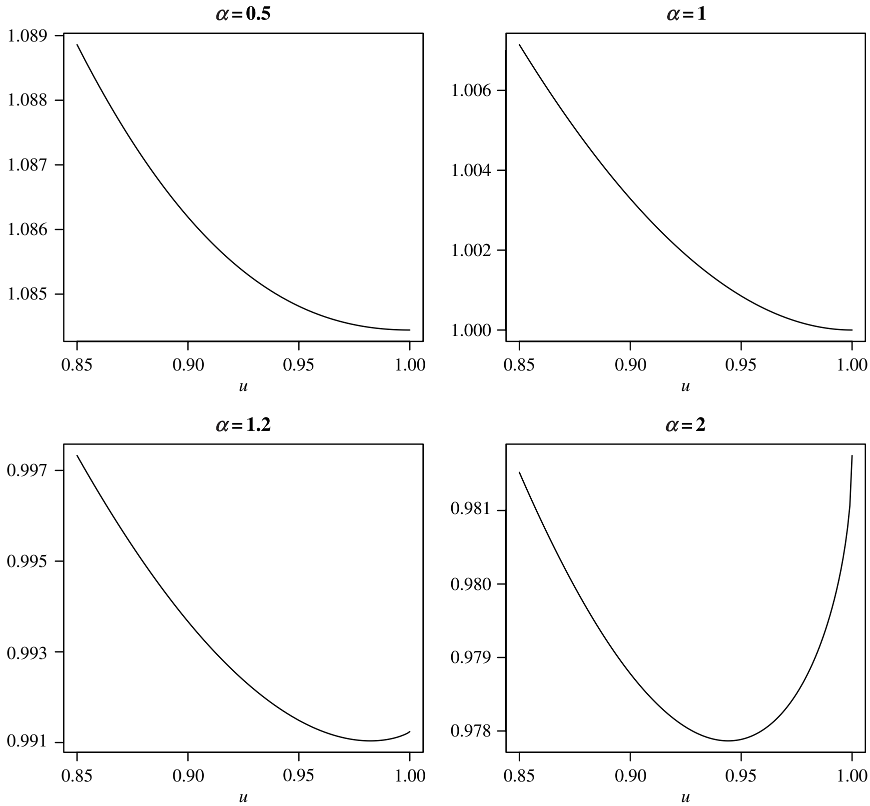

FIGURE 1: The plots show the function (36) when $n = 3$, survival signatures are $\boldsymbol{a} = (0.03, 0.58, 1)$ and $\boldsymbol{b} = (0.22, 0.39, 1.00)$ (which satisfy the mrl order), for component lifetime distributions given by $F(t) = t^{\alpha}$ for different values of α.

In a study of the case $\alpha > 1$, we will use the representation of $\mathcal{F}_n$ given in Proposition 3 in terms of expected sample spacings. If $X_{1:n}, \ldots, X_{n:n}$ is an ordered sample from F_{α}, it is well known that $X^{\alpha}_{1:n}, \ldots, X^{\alpha}_{n:n}$ is an ordered sample from the standard exponential distribution. Thus, for $i = 1, \ldots, n$, we have

$$\mathbb{E}[X_i^{\alpha}] - \mathbb{E}[X_{i-1}^{\alpha}] = \frac{1}{n - i + 1}. \tag{37}$$

For a distribution to be contained in $\tilde{\mathcal{F}}_n$, Proposition 3 requires that $\mathbb{E}[D_{i,n}] = \mathbb{E}[X_i] - \mathbb{E}[X_{i-1}]$ is increasing in i. By (37), this holds for the Weibull case if $\alpha = 1$, with strict inequalities. Thus, for a fixed n, by continuity with respect to α of the left-hand side in (37), we conclude that the increasing property of the $\mathbb{E}[D_{i:n}]$ will hold also for some $\alpha > 1$, sufficiently close to 1. Hence, each set $\tilde{\mathcal{F}}_n$ will contain Weibull distributions with shape parameter strictly larger than 1, and thus include densities which are not everywhere decreasing.

To pursue a more formal study of the above phenomenon, we apply the formula for expected values of order statistics of the Weibull distribution given in [16]. Using our notation, it is known that

$$\mathbb{E}[X_{i:n}] = n\binom{n-1}{i-1}\Gamma\left(1 + \frac{1}{\alpha}\right)\sum_{j=0}^{i-1}(-1)^{j-i+1}\binom{i-1}{j}\frac{1}{(n-j)^{1+1/\alpha}}. \tag{38}$$

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TABLE 1: Weibull and gamma distributions with shape parameter $\alpha > 1$ contained in $\tilde{\mathcal{F}}_n$ for different values of α. The dashes mean that $F_\alpha \notin \tilde{\mathcal{F}}_n$ for all $n \geq 2$.

α	Weibull $F_\alpha \in \tilde{\mathcal{F}}_n$ if $n \leq$	Gamma $F_\alpha \in \tilde{\mathcal{F}}_n$ if $n \leq$
1.1	10	13
1.2	5	6
1.3	3	5
1.4	3	4
1.5	2	3
1.6	2	3
1.7	2	2
1.8–2.6	—	2
≥ 2.7	—	—

For our purpose, we used (38) to calculate the differences $\Delta_{i,n} = \mathbb{E}[D_{i+1,n}] - \mathbb{E}[D_{i,n}]$, obtaining

$$\begin{aligned}
\Delta_{i,n} &= \mathbb{E}[X_{i+1:n}] - 2\mathbb{E}[X_{i:n}] + \mathbb{E}[X_{i-1:n}] \\
&= \binom{n}{i} \frac{\Gamma(1+1/\alpha)}{n-i+1} \sum_{j=0}^{i} (-1)^{j-i} \binom{i}{j} \frac{1}{(n-j)^{1+1/\alpha}} [(n-i)(n-i+1)] \\
&\quad + \binom{n}{i} \frac{\Gamma(1+1/\alpha)}{n-i+1} \sum_{j=0}^{i} (-1)^{j-i} \binom{i}{j} \frac{1}{(n-j)^{1+1/\alpha}} [2(i-j)(n-i+1)] \\
&\quad + \binom{n}{i} \frac{\Gamma(1+1/\alpha)}{n-i+1} \sum_{j=0}^{i} (-1)^{j-i} \binom{i}{j} \frac{1}{(n-j)^{1+1/\alpha}} [(i-j)(i-j-1)] \\
&= \binom{n}{i} \frac{\Gamma(1+1/\alpha)}{n-i+1} \sum_{j=0}^{i} (-1)^{j-i} \binom{i}{j} \frac{n-j+1}{(n-j)^{1/\alpha}}. \qquad (39)
\end{aligned}$$

Clearly, $F_\alpha \in \tilde{\mathcal{F}}_n$ if and only if $\Delta_{i,n} \geq 0$ for $i = 1, \ldots, n-1$. We have used expression (39) to generate the Weibull part of Table 1, which shows, for some values of $\alpha > 1$, the values of n for which we have $F_\alpha \in \tilde{\mathcal{F}}_n$.

Numerical experience with (39) clearly indicates that $\Delta_{i,n}$ increases with $i = 1, \ldots, n-1$ for fixed α. We do not have, however, a formal proof of this monotonicity. Assuming that this property holds, it would follow that $F_\alpha \in \tilde{\mathcal{F}}_n$ if and only if

$$\alpha \leq \frac{\log(n/(n-1))}{\log((n+1)/n)},$$

which is approximately $1 + 1/(n-1)$ for large n.

We also did a limited computer study to find Weibull distributions F_α with $\alpha > 1$ such that, for some n and signatures $\boldsymbol{s}$ and $\boldsymbol{t}$ with $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$, we have $S \not\leq_{\text{mrl}} T$. Considering for simplicity the $\boldsymbol{s}$ and $\boldsymbol{t}$ of Example 1, for which $n = 3$ and $\boldsymbol{a} = (\frac{1}{2}, \frac{3}{4}, 1)$, $\boldsymbol{b} = (\frac{5}{8}, \frac{5}{8}, 1)$, we found that $S \not\leq_{\text{mrl}} T$

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when the component distributions are Weibull with $\alpha \geq 3.71$. Considering $n = 4$ by adding a zero at the beginning of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we obtained $S \not\leq_{\text{mrl}} T$ for F_α with $\alpha \geq 2.17$. Adding new zeros to $\boldsymbol{a}$ and $\boldsymbol{b}$, to have $n = 5$ and $n = 6$, we obtained $S \not\leq_{\text{mrl}} T$ for $\alpha \geq 1.77$ and 1.59, respectively.

Example 4. (*The gamma distribution.*) The gamma distribution with scale parameter 1 and shape parameter $\alpha > 0$ has density $f_\alpha(t) = (\Gamma(\alpha))^{-1} t^{\alpha-1} \mathrm{e}^{-t}$, which is decreasing if and only if $\alpha \leq 1$. Hence, $F_\alpha \in \tilde{\mathcal{F}}_n$ for all n if $\alpha \leq 1$. In the same way as for the Weibull distribution, it can be shown by using Proposition 3 that the $\tilde{\mathcal{F}}_n$ will include gamma distributions with $\alpha > 1$. While Nadarajah [20] presented expected values of order statistics for the gamma distribution, because of the complexity of the formula, we decided to calculate the expected values of order statistics from the gamma distribution by simulation instead, using 100 000 iterations for each simulated value. The results are shown in Table 1 and are very similar to what we obtained for the Weibull distribution.

Example 5. (*The relation to stochastically ordered signatures.*) In the counterexample in Example 1, as well as in the examples with $S \not\leq_{\text{mrl}} T$ in Examples 2 and 3, we considered signatures with $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ and $\boldsymbol{s} \not\leq_{\text{hr}} \boldsymbol{t}$. By inspection, it is seen that, in all these cases, we had $\boldsymbol{s} \not\leq_{\text{st}} \boldsymbol{t}$. As noted in Section 2, the assumption that $\boldsymbol{s} \leq_{\text{hr}} \boldsymbol{t}$ would trivially imply that $S \leq_{\text{mrl}} T$. The question thus emerges whether the two conditions $\boldsymbol{s} \leq_{\text{st}} \boldsymbol{t}$ and $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ together would imply that $S \leq_{\text{mrl}} T$ for *all* distributions F.

First of all, no counterexample to such a claim can occur if F has support in two points as in Example 1. This follows since (13) would hold for any values p if $\boldsymbol{s} \leq_{\text{st}} \boldsymbol{t}$, since then $a_j \leq b_j$ for all j. Thus, to search for counterexamples, we need to consider F with support in at least three points. The following is the result of a computer search. Let F give positive mass to the three time points {0, 50, 70}, with respective probabilities 0.35, 0.02, 0.63. Next, let $n = 7$, and let two mixed 7-systems have survival signatures

$$\boldsymbol{a} = (0.03, 0.53, 0.57, 0.67, 0.69, 0.74, 1.00), \tag{40}$$

$$\boldsymbol{b} = (0.44, 0.61, 0.61, 0.86, 0.92, 0.99, 1.00). \tag{41}$$

In order to check the presence of the various stochastic orders for these signatures, we use Definition 3. It is straightforward to show that $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ and $\boldsymbol{s} \leq_{\text{st}} \boldsymbol{t}$, but that $\boldsymbol{s} \not\leq_{\text{hr}} \boldsymbol{t}$.

Let S and T respectively be the lifetimes of the two systems with the given F above. In order to show that $S \not\leq_{\text{mrl}} T$, we use the necessary and sufficient condition (10). This fraction, when calculated at $t = 0$ and $t = 50$, respectively takes the values 1.2630 and 1.2621, and is *not* increasing. Thus we conclude that $S \not\leq_{\text{mrl}} T$.

The result might be even more convincing if we can find an absolutely continuous $F \in \mathcal{C}$ which leads to the same conclusion. The above three-point distribution F suggests a bathtub-shaped density for such a distribution. We therefore looked for beta distributions with parameters α and β both being less than one. Our search identified several possible candidates, two of which are described below.

In Figure 2 we plot the function (31) for the systems with survival signatures (40) and (41), letting the component lifetime distribution F be beta distributions with respective parameters (0.58, 0.05) and (0.13, 0.07). The curves show a clear nonmonotonicity.

The conclusion to be drawn is that the conditions $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ *and* $\boldsymbol{s} \leq_{\text{st}} \boldsymbol{t}$ are not sufficient to ensure that $S \leq_{\text{mrl}} T$ holds for any component lifetime distribution F. Thus, the 'problem' that $S \leq_{\text{mrl}} T$ does not hold for all F cannot be resolved by additionally requiring the stochastic ordering of the signatures.

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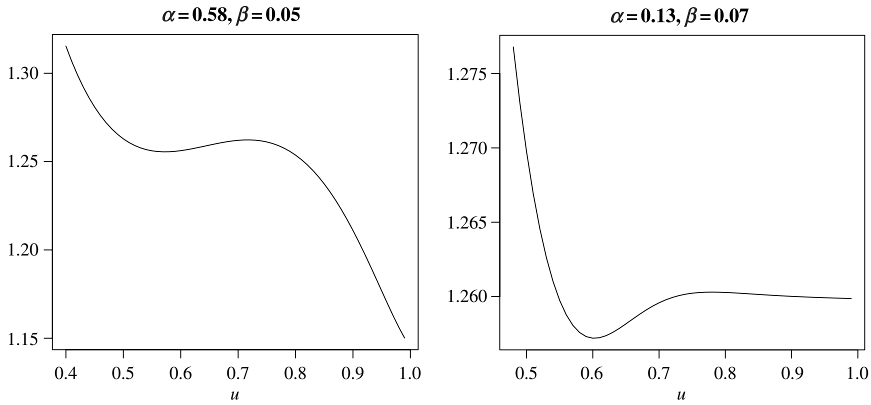

FIGURE 2: The plots show the function (31) when $n = 7$, survival signatures are $\boldsymbol{a} = (0.03, 0.53, 0.57, 0.67, 0.69, 0.74, 1.00)$ and $\boldsymbol{b} = (0.44, 0.61, 0.61, 0.86, 0.92, 0.99, 1.00)$ (which satisfy both the mrl order and st order), for two component lifetime distributions F, given by beta distributions with different parameters.

7. Final remarks

Remark 1. The result of Lemma 5 was stated for the case of general distributions in the final section of [13]. As we did in our approach, they suggested combining it with Caperaa's result in order to have a tool for studies of the mean residual life order.

Remark 2. In order to shed some light on the main idea of the proof of Theorem 1, it is instructive to consider the proof of the implication $\boldsymbol{s} \leq_{\text{hr}} \boldsymbol{t} \Rightarrow S \leq_{\text{hr}} T$ (see (8)) as given in [25, Theorem 4.4]. While we use representation (3) for the survival function of a system lifetime T, the latter proof uses representation (2), which is given in terms of the signatures s_i themselves. The proof then considers $\boldsymbol{s}$ and $\boldsymbol{t}$ with $\boldsymbol{s} \leq_{\text{hr}} \boldsymbol{t}$ and, in our notation, uses Lemma 4 with $k(i) = s_i$ and $l(i) = t_i$. The corresponding cumulative distribution functions $K(i)$ and $L(i)$ will hence, by assumption, satisfy $K \leq_{\text{hr}} L$. Furthermore, the proof of Samaniego [25] uses $\alpha(i) = \mathbb{P}(X_{i:n} > t)$, $\beta(i) = \mathbb{P}(X_{i:n} > s)$, with $s < t$. These are shown to have the desired monotonicity properties using known results on order statistics. Implication (8) then follows from Lemma 4.

The main difference in the proof of Theorem 1 of the present paper is that our proof is based on cumulative signatures instead of the signatures themselves. The key to our approach is the fact (established in Lemma 5) that $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ implies that $K \leq_{\text{hr}} L$, which allows the use of Lemma 4. Finally, we note that the proof of the present paper involves more complex expressions for $\alpha(i)$ and $\beta(i)$ than those displayed above for the proof in [25]. In terms of order statistics it is seen, following the proof of Proposition 3, that $\alpha(i)$ in (25) may be written as

$$\alpha(i) = \int_t^{\infty} \mathbb{P}(X_{(i-1):n} \leq u < X_{i:n}) \, \mathrm{d}u = \int_t^{\infty} [\mathbb{P}(X_{i:n} > u) - \mathbb{P}(X_{(i-1):n} > u)] \, \mathrm{d}u,$$

while the expression for $\beta(i)$ is similar with t replaced by s. Both proofs use, on the other hand, the result of Lemma 4 to obtain the final conclusion.

Remark 3. Lemma 3, due to Caperaa, is formulated as a necessary and sufficient condition for the given order between F and G. In Lemma 4, this would correspond to having $K \leq_{\text{hr}} L$ if and only if (20) holds for all $\alpha(i)$ and $\beta(i)$ satisfying the monotonicity requirements. Now, in our application, $K \leq_{\text{hr}} L$ if and only if $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$ by Lemma 5. This might suggest that if (10)

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holds for all $F \in \mathcal{F}_n$, then $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$. This would indeed be the case if all possible functions $\alpha(i)$ and $\beta(i)$ with $\beta(i)$ increasing and $\alpha(i)/\beta(i)$ increasing can be represented as in (25) and (26) for some F, s, and t. This is presumably not the case, but since one may, by varying F, s, and t, obtain a fairly rich class of functions $\alpha(i)$, $\beta(i)$, we can state the following result as a conjecture.

Let there be given two n-systems, with signatures $\boldsymbol{s}$ *and* $\boldsymbol{t}$*, i.i.d. component lifetimes with common distribution F, and system lifetimes denoted by S and T, respectively. If* $S \leq_{\text{mrl}} T$ *whenever* $F \in \mathcal{F}_n$*, then* $\boldsymbol{s} \leq_{\text{mrl}} \boldsymbol{t}$*. Equivalently, if* $\boldsymbol{s} \not\leq_{\text{mrl}} \boldsymbol{t}$ *then there is an* $F \in \mathcal{F}_n$ *such that* $S \not\leq_{\text{mrl}} T$.

Remark 4. The comparison of systems considered in the paper has been restricted to cases where two systems with signatures $\boldsymbol{s}$ and $\boldsymbol{t}$ have the same size. Now suppose that we are interested in comparing two systems that are not of the same size. Definition 3 can obviously not be used directly to determine ordering properties of signatures of different sizes. The approach taken by Samaniego [25, p. 32] is to 'convert' the smaller of two systems into an equivalent system of the same size as the larger one, thus allowing the use of comparison results for systems of the same size. Equivalent systems here means systems that have the same system lifetime distribution, for any (common) component distribution F. Samaniego [25, Theorem 3.2] gave an explicit formula for the signature of an $(n+1)$-system which is equivalent to a given n-system. This formula may be applied several times in succession depending on the difference in size of the two systems. Lindqvist *et al.* [17] studied properties of equivalent systems and showed, in particular, how to construct a system equivalent to a given system of a different size.

Remark 5. Coolen and Coolen-Maturi [7] generalized (3) to the case where there are $K > 1$ types of components. Let there be n_k components of type k, $k = 1, 2, \ldots, K$, with $\sum_{k=1}^{K} n_k = n$. Assuming that all component lifetimes are independent, where the lifetimes of components of type k have distribution F_k, they obtained

$$\mathbb{P}(T > t) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_K=0}^{n_K} a(j_1, \ldots, j_K) \prod_{k=1}^{K} \binom{n_k}{j_k} \bar{F}_k(t)^{j_k} F_k(t)^{n_k - j_k}. \tag{42}$$

Here $a(j_1, \ldots, j_K)$ is the *survival signature*, which is the probability that the system functions when j_k of the n_k components of type k function, while the product

$$\prod_{k=1}^{K} \binom{n_k}{j_k} \bar{F}_k(t)^{j_k} F_k(t)^{n_k - j_k}$$

is the probability that exactly j_k out of n_k components of type k function at time t, for $k = 1, \ldots, K$.

The concept of survival signature has in the recent literature been used in various applications. For example, Aslett *et al.* [1] presented an application to networks; Huang *et al.* [12] used the survival signature in an analysis of phased mission systems, while Eryilmaz *et al.* [9] studied joint reliability importance measures for system components. Samaniego and Navarro [26] investigated methods for comparison of coherent systems with heterogeneous components. They gave in particular a sufficient condition for stochastic ordering of the lifetimes of two such systems which generalizes (7). Specifically, it is seen from (42) that, if two systems have the same component types and corresponding component lifetime distributions, as well as the same number of components of each type, then their lifetimes are stochastically ordered if the survival signatures are ordered accordingly with respect to the componentwise ordering of vectors. We are, however, not aware of similar generalizations of (8) and (9) that would give sufficient conditions for the hazard rate and mean residual life ordering of systems with heterogeneous components. Such conditions might possibly be found in terms of suitable multivariate orders as considered, e.g. in [27, Chapter 6].

Remark 6. Throughout the paper we have considered the situation where the component lifetimes $X_1, X_2, \ldots, X_n$ are i.i.d. As is clear from the literature on system signatures, many results for the i.i.d. case can be extended to the case where $X_1, X_2, \ldots, X_n$ are exchangeable. For example, Navarro *et al.* [23] proved that implication (8) for the hazard rate order holds for exchangeable component lifetimes $X_1, X_2, \ldots, X_n$ provided $X_{1:n} \leq_{\text{hr}} X_{2:n} \leq_{\text{hr}} \cdots \leq_{\text{hr}} X_{n:n}$. Navarro and Rubio [22] showed that corresponding reverse implications hold for the st, hr, and lr orders. More precisely, they showed that the ordering of two system lifetimes for all exchangeable component distributions, with similarly ordered order statistics, implies the ordering of the respective signatures.

For the mean residual life order, it was proved in Navarro *et al.* [23] that, for two systems with signatures $\boldsymbol{s}$ and $\boldsymbol{t}$ satisfying $\boldsymbol{s} \leq_{\text{hr}} \boldsymbol{t}$, and with exchangeable component lifetimes satisfying $X_{1:n} \leq_{\text{mrl}} X_{2:n} \leq_{\text{mrl}} \cdots \leq_{\text{mrl}} X_{n:n}$, the corresponding system lifetimes S and T satisfy $S \leq_{\text{mrl}} T$. Recall from the discussion in Section 2 that in the i.i.d. case, $\boldsymbol{s} \leq_{\text{hr}} \boldsymbol{t}$ implies that $S \leq_{\text{hr}} T$ and, hence, trivially implies that $S \leq_{\text{mrl}} T$.

Acknowledgements

The authors are grateful for valuable comments and suggestions from two reviewers and an Associate Editor. This work was done while the first author was visiting the Department of Statistics, University of California, Davis in the academic year 2017/2018. The kind hospitality of the faculty and staff is gratefully acknowledged. The second author's research was supported in part by grant W911NF-17-1-0381 from the US Army Research Office.

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