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Riemann–Roch for Algebraic Stacks: I

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Abstract. In this paper we establish Riemann–Roch and Lefschetz–Riemann–Roch theorems for arbitrary proper maps of finite cohomological dimension between algebraic stacks in the sense of Artin. The Riemann–Roch theorem is established as a natural transformation between the G-theory of algebraic stacks and topological G-theory for stacks: we define the latter as the localization of G-theory by topological K-homology. The Lefschetz–Riemann–Roch is an extension of this including the action of a torus for Deligne–Mumford stacks. This generalizes the corresponding Riemann–Roch theorem (Lefschetz–Riemann–Roch theorem) for proper maps between schemes (that are also equivariant for the action of a torus, respectively) making use of some fundamental results due to Vistoli and Toen. A key result established here is that topological G-theory (as well as rational G-theory) has cohomological descent on the isovariant étale site of an algebraic stack. This extends cohomological descent for topological G-theory on schemes as proved by Thomason.

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1. Introduction

In this paper we consider the general Riemann–Roch problem for arbitrary proper maps of finite cohomological dimension between algebraic stacks in the sense of Artin. Even in the case of Deligne–Mumford stacks, the problem was only recently solved in [Toe-1] and the difficulties that can come up in general may be seen already in the case of finite group actions on schemes. Let G denote a finite group, viewed as a group scheme over a field k: we assume the order of G is prime to the characteristic of k. Now the Grothendieck group of vector bundles on the stack $[\mathrm{Spec}\ k/G]$ may be identified with the representation ring of the finite group, namely $R(G)$ or equivalently $K_G^0(\mathrm{Spec}\ k)$. Moreover, $H^*_{\mathrm{et}}([\mathrm{Spec}\ k/G]; \mathbb{Q}) \cong H^*_{\mathrm{et}}(BG; \mathbb{Q})$. Though $R(G)$ is far from being trivial (even when tensored with $\mathbb{Q}$), the cohomology ring $H^*(BG; \mathbb{Q}) \cong \mathbb{Q}$. Therefore, the diagram

$$\begin{array}{ccc} K_G^0(\mathrm{Spec}\ k) & \xrightarrow{ch^G} & H^*_{\mathrm{et}}(BG; \mathbb{Q}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ K^0(\mathrm{Spec}\ k) & \xrightarrow{ch} & H^*_{\mathrm{et}}(\mathrm{Spec}\ k; \mathbb{Q}) \end{array}$$

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fails to commute, where $p\colon [\mathrm{Spec}\ k/G] \to \mathrm{Spec}\ k$ is the obvious (nonrepresentable) map of algebraic stacks. (The top row is the G-equivariant Chern character, whereas the bottom row is the usual Chern character which one may identify with the rank map. One may identify the left most column with the map, sending a representation of G to its G invariant part.) This problem was solved in [Toe-1] by a rather elaborate procedure, ultimately making use of a theorem of Vistoli which says the equivariant higher algebraic K-theory of a regular scheme provided with the action of a finite group is isomorphic to the higher étale K-theory of the inertia stack provided everything is tensored with $\mathbb{C}$. (See the discussion below for more details.) Though, the corresponding result is known for compact lie group actions on manifolds, the techniques involved (especially induction) do not generalize to the actions of reductive groups on regular schemes.

In fact the difficulty with Riemann–Roch for algebraic stacks may already be seen by the lack of commutativity of the following diagram:

$$\begin{array}{ccc} K^0_G(\mathrm{Spec}\ k) & \longrightarrow & \mathbb{H}^0_{\mathrm{et}}([\mathrm{Spec}\ k/G]; \mathcal{K}_{\mathbb{Q}}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ K^0(\mathrm{Spec}\ k) & \longrightarrow & \mathbb{H}^0_{\mathrm{et}}(\mathrm{Spec}\ k; \mathcal{K}_{\mathbb{Q}}) \end{array}$$

where the last terms in each row denote the étale hyper-cohomology of the corresponding stack computed with respect to the presheaf $\mathcal{K}_{\mathbb{Q}}$; this is the presheaf defined by $U \to K(U)_{\mathbb{Q}}$ = the localization of the algebraic K-theory spectrum $K(U)$ at $\mathbb{Q}$, U on the étale site of the appropriate stack. One of the key ideas in this paper may now be stated in the above context as follows: if one replaces the étale topology above with another topology (called the isovariant étale topology) we define in Section 3 (and the presheaf $\mathcal{K}$ is replaced by the equivariant version $\mathcal{K}^G$), then the corresponding diagram *does commute*.

We will adopt the following terminology in the statement of Theorems 1.1 and 1.2. Let J denote a set of primes in $\mathbb{Z}$. Assume that the base scheme S is Noetherian of finite Krull dimension and that there is a uniform bound on the l-torsion étale cohomological dimension of the residue fields $k(s)$ for all points s in S and all $l \varepsilon J$. (Observe that this hypothesis holds if S is of finite type over an algebraically closed field or over $\mathbb{Z}[\sqrt{-1}]$ or if 2 does not belong to J and S is of finite type over $\mathbb{Z}$.) Assume also that l is *invertible* in $\mathcal{O}_X$, for any X which is an object over S (i.e. a scheme, an algebraic space or an algebraic stack) that we consider and for all primes $l \varepsilon J$. Assume also the hypotheses in (5.1) and that all the objects we consider are locally Noetherian over the given base scheme. (However, most of our basic results will hold only for algebraic stacks that are finitely presented over the given base scheme.)

We may summarize the main theorems of the paper as follows:

THEOREM 1.1 (see Theorem 5.10 and Corollary 5.12). *Let* $\mathbf{G}$ *denote the presheaf of spectra corresponding to the G-theory defined in Definition* 5.4 *and let* $\mathbf{G}_K \otimes \mathbb{Z}_{(J)}$ *denote the localization of the presheaf* $\mathbf{G}$ *first in the sense of Bousfield by topological*

K-homology followed by inverting the primes not in J. Let $\mathcal{S}$ denote an algebraic stack, finitely presented over the base scheme S, with $\mathcal{S}_{\mathrm{iso.et}}$ denoting the isovariant étale site of the stack $\mathcal{S}$ defined in Section 3. Then the obvious augmentation

$$G(\mathcal{S})_K \otimes \mathbb{Z}_{(J)} \to \mathbb{H}_{\mathrm{iso.et}}(\mathcal{S}, \mathbf{G}_K \otimes \mathbb{Z}_{(J)}) \tag{1.0.1}$$

is a weak-equivalence of spectra where the right-hand side denotes the hyper-cohomology spectrum computed on the isovariant étale site. (*One may restate the above result as: the presheaf $\mathbf{G}_K \otimes \mathbb{Z}_{(J)}$ has cohomological descent on the isovariant étale site.*) *Moreover, there exists a strongly-convergent spectral sequence*

$$E_2^{s,t} = H^s_{\mathrm{iso.et}}(\mathcal{S}, \pi_t(\mathbf{G}_K \otimes \mathbb{Z}_{(J)})) \Rightarrow \pi_{-s+t}(G(\mathcal{S})_K \otimes \mathbb{Z}_{(J)}). \tag{1.0.2}$$

In view of the above theorem we will call $G(\mathcal{S})_K \otimes \mathbb{Z}_{(J)}$ topological G-theory. This will be denoted $G^{\mathrm{top}}(\mathcal{S})$. The presheaf $\mathbf{G}_K \otimes \mathbb{Z}_{(J)}$ of spectra will be called the presheaf of *topological G-theory*. (We may also use $G(\mathcal{S})/l^\nu[\beta^{-1}]$ for $G^{top}(\mathcal{S})$ where l is as above, $\nu \gg 0$ and β denotes *the Bott element*.)

THEOREM 1.2 (Riemann–Roch from algebraic to topological G-theory). *Let $f\colon \mathcal{S}' \to \mathcal{S}$ denote any proper map between two algebraic stacks finitely presented over S and of finite cohomological dimension. Then the direct image map f_* fits in the following homotopy commutative square:*

$$\begin{tikzcd}
G(\mathcal{S}') \arrow[r] \arrow[d, "f_*"'] & G^{top}(\mathcal{S}') \arrow[d, "f_*"] \\
G(\mathcal{S}) \arrow[r] & G^{top}(\mathcal{S})
\end{tikzcd}$$

The above theorem might seem like a tautology, since the right-hand side is a suitable localization of the left-hand side. However, as in [T-2], [T-3], it is the right-hand side that can be computed by the spectral sequence in the above theorem, whereas there is no such spectral sequence for computing the left-hand side. We will in fact prove a stronger version of the above two theorems including the action of a smooth group scheme on the stacks $\mathcal{S}$ and $\mathcal{S}'$.

As an application of cohomological descent for $\mathbf{G}_K \otimes \mathbb{Z}_{(J)}$, one obtains the following Lefschetz–Riemann–Roch theorem where $G(\mathcal{S})_K \otimes \mathbb{Z}_{(J)}$ is denoted by $G^{top}(\mathcal{S})$. We will assume the base scheme S is the spectrum of an algebraically closed field k, all the stacks we consider are Deligne–Mumford and finitely presented over k and that the orders of the stabilizers on all the stacks we consider are different from the characteristic of k in the following. Moreover, $\mathbb{Q}(\mu_\infty)$ will denote the algebra over $\mathbb{Q}$ generated by $\mathbb{Q}$ and μ_∞, with μ_∞ denoting the roots of unity in k imbedded in $\mathbb{C}^*$. Let T denote a torus, let $R(T)$ denote the representation ring of T and let $\mathfrak{p}$ denote a prime ideal in $R(T)$ corresponding to a sub-torus T'. Given an action of a sub-torus T'' of T (which may be either T itself or the given sub-torus T') on an algebraic stack $\mathcal{S}$ as in Definition 5.1, one lets $\mathrm{Coh}(\mathcal{S}, T'')$ = the category of coherent sheaves on the stack $\mathcal{S}$ with a T''-action.

We let $G(\mathcal{S}, T'') = K(\mathrm{Coh}(\mathcal{S}, T'')) =$ the K-theory spectrum of the category $\mathrm{Coh}(\mathcal{S}, T'')$ and similarly $G^{top}(\mathcal{S}, T') =$ the topological K-theory of the above category (defined as above by localizing with respect to topological K-homology followed by inverting the primes not in J.) In this case we define the fixed point stack $\mathcal{S}^{T'}$ as in Definition 6.4 so that the induced map $i: \mathcal{S}^{T'} \to \mathcal{S}$ is a closed immersion. Let $I_{\mathcal{S}^{T'}}$ denote the inertia stack associated to $\mathcal{S}^{T'}$: there is an obvious map $\pi^{T'}: I_{\mathcal{S}^{T'}} \to \mathcal{S}^{T'}$ that is unramified (or a local imbedding) since the stack $\mathcal{S}$ is assumed to be Deligne–Mumford. It is shown in 6.6 below that one may find a finite étale cover $\tilde{T}' \to T'$, so that when $\tilde{T}'$ acts on $\mathcal{S}^{T'}$ through the action of T', this action is *trivial*. Moreover, $\mathcal{S}^{\tilde{T}'} = \mathcal{S}^{T'}$ and when $\mathcal{S}$ is a smooth Deligne–Mumford stack, $\mathcal{S}^{\tilde{T}'}_{red}$ is also smooth.

Given a presheaf of spectra P, we let $P \otimes \mathbb{Q}$ the localization of P at $\mathbb{Q}$ in the sense of [B-K]. Next we follow [Toe-1] and let $G_{\mathrm{et}}(\mathcal{S}) \otimes \mathbb{Q} = \mathbb{H}_{\mathrm{et}}(\mathcal{S}, \mathbf{G} \otimes \mathbb{Q})$ which is the étale hypercohomology of the stack $\mathcal{S}$ with respect to the presheaf $\mathbf{G} \otimes \mathbb{Q}$. We also let $G_{\mathrm{et}}(\mathcal{S}, T) \otimes \mathbb{Q} = \mathbb{H}_{\mathrm{et}}(\mathcal{S}, \mathbf{G}(\quad, T) \otimes \mathbb{Q})$ where $\mathbf{G}(\quad, T) \otimes \mathbb{Q}$ denotes the presheaf of spectra associated to T-equivariant coherent sheaves on $\mathcal{S}$. Similarly $K_{\mathrm{et}}(\mathcal{S}, T) \otimes \mathbb{Q} = \mathbb{H}_{\mathrm{et}}(\mathcal{S}, \mathbf{K}(\quad, T) \otimes \mathbb{Q})$ where $\mathbf{K}(\quad, T)$ denotes the presheaf of spectra associated to T-equivariant locally free coherent sheaves.

We will assume, henceforth, that $\mathcal{S}$ is a smooth Deligne–Mumford stack. Next, let $\mathcal{N}_{\mathcal{S}^{\tilde{T}'}}$ denote the conormal sheaf associated to the local imbedding $I_{\mathcal{S}^{\tilde{T}'}_{red}} \to \mathcal{S}^{\tilde{T}'}_{red}$; Toen associates to the class $\lambda_{-1}(\mathcal{N}_{\mathcal{S}^{\tilde{T}'}})$ a class $\alpha_{\mathcal{S}^{\tilde{T}'}} \varepsilon \pi_0(K_{\mathrm{et}}(I_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$ which is *invertible*.

Recall that Toen (see [Toe-1] Théorèm 3.15) defines a natural isomorphism

$$\phi_{\mathcal{S}^{\tilde{T}'}}: \pi_*(G(\mathcal{S}^{\tilde{T}'})) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}(\mu_\infty) \to \pi_*(G_{\mathrm{et}}(I_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q}) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(\mu_\infty).$$

(Here $\mathbb{Q}(\mu_\infty) =$ the $\mathbb{Q}$-algebra generated by the roots of unity of the field k; one may choose an imbedding of this into $\mathbb{C}^*$.) In view of the isomorphisms

$$\pi_*(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}')) \cong \mathbb{Z}[M'] \underset{\mathbb{Z}}{\otimes} \pi_*(G(\mathcal{S}^{\tilde{T}'})), \tag{1.0.3}$$

$$\pi_*(G_{\mathrm{et}}(I_{\mathcal{S}^{\tilde{T}'}}, \tilde{T}')) \cong \mathbb{Z}[M'] \underset{\mathbb{Z}}{\otimes} \pi_*(G_{\mathrm{et}}(I_{\mathcal{S}^{\tilde{T}'}})) \tag{1.0.4}$$

this extends to define an isomorphism $\phi_{\mathcal{S}^{\tilde{T}'}}: \pi_*(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}')) \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) \to \pi_*(G_{\mathrm{et}}(I_{\mathcal{S}^{\tilde{T}'}}, \tilde{T}') \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty)$. Moreover, it is shown in [Toe-1] Lemme 4.12 (see also 6.0.18 which shows some of the hypotheses in [Toe-1] may be relaxed) that the composition $\psi_{\mathcal{S}} = \alpha^{-1}_{\mathcal{S}^{\tilde{T}'}} \cap (\quad) \circ \phi_{\mathcal{S}^{\tilde{T}'}}$ commutes with proper push-forward. Assume in addition to the above situation that the prime ideal $\mathfrak{p}$ in $R(T)$ corresponds to the subtorus $\tilde{T}'$. In this case, we prove (see Proposition 6.9 below) that if N is the conormal sheaf associated to the closed immersion $i: \mathcal{S}^{\tilde{T}'}_{red} \to \mathcal{S}$, the class $\lambda_{-1}(N) \varepsilon \pi_0(K(\mathcal{S}^{\tilde{T}'}, \tilde{T}'))_{(\mathfrak{p})}$ is a unit and that the Gysin map $i_*: \pi_*(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}'))_{(\mathfrak{p})} \to \pi_*(G(\mathcal{S}, \tilde{T}'))_{(\mathfrak{p})}$ is an isomorphism with inverse defined by $i^*(\quad) \cap \lambda_{-1}(N)^{-1}$.

Combining the above isomorphisms, we obtain the isomorphism:

$$\pi_*(G(\mathcal{S}, \tilde{T}'))_{(\mathfrak{p})} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}(\mu_\infty) \xrightarrow{i^*(\quad) \cap \lambda_{-1}(N)^{-1}} \pi_*(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}'))_{(\mathfrak{p})} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}(\mu_\infty)$$
$$\xrightarrow{\psi_{\mathcal{S}}} \pi_*(G_{\mathrm{et}}(I_{\mathcal{S}^{\tilde{T}'}}, \tilde{T}') \otimes \mathbb{Q})_{(\mathfrak{p})} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(\mu_\infty) \tag{1.0.5}$$

We will denote this isomorphism by $\Psi_{\mathcal{S}}$.

THEOREM 1.3 (Lefschetz–Riemann–Roch). *Assume that T' is a subtorus of the torus T acting on the smooth Deligne–Mumford stacks $\mathcal{S}$ and $\mathcal{S}'$ and that $f: \mathcal{S}' \to \mathcal{S}$ is a T'-equivariant proper map of finite cohomological dimension. Let $\tilde{T}' \to T'$ denote a finite étale cover so that $\tilde{T}'$ acts trivially on the stack $\mathcal{S}'^{\tilde{T}'}$ and $\mathcal{S}^{\tilde{T}'}$. Let $i: \mathcal{S}^{\tilde{T}'} \to \mathcal{S}$ and $i': \mathcal{S}'^{\tilde{T}'} \to \mathcal{S}'$ denote the associated closed immersions. Then the following diagram commutes:*

$$\begin{tikzcd}
\pi_*(G(\mathcal{S}', \tilde{T}')) \arrow[dr] \arrow[dd, "f_*"] & & & \pi_*(G_{et}(I_{\mathcal{S}'\tilde{T}'}, \tilde{T}')) \arrow[dl] \arrow[dd, "f^{I,\tilde{T}'}"] \\
& \pi_*(G(\mathcal{S}', \tilde{T}'))_{(\mathfrak{p})} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}(\mu_\infty) \arrow[r, "\Psi_{\mathcal{S}'}", "\cong"'] \arrow[dd, "f_*"] & \pi_*(G_{et}(I_{\mathcal{S}'\tilde{T}'}, \tilde{T}') \otimes \mathbb{Q})_{(\mathfrak{p})} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(\mu_\infty) \arrow[dd, "f_*^{I,\tilde{T}'}"] & \\
\pi_*(G(\mathcal{S}, \tilde{T}')) \arrow[dr] & & & \pi_*(G_{et}(I_{\mathcal{S}\tilde{T}'}, \tilde{T}')) \arrow[dl] \\
& \pi_*(G(\mathcal{S}, \tilde{T}'))_{(\mathfrak{p})} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}(\mu_\infty) \arrow[r, "\Psi_{\mathcal{S}}", "\cong"'] & \pi_*(G_{et}(I_{\mathcal{S}\tilde{T}'}, \tilde{T}') \otimes \mathbb{Q})_{(\mathfrak{p})} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(\mu_\infty) &
\end{tikzcd}$$

COROLLARY 1.4. (i) *Let $\mathcal{S}'$ denote a smooth Deligne–Mumford stack that is provided with a proper map $f: \mathcal{S}' \to X$ of finite cohomological dimension where X is a regular scheme. Assume $\mathcal{S}'$ is provided with the action of a torus T, T' is subtorus and that the map f is T'-equivariant for the trivial action of T' on X. Assume further that X has an ample family of line bundles, so that one obtains the weak equivalence $G(X) \simeq K(X)$. Let F denote a T'-equivariant coherent sheaf on the stack $\mathcal{S}'$. Now we obtain the equality in $\pi_0(K(X, \tilde{T}'))_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \otimes \mathbb{Q}(\mu_\infty) \cong \pi_0(K_{\mathrm{et}}(X, \tilde{T}'))_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty)$:*

$$Rf_*(F) = \Sigma_i(-1)^i R^i f_* F = \Sigma_i(-)^i R^i f_*^{I, \tilde{T}'}(\Psi_{\mathcal{S}'}(F)). \tag{1.0.6}$$

(ii) *Taking $X = \operatorname{Spec} k$, we obtain*

$$\Sigma_i(-1)^i H^i(\mathcal{S}'; F) = \Sigma_i(-1)^i H^i(I_{\mathcal{S}'\tilde{T}'}, \Psi_{\mathcal{S}'}(F)) \tag{1.0.7}$$

in the ring $R(\tilde{T}')_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty)$.

As has been noticed for sometime now, there is close connection between equivariant algebraic topology in the sense introduced by Bredon and studied extensively by May *et al.* and the cohomology theory of algebraic stacks (see, for example, [Vi] or [Toe-1]). This was explained very nicely in [T-3] and we recall this in Section 2 of the paper. We hope this serves to nicely explain the leading ideas of this paper, in a rather elementary manner. In Section 3, we define the isovariant étale site and study it in great detail, concluding with Theorems 3.13, 3.26 and 3.27. These show that the isovariant étale site of an algebraic stack is a good substitute for the étale topology of its coarse moduli space: the main advantage is that the isovariant étale site is defined for all algebraic stacks irrespective of whether a coarse moduli space exists or not.

In the fourth section we define and study hyper-cohomology on the isovariant étale site with respect to presheaves of spectra. Section 5 is devoted to cohomological descent on the isovariant étale site. The main results are Theorem 5.10 and Proposition 5.15: Theorem 5.10 provides cohomological descent for suitable localizations of G-theory which may be viewed as variants of topological G-theory. Proposition 5.15 provides the identification of the stalks of the topological G-theory presheaf on the isovariant étale site and finds application in the proof of the Lefschetz–Riemann–Roch. The last section discusses several forms of Riemann–Roch as a natural transformation between G-theory and suitable topological G-theory and concludes with a Lefschetz–Riemann–Roch for the actions of tori on Deligne–Mumford stacks.

In a sequel to this paper, we define cohomology and homology theories generalizing those of Bredon (i.e. Bredon-style equivariant theories as in [Br], [LMS]) on the isovariant étale site of algebraic stacks. In the case where the stack has finite diagonal (observe that these are in general Artin stacks), we obtain Riemann–Roch and Lefschetz–Riemann–Roch theorems in this setting.

2. Equivariant Algebraic Topology

First of all, one needs to point out that there are two distinct notions of equivariant cohomology theories, one originally due to Bredon (see [Br], [LMS]) and another due to Borel (see [Bo], [Hs]). Though the latter is a coarser invariant, it is easier to define and this often accounts for its popularity. In fact, in the algebraic setting (i.e. for studying algebraic group actions on schemes) no one has even defined an analogue of the former theory. A key difference between the two types of theories can be seen in the definition of a map to be a weak homotopy equivalence. Let X denote a G-space where G is a compact topological group. In the Bredon style theories, one defines the *G-topology on X* with the closed subsets of X given by G-stable closed subspaces of X. The points in this topology therefore correspond to the orbits of G on X, all of which are closed since the group G is compact. One may readily see that, therefore, the G-topology on X is equivalent to the topology on the quotient space X/G. In Borel style theories, one defines a simplicial space $EG\times_G X$, then takes its realization, $|EG\times_G X|$, to obtain a space and defines the topology to be the topology on the above realization.

The difference between the two is clearly seen in the definition of equivariant K-theory. The Atiyah–Segal equivariant K-theory of X is the Grothendieck group of the category of all G-equivariant vector bundles on X. This is a Bredon style theory, since it is defined only on G-stable subsets of X and a map $f\colon X \to Y$ between two G-spaces induces an isomorphism on Atiyah–Segal G-equivariant K-theory, in general, only if there is a G-equivariant map $g\colon Y \to X$ and G-equivariant homotopy equivalences $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. On the other hand one may consider $K^0(|EGX|)$. This is a Borel style equivariant cohomology theory. A G-equivariant map $f\colon\ X \to Y$ induces an isomorphism on these groups, if there is a map $g\colon\ Y \to X$, not necessarily G-equivariant, so that the compositions $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$ by homotopies that are once again not necessarily G-equivariant. Moreover, one knows that the Borel-style equivariant K-theory of X is the completion of the Atiyah–Segal equivariant K-theory of X (see [A.S2]) and is therefore a coarser invariant of X.

Next one considers the definition of equivariant cohomology in the sense of Bredon. We may define this concisely as follows. (The definitions in [Br] and [LMS] are essentially equivalent to this, though the definitions seem a bit more complicated as they are not stated in terms of sheaf cohomology.) First, define a presheaf $\mathcal{R}^G$: $G - topology \quad of \quad X \to (Abelian \quad groups)$ by $\Gamma(U, \mathcal{R}^G) = K^0_G(U) =$ the G equivariant Atiyah–Segal K-theory of U. One may observe that if G/H is a point on the above topology of X, the stalk $\mathcal{R}^G_{G/H} \cong R(H)$, at least for suitable X. Given an Abelian presheaf P on the G-topology of X, one defines the Bredon equivariant cohomology of X, $H^*_{G,Br}(X; P) = R\Gamma(X, (P \otimes \mathcal{R})\tilde{})$ where $\tilde{}$ denotes the functor sending a presheaf to its associated sheaf and $R\Gamma(X, \quad)$ denotes the derived functor of the global section functor computed on the G-topology of X. So defined, $H^*_{G,Br}(X; P)$ is a module over $K^0_G(X)$ and hence over $R(G)$. Our procedure for defining Bredon style equivariant cohomology may be therefore summarized as follows: define a topology where the open sets are G-stable open sets and modify the Abelian presheaf P on this site by the sheaf $\mathcal{R}^G$ that contains information on the representations of G. One may now contrast this with the definition of the usual G-equivariant cohomology of X (which is a Borel style equivariant cohomology). Let P denote an Abelian presheaf on the simplicial space $EG \times_G X$. Then one defines $H^*_G(X; P) = R\Gamma(EG \times_G X, P)$. This is a module over $H^*_G(X, \mathbb{Z})$ and, hence, over $H^*(BG; \mathbb{Z})$.

Finally consider the case where G is a group scheme acting on a scheme X. One runs into various difficulties, if one tries to define a Bredon style equivariant étale cohomology in this setting. The main difficulties are in the definition of the G-topology. The discussion in [T-3], Section 2, shows how to define an appropriate topology in this setting so that the definition of a Bredon style equivariant étale cohomology is still possible. Guided by this example, we define and study a site (or topology) for any Artin stack in the next section which may be used to define a finer variant of the cohomology of a stack.

3. The Isovariant Étale Site

3.0.8. BASIC FRAMEWORK

Let S denote a Noetherian separated scheme which will serve as the base scheme. All objects (i.e. schemes, algebraic spaces and algebraic stacks) we consider will be *defined over the base scheme S and locally Noetherian*. In particular, they are all locally quasi-compact. Fibered products over the base scheme will be often denoted

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just as a product. (For the most part we may restrict to finitely presented objects over the base scheme S; but it will often be necessary to consider filtered inverse limits of such objects with affine structure maps for the inverse system.)

Let $\mathcal{S}$ denote an algebraic stack. We define and study several new sites associated to stacks in this section. Given an algebraic stack $\mathcal{S}$, recall the *inertia stack* $I_{\mathcal{S}}$ associated to $\mathcal{S}$ is defined by the fibered product $\mathcal{S} \times_{\Delta, \mathcal{S}\times\mathcal{S}, \Delta} \mathcal{S}$. Since $\Delta\colon\ \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable, so is the obvious induced map $I_{\mathcal{S}} \to \mathcal{S}$.

DEFINITIONS 3.1. (i) Let $f\colon\ \mathcal{S}' \to \mathcal{S}$ be a map of algebraic stacks. We say f is *isovariant* if the natural map $I_{\mathcal{S}'} \to I_{\mathcal{S}} \times_{\mathcal{S}} \mathcal{S}'$ is a 1−isomorphism, where $I_{\mathcal{S}'}$ ($I_{\mathcal{S}}$) denotes the inertia stack of $\mathcal{S}'$ ($\mathcal{S}$, respectively).

(ii) *The smooth and étale sites*. Given an algebraic stack $\mathcal{S}$, we let $\mathcal{S}_{smt}$ ($\mathcal{S}_{\underline{smt}}$) denote the site whose objects are smooth maps $u\colon\ \mathcal{S}' \to \mathcal{S}$ of algebraic stacks (smooth maps $u\colon\ U \to \mathcal{S}$ with U an algebraic space). Given two such objects $u\colon\ \mathcal{S}' \to \mathcal{S}$ and $v\colon\ \mathcal{S}'' \to \mathcal{S}$, a morphism $u \to v$ is a commutative triangle of stacks

$$\begin{tikzcd} \mathcal{S}' \arrow[rr, "\phi"] \arrow[dr, "u"'] & & \mathcal{S}'' \arrow[dl, "v"] \\ & \mathcal{S} & \end{tikzcd}$$

(i.e. There is *given* a 2-isomorphism $\alpha\colon u \to v \circ \phi$.) The site $\mathcal{S}_{\mathrm{et}}$ is the full subcategory of $\mathcal{S}_{smt}$ consisting of étale representable maps $u\colon \mathcal{S}' \to \mathcal{S}$, where $\mathcal{S}'$ is an algebraic stack. Finally, when $\mathcal{S}$ is a Deligne–Mumford stack, $\mathcal{S}_{\underline{et}}$ will denote the full subcategory of $\mathcal{S}_{\mathrm{et}}$ consisting of étale maps $u\colon U \to \mathcal{S}$ with U an algebraic space as objects.

(iii) *The isovariant étale and smooth sites*. If $\mathcal{S}$ is an algebraic stack, $\mathcal{S}_{iso.et}$ will denote the full subcategory of $\mathcal{S}_{\mathrm{et}}$ consisting of (representable) maps $u\colon \mathcal{S}' \to \mathcal{S}$ that are also *isovariant*. $\mathcal{S}_{iso.smt}$ is defined similarly as a full subcategory of $\mathcal{S}_{smt}$. For the most part we will only consider the site $\mathcal{S}_{iso.et}$. (It follows from the lemma below that these indeed define pretopologies (or sites) in the sense of Grothendieck.)

(iv) We will consider sheaves on any of the above sites with values in the category of Abelian groups, or modules over a ring, etc. If $\mathcal{C}$ is any one of the above sites, we will denote the corresponding category of sheaves on $\mathcal{C}$ by $\mathrm{Sh}(\mathcal{C})$.

LEMMA 3.2. (i) *Isovariant maps are representable.*

(ii) *Isovariant maps are stable by base-change and composition.*

Proof. (i) Let $f\colon \mathcal{S}' \to \mathcal{S}$ denote an isovariant map. Let $\phi\colon V \to U$ denote a map of schemes and let $y \in \mathrm{ob}(\mathcal{S}_U)$. To prove (i), it suffices to show that for each such pair (ϕ, y), the category $\mathcal{S}'_{y,V}$ whose objects are pairs $(x \in \mathrm{ob}(\mathcal{S}'_V),\ g \in \mathrm{Hom}_{\mathcal{S}_V}(f(x), \phi^*(y))$ and where a morphism $(x_1, g_1) \to (x_2, g_2)$ is a morphism $h\colon x_1 \to x_2$ *in* $\mathcal{S}'_V$ so that $g_1 = g_2 \circ f(h)$ is *discrete*. Let $h_1, h_2\colon x_1 \to x_2$ denote two such morphisms. We will show that $h_2 = h_1$. Observe that $f(h_1) = g_2^{-1} \circ g_1 = f(h_2)$ and therefore $f(h_2^{-1}) \circ f(h_1) = f(h_2^{-1} \circ h_1) = \mathrm{id}$; since f induces an isomorphism on the inertia stacks, it follows that $h_2^{-1} \circ h_1 = \mathrm{id}$, i.e. $h_2 = h_1$. This proves the category $\mathcal{S}'_{y,V}$ is equivalent to a set, i.e. is a discrete category (i).

(ii) Recall the inertia stack $I_{\mathcal{S}} = \mathcal{S} \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ where both the maps $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ are the diagonal maps. Now one may show readily that an atlas for $I_{\mathcal{S}} =$ the equalizer of the two maps

$$X \underset{\mathcal{S}}{\times} X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$$

where $x\colon X \to \mathcal{S}$ is an atlas for the stack $\mathcal{S}$. Since equalizers are preserved by pullbacks it follows readily that isovariant maps are stable under base-change. It is clear that isovariant maps are also stable under composition. $\square$

EXAMPLE 3.3 (Quotient stacks). Let G denote a smooth group scheme acting on an algebraic space X. The objects of $[X/G]_{iso.et}$ may be identified with maps $u\colon U \to X$ where U is an algebraic space provided with a G-action so that u is étale and induces an isomorphism on the isotropy groups. Observe that any representable map $\mathcal{S}' \to [X/G]$ of algebraic stacks may identified with a G-equivariant map $u\colon U \to X$, with U an algebraic space. The iso-variance forces isomorphism of the isotropy subgroups.

DEFINITION 3.4 (see [L-MB] (1.4.3)). An *algebraic groupoid* $\mathcal{X}$ consists of a triple (X_0, X_1, X_2) of algebraic spaces provided with the following data:

(i) maps $s, t\colon X_1 \to X_0$ ($s =$ the *source*, $t =$ the *target*), $X_2 = X_1 \times_{s,X_0,t} X_1$

(ii) a map $m\colon X_1 \times_{s,X_0,t} X_1 \to X_1$ which is associative in the obvious sense (which we call the *groupoid law*)

(iii) a map $e\colon X_0 \to X_1$ so that the composition $s \circ e = \mathrm{id}_{X_0} = t \circ e$, a map $in : X_1 \to X_1$ so that, $in^2 = \mathrm{id}_{X_1}$, $s \circ in = t$, $t \circ in = s$, $t \circ m = s \circ pr_2$ and $s \circ m = t \circ pr_1$. (Observe that, since $in^2 = \mathrm{id}_{X_1}$, in must be an isomorphism.) Moreover

(iv) $m \circ (\mathrm{id}_{X_1} \times e) = m \circ (e \times \mathrm{id}_{X_1}) = \mathrm{id}_{X_1}$, $m \circ (in \times \mathrm{id}) = e \circ s$ and $m \circ (\mathrm{id} \times in) = e \circ t$.

DEFINITION 3.5. Let χ denote an algebraic groupoid. Given an algebraic space $y\colon Y \to X_0$, a left-action of the algebraic groupoid χ on Y is given by an isomorphism $\Phi\colon X_1 \times_{s,X_0,y} Y \cong X_1 \times_{t,X_0,y} Y$ so that $\mathcal{Y} = (Y, X_1 \times_{s,X_0,y} Y)$ with

$$s_Y = s \times \mathrm{id}_Y, \qquad t_Y = t \times \mathrm{id}_Y, \qquad e_Y = e \times \mathrm{id}_Y, \qquad in_Y = in \times \mathrm{id}_Y,$$

$$m_Y\colon X_1 \underset{s,X_0,y}{\times} Y \underset{Y}{\times} X_1 \underset{s,X_0,y}{\times} Y \cong X_1 \underset{s,X_0,t}{\times} X_1 \underset{s,X_0,y}{\times} Y \to X_1 \underset{s,X_0,y}{\times} Y = m \times \mathrm{id}_Y$$

defines an algebraic groupoid. We say $y\colon Y \to X_0$ has *trivial action* by the groupoid if the following conditions are satisfied: $X_1 \times_{s,X_0,y} Y = X_1 \times_{t,X_0,y} Y$ and the isomorphism $\Phi = \mathrm{id}$. (See Proposition 3.7 for a some what different explanation of groupoid actions. The above definition of an action being trivial, though sufficient for our purposes (since we consider triviality for actions only by inertia groupoids) is not the most general.)

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3.1. CLASSIFYING SIMPLICIAL ALGEBRAIC SPACES

3.1.1. *Sites*

Let $\mathcal{S}$ denote an algebraic stack and let $x\colon X \to \mathcal{S}$ denote an atlas. Let $B_x\mathcal{S}$ denote the classifying simplicial algebraic space associated to x: i.e. $(B_x\mathcal{S})_n = (\mathrm{cosk}_0^{\mathcal{S}} X)_n = X \times_{\mathcal{S}} X \cdots \times_{\mathcal{S}} X$. One defines the *small* smooth (étale) site of $B_x\mathcal{S}$ as in [Fr], p. 7. Recall each object in this site will be an object in the smooth (étale site) of some $(B_x\mathcal{S})_n$ for some n and a morphism between two such objects will be a map lying over some structure map of $B_x\mathcal{S}$. We will denote these sites by $B_x\mathcal{S}_{smt}$ ($B_x\mathcal{S}_{\mathrm{et}}$, respectively). The corresponding *big* sites will be denoted $\mathrm{SMT}(B_x\mathcal{S})$ ($\mathrm{ET}(B_x\mathcal{S})$, respectively). Recall that an object in the corresponding big site consists of an object U in $\mathrm{SMT}(B_x\mathcal{S}_n)$ ($\mathrm{ET}(B_x\mathcal{S}_n)$) for some fixed integer n with morphisms between two such objects defined as morphisms lying over some structure map of the simplicial space $B_x\mathcal{S}$. Coverings are defined in the obvious manner and coincide in the small and the corresponding big sites.

3.1.2. *Topoi*

Given a site as above associated to a simplical algebraic space X, a sheaf F on X in the above site will be given by a collection $F = \{F_n|n\}$ of sheaves F_n on the corresponding site of X_n along with maps $\Phi_\alpha\colon \alpha^*(F_n) \to F_m$ for any structure map $\alpha\colon X_m \to X_n$. Moreover, the maps $\{\Phi_\alpha|\alpha\}$ are required to satisfy an obvious compatibility condition. The category of all sheaves of sets on the small smooth site (the small étale site, the big smooth site, the big étale site) of X will be denoted $\mathrm{Sh}_{sets}(X_{smt})$ ($\mathrm{Sh}_{sets}(X_{\mathrm{et}})$, $\mathrm{Sh}_{sets}(SMT(X))$, $\mathrm{Sh}_{sets}(ET(X))$, respectively). A sheaf $F = \{F_n|n\}$ on a simplicial space X has *descent* if the maps Φ_α are all isomorphisms. The category of sheaves with descent forms a *full* subcategory closed under extensions. For example, the category of sheaves of sets with descent on the small smooth site will be denoted $\mathrm{Sh}^{des}_{sets}(X_{smt})$. If $\mathcal{C}$ is any of the above sites, $\mathrm{Presh}_{sets}(\mathcal{C})$ will denote the corresponding category of presheaves of sets.

3.1.3. The above discussion also applies to truncated simplicial algebraic spaces and in particular to algebraic groupoids. Given an algebraic groupoid χ, one defines the associated small (big) smooth and étale sites as the corresponding sites of the truncated simplicial space consisting of the X_0, X_1 and $X_2 = X_1 \times_{X_0} X_1$ along with the given structure maps between them. A sheaf on such a site will consist of a collection of sheaves $F = \{F_n|n = 0, 1, 2\}$, with F_i on X_i along with structure maps $\{\Phi_\alpha\colon \alpha^*(F_n) \to F_m|\alpha\}$ as above. For example, the category of sheaves of sets on the small étale site of χ will be denoted $\mathrm{Sh}_{sets}(\chi_{\mathrm{et}})$. The corresponding *full* sub-category of sheaves with descent will be denoted $\mathrm{Sh}^{des}_{sets}(\chi_{\mathrm{et}})$. ($\mathrm{Presh}^{des}_{sets}(\chi_{\mathrm{et}})$ will denote the full subcategory of $\mathrm{Presh}_{sets}(\chi_{\mathrm{et}})$ where the corresponding maps Φ_α are isomorphisms of presheaves.)

3.1.4. Observe that there exists an equivalence

$$\mathrm{Sh}_{\mathrm{sets}}(X_{0,\mathrm{et}}) \simeq \begin{matrix}(\text{algebraic spaces etale and locally})\\(\text{of finite type over } X_0)\end{matrix}$$

This extends to an equivalence

$$\mathrm{Sh}^{\mathrm{des}}_{\mathrm{sets}}((B_x\mathcal{S})_{\mathrm{et}}) \simeq \begin{matrix}(\text{algebraic spaces Y in } (\mathrm{X}_0)_{\mathrm{et}})\\(\text{with an action by the groupoid } \chi)\end{matrix}$$

Remark 3.6. All the above definitions apply to Abelian sheaves or sheaves of R-modules, where R is a commutative ring. However, for the most part, we will be concerned with the topoi of sheaves of sets. We will also consider mostly the étale sites.

3.1.5. One may obtain the following alternate description of sheaves with descent on the *big* étale site of an algebraic groupoid χ. Let χ denote an algebraic groupoid. A sheaf F of sets on $ET(X_0)$ has an action by χ if there is a given a pairing: μ: $X_1 \times_{s,X_0,f} F \to F$ (where f: $F \to X_0$ is the obvious structure map) which makes the square

$$\begin{tikzcd} X_1 \underset{s,X_0,f}{\times} F \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_1"'] & F \arrow[d, "f"] \\ X_1 \arrow[r, "t"] & X_0 \end{tikzcd}$$

Cartesian and which is associative in the sense that the diagram

$$\begin{tikzcd} X_1 \underset{s,X_0,t}{\times} X_1 \underset{s,X_0,f}{\times} F \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "m\times\mathrm{id}"'] & X_1 \underset{s,X_0,f}{\times} F \arrow[d, "\mu"] \\ X_1 \underset{s,X_0,f}{\times} F \arrow[r, "\mu"] & F \end{tikzcd}$$

commutes. (Here we view X_i as the obvious sheaves represented by the algebraic spaces X_i, $i = 1, 2$.) We denote this full subcategory of sheaves of sets on $\mathrm{ET}(X_0)$ by $\mathrm{Sh}^{\chi}_{\mathrm{sets}}(\mathrm{ET}(X_0))$.

PROPOSITION 3.7. *There exists an equivalence* $\mathrm{Sh}^{\mathrm{des}}_{\mathrm{sets}}(\mathrm{ET}(\chi)) \simeq \mathrm{Sh}^{\chi}_{sets}(\mathrm{ET}(X_0))$.

Proof. Let $F = (F_0, F_1, F_2, \Phi)$ denote a sheaf of sets with descent on $\mathrm{ET}(\chi)$. Let f_0: $F_0 \to X_0$ denote the given map. Now one obtains the diagram

$$\begin{tikzcd} F_0 \arrow[d] & pr_2^*(F_0) \arrow[l] \arrow[r, "\Phi"] \arrow[d] & \mu^*(F_0) \arrow[r] \arrow[dl] & F_0 \arrow[dl] \\ X_0 & X_1 \arrow[l, "s"] \arrow[r, "t"'] & X_0 & \end{tikzcd}$$

where the first and last squares are Cartesian. Given a scheme Y over X_0 and maps $\alpha\colon Y \to F_0$, $g_0\colon Y \to X_0$ and $g_1\colon Y \to X_1$ so that $s \circ g_1 = g_0$ and $f_0 \circ \alpha = g_0$, one defines the map $\mu(g_1, g_0, \alpha)\colon Y \to F_0$ by first taking the induced map $Y \to pr_2^*(F_0) \overset{\Phi}{\to} \mu^*(F_0)$ and then following it by the map $\mu^*(F_0) \to F_0$ forming the last map in the top row of the above diagram. Now the associativity condition above follows from the co-cycle condition on the isomorphism Φ. Conversely maps $Y \to X_1 \times_{s,X_0,f} F$ correspond under the action μ to a unique map $Y \to X_1 \times_{t,X_0,f} F$ thereby providing an isomorphism $pr_2^*(F) \cong \mu^*(F)$. The associativity of the action will provide the necessary co-cycle conditions. $\square$

PROPOSITION 3.8. *Let $\mathcal{S}$ denote an algebraic stack, $x\colon X \to \mathcal{S}$ an atlas, $\chi =$ the associated algebraic groupoid and $B_x\mathcal{S} =$ the associated classifying simplicial algebraic space.*

(i) *There exist maps $\bar{x}\colon (B_x\mathcal{S})_{\mathrm{et}} \to \mathcal{S}_{\mathrm{et}}$ and $\tilde{x}\colon \chi = \mathrm{tr}_2(B_x\mathcal{S})_{\mathrm{et}} \to \mathcal{S}_{\mathrm{et}}$ of sites*
(ii) *One obtains an equivalence of categories:*
$\mathrm{Sh}_{sets}(\mathcal{S}_{smt}) \simeq \mathrm{Sh}^{des}_{sets}(\chi_{smt}) \simeq \mathrm{Sh}^{des}_{sets}(B_x\mathcal{S}_{smt}) \simeq \mathrm{Sh}^{des}_{sets}(B_x\mathcal{S}_{\mathrm{et}}) \simeq \mathrm{Sh}^{des}_{sets}(\chi_{\mathrm{et}})$.

(*Here* tr_2 *denotes the truncation of the classifying simplicial algebraic space $B_x\mathcal{S}$ above degree* 2.)

Proof. The first assertion is clear. The first equivalence in (ii) is provided by descent theory, while the second follows readily by the identities relating the compositions of the structure maps of the simplicial space $B_x\mathcal{S}$. For an algebraic space, any smooth cover has a refinement by an étale cover. Therefore, if $\epsilon\colon \chi_{smt} \to \chi_{\mathrm{et}}$ is the obvious map of sites, $\epsilon_* \circ \epsilon^*$ is naturally isomorphic to the identity showing ϵ^* is fully-faithful. One may also show the composition $\epsilon^* \circ \epsilon_*$ is naturally isomorphic to the identity showing the functor ϵ^* is an equivalence. $\square$

PROPOSITION 3.9. *Let $\mathcal{S}$ denote an algebraic stack, $x\colon X \to \mathcal{S}$ an atlas and $B_x\mathcal{S}$ the corresponding simplicial algebraic space. Then there exists a map of simplicial algebraic spaces $\pi_\bullet\colon B(X \times_{\mathcal{S}} I_{\mathcal{S}}) \to B_x\mathcal{S}$ where the first is the classifying simplicial algebraic space associated to the group-scheme $X \times_{\mathcal{S}} I_{\mathcal{S}}$ over X.*

Proof. Observe that $B(X \times_{\mathcal{S}} I_{\mathcal{S}})_n =$ the n-fold fibered product of $X \times_{\mathcal{S}} I_{\mathcal{S}}$ over X. If $n = 0$, this is just X and in this case the map $\pi_0 = \mathrm{id}_X$. Both the structure maps

$$B(X \underset{\mathcal{S}}{\times} I_{\mathcal{S}})_1 = X \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \to X = B(X \underset{\mathcal{S}}{\times} I_{\mathcal{S}})_0$$

are the same and are given by the projection to the first factor. Observe that if T is scheme,

$$X \underset{\mathcal{S}}{\times} X(T) = \{(\psi_1, \psi_2, \phi) | \psi_i \in X(T), \quad \phi\colon x(\psi_1) \overset{\cong}{\to} x(\psi_2) \quad \text{in} \quad S(T)\}$$

while

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$$X\underset{\mathcal{S}}{\times}I_{\mathcal{S}}(T) = \{(\psi'_1, (\psi'_{20}, \psi'_{21}), \eta) | \psi'_1 \in X(T), \quad \psi'_{20} \in \mathcal{S}(T), \quad \psi'_{21} \in \mathrm{Aut}(\psi'_{20}),$$
$$\eta\colon x(\psi'_1) \overset{\cong}{\to} \psi'_{20} \quad \text{in} \quad \mathcal{S}(T)\}.$$

Now $\pi_1(\psi'_1, (\psi'_{20}, \psi'_{21}), \eta) = (\psi'_1, \psi'_1, \eta^{-1} \circ \psi'_{21} \circ \eta)$. The remaining maps $\{\pi_n | n \geqslant 1\}$ are defined similarly and one may readily verify that the maps π_n commute with the structure maps of the simplicial algebraic spaces. □

PROPOSITION 3.10. *Let $\mathcal{S}$ denote an algebraic stack and let F denote a sheaf on $\mathcal{S}_{smt}$. Let $\epsilon\colon I_{\mathcal{S}} \to \mathcal{S}$ denote the obvious map, let μ, pr_1, $pr_2\colon I_{\mathcal{S}} \times_{\mathcal{S}} I_{\mathcal{S}} \to I_{\mathcal{S}}$ denote the group action and the obvious projections and let $e\colon \mathcal{S} \to I_{\mathcal{S}}$ denote the unit.*

(i) *Let $x\colon X \to \mathcal{S}$ denote an atlas for $\mathcal{S}$. The map $\pi = \pi_1\colon X \times_{\mathcal{S}} I_{\mathcal{S}} \to X \times_{\mathcal{S}} X$ makes the triangle commute.*

$$\begin{tikzcd} X \underset{\mathcal{S}}{\times} X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \\ & X \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \arrow[ul, "\pi"] \arrow[u, "pr_1"'] \end{tikzcd}$$

(ii) *Let $\bar{x}\colon X \times_{\mathcal{S}} I_{\mathcal{S}} \to I_{\mathcal{S}}$ denote the obvious map induced by x. Let $\bar{\mu}$, $\overline{pr}_1$, $\overline{pr}_2\colon X \times_{\mathcal{S}} I_{\mathcal{S}} \times_{\mathcal{S}} I_{\mathcal{S}} \to X \times_{\mathcal{S}} I_{\mathcal{S}}$ denote the obvious maps induced by μ, pr_1 and pr_2. Let $\bar{e}\colon X \to X \times_{\mathcal{S}} I_{\mathcal{S}}$ denote the map induced by e. Then there exists an isomorphism $\phi: \bar{x}^*\epsilon^*(F) \to \bar{x}^*\epsilon^*(F)$ satisfying a cocycle condition between the pull-backs by $\overline{\mu}$, $\overline{pr}_1$ and $\overline{pr}_2$ and so that the pull-back by $\bar{e}$ is the identity.*

Proof. Let $x\colon X \to \mathcal{S}$ denote an atlas for $\mathcal{S}$. The last proposition shows we obtain the commutative diagram:

$$\begin{tikzcd} X \underset{\mathcal{S}}{\times} X \underset{\mathcal{S}}{\times} X \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & X \underset{\mathcal{S}}{\times} X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \\ X \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \arrow[u, "\tilde{pr}"] \arrow[r, shift left=2, "d'_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d'_2"'] & X \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \arrow[u, "\pi"] \arrow[r, "pr_1"] & X \arrow[u, "id"'] \end{tikzcd} \tag{3.1.6}$$

The maps d_i are the obvious maps of the simplicial algebraic spaces above. Now there exists an isomorphism $\phi\colon p_1^* x^*(F) \to p_2^* x^*(F)$ satisfying an obvious co-cycle condition. Consider $\pi^*(\phi)$. Observe that $\pi^* p_1^* x^*(F) = pr_1^* x^*(F) = \bar{x}^*\epsilon^*(F)$ and similarly $\pi^* p_2^* x^*(F) = pr_1^* x^*(F) = \bar{x}^*\epsilon^*(F)$. Therefore, $\pi^*(\phi)$ defines an isomorphism $\bar{x}^*\epsilon^*(F) \to \bar{x}^*\epsilon^*(F)$. Moreover, the commutative diagram on the left provides the cocycle condition between the three pull-backs of this to $X \times_{\mathcal{S}} I_{\mathcal{S}} \times_{\mathcal{S}} I_{\mathcal{S}}$. The map $\bar{e}$ is a section to pr_1 and if $\delta\colon X \to X \times_{\mathcal{S}} X$ is the diagonal, $\delta = \pi \circ \bar{e}$. □

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EXAMPLE 3.11. Let $\mathcal{S} = [X/G]$ where G is a finite group acting on a scheme. Now $X \times_{\mathcal{S}} I_{\mathcal{S}} = \cup_{x \in X} G_x \times X^{G_x}$ and it is clear that there is an action by $X \times_{\mathcal{S}} I_{\mathcal{S}}$ on any G-equivariant sheaf F on X: in fact this corresponds to a representation of G_x on each stalk F_x.

DEFINITION 3.12. Assume as in the above situation that $\mathcal{S}$ is an algebraic stack, $x: X \to \mathcal{S}$ is a given atlas and $B_x\mathcal{S}$ the associated classifying simplicial algebraic space. We let $\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ denote the *full-sub-category* of $\mathrm{Sh}^{des}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ where the isomorphism ϕ given in the last proposition is the identity. $\mathrm{Sh}^{tr.in}_{sets}(tr_2(B_x\mathcal{S})_{\mathrm{et}})$ is defined similarly. (One defines $\mathrm{Presh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ similarly.)

The following result should be taken as the key to understanding and working with the isovariant étale sites.

THEOREM 3.13. *Assume that $\mathcal{S}$ is finitely presented over the base scheme S, a coarse moduli space $\mathfrak{m}$ exists (as an algebraic space) for the stack $\mathcal{S}$ and that $\mathcal{S}$ is a gerbe over* $\mathfrak{M}$. (i) *Then the functor $V \mapsto V \times_{\mathfrak{M}} \mathcal{S}$, $\mathfrak{M}_{\mathrm{et}} \to \mathcal{S}_{iso.et}$ is an equivalence of sites.* (ii) *Let $m: \mathcal{S} \to \mathfrak{m}$ denote the obvious map. Then the functor $F \mapsto \bar{x}^* m^*(F)$ defines an equivalence of categories* $\mathrm{Sh}(\mathfrak{M}_{\mathrm{et}}) \to \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$

Proof. We will prove the second part of the theorem first. We consider the following commutative diagram

$$\begin{array}{ccccc} X \underset{\mathfrak{M}}{\times} X & \overset{p_1'}{\underset{p_2'}{\rightrightarrows}} & X & \xrightarrow{mox} & \mathfrak{M} \\ \uparrow \eta & & \uparrow id & & \uparrow m \\ X \underset{\mathcal{S}}{\times} X & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & X & \xrightarrow{x} & \mathcal{S} \end{array} \tag{3.1.7}$$

Let F denote a sheaf on $B_x\mathcal{S}_{\mathrm{et}}$ so that $F = \bar{x}^* m^*(\bar{F})$ for some sheaf $\bar{F}$ on $\mathfrak{M}_{\mathrm{et}}$. Now we retrace our arguments above showing the existence of the isomorphism $\pi^*(\phi): \pi^* p_1^* x^*(F) \to \pi^* p_2^* x^*(F)$ (see (3.1.6).). The key observation is that the composition $X \times_{\mathcal{S}} I_{\mathcal{S}} \to^{\pi} X \times_{\mathcal{S}} X \overset{\eta}{\to} X \times_{\mathfrak{M}} X$ factors as $X \times_{\mathcal{S}} I_{\mathcal{S}} \to^{pr_1} X \to^{\Delta} X \times_{\mathfrak{M}} X$. Since $\Delta^*(\phi)$ is the identity, it follows that so is $\pi^*(\phi)$. This proves that if $\bar{F}$ is a sheaf on $\mathfrak{m}_{\mathrm{et}}$, then $\bar{x}^* m^*(\bar{F}) \in \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$.

To see the converse suppose F is a sheaf on $B_x\mathcal{S}_{\mathrm{et}}$ with descent. Using the notation as in (3.1.6), there exists an isomorphism $\phi: p_1^*(F) \to p_2^*(F)$ satisfying an obvious cocycle condition and whose pull-back by the diagonal to X is the identity. We will first show that there exists an isomorphism $\bar{\phi}: p_1'^*(F) \to p_2'^*(F)$ so that $\phi = \eta^*(\bar{\phi})$. To see this one needs to observe that the map induced by m, $X \times_{\mathcal{S}} X \overset{\eta}{\to} X \times_{\mathfrak{M}} X$ is faithfully-flat. (Since this is local on $\mathfrak{m}$ in the fppf topology, one may readily reduce to the case where the stack is a neutral gerbe in which case the map m and therefore the above induced map has a section. To be precise, consider the pull-back of the stack to X by the map $X \to \mathcal{S} \to \mathfrak{m}$ where $X \to \mathcal{S}$ denotes an atlas. The pull-backed

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stack is a neutral gerbe over X and X is flat over $\mathfrak{M}$.) Therefore, by faithfully-flat descent it suffices to show that $\pi_1^*(\phi) = \pi_2^*(\phi)$ where

$$\pi_i\colon X \underset{\mathcal{S}}{\times} X \underset{\underset{\mathfrak{M}}{X \times X}}{\times} X \underset{\mathcal{S}}{\times} X \to X \underset{\mathcal{S}}{\times} X$$

denotes the projection to the ith factor. (Recall that faithfully flat maps between algebraic spaces satisfy the following condition (see [Mur] p.121, p.124): let $X \to Y$ denote a faithfully flat map and which is also locally of finite type between algebraic spaces. Then a map $f\colon X \to X$ descends to a map $g\colon Y \to Y$ if $\pi_1^*(f) = \pi_2^*(f)$, where $\pi_i\colon X \times_Y X \to X$ is the projection to the ith factor.) Observe from the diagram (3.1.7) that $p_i = p_i' \circ \eta$, $i = 1, 2$. Therefore, $\pi_i^*(\phi)\colon \pi_i^*\eta^* p_1'^*(F) \to \pi_i^*\eta^* p_2'^*(F)$. Now $\eta \circ \pi_1 = \eta \circ \pi_2$. It follows therefore that both $\pi_1^*(\phi)$ and $\pi_2^*(\phi)$ map $\pi_1^*\eta^* p_1'^*(F)$ to $\pi_1^*\eta^* p_2'^*(F)$. Recall that the fibres of π_i are the orbits of $X \times_{\mathcal{S}} I_{\mathcal{S}}$ and that the map $\pi_1^*(\phi)$ is an equivariant map between two equivariant sheaves for the action of the group-scheme $X \times_{\mathcal{S}} I_{\mathcal{S}}$. Therefore, it suffices to show that the maps $\pi_1^*(\phi)$ and $\pi_2^*(\phi)$ agree at the stalk at a point in each fibre. Since the maps π_i have a section, namely the diagonal map, it follows that this is indeed the case. Therefore, $\pi_1^*(\phi) = \pi_2^*(\phi)$ and therefore there exists a map $\bar{\phi}\colon p_1'^*(F) \to p_2'^*(F)$ so that $\phi = \eta^*(\bar{\phi})$.

Observe that the projection $pr_1\colon X \times_{\mathcal{S}} I_{\mathcal{S}} \to X$ is faithfully flat by the hypotheses and that $\eta \circ \pi = \Delta \circ pr_1$. The hypothesis that $F \in \mathrm{Sh}^{tr.in}(B_x \mathcal{S}_{\mathrm{et}})$ implies that the isomorphism $\pi^*\eta^*(\bar{\phi})$ is the identity. But $pr_1^*\Delta^*(\bar{\phi}) = \pi^*\eta^*(\bar{\phi})$ and pr_1 is faithfully flat; therefore, $\Delta^*(\bar{\phi})$ itself is the identity. The faithful flatness of η readily implies that the pull-backs of $\bar{\phi}$ to $X \times_{\mathfrak{M}} X \times_{\mathfrak{M}} X$ satisfy the required co-cycle condition. This completes the proof of the second part of the theorem.

Now we consider the first part. Observe that any isovariant étale map $\mathcal{S}' \to \mathcal{S}$ in $\mathcal{S}_{iso.et}$ is a representable étale map. We will show that $\mathcal{S}' = \mathfrak{M}' \times_{\mathfrak{M}} \mathcal{S}$ for some étale map $\mathfrak{M}' \to \mathfrak{M}$. Let $x\colon X \to \mathcal{S}$ denote an atlas for the stack $\mathcal{S}$ and let $x'\colon X' = X \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{S}'$ denote the induced atlas for $\mathcal{S}'$. Observe that $X \times_{\mathcal{S}} I_{\mathcal{S}}$ is a group scheme over X and that it acts on $X \times_{\mathcal{S}} X$ as in Proposition 3.10 with the geometric quotient being $X \times_{\mathfrak{M}} X$. By isovariance, $I_{\mathcal{S}'} \cong I_{\mathcal{S}} \times_{\mathcal{S}} \mathcal{S}'$ and $X' \times_{\mathcal{S}'} I_{\mathcal{S}'} \cong X' \times_X X \times_{\mathcal{S}} I_{\mathcal{S}}$. Therefore, we obtain the Cartesian square:

$$\begin{tikzcd}
X' \underset{\mathcal{S}'}{\times} X'/(X' \underset{X}{\times} X \underset{\mathcal{S}}{\times} I_{\mathcal{S}'}) \arrow[r] \arrow[d, shift right, "pr_2'"'] \arrow[d, shift left, "pr_1'"] & X \underset{\mathcal{S}}{\times} X/X \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \arrow[d, shift right, "pr_2"'] \arrow[d, shift left, "pr_1"] \\
X' \arrow[r] & X
\end{tikzcd}$$

The two left columns define a flat equivalence relation on X'. (They are flat since they are obtained by base-change from the two right-most columns: now one may identify $X \times_{\mathcal{S}} X/(X \times_{\mathcal{S}} I_{\mathcal{S}})$ with $X \times_{\mathfrak{M}} X$ and the two projections pr_i with the corresponding projections from the latter to X.) Therefore, the quotient of this flat equivalence relation exists as an algebraic space $\mathfrak{M}'$. Moreover, the map $X' \to X$ induces a map

$\mathfrak{M}' \to \mathfrak{M}$ so that one obtains a Cartesian square:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \mathfrak{M}' \arrow[r] & \mathfrak{M} \end{tikzcd}$$

Now the bottom map is also étale by descent theory. Observe that

$$X' \underset{\mathcal{S}'}{\times} X' \cong X \underset{\mathcal{S}}{\times} X \underset{\mathcal{S}}{\times} \mathcal{S}' = (X \underset{\mathcal{S}}{\times} X) \underset{\mathfrak{M}}{\times} \mathfrak{M}' \quad \text{and} \quad X \underset{\mathcal{S}}{\times} \mathcal{S}' \cong X \underset{\mathfrak{M}}{\times} \mathfrak{M}'.$$

Therefore, $\mathcal{S}' = \mathfrak{M}' \times_{\mathfrak{M}} \mathcal{S}$. This completes the proof of the first assertion in the theorem $\square$

COROLLARY 3.14. *Assume the hypotheses of Theorem* 3.13. *Then one obtains an equivalence of the following categories of sheaves* $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$, $\mathrm{Sh}(\mathcal{S}_{iso.et})$ *and* $\mathrm{Sh}(\mathfrak{m}_{\mathrm{et}})$

Proof. This is clear from the last theorem. $\square$

Remark 3.15. Let $\mathrm{Sh}^{tr.in}_{sets}(\mathcal{S}_{\mathrm{et}})$ denote the category of all sheaves of *sets* on $\mathcal{S}_{\mathrm{et}}$ with trivial action by the inertia stack $I_{\mathcal{S}}$ as in Definition 3.12. It is necessary for us (see Proposition 3.18) below to show that this is a *Grothendieck topos* and therefore that there exists a site $\mathcal{S}^{tr.in}_{\mathrm{et}}$ so that the category of sheaves of sets on $\mathcal{S}^{tr.in}_{\mathrm{et}}$ is $\mathrm{Sh}^{tr.in}_{sets}(\mathcal{S}_{\mathrm{et}})$. We will begin by recalling the situation in Definition 3.5.

PROPOSITION 3.16. *Let* $\chi = (X_0, X_1)$ *denote an algebraic groupoid associated to an algebraic stack* $\mathcal{S}$ *with* $x\colon X_0 \to \mathcal{S}$ *an atlas. Then* $\mathrm{Sh}^{des}_{sets}(\chi_{\mathrm{et}})$ *and* $\mathrm{Sh}^{des}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ *are Grothendieck topoi.*

Proof. Observe that the small étale topos on X_0, $\mathrm{Sh}_{sets}(X_{0,et})$ is a Grothendieck topos. By using suitable universes one may also ensure that so is the big étale topos on X_0, i.e. $\mathrm{Sh}_{sets}(\mathrm{ET}(X_0))$. Let $\chi = \mathrm{tr}_2(B_x\mathcal{S})$ denote the algebraic groupoid obtained by truncating the simplicial algebraic space $B_x\mathcal{S}$. Since the obvious functor from the category of sheaves on the groupoid to the category of sheaves on X_0 preserves and reflects colimits and finite limits the conditions in [SGA]4, IV, 1.1.2(a), (b) and (c) hold. Now it suffices to show that the categories $\mathrm{Sh}^{des}_{sets}(\chi_{\mathrm{et}})$ and $\mathrm{Sh}^{des}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ have a small family of generators.

We begin with the observation that the category $\mathrm{Sh}_{sets}(\mathcal{S}_{smt})$ is a Grothendieck topos and therefore has a small family of generators. Now the equivalences of categories in Proposition 3.8 completes the proof. $\square$

PROPOSITION 3.17. *Let* $i\colon \mathcal{S}_0 \to \mathcal{S}$ *denote a closed immersion of algebraic stacks with open complement* $j\colon \mathcal{S}_1 \to \mathcal{S}$. *Now* j *induces an open immersion of the topoi with complementary closed immersion* i *(in the sense of* [*SGA*] *4, IV,* (9.3.5)*, i.e.* i_* *and* j_* *are fully-faithful and that the image of* i_* *is the subcategory of objects that* j^* *sends to* ϕ):

$$\mathrm{Sh}(B_{x_0}\mathcal{S}_{0,et}) \xrightarrow{i_*} \mathrm{Sh}(B_x\mathcal{S}_{\mathrm{et}}) \xleftarrow{j_*} \mathrm{Sh}(B_{x_1}\mathcal{S}_{1,et}), \tag{3.1.8}$$

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$$\mathrm{Sh}^{tr.in}(B_{x_0}\mathcal{S}_{0,et}) \xrightarrow{i_*} \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}}) \xleftarrow{j_*} \mathrm{Sh}^{tr.in}(B_{x_1}\mathcal{S}_{1,et}). \tag{3.1.9}$$

Moreover, the functor

$$j_!\colon \mathrm{Sh}(B_{x_1}\mathcal{S}_{1,et}) \to \mathrm{Sh}(B_x\mathcal{S}_{\mathrm{et}}) \quad (i_*\colon \mathrm{Sh}(B_{x_0}\mathcal{S}_{0,et}) \to \mathrm{Sh}(B_x\mathcal{S}_{\mathrm{et}}))$$

induces a functor

$$j_!\colon \mathrm{Sh}^{tr.in}(B_{x_1}\mathcal{S}_{1,et}) \to \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}}) \; (i_*\colon \mathrm{Sh}^{tr.in}(B_{x_0}\mathcal{S}_{0,et}) \to \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}}),$$

respectively) with $j_!$ (i_*) left-adjoint to j^* (right-adjoint to i^*, respectively).

Proof. The results of [SGA] 4, VIII, (6.3) extended to algebraic spaces and then to simplicial algebraic spaces readily proves the assertion for (3.1.8). The observation that in the diagram

$$X \underset{\mathcal{S}_0}{\times} I_{\mathcal{S}_0} \xrightarrow{i} X \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \xleftarrow{j} X \underset{\mathcal{S}_1}{\times} I_{\mathcal{S}_1}$$

j is an open immersion with i its complementary closed immersion, along with (3.1.8) shows that (3.1.9) is also true. The last assertion regarding $j_!$ and i_* may be verified readily. $\square$

PROPOSITION 3.18. *Let $\mathcal{S}$ denote an algebraic stack that is finitely presented over the base scheme S with an atlas $x\colon X \to \mathcal{S}$. Then the topos $\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ is a Grothendieck topos.*

Proof. Since the obvious functor $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}}) \to \mathrm{Sh}(B_x\mathcal{S}_{\mathrm{et}})$ preserves and reflects colimits and finite limits, the conditions in [SGA]4, IV, 1.1.2(a), (b) and (c) hold. Now it suffices to show the existence of a small family of generators to satisfy the condition of [SGA]4, IV, 1.1.2(d). Observe that there exists a finite filtration $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_n = \mathcal{S}$ by locally closed algebraic substacks $\mathcal{S}_i$ so that each $(\mathcal{S}_i - \mathcal{S}_{i-1})_{red}$ is a gerbe over its coarse moduli-space. Let $x_i\colon X_i \to \mathcal{S}_i - \mathcal{S}_{i-1}$ denote the induced atlas for $\mathcal{S}_i - \mathcal{S}_{i-1}$. By Theorem 3.13, each of the topos $\mathrm{Sh}^{tr.in}(B_{x_i}(\mathcal{S}_i - \mathcal{S}_{i-1})_{\mathrm{et}})$ is a Grothendieck topos. The last proposition shows that the category $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$ is obtained by gluing the sub-categories $\mathrm{Sh}^{tr.in}(B_{x_i}(\mathcal{S}_i - \mathcal{S}_{i-1})_{\mathrm{et}})$. For each $i = 1, \ldots, n$, let $j_i\colon \mathcal{S}_i - \mathcal{S}_{i-1} \to \mathcal{S}$ denote the obvious locally closed immersion. Clearly, if $\{G_i^\alpha | \alpha\}$ is a set of generators for $\mathrm{Sh}^{tr.in}(B_{x_i}(\mathcal{S}_i - \mathcal{S}_{i-1})_{\mathrm{et}})$, the collection $\{j_{i!}(G_i^\alpha) | \alpha, i\}$ will be a set of generators for $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$. $\square$

COROLLARY 3.19. (i) *There exists a site $B_x\mathcal{S}^{tr.in}_{\mathrm{et}}$ so that the category of sheaves of sets on the latter is equivalent to $\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$.*
(ii) *Now there exist a map*

$$B_x\mathcal{S}^{tr.in}_{\mathrm{et}} \xrightarrow{e} \mathcal{S}_{iso.et}$$

of sites. The corresponding inverse-image functor e^{-1} sends coverings to coverings. $e^*\colon \mathrm{Sh}_{sets}(\mathcal{S}_{iso.et}) \to \mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ is fully-faithful.

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(iii) *There exists a map of topoi* $f^*: \mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}}) \to \mathrm{Sh}^{des}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ *which is also faithful and conservative.*

Proof. The first assertion follows from Giraud's Theorem as in [SGA]4, IV Theorem (1.2), in view of the proposition above. It may be worthwhile recalling the construction of the associated site starting with the given set of generators for the category $\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$. First, one enlarges the given set of generators by taking all finite inverse limits among them. These form the objects of the site. The topology on this site is the one induced by the *canonical topology* on the given category $\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$. Now observe that coverings are given by universal epimorphisms.

Given an object $\mathcal{S}' \to \mathcal{S}$ in $\mathcal{S}_{iso.et}$, observe that $I_{\mathcal{S}'} \cong \mathcal{S}' \times_{\mathcal{S}} I_{\mathcal{S}}$. Therefore, the sheaf represented by $X' = \mathcal{S}' \times_{\mathcal{S}} X$ on $B_x\mathcal{S}_{\mathrm{et}}$ has trivial action by $X \times_{\mathcal{S}} I_{\mathcal{S}}$. Clearly it has descent. Therefore, it defines a sheaf in $\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$. The functor $\mathcal{S}' \mapsto X' \mapsto h_{X'} =$ the sheaf represented by X', preserves pull-backs and sends coverings to epimorphisms. This defines the map of sites e. To show that e^* is fully-faithful, it suffices to show that $e_* \circ e^* = id$. We will establish this as follows.

First consider the functor $e^{-1}: \mathcal{S}_{iso.et} \to B_x\mathcal{S}^{tr.in}_{\mathrm{et}}$. We observe this is fully faithful as follows. Suppose $f, g: \mathcal{S}' \to \mathcal{S}''$ are maps in $\mathcal{S}_{iso.et}$ so that $e^{-1}(f) = e^{-1}(g): e^{-1}(\mathcal{S}') \to e^{-1}(\mathcal{S}'')$. This being a map of sheaves in $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}}) \subseteq \mathrm{Sh}^{des}(\mathcal{B}_x\mathcal{S}_{\mathrm{et}})$ satisfies descent conditions to descend to a unique map $\mathcal{S}' \to \mathcal{S}''$. i.e. f and g must be equal to begin with. This shows the functor e^{-1} is faithful. To see it is full, let $f: e^{-1}(\mathcal{S}') = h_{X'} \to e^{-1}(\mathcal{S}'') = h_{X''}$ denote a map in $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$. By the Yoneda lemma, f is induced by a map $g: X' \to X''$ which satisfies descent conditions to descend to a unique map $\mathcal{S}' \to \mathcal{S}''$. This shows e^{-1} is also full.

Now consider $\Gamma(U, e_*e^*(F)$, for $U \in \mathcal{S}_{iso.et}$ and $F \in \mathrm{Sh}(\mathcal{S}_{iso.et})$. $e^*(F)$ is the sheafification of the presheaf $e^{\#}(F)$ and

$$\Gamma(U, e_*e^{\#}(F)) = \Gamma(e^{-1}(U), e^{\#}(F)) = \varinjlim_{e^{-1}(U)\to e^{-1}(W)} \Gamma(W, F).$$

By the arguments in the above paragraph, the last colimit identifies with $\varinjlim_{U\to W} \Gamma(W, F) = \Gamma(U, F)$.

This shows $e_* \circ e^{\#}(F) = F$ for any sheaf $F \in \mathrm{Sh}(\mathcal{S}_{iso.et})$. Therefore (denoting the functor sending a presheaf to the associated sheaf by a) and making use of Proposition 3.25 (below) we obtain

$$e_* \circ e^*(F) = e_* \circ a \circ e^{\#}(F) = a \circ e_* \circ e^{\#}(F) = e_* \circ e^{\#}(F) = F.$$

It follows that e^* is fully-faithful.

Now we consider (iii). The obvious (inclusion) functor $\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}}) \to \mathrm{Sh}^{des}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ preserves all colimits and finite limits and therefore by [SGA]4, IV, 3.13, may be written as f^* for a map f of the corresponding topoi. Clearly this functor is (fully)-faithful and, hence, conservative. □

Remark 3.20. In Theorem 3.27 we will prove that the functor e^* is an equivalence of categories, in general. Observe that the forgetful functor sending a presheaf of

Abelian groups, or modules over a ring to the corresponding presheaf of sets preserves limits. Therefore, it sends sheaves to sheaves and induces an equivalence between the category of sheaves of, say Abelian groups and the sub-category of Abelian-group objects of the category of sheaves of sets. This observation shows that there exists a functor e^* on the corresponding categories of sheaves of, say Abelian groups. This will also be fully-faithful on the corresponding categories.

PROPOSITION 3.21. *Let $f\colon \mathcal{S}_1 \to \mathcal{S}_0$ denote a representable map of algebraic stacks finitely presented over S that is integral, radicial and surjective. Let $x_0\colon X_0 \to \mathcal{S}_0$ denote an atlas, $B_{x_0}\mathcal{S}_0$ the corresponding simplicial algebraic space, $x_1\colon X_1 = X_0 \times_{\mathcal{S}_0} \mathcal{S}_1 \to \mathcal{S}_1$ the induced atlas and $B_{x_1}\mathcal{S}_1$ the corresponding simplicial algebraic space. Then f^* defines equivalences*

$$\mathrm{Sh}_{sets}(B_{x_0}\mathcal{S}_{0,et}) \to \mathrm{Sh}_{sets}(B_{x_1}\mathcal{S}_{1,et}),\ \mathrm{Sh}^{tr.in}_{sets}(B_{x_0}\mathcal{S}_{0,et}) \to \mathrm{Sh}^{tr.in}_{sets}(B_{x_1}\mathcal{S}_{1,et}).$$

Moreover, f^* also induces an equivalence:

$$\mathrm{Sh}_{sets}(\mathcal{S}_{0,iso.et}) \to \mathrm{Sh}_{sets}(\mathcal{S}_{1,iso.et}).$$

Proof. The induced map $Bf_\bullet\colon B_{x_1}\mathcal{S}_1 \to B_{x_0}\mathcal{S}_0$ is integral, radicial and surjective in each degree. Moreover, so is the induced map $BX_1 \times_{\mathcal{S}_1} I_{\mathcal{S}_1} \to BX_0 \times_{\mathcal{S}_0} I_{\mathcal{S}_0}$. One may verify the latter by observing the Cartesian square where π is the map defined in Proposition 3.9:

$$\begin{tikzcd}
X \underset{\mathcal{S}}{\times} I_{\mathcal{S}} \arrow[r, "\pi"] \arrow[d] & X \underset{\mathcal{S}}{\times} X \arrow[d] \\
X \arrow[r, "\delta"] & X \times X
\end{tikzcd}$$

This proves the first assertion. Now f^* induces a map $\mathrm{Sh}_{sets}(\mathcal{S}_{0,iso.et}) \to \mathrm{Sh}_{sets}(\mathcal{S}_{1,iso.et})$. Since the functors

$$e^*\colon \mathrm{Sh}_{sets}(\mathcal{S}_{0,iso.et}) \to \mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{0,et}), \qquad e^*\colon \mathrm{Sh}_{sets}(\mathcal{S}_{1,iso.et}) \to \mathrm{Sh}^{tr.in}_{sets}(B_{x_1}\mathcal{S}_{1,et})$$

are fully-faithful, it follows from the first assertion that $f^*\colon \mathrm{Sh}_{sets}(\mathcal{S}_{0,iso.et}) \to \mathrm{Sh}_{sets}(\mathcal{S}_{1,iso.et})$ is also fully-faithful. Therefore, it suffices to show the following: given $\mathcal{S}'_1 \to \mathcal{S}_1$ isovariant and étale in the site $\mathcal{S}_{1,iso.et}$, there exists an isovariant étale map $\mathcal{S}'_0 \to \mathcal{S}_0$ so that $\mathcal{S}'_1 = \mathcal{S}'_0 \times_{\mathcal{S}_0} \mathcal{S}_1$.

For this, observe that f^* induces an equivalence of the étale sites

$$X_{0,et} \to X_{1,et} \quad \text{and} \quad (X_0 \underset{\mathcal{S}_0}{\times} X_0)_{\mathrm{et}} \to (X_1 \underset{\mathcal{S}_1}{\times} X_1)_{\mathrm{et}}.$$

Therefore, one obtains equivalences of categories:

$$\mathrm{Sh}_{sets}(X_{0,et}) \simeq \mathrm{Sh}_{sets}(X_{1,et}),\ \mathrm{Sh}_{sets}((X_0 \underset{\mathcal{S}_0}{\times} X_0)_{\mathrm{et}}) \simeq \mathrm{Sh}_{sets}((X_1 \underset{\mathcal{S}_1}{\times} X_1)_{\mathrm{et}}).$$

In view of the equivalence of categories in Proposition 3.8, one observes that the functor

$$f^*\colon \mathrm{Sh}_{sets}(X_{0,smt}) \to \mathrm{Sh}_{sets}(X_{1,smt}) \tag{3.1.10}$$

is also fully faithful.

Let $\mathcal{S}'_1 \to \mathcal{S}_1$ denote an isovariant étale map in the site $\mathcal{S}_{1,iso.et}$; let $X'_1 = X_1 \times_{\mathcal{S}_1} \mathcal{S}'_1$. The equivalence of the étale sites of X_0 and X_1 shows that there exists an étale map $X'_0 \to X_0$ so that $f^*(X'_0) = X'_1$. Similarly, there exists an étale map $X''_0 \to X_0 \times_{\mathcal{S}_0} X_0$ so that $f^*(X''_0) = X'_1 \times_{\mathcal{S}'_1} X'_1 = X''_1$. Let a_1, b_1 denote the two obvious maps $X''_1 \to X'_1$: the map $X''_1 \xrightarrow{a_1 \times b_1} X'_1 \times X'_1$ is separated and quasi-compact. By (3.1.10), it follows that there exist two smooth maps $a_0, b_0\colon X''_0 \to X'_0$ so that $f^*(a_0) = a_1$ and $f^*(b_0) = b_1$. Since the induced maps $X'_1 \to X'_0$ and $X''_1 \to X''_0$ are radicial and surjective (and hence universal homeomorphisms), one may see that the induced map $X''_0 \overset{a_0,b_0}{\to} X'_0 \times X'_0$ is separated and quasi-compact.

Therefore, $X''_0 \underset{b_0}{\overset{a_0}{\rightrightarrows}} X'_0$ defines an algebraic groupoid. (The groupoid law is defined by requiring that f^* applied to the the composition $X''_0 \times_{X'_0} X''_0 \to X''_0$ is the composition $X''_1 \times_{X'_1} X''_1 \to X''_1$. Similarly the remaining structure maps of the groupoid are defined by requiring f^* applied to a structure map of the groupoid is the corresponding structure map of the groupoid $X''_0 \underset{b_1}{\overset{a_1}{\rightrightarrows}} X'_0$ that corresponds to the algebraic stack $\mathcal{S}'_1$.) Let $\mathcal{S}'_0$ denote the corresponding algebraic stack. Clearly $f^*(\mathcal{S}'_0) = \mathcal{S}'_1$ since $f^*(X'_0) = X'_1$ and $f^*(X''_0) = X''_1$.

Now we proceed to show that $\mathcal{S}'_0 \to \mathcal{S}_0$ is isovariant étale and that the induced map $\mathcal{S}'_1 \to \mathcal{S}'_0$ is integral, radicial and surjective. The last assertion follows by faithfully flat descent since the maps $X'_1 \to X'_0$ and $X''_1 \to X''_0$ are both integral, radicial and surjective. One may show the isovariance of $\mathcal{S}'_0 \to \mathcal{S}_0$ as follows. First the isovariance of $\mathcal{S}'_1 \to \mathcal{S}_1$ implies $X_1 \times_{\mathcal{S}_1} I_{\mathcal{S}_1}$ acts trivially on the sheaf X'_1 where the action is defined as in Proposition 3.10. The diagrams in (3.1.6) for $X = X_1$ and X_0 (with the corresponding stack $\mathcal{S} = \mathcal{S}_1$ and $\mathcal{S}_0$) correspond under pull-back by maps that are integral radicial and surjective: therefore, the action of $X_0 \times_{\mathcal{S}_0} I_{\mathcal{S}_0}$ on X'_0 is also trivial. This shows $X'_0 \times_{\mathcal{S}_0} I_{\mathcal{S}_0} \subseteq X'_0 \times_{\mathcal{S}'_0} I_{\mathcal{S}'_0}$. Since the map $\mathcal{S}'_0 \to \mathcal{S}_0$ sends $I_{\mathcal{S}'_0}$ to $I_{\mathcal{S}_0}$, clearly $X'_0 \times_{\mathcal{S}'_0} I_{\mathcal{S}'_0} \subseteq X'_0 \times_{\mathcal{S}_0} I_{\mathcal{S}_0}$. Therefore $X'_0 \times_{\mathcal{S}'_0} \mathcal{S}'_0 \times_{\mathcal{S}_0} I_{\mathcal{S}_0} = X'_0 \times_{\mathcal{S}'_0} I_{\mathcal{S}'_0}$. Since $X'_0 \to \mathcal{S}'_0$ is faithfully flat, it follows that $\mathcal{S}'_0 \times_{\mathcal{S}_0} I_{\mathcal{S}_0} = I_{\mathcal{S}'_0}$. This proves the map $\mathcal{S}'_0 \to \mathcal{S}_0$ is isovariant. To see it is also étale, observe the commutative diagram:

$$\begin{tikzcd}
X''_0 \arrow[r, shift left, "a_0"] \arrow[r, shift right, "b_0"'] \arrow[d] & X'_0 \arrow[r] \arrow[d] & \mathcal{S}'_0 \arrow[d] \\
X_0 \underset{\mathcal{S}_0}{\times} X_0 \arrow[r, shift left, "a_1"] \arrow[r, shift right, "b_1"'] & X_0 \arrow[r] & \mathcal{S}_0
\end{tikzcd}$$

By comparison with the corresponding diagram involving $\mathcal{S}'_1$ and $\mathcal{S}_1$, one may see that the squares in the above diagram are in fact Cartesian; similarly, $X''_0 = X'_0 \times_{\mathcal{S}'_0} X'_0$. The two left vertical maps are étale and therefore, by faithfully flat descent, the induced map $\mathcal{S}'_0 \to \mathcal{S}_0$ is also étale. □

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THEOREM 3.22. *Let* $i\colon \mathcal{S}_0 \to \mathcal{S}$ *denote a closed immersion of algebraic stacks finitely presented over the base scheme* S. *Let* $\alpha\colon \mathcal{S}'_0 \to \mathcal{S}_0$ *denote an isovariant étale map in* $\mathcal{S}_{0,iso.et}$. *Then there exists an isovariant étale map* $\mathcal{S}' \to \mathcal{S}$ *in* $\mathcal{S}_{iso.et}$ *so that* $i^*(\mathcal{S}') = \mathcal{S}'_0$.

Proof. Let $x\colon X \to \mathcal{S}$ denote a fixed atlas for $\mathcal{S}$ which we will assume is a separated and quasi-compact scheme. Let $\mathcal{S}_1 = \mathcal{S} - \mathcal{S}_0$ and let $x_i\colon X_i \to \mathcal{S}_i$ denote the induced atlases for $\mathcal{S}_i$, $i = 0, 1$. Clearly X_0 is a closed sub-scheme of X with open complement X_1.

Step 1. Next we begin with the following diagram:

$$\begin{array}{ccc} X'_0 & & \\ \downarrow{\scriptstyle \phi_0} & & \\ X_0 & \xrightarrow{\bar{i}} & X \end{array}$$

where $X'_0 = \mathcal{S}'_0 \times_{\mathcal{S}} X$. In this diagram, the map ϕ_0 is étale, while $\bar{i}$ is a closed immersion. By [EGA]IV, 18.1.1 the following hold: there exists a family $\{U_i \to X|i\}$ of étale maps so that $\{U_i \times_X X_0 \to X_0|i\}$ forms an étale cover of X_0 and each of the maps $U_i \times_X X_0 \to X_0$ factors through the map $\phi_0\colon X'_0 \to X_0$, with the corresponding map $U_i \times_X X_0 \to X'_0$ a Zariski open immersion. Let $\tilde{X}' = \sqcup_i U_i$ and $\tilde{x}'\colon \tilde{X}' \to X$ be the obvious induced map. This map is étale and the map $\tilde{X}' \times_X X_0 \to X_0$ factors through an étale surjective map to X'_0.

Observe that $\tilde{X}' \times_X X_0 = \tilde{X}' \times_X X \times_{\mathcal{S}} \mathcal{S}_0 \cong \tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0$. Therefore, the induced map of this to $\mathcal{S}_0$ factors through a smooth surjective map to $\mathcal{S}'_0$. i.e. $\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0$ is also an atlas for the stack $\mathcal{S}'_0$.

Let

$$R_0 = \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_0 \underset{\mathcal{S}'_0}{\times} \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_0 \tag{3.1.11}$$

Since the map $\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0 \to \mathcal{S}'_0$ is smooth surjective,

$$\mathcal{R}_0 : R_0 \underset{t}{\overset{s}{\rightrightarrows}} \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_0 \tag{3.1.12}$$

defines an algebraic groupoid. (Observe that the map $\delta = (s, t)\colon R_0 \to \tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0 \times \tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0$ is quasi-compact and separated in view of the hypotheses. The separatedness follows from the observation that $\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0$ is a separated scheme. The groupoid law is the obvious one.) Therefore, $\mathcal{R}_0$ defines an algebraic stack with $\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0$ as an atlas. By [L-MB] Remarque (4.8) this stack may be identified with the stack $\mathcal{S}'_0$.

Next consider $\tilde{X}' \times_X X_1$ where $X_1 = X - X_0$. Now the Cartesian square

$$\begin{array}{ccc} \tilde{X}' \underset{X}{\times} X_1 & \longrightarrow & \tilde{X}' \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X \end{array}$$

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and the observation that the map $\tilde{X}' \to X$ is étale, shows the induced map $\tilde{X}' \times_X X_1 \to X_1$ is also étale. Therefore, the image of this map is an open dense subscheme of X_1: call it W. Observe again that $\tilde{X}' \times_X X_1 = \tilde{X}' \times_X X \times_{\mathcal{S}} \mathcal{S}_1 \cong \tilde{X}' \times_{\mathcal{S}} \mathcal{S}_1$. Let

$$R_1 = (\tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_1) \underset{\mathcal{S}_1}{\times} (\tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_1) \tag{3.1.13}$$

Now

$$\mathcal{R}_1 : R_1 \underset{t}{\overset{s}{\rightrightarrows}} \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_1 \tag{3.1.14}$$

defines an algebraic groupoid and therefore an algebraic stack with $\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_1$ as an atlas. (Once again the separatedness and quasi-compactness of $\delta = (s, t)$ follows from that of $\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_1$.) We will denote this algebraic stack by $\mathcal{S}_1'$. Clearly this maps to $\mathcal{S}_1$. In order to show this defines an open sub-stack of $\mathcal{S}_1$ one may proceed as follows. First, using the construction of the algebraic stack $\mathcal{S}_1'$, starting with the algebraic groupoid $\mathcal{R}_1$, one may observe that the map $\mathcal{S}_1' \to \mathcal{S}_1$ is a monomorphism and hence also representable. (See [L-MB] Proposition (1.4.1.2).) The map from the groupoid $\mathcal{R}_1$ to the groupoid $(X_1 \times_{\mathcal{S}_1} X_1 \rightrightarrows X_1)$ factors through the sub-groupoid given by the images of R_1 in $X_1 \times_{\mathcal{S}_1} X_1$ and W; this sub-groupoid also defines the stack $\mathcal{S}_1'$. Therefore, $\mathcal{S}_1'$ is an open sub-stack of $\mathcal{S}_1$.

Step 2. Next we consider $R = R_0 \cup R_1$.

We *claim* that R defines an algebraic groupoid $\mathcal{R}$ on $\tilde{X}'$ and that it is in fact an open sub-algebraic space of $\tilde{X}' \times_{\mathcal{S}} \tilde{X}'$.

We consider the induced map:

$$s: \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_0 \underset{\mathcal{S}_0'}{\times} \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_0 \to \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_0 \underset{\mathcal{S}_0}{\times} \tilde{X}' \underset{\mathcal{S}}{\times} \mathcal{S}_0. \tag{3.1.15}$$

Observe that the last term above may be identified with $\tilde{X}' \times_{\mathcal{S}} \tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0 \cong (\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0') \times_{\mathcal{S}_0'} (\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0)$. Clearly the latter maps by $p = (id \times \alpha) \times id$ to $(\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0) \times_{\mathcal{S}_0'} (\tilde{X}' \times_{\mathcal{S}} \mathcal{S}_0)$. One may now readily verify that the composition $p \circ s$ is the identity. (One may verify this, for example, on the points of the algebraic spaces we are considering.) Clearly p, being induced by α, is étale. Therefore, it follows that s (being a section to an étale map) is an open immersion.

Now observe that $R_1 = \tilde{X}' \times_{\mathcal{S}} \mathcal{S}_1 \times_{\mathcal{S}_1} \tilde{X}' \times_{\mathcal{S}} \mathcal{S}_1 \cong (\tilde{X}' \times_{\mathcal{S}} \tilde{X}') \times_{\mathcal{S}} \mathcal{S}_1$. Let $\Phi: R_0 \cup R_1 \to (\tilde{X}' \times_{\mathcal{S}} \tilde{X}') \times_{\mathcal{S}} \mathcal{S}_0 \cup (\tilde{X}' \times_{\mathcal{S}} \tilde{X}') \times_{\mathcal{S}} \mathcal{S}_1 = (\tilde{X}' \times_{\mathcal{S}} \tilde{X}') \times_{\mathcal{S}} \mathcal{S} = \tilde{X}' \times_{\mathcal{S}} \tilde{X}'$ be the map induced by s on R_0 to $(\tilde{X}' \times_{\mathcal{S}} \tilde{X}') \times_{\mathcal{S}} \mathcal{S}_0$ and by the identity on R_1 to $(\tilde{X}' \times_{\mathcal{S}} \tilde{X}') \times_{\mathcal{S}} \mathcal{S}_1$. Using the observation that s is an open immersion and that R_1 maps by the identity to its image, one may readily see that the map Φ is in fact an open immersion and hence in particular étale. Therefore, the compositions given by Φ and the two projections $\tilde{X}' \times_{\mathcal{S}} \tilde{X}' \to \tilde{X}'$ are also smooth.

Therefore,

$$R \xrightarrow{\Phi} \tilde{X}' \underset{\mathcal{S}}{\times} \tilde{X}' \underset{t}{\overset{s}{\rightrightarrows}} \tilde{X}'$$

defines an algebraic groupoid and an associated algebraic stack. (Once again the groupoid law is the obvious one.) We denote this stack by $\mathcal{S}'$. The observation that

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$\mathcal{R} \times_{\mathcal{S}} \mathcal{S}_0 = \mathcal{R}_0$ and $\mathcal{R} \times_{\mathcal{S}} \mathcal{S}_1 = \mathcal{R}_1$ show that $\mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_0 \cong \mathcal{S}'_0$ and $\mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_1 \cong$ an open sub-stack of $\mathcal{S}_1$.

Finally it suffices to show that the map $\mathcal{S}' \to \mathcal{S}$ is representable and is isovariant. For the first it suffices to show that if $z: Z \to \mathcal{S}$ is a map from an algebraic space, $Z \times_{\mathcal{S}} \mathcal{S}'$ is an algebraic space. Now $Z \times_{\mathcal{S}} \mathcal{S}' \times_{\mathcal{S}'} \mathcal{S}'_0 \to Z \times_{\mathcal{S}} \mathcal{S}'$ is a closed immersion while $Z \times_{\mathcal{S}} \mathcal{S}' \times_{\mathcal{S}'} \mathcal{S}'_1 \to Z \times_{\mathcal{S}} \mathcal{S}'$ is the complimentary open immersion. Both $Z \times_{\mathcal{S}} \mathcal{S}' \times_{\mathcal{S}'} \mathcal{S}'_0 \cong Z \times_{\mathcal{S}} \mathcal{S}'_0$ and $Z \times_{\mathcal{S}} \mathcal{S}' \times_{\mathcal{S}'} \mathcal{S}'_1 \cong Z \times_{\mathcal{S}} \mathcal{S}'_1$ are algebraic spaces: recall that $\mathcal{S}'_0 \to \mathcal{S}_0 \to \mathcal{S}$ and $\mathcal{S}'_1 \to \mathcal{S}_1 \to \mathcal{S}$ are both representable morphisms. Therefore, it follows that $Z \times_{\mathcal{S}} \mathcal{S}'$ is also an algebraic space proving the map $\mathcal{S}' \to \mathcal{S}$ is representable.

Observe also that the maps $\mathcal{S}'_i \to \mathcal{S}_i$ are isovariant: for $i = 0$ this follows from the hypothesis that $\mathcal{S}'_0 \to \mathcal{S}_0$ is isovariant while for $i = 1$ this follows from the observation that $\mathcal{S}'_1 \to \mathcal{S}_1$ is an open immersion. $\square$

Remark 3.23. The above theorem is established for quotient stacks in [T-3] Lemma 2.14. Even for the action of a trivial group, such a result seems relatively unknown and seems to hold only in the setting of algebraic spaces and not schemes. The only result for schemes that holds in general, seems to be the result from [EGA] IV, 18.1.1 that we used in Step 1 of the proof.

Throughout the next proposition $\mathrm{Presh}(\mathcal{C})$ will denote the category of presheaves of sets on the site $\mathcal{C}$. $\mathrm{Sh}(\mathcal{C})$ will denote the corresponding category of sheaves of sets and $a: \mathrm{Presh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ will denote the functor sending a presheaf to its associated sheaf.

PROPOSITION 3.24. *Let $i: \mathcal{S}_0 \to \mathcal{S}$ denote a closed immersion of algebraic stacks. Let $i_\#: \mathrm{Presh}(\mathcal{S}_{0,iso.et}) \to \mathrm{Presh}(\mathcal{S}_{iso.et})$ be defined by $\Gamma(V, i_\# P) = \Gamma(V \times_{\mathcal{S}} \mathcal{S}_0, P)$. $i_\#: \mathrm{Presh}_{sets}(B_x\mathcal{S}_{0,et}) \to \mathrm{Presh}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ will denote the corresponding functor defined similarly. Let $i_*: \mathrm{Sh}(\mathcal{S}_{o,iso.et}) \to \mathrm{Sh}(\mathcal{S}_{iso.et})$, $i_*: \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{0,et}) \to \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$ denote the corresponding functors at the level of sheaves. Now $i_\#$ induces a functor $\mathrm{Presh}^{tr.in}_{sets}(B_x\mathcal{S}_{0,et}) \to \mathrm{Presh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$ and one obtains the equality*

$$a \circ i_\# = i_* \circ a. \tag{3.1.16}$$

Moreover, if

$$e^*: \mathrm{Sh}(\mathcal{S}_{iso.et}) \to \mathrm{Sh}^{tr.in}(B_x\mathcal{S}) \quad \text{and} \quad e^*: \mathrm{Sh}(\mathcal{S}_{0,iso.et}) \to \mathrm{Sh}^{tr.in}(B_{x_0}\mathcal{S}_0)$$

are the functors in Corollary 3.19, one also obtains the equality

$$e^* \circ i_* = i_* \circ e^* \tag{3.1.17}$$

Proof. According to [SGA]4, II, Section 3, the sheafification functor a on any site $\mathcal{C}$ is defined by

$$a(P) = LL(P), \quad P \in \mathrm{Presh}(\mathcal{C}), \tag{3.1.18}$$

where $\mathrm{Presh}(\mathcal{C})$ denotes the category of presheaves of sets on the site $\mathcal{C}$,

$$\Gamma(U, L(P)) = \varinjlim_{R \in J(U)} \mathrm{Hom}_{\mathrm{Presh}(\mathcal{C})}(R, P) \tag{3.1.19}$$

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and $J(U)$ is the category of covering sieves of U, $U \in \mathcal{C}$. If R is generated by $\{u_i : U_i \to U | i\}$,

$$\mathrm{Hom}_{\mathrm{Presh}(\mathcal{C})}(R, P) = \mathrm{Equalizer}\left(\prod_i \Gamma(U_i, P) \underset{s}{\overset{s}{\rightrightarrows}} \prod_{i,j} \Gamma(U_i \underset{U}{\times} U_j, P)\right).$$

Recall that i_* is just $i_\#$ restricted to the category of sheaves. Therefore, it suffices to show that $i_\# L = L i_\#$. For $U \to \mathcal{S}$ in $\mathcal{S}_{iso.et}$,

$$\Gamma(U, Li_\# P) = \varinjlim_{R \in J(U)} \mathrm{Hom}(R, i_\# P) = \varinjlim_{R \in J(U)} \mathrm{Hom}(R \underset{\mathcal{S}}{\times} \mathcal{S}_0, P) \tag{3.1.20}$$

By Theorem 3.22, $J(U) \underset{\mathcal{S}}{\times} \mathcal{S}_0$ is cofinal in $J(U) \times_{\mathcal{S}} \mathcal{S}_0$. Therefore, the above colimit is equal to the corresponding colimit $\varinjlim_{R' \in J(U \times_{\mathcal{S}} \mathcal{S}_0)} \mathrm{Hom}(R', P)$. One may identify this with $\Gamma(U \times_{\mathcal{S}} \mathcal{S}_0, L(P)) = \Gamma(U, i_\# L(P))$ as required. This proves the first assertion for the functor $i_* : \mathrm{Sh}(\mathcal{S}_{0,iso.et}) \to \mathrm{Sh}(\mathcal{S}_{iso.et})$. The remarks in 3.23 first show that the results of the last theorem hold on the étale site of algebraic spaces and that functor $i_\# : \mathrm{Presh}_{sets}(B_x \mathcal{S}_{0,et}) \to \mathrm{Presh}_{sets}(B_x \mathcal{S}_{\mathrm{et}})$ preserves presheaves with descent and induces a functor $i_\# : \mathrm{Presh}_{sets}^{tr.in}(B_x \mathcal{S}_{0,et}) \to \mathrm{Presh}_{sets}^{tr.in}(B_x \mathcal{S}_{\mathrm{et}})$. Now the identity in (3.1.16) follows for the functor $i_* : \mathrm{Sh}^{tr.in}(B_x \mathcal{S}_{0,et}) \to \mathrm{Sh}^{tr.in}(B_x \mathcal{S}_{\mathrm{et}})$ by entirely similar arguments as above.

Next we consider the second assertion. Let F denote a sheaf on $\mathcal{S}_{iso.et}$ or on $\mathcal{S}_{0,iso.et}$. According to [SGA] 4, III (1.3), $e^*(F)$ is the sheafification of the presheaf $e^\# F$ defined by $\Gamma(U, e^\# F) = \varinjlim_{U \to e^{-1}(W)} \Gamma(W, F)$. Here $e^{-1} : \mathcal{S}_{iso.et} \to B_x \mathcal{S}_{\mathrm{et}}^{tr.in}$ is the inverse-image functor associated to the map of sites e in Corollary 3.19. The colimit is taken over the filtered category which is the opposite of the comma category U/e. (Recall the objects of the category U/e are $w : W \to \mathcal{S}$ in $\mathcal{S}_{iso.et}$ along with a map $w' : U \to e^{-1}(W)$. Morphisms from $(w_1 : W_1 \to \mathcal{S}, w_1')$ to $(w_2 : W_2 \to \mathcal{S}, w_2')$ are given by maps $\phi : W_1 \to W_2$ in $\mathcal{S}_{iso.et}$ so that $w_2' = w_1' \circ e^{-1}(\phi)$. (A similar description applies to the functor $e^\#$ for $e^{-1} : \mathcal{S}_{0,iso.et} \to B_x \mathcal{S}_{0,et}^{tr.in}$.)

Next apply the identity in (3.1.16) for the map of sites $B_{x_0} \mathcal{S}_{0,et}^{tr.in} \to B_x \mathcal{S}_{\mathrm{et}}^{tr.in}$. Therefore, $i_* \circ e^* = i_* \circ a \circ e^\# = a \circ i_\# \circ e^\#$, i.e. $i_* \circ e^*(F)$ is the sheaf associated to the presheaf

$$\Gamma(U, i_\# e^\# F) = \Gamma(U \underset{\mathcal{S}}{\times} \mathcal{S}_0, e^\# F) = \varinjlim_{(U \underset{\mathcal{S}}{\times} \mathcal{S}_O) \to e^{-1}(w')} \Gamma(W', F) \tag{3.1.21}$$

On the other hand, $e^* i_*(F) = a e^\# i_* F = a e^\# i_\# F)$ which is the sheaf associated to the presheaf

$$\Gamma(U, e^\# i_\# F) = \varinjlim_{(U \to e^{-1}(w)} \Gamma(W, i_\# F) = \varinjlim_{(U \to e^{-1}(w)} \Gamma(W \underset{\mathcal{S}}{\times} \mathcal{S}_0, F)$$

The colimit is taken over the isovariant étale $W \to \mathcal{S}$ provided with a map $U \to e^{-1}(W)$. By Theorem 3.22, the filtered category appearing in the last colimit is cofinal in the filtered system appearing in the colimit in (3.1.21). Therefore, the colimits in (3.1.21) and (3.1.22) are isomorphic. This proves the second assertion of the proposition. □

PROPOSITION 3.25. *Let $\mathcal{S}$ denote an algebraic stack finitely presented over the base scheme S. Let* $e_*: \mathrm{Presh}_{sets}(B_x\mathcal{S}^{tr.in}) \to \mathrm{Presh}_{sets}(\mathcal{S}_{iso.et})$ *denote the direct image functor associated to the map of sites* $e: B_x\mathcal{S}^{tr.in} \to \mathcal{S}_{iso.et}$*. If a denotes the functor sending a presheaf to the associated sheaf, then there is a natural isomorphism* $a \circ e_* \cong e_* \circ a$*.*

Proof. Observe that if L is the functor as defined in (3.1.18), then $a = L \circ L$. Therefore, it suffices to show that e_* commutes with the functor L. This will follow, once we show that given any cover $v: V \to e^{-1}(U)$ in $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$ for any $U \in \mathcal{S}_{iso.et}$, one may find an isovariant étale $u': U' \to U_{iso.et}$ so that $v = e^{-1}(u'): V = e^{-1}(U') \to e^{-1}(U)$. Let $V \to e^{-1}(U)$ be a given cover. Observe that V is a sheaf of sets with descent and with trivial action by the inertia on B_xU_{et}. Observe also that $e^{-1}(U) \to U$ is an atlas and, therefore,

$$e^{-1}(U) \underset{U}{\times} e^{-1}(U) \underset{t}{\overset{s}{\rightrightarrows}} e^{-1}(U)$$

is a presentation for the algebraic stack U. V is represented by an algebraic space which will be denoted by V itself. Since V is sheaf with descent, there is an action by the above groupoid on V (in the sense of 3.1.5) so that there is an isomorphism $\Phi: s^*(V) \to t^*(V)$ satisfying cocycle conditions. Therefore one obtains an algebraic groupoid $s^*(V) \rightrightarrows V$ which defines an algebraic stack $u': U' \to U$. (The quasi-compactness of the map $\delta = (s, t): s^*(V) \to V \times V$ follows from hypothesis on $\mathcal{S}$. Observe that we may assume the atlas X is quasi-compact. The separatedness of δ may be deduced from that of $(s, t): e^{-1}(U) \times_U e^{-1}(U) \to e^{-1}(U) \times e^{-1}(U)$.) The map u' will be étale by descent theory and the hypothesis that V has trivial action by the inertia implies u' is isovariant. (Observe that V has trivial action by the inertia implies $I_U \times_U V = I_{U'} \times_{U'} V$.) One may now show that $v = e^{-1}(u')$ as in the proof of Corollary 3.19. □

THEOREM 3.26. *Let* $i: \mathcal{S}_0 \to \mathcal{S}$ *denote a closed immersion of algebraic stacks finitely presented over the given base scheme S with open complement* $j: \mathcal{S}_1 \to \mathcal{S}$*. Now j induces an open immersion of the topoi with complementary closed immersion i (in the sense of* [*SGA*] 4*, IV,* (9.3.5))*:*

$$\mathrm{Sh}(\mathcal{S}_{0,iso.et}) \xrightarrow{i_*} \mathrm{Sh}(\mathcal{S}_{iso.et}) \xleftarrow{j_*} \mathrm{Sh}(\mathcal{S}_{1,iso.et}) \tag{3.1.23}$$

Proof. By [SGA] 4, IV, (9.3.5), the assertions are equivalent to proving that i_* and j_* are fully-faithful and that the image of i_* is the subcategory of objects that j^* sends to ϕ. Recall that we already established the corresponding assertions for the étale topos of $B_x\mathcal{S}$ and for the corresponding full subcategory of sheaves with trivial action by the inertia – see (3.1.8) and (3.1.9). We will now use this to deduce that (3.1.23) also holds. As $j: \mathcal{S}_1 \to \mathcal{S}$ is a mono-morphism in the site $\mathcal{S}_{iso.et}$, it is clear that

$j_*: \mathrm{Sh}(\mathcal{S}_{1,iso.et}) \to \mathrm{Sh}(\mathcal{S}_{iso.et})$ is an open immersion of topoi and therefore is fully-faithful. (See [SGA]4, IV, 9.2 and VIII, 6.2.)

Showing that i_* is fully-faithful is equivalent to showing the adjunction map $i^* \circ i_* \to id$ is an isomorphism in $\mathrm{Sh}(\mathcal{S}_{0,iso.et})$. As $e^*: \mathrm{Sh}(\mathcal{S}_{0,iso.et}) \to \mathrm{Sh}^{tr.in}(B_{x_0}\mathcal{S}_{0.et})$ is faithful, it suffices to show that $e^*i^*i_* \to e^*$ is an isomorphism in $\mathrm{Sh}^{tr.in}(B_{x_0}\mathcal{S}_{0.et})$. By the proposition 3.24, $e^*i^*i_* = i^*e^*i_* = i^*i_*e^*$. Therefore, it suffices to show that $i^* \circ i_* \to id$ is an isomorphism as functors on $\mathrm{Sh}^{tr.in}(B_{x_0}\mathcal{S}_{0.et})$, which is true by (3.1.9). Finally, it remains to show that if $F \in \mathrm{Sh}(\mathcal{S}_{iso.et})$ and $j^*F = \phi$, then the natural map $F \to i_*i^*F$ is an isomorphism. However, if $j^*(F) = \phi$, $j^*e^*(F) = e^*j^*(F) = \phi$ and therefore $e^*(F) \to i_*i^*e^*(F)$ is an isomorphism in $\mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}})$. Since e^* is faithful, the map $F \to i_*i^*(F)$ is an isomorphism as required. □

THEOREM 3.27. *Let $\mathcal{S}$ denote a finitely presented algebraic stack over the base scheme S with $x: X \to \mathcal{S}$ an atlas. Then the map*

$$e^*: \mathrm{Sh}(\mathcal{S}_{iso.et}) \to \mathrm{Sh}^{tr.in}(B_x\mathcal{S}_{\mathrm{et}}) \tag{3.1.24}$$

is an equivalence of topoi. There is a finite filtration of $\mathcal{S}$

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_n = \mathcal{S} \tag{3.1.25}$$

by locally closed algebraic substacks so that each $(\mathcal{S}_i - \mathcal{S}_{i-1})_{red}$ is a gerbe over its coarse moduli-space $\mathfrak{M}_i$ (which exists as an algebraic space) and $\mathrm{Sh}((\mathcal{S}_i - \mathcal{S}_{i-1})_{iso.et})$ is equivalent to the topos of sheaves on $\mathfrak{M}_{i,et}$. The isovariant étale site has a conservative family of points and the points correspond to the geometric points of the coarse-moduli space of $\mathfrak{M}_i$ for all i.

Proof. First observe that a filtration as in (3.1.25) with each $\mathcal{S}_i - \mathcal{S}_{i-1}$ a gerbe over its coarse moduli space exists for any reduced algebraic stack. Therefore, the second statement follows immediately from Theorem 3.13. Moreover, the same theorem shows that the functor e^* induces an equivalence $\mathrm{Sh}_{sets}((\mathcal{S}_i - \mathcal{S}_{i-1})_{iso.et}) \to \mathrm{Sh}^{tr.in}_{sets}(B_{x_i}(\mathcal{S}_i - \mathcal{S}_{i-1})_{\mathrm{et}})$ where x_i is the induced atlas for $\mathcal{S}_i - \mathcal{S}_{i-1}$. By the previous theorem (and by Proposition 3.17), $\mathrm{Sh}_{sets}(\mathcal{S}_{iso.et})$ ($\mathrm{Sh}^{tr.in}_{sets}(B_x\mathcal{S}_{\mathrm{et}})$) is obtained by gluing the topoi $\mathrm{Sh}_{sets}((\mathcal{S}_i - \mathcal{S}_{i-1})_{iso.et})$ ($\mathrm{Sh}^{tr.in}_{sets}(B_{x_i}(\mathcal{S}_i - \mathcal{S}_{i-1})_{\mathrm{et}})$, respectively). This proves the first statement.

As shown in Theorem 3.13, the isovariant étale site of $\mathcal{S}_i - \mathcal{S}_{i-1}$ is equivalent to the étale site of its coarse-moduli space $\mathfrak{M}_i$. Since the topos $\mathrm{Sh}_{sets}(\mathcal{S}_{iso.et})$ is obtained by gluing the topoi $\mathrm{Sh}_{sets}((\mathcal{S}_i - \mathcal{S}_{i-1})_{iso.et})$, it follows that the geometric points of the coarse moduli-space of all the $\mathcal{S}_i - \mathcal{S}_{i-1}$, all i, form a conservative family of points. □

Remark 3.28. In view of Remark 3.20, the results of both the above theorems extend to sheaves with values in other categories, like Abelian sheaves, sheaves of modules over a ring etc.

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4. Hypercohomology on the Isovariant Étale Site

In this section we define and establish several properties for the hyper-cohomology computed on the isovariant site with respect to a presheaf of spectra. In view of Theorem 3.27, we may use the Godement resolutions to define this: in fact the general framework of presheaves on a site with values in a complete pointed simplicial category adopted in [J-2] and [J-3] is perfectly suitable for us. We begin by adding a few more basic hypotheses.

4.1. FURTHER HYPOTHESES

Let J denote a set of primes in $\mathbb{Z}$. Assume that the base scheme S is of finite Krull dimension and that there is a uniform bound on the étale cohomological dimension of the residue fields $k(s)$ for all points s in S with respect to all l-torsion sheaves and all $l \in J$. (Observe that this hypothesis holds if S is of finite type over an algebraically closed field or over $\mathbb{Z}[\sqrt{-1}]$ or if 2 does *not* belong to J and S is of finite type over $\mathbb{Z}$.) Assume also that l is *invertible* in $\mathcal{O}_X$, for any X which is a finitely presented object over the base scheme S that we consider.

4.2. CONVENTIONS

Let $\mathcal{C}$ denote a site which is *closed under all finite inverse limits*, let $\bar{\mathcal{C}}$ denote a set, let (*sets*) denote the category of all small sets and let $(sets)^{\bar{\mathcal{C}}}$ denote the product of the category (sets) indexed by $\bar{\mathcal{C}}$. Assume that we are given a conservative family of points of $\mathcal{C}$ indexed by $\bar{\mathcal{C}}$: recall this means we are given a morphism $\bar{\pi}\colon (sets)^{\bar{\mathcal{C}}} \to \mathcal{C}$ of sites so that a sequence of sheaves $F' \to F \to F''$ (with values in any Abelian category) is short-exact if and only if $0 \to \bar{\pi}^*(F') \to \bar{\pi}^*(F) \to \bar{\pi}^*(F'') \to 0$ is exact. For the most part $\mathbf{S}$ will denote the category of fibrant spectra, though any of the other categories appearing in [J-2] may also be used. Now Presh$(\mathcal{C}, \mathbf{S})$ will denote the category of all presheaves on the site $\mathcal{C}$ taking values in $\mathbf{S}$. If $\mathbf{S}$ denotes the category of fibrant spectra and $P \in \text{Presh}(\mathcal{C}, \mathbf{S})$, $\pi_n(P)\tilde{\ }$ will denote the sheaf associated to the Abelian presheaf on $\mathcal{C}\colon U \to \pi_n(\Gamma(U, P))$ where π_n is the nth (stable) homotopy group. A map of presheaves $f\colon P \to P'$ in Presh$(\mathcal{C}, \mathbf{S})$ will be called a *quasi-isomorphism* if it induces an isomorphism on $\pi_n(f)\tilde{\ }$. (*Throughout we will denote quasi-isomorphisms by* $\simeq$ *while isomorphisms will be denoted by* $\cong$.)

4.2.1. *Cohomology Truncation*

In all cases, $\tau_{\leqslant n}P$ will denote an object in Presh$(\mathcal{C}, \mathbf{S})$ defined by $\pi_i(\Gamma(U, \tau_{\leqslant n}P)) \cong \pi_i(\Gamma(U, P))$ if $i \leqslant n$ and $\cong 0$ otherwise, for any U in the site $\mathcal{C}$. In the case of fibrant spectra, the above truncation functors are defined by the canonical Postnikov truncation functors. (See [T-5] Lemma (5.51), for example). One may observe that $\{\Gamma(U, \tau_{\leqslant n}P)|n\}$ is an inverse system of fibrations for each U in this case. Moreover, the natural map $P \to \lim_{\infty \leftarrow n} \tau_{\leqslant n}P$ is an *isomorphism* of presheaves.

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4.2.2. *Homotopy Inverse Limits*

Observe that there exists a bi-functor:

$$\otimes\colon (\text{pointed simplicial sets}) \times \operatorname{Presh}(\mathcal{C}, \mathbf{S}) \to \operatorname{Presh}(\mathcal{C}, \mathbf{S})$$

(The functor $\otimes$ is defined in [J-2] Section 6 as a colimit and therefore commutes with colimits in either argument.) Let $L(Const)\colon \operatorname{Presh}(\mathcal{C}, \mathbf{S}) \to \operatorname{Presh}(\mathcal{C}, \mathbf{S})^{\Delta}$ denote the functor sending an object $M \in \mathbf{S}$ to the cosimplicial object $n \mapsto \Delta[n]_+ \otimes M$.

The above functor has a right adjoint which is called the homotopy inverse limit along Δ and denoted $\underset{\Delta}{\operatorname{holim}}$. This will be defined as an *end* and therefore will commute with inverse limits. (See [J-2], Section 6, for details on the homotopy inverse limit.)

In the above situation, a map $f\colon X^\bullet \to Y^\bullet$ between two cosimplicial objects in *Presh*$(\mathcal{C}, \mathbf{S})$ will be called a quasi-isomorphism if for each n, the map $f^n\colon X^n \to Y^n$ is a quasi-isomorphism. In the above situations, the functor $\underset{\Delta}{\operatorname{holim}}$ preserves quasi-isomorphisms (and therefore defines a functor at the level of the associated derived categories). (See [J-2] (6.3.4) for a discussion of these.)

4.2.3. *The Canonical Resolutions of Godement*

We will assume the situation of 4.2. Let $\mathcal{C}$ denote a site as there. Assume that we are given a conservative family of points of $\mathcal{C}$ indexed by $\bar{\mathcal{C}}$ as above. (For each point p of $\bar{\mathcal{C}}$ is associated a point of the site $\mathcal{C}$ indexed by p itself.) Let a denote the functor sending a presheaf on $(sets)^{\bar{\mathcal{C}}}$ to its associated sheaf and let U denote the forgetful functor sending a sheaf on the site $\mathcal{C}$ to its underlying presheaf. Now the functors $U \circ \bar{\pi}_*$ and $a \circ \bar{\pi}^*$ define a triple; let $G = U \circ \bar{\pi}_* \circ a \circ \bar{\pi}^* = \bar{\pi}_* \circ U \circ a \circ \bar{\pi}^*$. Observe that $G = \prod_{p \in \bar{\mathcal{C}}} p_* \circ U \circ a \circ p^*$ where, for each point p of $\bar{\mathcal{C}}$ is the associated map of sites $p\colon (sets) \to \mathcal{C}$. Let $P \in \operatorname{Presh}(\mathcal{C}; \mathbf{S})$.

The above triple defines an augmented cosimplicial object $G^\bullet P\colon P \xrightarrow{d^{-1}} GP \dots G^{n+1}P$ in $\operatorname{Presh}(\mathcal{C}; \mathbf{S})$. We define

$$\mathcal{G}P = \underset{\Delta}{\operatorname{holim}}\{G^n P|n\}, \quad \text{i.e. } \Gamma(U, \mathcal{G}P) = \underset{\Delta}{\operatorname{holim}}\{\Gamma(U, G^n P)|n\}$$

for any U in the site $\mathcal{C}$.

Let $\mathcal{C}$, $\mathcal{C}'$ denote two sites and let $\phi_*\colon \operatorname{Presh}(\mathcal{C}'; \mathbf{S}) \to \operatorname{Presh}(\mathcal{C}; \mathbf{S})$ denote a *left-exact functor*. We define *the right-derived functor* $R\phi_*\colon \operatorname{Presh}(\mathcal{C}'; \mathbf{S}) \to \operatorname{Presh}(\mathcal{C}; \mathbf{S})$ by

$$R\phi_*(P) = \underset{\Delta}{\operatorname{holim}}\{\phi(G^n P)|n\}. \tag{4.2.4}$$

This is the presheaf defined by

$$U \to \Gamma(U, R\phi_*(P)) = \underset{\Delta}{\operatorname{holim}}\{\Gamma(U, \phi_*(G^n P))|n\}.$$

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The spectral sequence of [J-2], (6.3.6) provides a spectral sequence

$$E_2^{s,t} = R^s\phi_*(\pi^{-t}(P)) \Rightarrow R^{s+t}\phi_*(P). \tag{4.2.5}$$

We also define the global section functor for presheaves. For this purpose let pt denote the site with one object, pt, and one morphism which is the identity map of pt. (This category is made into a site in the obvious trivial manner.) Now one may identify presheaves on pt with values in a category $\mathbf{S}$ with the category $\mathbf{S}$ itself. If $\mathcal{C}$ is a site with a terminal object X, we define a map of sites $\pi: \mathcal{C} \to pt$ by sending pt to X. We let $\Gamma(\mathcal{C}, P) = \Gamma(X, P) = \pi_*(P)$ for any $P \in \mathrm{Presh}(\mathcal{C}, \mathbf{S})$ and

$$\mathbb{H}_{\mathcal{C}}(X, P) = R\Gamma(X, -)(P), \qquad 4.2.6$$

where the right-hand side is defined as in (4.2.4). This defines the hyper-cohomology on the isovariant étale site with respect to any presheaf of spectra P. This will be denoted $\mathbb{H}_{iso.et}(X, P)$.

PROPOSITION 4.1. *Assume in addition to the above situation that there exists a functor ϕ^* left adjoint to ϕ_*. Then the obvious map $R\phi_*(P) \to \lim_{\infty\leftarrow n} R\phi_*(\tau_{\leqslant n}P)$ is a quasi-isomorphism for any $P \in \mathrm{Presh}(\mathcal{C}', \mathbf{S})$.*

Proof. See [J-2] (3.4.1) for a proof. □

COROLLARY 4.2. *Assume that both the sites above are closed under finite inverse limits.*

(i) *Next assume the following in addition to the hypothesis of* (4.2.4). *Let $\mathcal{C}$ be a full sub-category of $\mathcal{C}'$, let $\phi : \mathcal{C}' \to \mathcal{C}$ be the* map *of sites associated to a fully-faithful functor $\bar{\phi}: \mathcal{C} \to \mathcal{C}'$ and let ϕ_* be the direct image functor of presheaves associated to ϕ. Assume that every $\mathcal{C}$-covering of any object U in $\mathcal{C}$ is a $\mathcal{C}'$-covering and that every $\mathcal{C}'$-covering of such an object is dominated by a $\mathcal{C}$-covering. If $P \in \mathrm{Presh}(\mathcal{C}', \mathbf{S})$, the natural* map *$\phi_*(P) \to R\phi_*(P)$ is a quasi-isomorphism.*

(ii) *Assume the following in addition to the hypotheses of* (4.2.4). *There exists a* map *of sites $\phi : \mathcal{C}' \to \mathcal{C}$ so that ϕ^* is the inverse image functor of presheaves associated to ϕ. If $P \in \mathrm{Presh}(\mathcal{C}, \mathbf{S})$, the obvious* map *$P \to R\phi_*\phi^*(P)$ is a quasi-isomorphism if the corresponding* map *$F \to R\phi_*\phi^*(F)$ is a quasi-isomorphism for any Abelian sheaf F on the site $\mathcal{C}$.*

Proof. We consider (i) first. The hypotheses readily imply that the functor ϕ_* on Abelian sheaves is exact. (See [Mi] p. 111.) It follows also that the spectral sequence in (4.2.5) degenerates identifying $\pi_k(R\phi_*(P))\tilde{}$ with $\phi_*(\pi_k(P))\tilde{}$. Since the sites are all closed under finite inverse limits, the direct limits involved in the definition of the stalks are all filtered direct limits and commute with taking π_k. The hypotheses imply that the stalks of $\pi_k(\phi_*(P))$ and $\phi_*\pi_k(P)$ are both isomorphic to the stalks of the presheaf $\pi_*(P)$. It follows that the natural map $\pi_k(\phi_*(P))\tilde{} \to \phi_*(\pi_k(P))\tilde{}$ is an isomorphism. This proves (i).

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First we show that (ii) holds when P is replaced by $\tau_{\leqslant n}P$ for any fixed integer n. Recall ϕ^* is exact in the sense it commutes with finite direct and inverse limits. (This follows from the hypothesis that the sites are closed under finite inverse limits.) It follows that the spectral sequence in [J -1](6.3.6) for $R\phi_* \circ \phi^*(P)$ now reduces to the spectral sequence in (4.2.5) for $R\phi_*$ applied to $\phi^*(P)$. The hypothesis on P ensures that this spectral sequence converges strongly. Therefore, we reduce to showing that the map $\pi_t(P)\tilde{\ } \to R\phi_*\phi^*(\pi_t(P)\tilde{\ })$ is a quasi-isomorphism for all t. This proves (ii) holds when P is replaced by any $\tau_{\leqslant n}P$.

Now $P \cong \lim_{\infty \leftarrow n} \tau_{\leqslant n} P$. Applying Proposition 4.1 to P replaced by $\phi^*(P)$, it suffices to show that $\phi^*(\tau_{\leqslant n}P) \simeq \tau_{\leqslant n}(\phi^*(P))$ as presheaves. Since the functor $\tau_{\leqslant n}$ is characterized by $\pi_k(\tau_{\leqslant n}P) \cong \pi_k(P)$ if $k \leqslant n$ and $\cong 0$ otherwise, it suffices to show $\pi_k(\phi^*(P)) \cong \phi^*(\pi_k(P))$ as Abelian presheaves. Since ϕ^* is assumed to be the inverse image functor associated to a map of sites it is defined by a filtered direct limit which commutes with taking π_k. □

PROPOSITION 4.3. *Let $\mathcal{S}$ denote an algebraic stack finitely presented over the base scheme S. Under the above hypotheses, there is a uniform bound $M >> 0$ so that for every $\mathcal{S}' \to \mathcal{S}$ in the site $\mathcal{S}_{iso.et}$, $H^n_{iso.et}(\mathcal{S}', F) = 0$ for all $n > M$ and all sheaves F of $\mathbb{Z}_{(J)}$-modules on $\mathcal{S}_{iso.et}$. (Here $\mathbb{Z}_{(J)}$ denotes the localization of $\mathbb{Z}$ by inverting all primes not in J.)*

Proof. The proof is by Noetherian induction. We will assume inductively that the proposition is true for every proper closed immersion $\mathcal{S}_0 \to \mathcal{S}$ of algebraic stacks. By Theorem 3.27, we may assume without loss of generality that $\mathcal{S}$ is reduced and that there exists such a closed immersion so that if $\mathcal{S}_1$ denotes the complement of $\mathcal{S}_0$, $\mathcal{S}_1$ is a gerbe over its coarse moduli space $\mathfrak{M}_1$. Now $\mathfrak{M}_1$ is an algebraic space finitely presented over the base scheme S and therefore, there exists a uniform bound on the étale cohomological dimension of $\mathfrak{M}'_1 \to \mathfrak{M}_1$ in the étale site of $\mathfrak{M}_1$. By the equivalence of topoi as in Theorem 3.13, the conclusion of the proposition now holds for $\mathcal{S}_1$. Let M_1 denote the uniform bound on the cohomological dimension here and let M_0 denote the uniform bound on the cohomological dimension on $\mathcal{S}_0$. Now $M = M_1 + M_0 + 1$ will be a uniform cohomological bound on $\mathcal{S}_{iso.et}$. This argument follows exactly as in [T-3] pp. 607–608 and is therefore skipped. □

PROPOSITION 4.4. *Let $\mathcal{S}$ denote an algebraic stack that is Noetherian. Then the isovariant étale site of $\mathcal{S}$ as well as the corresponding topos is algebraic and coherent in the sense of* [*SGA*] 4, *VI*, 2.3. *Therefore, if $\{P_\alpha|\alpha\}$ is a filtered direct limit of Abelian presheaves,*

$$\operatorname*{colim}_{\alpha} \mathbb{H}_{iso.et}(\mathcal{S}, P_\alpha) \simeq \mathbb{H}_{iso.et}(\mathcal{S}, \operatorname*{colim}_{\alpha} P_\alpha).$$

The same conclusion holds if $\{P_\alpha|\alpha\}$ is a filtered direct system of presheaves taking values in **S** *so that $\pi_*(P_\alpha)\tilde{\ }$ are all sheaves of modules over $\mathbb{Z}_{(J)}$.*

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Proof. The site $\mathcal{S}_{iso.et}$ consists of Noetherian algebraic stacks and is closed under fibered products. Moreover, every isovariant étale cover for an object in $\mathcal{S}_{iso.et}$ has a finite subcover. Therefore, every object U in this site is both quasi-compact and quasi-separated both in the site and also when viewed as an element of the topos $Sh_{sets}(\mathcal{S}_{iso.et})$. (See [SGA] 4, VI, 2.1.1, 1.1 and 1.2.) By [SGA] 4, 2.3, 2.4.1, the site $\mathcal{S}_{iso.et}$ and the topos $Sh_{sets}(\mathcal{S}_{iso.et})$ are both algebraic and coherent. This proves the first assertion. Now the second assertion holds when $\{P_\alpha|\alpha\}$ is a filtered direct system of Abelian sheaves by [SGA] 4, 8.7.3, 3.1 and VII, 5.7.

Observe that the spectral sequence (4.2.5) with $\phi = \mathbb{H}_{iso.et}(\mathcal{S}, \quad)$ converges strongly for every presheaf P with values in $\mathbf{S}$ so that $\pi_*(P)\tilde{}$ is a sheaf of modules over $\mathbb{Z}_{(J)}$. Therefore, the hyper-cohomology also commutes with filtered colimits of presheaves taking values in $\mathbf{S}$ satisfying the hypotheses of the proposition. □

PROPOSITION 4.5. *Let* $\{\mathcal{S}_\alpha \stackrel{f_{\alpha,\beta}}{\leftarrow} \mathcal{S}_\beta|\alpha, \beta \in I\}$ *denote a filtered direct system of Noetherian algebraic stacks where each* map $f_{\alpha,\beta}$ *is representable and affine.*

(i) *Then the inverse limit* $\lim_\alpha \mathcal{S}_\alpha = \mathcal{S}$ *exists as an algebraic stack. There exists a compatible system of projections* $\{p_\alpha : \mathcal{S} \to \mathcal{S}_\alpha|\alpha\}$.
(ii) *For each* α*, let* P_α *denote a presheaf on* $\mathcal{S}_{\alpha,iso.et}$ *with values in* $\mathbf{S}$ *so that* $\pi_*(P_\alpha)\tilde{}$ *are sheaves of modules over* $\mathbb{Z}_{(J)}$. *Assume further that for each* $\beta \geqslant \alpha$*, there is given a* map $f^*_{\alpha,\beta}(P_\alpha) \to P_\beta$ *so that the collection of such maps are compatible (in the obvious sense). Let* P = *the direct limit of the filtered direct system* $\{p^*_\alpha(P_\alpha)|\alpha\}$ *of presheaves on* $\mathcal{S}_{iso.et}$. *Now the canonical* map

$$\operatorname*{colim}_\alpha \mathbb{H}_{iso.et}(\mathcal{S}_\alpha, P_\alpha) \xrightarrow{\simeq} \mathbb{H}_{iso.et}(\mathcal{S}, \operatorname*{colim}_\alpha p^*_\alpha P_\alpha)$$

is a quasi-isomorphism.

Proof. We will first show that $\mathcal{S}$ exists. Pick an $\alpha_0 \epsilon I$ and consider the cofinal system of $\beta \epsilon I$ so that $\beta \mapsto \alpha_0$. Let $x_{\alpha_0} : X_{\alpha_0} \to \mathcal{S}_{\alpha_0}$ denote an atlas and let

$$X_{\alpha_0} \underset{\mathcal{S}_{\alpha_0}}{\times} X_{\alpha_0} \underset{t_{\alpha_0}}{\overset{s_{\alpha_0}}{\rightrightarrows}} X_{\alpha_0} \tag{4.2.7}$$

denote the corresponding algebraic groupoid. Since each $f_{\alpha_0,\beta}: \mathcal{S}_\beta \to \mathcal{S}_{\alpha_0}$ is representable, one may take pull-backs by this map of the algebraic groupoid in (4.2.7) to obtain the algebraic groupoid

$$X_\beta \underset{\mathcal{S}_\beta}{\times} X_\beta \underset{t_\alpha}{\overset{s_\beta}{\rightrightarrows}} X_\beta, \quad \beta \geqslant \alpha_0 \tag{4.2.8}$$

The induced maps of the corresponding algebraic groupoids are affine and therefore, one may take the inverse limit to define an algebraic groupoid

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$$X_1 = \lim_{\beta}(X_\beta \underset{\mathcal{S}_\beta}{\times} X_\beta) \underset{\underset{\beta}{\lim s_\alpha}}{\overset{\underset{\beta}{\lim s_\beta}}{\rightrightarrows}} X_0 = \lim_{\beta} X_\beta \tag{4.2.9}$$

Since $s = \lim_\beta s_\beta$ and $t = \lim_\beta t_\beta$ the diagonal map $\delta = (s, t) = \lim_\beta(s_\beta, t_\beta) : X_1 \to X_0 \times X_0$ is also quasi-compact and separated. Therefore, by [L-MB] Corollaire (4.7), the above groupoid defines an algebraic stack $\mathcal{S}$. Clearly the projections $p_\alpha \colon \mathcal{S} \to \mathcal{S}_\alpha$ exist. These prove the first assertion. The second assertion for the case of Abelian sheaves follows readily from [SGA] IV, Exposé VI, 8.7.4. The general case follows as in the proof of Proposition 4.4. □

We end this section by briefly considering Čech hyper-cohomology on the isovariant site.

DEFINITION 4.6. *Weakly cofinal system of coverings*. Let $\mathcal{S}$ denote an algebraic stack. A system, $\{\mathcal{S}_\alpha | \alpha\}$, of *isovariant étale coverings* of $\mathcal{S}$ is weakly cofinal in the system of all isovariant étale coverings of $\mathcal{S}$, if each isovariant étale covering has a refinement in the given system.

PROPOSITION 4.7. *Let $\mathcal{S}$ denote an algebraic stack as before and let P denote a presheaf on $\mathcal{S}_{iso.et}$ with values in* $\mathbf{S}$ *so that $\pi_*(P)\tilde{\ }$ is a sheaf of modules over $\mathbb{Z}_{(J)}$. Let*

$$\check{\mathbb{H}}_{iso.et}(\mathcal{S}, P) = \underset{\Delta}{\mathrm{holim}}\ \underset{\rightarrow}{\lim}\Gamma(\mathrm{cosk}_0(u), P), \tag{4.2.10}$$

where the colimit is over a weakly cofinal system of isovariant étale coverings $u \colon U \to \mathcal{S}$ of $\mathcal{S}$. Now there exists quasi-isomorphisms

$$\mathbb{H}_{iso.et}(\mathcal{S}, P) \simeq \check{\mathbb{H}}_{iso.et}(\mathcal{S}, \mathbb{H}_{iso.et}(\quad, P)). \tag{4.2.11}$$

Let $(alg.stacks/S)_{iso.et}$ denote the big isovariant étale site of algebraic stacks over S, i.e. objects are algebraic stacks over S, morphisms are morphisms over S and coverings are isovariant étale coverings. Let P denote a presheaf on $(alg.stacks/S)_{iso.et}$ which has the localization property: i.e. for each closed immersion $\mathcal{S}_0 \to \mathcal{S}$ of algebraic stacks with open complement $\mathcal{S}_1 \to \mathcal{S}$, one obtains a fibration sequence of presheaves $\Gamma(\mathcal{S}_0, P) \to \Gamma(\mathcal{S}, P) \to \Gamma(\mathcal{S}_1, P)$ in the sense of Definition 5.6. In this case, one also obtains the quasi-isomorphism: $\check{\mathbb{H}}_{iso.et}(\mathcal{S}, \mathbb{H}_{iso.et}(\quad, P)) \simeq \check{\mathbb{H}}_{iso.et}(\mathcal{S}, P)$.

Proof. In view of the hypotheses, there is a uniform cohomological bound which shows that the hypotheses of [T-5] Theorem 1.46 are met. This proves the first quasi-isomorphism. Again, by the hypotheses, one has a uniform cohomological bound, which enables one to prove the last quasi-isomorphisms as in [T-5] Proposition 1.54. (Using the observation that $\check{\mathbb{H}}_{iso.et}(\quad, -)$ and $\mathbb{H}_{iso.et}(\quad, -)$ preserve fibration sequences, one may in fact use devissage as in Theorem 3.27 to reduce to the case where the isovariant étale site is replaced by the étale site of the coarse moduli space. At this point one may invoke [T-5] Proposition 1.54 to finish the proof.) □

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5. Isovariant Étale and Étale Cohomological Descent

5.1. ADDITIONAL HYPOTHESES

In this section we need to put additional hypotheses as in [T-3] (3.1) on the base scheme S, in addition to the ones in (4.1). We will assume the following: There is a Tate-Tsen filtration on the separable closure of the residue fields $k(s)$ at all points of S as in [T-5] 2.112. This hypothesis is satisfied if $k(s)$ is of finite transcendence degree over $\mathbb{Q}$, $\hat{\mathbb{Q}_p}$, $\mathbb{F}_p$, $\mathbb{F}_p((t))$ or over a separably closed field $\bar{k}$.

DEFINITION 5.1. Let $\mathcal{S}$ denote an algebraic stack and let G denote an *affine smooth group-scheme* both defined over a a Noetherian base ring S. A representable morphism $\mu: G \times \mathcal{S} \to \mathcal{S}$ defines *an action* of G on $\mathcal{S}$ if it satisfies the following conditions:

(*lax associativity and lax unit*): viewing $G \times G \times \mathcal{S}$, $G \times \mathcal{S}$ and $\mathcal{S}$ as lax-functors $(schemes/S)^{op} \to (groupoids)$ the obvious associativity and unit axiom for group-actions hold in the 2-category of lax-functors.

Remarks 5.2. (1) Recall a lax-functor $F: (schemes/S)^{op} \to (groupoids)$ is a not a functor, but the following data: for each $X \in (schemes/S)$, one is given a groupoid $F(X)$ and for each morphism $f: Y \to X$ of schemes over S, one is given a morphism $F(f): F(X) \to F(Y)$ so that if $g: Z \to Y$ is another morphism of schemes, one is given a natural isomorphism $\epsilon_{g,f}: F(g) \circ F(f) \xrightarrow{\simeq} F(g \circ f)$ so that the natural isomorphisms satisfy an obvious associativity and unital condition. (See [Hak], Chapitre I, for details: Lax-functors are called 2-functors there.) An algebraic stack may be viewed, therefore, as a lax-functor in the above sense satisfying certain other conditions.

(2) In general, there may not exist an atlas for the stack onto which the group-scheme action extends. This is similar to the situation where an algebraic group acts on a scheme, and in general, there may not be an affine cover of the scheme, which is stable by the group action. Assume that G is a torus or a diagonalizable group scheme acting on a stack $\mathcal{S}$ that is normal. By [Sum] and [J-2], we see that any atlas onto which the action extends may be refined to an atlas that is affine.

(3) Suppose in addition to the hypothesis in Definition 5.1, that a coarse moduli space $\mathfrak{M}$ exists (as an algebraic space) for the stack $\mathcal{S}$. Then $G \times \mathfrak{M}$ is a coarse-moduli space for the stack $G \times \mathcal{S}$. The universal property of the coarse-moduli space for maps from algebraic stacks to algebraic spaces shows that the composition $G \times \mathcal{S} \xrightarrow{\mu} \mathcal{S} \to \mathfrak{M}$ factors through $G \times \mathfrak{M}$, where μ denotes the group-action. It follows that one obtains an induced action of G on the coarse-moduli space $\mathfrak{M}$.

(4) A particularly simple example of a group action on an algebraic stack is the following. Assume that the stack $\mathcal{S}$ in Definition 5.1 is in fact the quotient stack $[X/H]$ associated to the action of a group-scheme H on the algebraic space X. We will, assume in this situation, that X itself is the atlas of $[X/H]$ onto which the

G-action lifts and that the actions of G and H on X commute. Therefore, we obtain an action of the group-scheme $G \times H$ on X.

DEFINITION 5.3. Let G denote an affine smooth group scheme acting on the algebraic stack $\mathcal{S}$. We say the action is *trivial* if there is a splitting to the top-row in the diagram (i.e. there is a 1-morphism $s: G \times \mathcal{S} \to P$ so that there is given a 2-isomorphism $\epsilon \circ s \overset{\simeq}{\to} \mathrm{id}_{G \times \mathcal{S}}$):

$$\begin{array}{ccc} P & \xrightarrow{\epsilon} & G \times S \\ \big\downarrow & & \big\downarrow{\scriptstyle(\mu, pr_2)} \\ \mathcal{S} & \xrightarrow{\Delta} & \mathcal{S} \times \mathcal{S} \end{array} \tag{5.1.1}$$

(Equivalently the two morphisms μ, $pr_2: G \times \mathcal{S} \to \mathcal{S}$ may be identified in the 2-category of lax-functors $(schemes/S)^{op} \to (groupoids)$ and lax-natural transformations between them.)

DEFINITION 5.4. Let $\mathcal{S}$ denote an algebraic stack and let $\mathcal{S}_{smt}$ denote the smooth site of $\mathcal{S}$. A sheaf F of $\mathcal{O}_{\mathcal{S}}$-modules on $\mathcal{S}_{smt}$ is a coherent sheaf (a vector bundle) if for any atlas $x: X \to \mathcal{S}$, $x^*(F)$ is a coherent sheaf (a vector bundle, respectively) on X_{smt}. One may see that the category of coherent sheaves (vector bundles) is *Abelian* (*exact*) and also symmetric monoidal under the direct sum operation. The former (latter) category will be denoted $\mathrm{Coh}(\mathcal{S})$ ($\mathrm{Vect}(\mathcal{S})$, respectively). We let $G(\mathcal{S})$ ($K(\mathcal{S})$) denote the algebraic K-theory spectrum of the category of coherent sheaves (vector bundles, respectively). One may also consider the corresponding presheaves of fibrant spectra on the site $\mathcal{S}_{iso.et}$: these are denoted $\mathbf{G}$ and $\mathbf{K}$, respectively. In addition, we may need to consider the situation where a smooth group scheme G acts on an algebraic stack $\mathcal{S}$ as in Definition 5.1. Making use of Proposition (7.1) in the appendix, one may observe that G-equivariant coherent sheaves (vector bundles) on the stack $\mathcal{S}$ correspond to coherent sheaves (vector bundles, respectively) on the quotient stack $[\mathcal{S}/G]$. i.e. If we let $\mathrm{Coh}(\mathcal{S}, G)$ ($\mathrm{Vect}(\mathcal{S}, G)$) denote the category of coherent sheaves (vector bundles, respectively) on the stack $\mathcal{S}$ that are equivariant with respect to the action of G, then there is an equivalence of categories $\mathrm{Coh}(\mathcal{S}, G) \simeq \mathrm{Coh}([\mathcal{S}/G])$ and $\mathrm{Vect}(\mathcal{S}, G) \simeq \mathrm{Vect}([\mathcal{S}/G])$. (Recall a coherent sheaf F on $\mathcal{S}$ is G-equivariant, if there exists an isomorphism $\phi: pr_2^*(F) \to \mu^*(F)$ satisfying the usual conditions. Here pr_2 (μ): $G \times \mathcal{S} \to \mathcal{S}$ is the projection to the second factor (group action, respectively). Now these conditions correspond to the descent data for a coherent sheaf (vector bundle) on the stack $\mathcal{S}$ to descend to the stack $[\mathcal{S}/G]$.) Therefore, making use of Proposition (7.1), one may incorporate the equivariant theory into the following discussion.

Remark 5.5. It is often advantageous to replace the presheaf $\mathbf{K}$ by the presheaf of K-theory spectra corresponding to perfect complexes on a stack. Then it is shown in [J-3] that, if the stack $\mathcal{S}$ is smooth, one obtains a weak-equivalence $\mathbf{K}(\mathcal{S}) \simeq \mathbf{G}(\mathcal{S})$ where $\mathbf{G}(\mathcal{S})$ is the same G-theory considered above.

DEFINITION 5.6. Let $\mathcal{C}$ denote a site. One may define fibration sequences of presheaves of spectra on $\mathcal{C}$ in the following manner. First one has the notion of a *Path object* associated to any presheaf P. One defines this as $\mathrm{Map}(\Delta[1]_+, P)$ along with the obvious maps $d^i = \mathrm{Map}(d_i, P)\colon \mathrm{Map}(\Delta[1]_+, P) \to \mathrm{Map}(\Delta[0]_+, P) \cong P$, $i = 0, 1$. We will denote this object as $\mathrm{Path}(P)$. Observe that since $\Gamma(U, P)$ is a fibrant spectrum, the maps $\Gamma(U, d^i)$ are fibrations for each U in the site $\mathcal{C}$; if $f\colon P' \to P$ is a map of presheaves, one defines $\mathrm{Path}(f) = \times_{P_{f,Y,d_0}} \mathrm{Path}(P)$ and $\Omega(f) =$ the kernel of the map $\mathrm{Path}(f) \to P$ induced by the map d^1. We call $\Omega(f)$ the *canonical homotopy fibre* of f. A diagram $Q \to P \xrightarrow{f} P''$ of presheaves of spectra is called a *fibration sequence of presheaves* if there exists a map $Q \to \Omega(f)$ which is a quasi-isomorphism and fitting in a commutative diagram

$$\begin{array}{ccccc}
Q & \longrightarrow & P & \xrightarrow{f} & P'' \\
\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\
\Omega(f) & \longrightarrow & P & \xrightarrow{f} & P''
\end{array}$$

PROPOSITION 5.7. (i) *Let $i\colon \mathcal{S}_0 \to \mathcal{S}$ denote a closed immersion of algebraic stacks finitely presented over the base scheme S with open complement $j\colon \mathcal{S}_1 \to \mathcal{S}$. Denoting by $i_\#$ ($j_\#$) the direct image functor for presheaves, one obtains a fibration sequence*

$$i_\# \mathbf{G}_{\mathcal{S}_0}(\quad) \to \mathbf{G}_{\mathcal{S}}(\quad) \to j_\# \mathbf{G}_{\mathcal{S}_1}(\quad), \tag{5.1.2}$$

where $\mathbf{G}_{\mathcal{S}_i}(\quad)$ ($\mathbf{G}_{\mathcal{S}}(\quad)$) denotes the presheaf of spectra defined by $\mathbf{G}(\quad)$ on $\mathcal{S}_{i,iso.et}$ ($\mathcal{S}_{iso.et}$, respectively).

(*ii*) *Assume the following in addition to the hypotheses of* (*i*): *$\mathcal{S}$ denotes an algebraic stack provided with the action of a smooth group scheme G and that $p\colon \mathcal{S} \to X$ is a G-equivariant map to an algebraic space X provided with an action of G. Let $i\colon X_0 \to X$ ($j\colon X_1 = X - X_0 \to X$) denote the G-equivariant closed immersion of a closed sub-algebraic space* (*the G-equivariant open immersion of its complement, respectively*). *Let $\mathcal{S}_i = X_i \times_X \mathcal{S}$ and $p_i\colon \mathcal{S}_i \to X_i$ denote the induced maps. Then one obtains the fibration sequence on $([X/G])_{iso.et}$*

$$i_\# p_{0\#} \mathbf{G}_{[\mathcal{S}_0/G]}(\quad) \to p_\# \mathbf{G}_{[\mathcal{S}/G]}(\quad) \to j_\# p_{1\#} \mathbf{G}_{[\mathcal{S}_1/G]}(\quad), \tag{5.1.3}$$

where $\mathbf{G}_{[\mathcal{S}_i/G]}(\quad)$ ($\mathbf{G}_{[\mathcal{S}/G]}(\quad)$) denotes the presheaf of spectra defined by $\mathbf{G}(\quad)$ on $[\mathcal{S}_i/G]_{iso.et}$ ($([\mathcal{S}/G])_{iso.et}$, respectively) and $p_\#$, $p_{i\#}$ are the obvious direct image functors of presheaves.

Proof. (i) Let $\mathcal{S}' \to \mathcal{S}$ denote an object in the site $\mathcal{S}_{iso.et}$. Now $\mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_0 \to \mathcal{S}'$ is a closed immersion with open complement $\mathcal{S}' \underset{\mathcal{S}}{\times} \mathcal{S}_1 \to \mathcal{S}'$. The commutative diagram

$$\begin{array}{ccccc}
\Gamma(\mathcal{S}', i_\# \mathbf{G}_{\mathcal{S}_0}(\,)) & \longrightarrow & \Gamma(\mathcal{S}', \mathbf{G}_{\mathcal{S}}(\,)) & \longrightarrow & \Gamma(\mathcal{S}', j_\# \mathbf{G}_{\mathcal{S}_1}(\,)) \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\
G(\mathcal{S}' \underset{\mathcal{S}}{\times} \mathcal{S}_0) & \longrightarrow & G(\mathcal{S}') & \longrightarrow & G(\mathcal{S}' \underset{\mathcal{S}}{\times} \mathcal{S}_1)
\end{array}$$

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shows it suffices to prove the bottom row is a fibration sequence of spectra. By invoking Quillen's localization theorem for Abelian categories, it suffices to show that the restriction map induces a weak-equivalence of K-theory spectra: $K(\mathrm{Coh}(\mathcal{S}')/\mathrm{Coh}(\mathcal{S}'_0)) \to K(\mathrm{Coh}(\mathcal{S}'_1))$, where $\mathcal{S}'_0 = \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_0$ and $\mathcal{S}'_1 = \mathcal{S}' \times_{\mathcal{S}} \mathcal{S}_1$. Let $\mathcal{J}$ be the sheaf of ideals defining $\mathcal{S}_0$ in $\mathcal{S}$. In order to apply Quillen's localization theorem one needs to show that every coherent sheaf F on the stack $\mathcal{S}_1$ admits an extension to a coherent sheaf on the stack $\mathcal{S}$: this follows from [L-MB] Proposition (8.5). This completes the proof of (i). An entirely similar argument applies to complete the proof of (ii). The only additional observation needed is that the map p (p_i) induces a map of sites $[\mathcal{S}/G]_{iso.et} \to [X/G]_{iso.et}$ ($[\mathcal{S}_i/G]_{iso.et} \to [X_i/G]_{iso.et}$, respectively). This is clear by Lemma 3.2 (ii). □

PROPOSITION 5.8. *Let* $i\colon \mathcal{S}_0 \to \mathcal{S}$ *denote a closed immersion of algebraic stacks and let* P *denote a presheaf of fibrant spectra on* $\mathcal{S}_{o,iso.et}$. *Denoting by* $i_\#$ *the direct image functor for presheaves, one obtains a weak-equivalence of spectra* $\mathbb{H}_{iso.et}(\mathcal{S}_0, P) \simeq \mathbb{H}_{iso.et}(\mathcal{S}, i_\# P)$ *that is natural in* P.

Proof. In view of Proposition 4.1, it suffices to prove this proposition with P replaced by $\tau_{\leq n}P$ for some n. In this case the spectral sequence in (4.2.5) applied to $\mathbb{H}_{iso.et}(\mathcal{S},\quad)$ and $\mathbb{H}_{iso.et}(\mathcal{S},\quad) \circ i_\#$ reduces the problem to showing an isomorphism at the E_2-terms of the corresponding spectral sequences. i.e. we obtain an isomorphism

$$H^n_{iso.et}(\mathcal{S}_0, a\mathcal{H}^m(P)) \xrightarrow{\simeq} H^n_{iso.et}(\mathcal{S}, a\mathcal{H}^m(i_\# P)). \tag{5.1.4}$$

Observe that $\mathcal{H}^m(i_\# P) \cong i_\#(\mathcal{H}^m(P))$ and by Proposition 3.24, $a \circ i_\# = i_* \circ a$ so that the right-hand side identifies with $H^n_{iso.et}(\mathcal{S}, i_* a\mathcal{H}^m(P))$. Observe from Theorem (3.26) that i_* is a closed immersion of topoi and therefore, by [SGA] 4, IV, Section 14, is an *exact* functor, i.e. one may identify i_* with Ri_*. Therefore, the isomorphism in (5.1.4) follows. □

5.1.5. *Localization of K-Theory Spectra and Other Variants*

Let $\mathcal{S}$ denote an algebraic stack as before and let $\mathbf{G}(\quad)$ denote the presheaf of G-theory spectra on $\mathcal{S}_{iso.et}$. Denoting by $K =$ topological K-homology, one obtains the presheaf of spectra $\mathbf{G}_K(\quad)$ which is a localization of $\mathbf{G}(\quad)$ by K in the sense of Bousfield. (See [Bous].) Given a set of primes J in $\mathbb{Z}$ as before, one may now localize the above presheaf by inverting all primes that are *not* in J. The resulting presheaf of spectra will be denoted $\mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)}$. One may also smash $\mathbf{G}(\quad)$ with the Moore spectrum $M(l^\nu)$, $\nu \gg 0$ to obtain the presheaf $\mathbf{G}/l^\nu(\quad)$. Finally, one may also invert *the Bott element* β to obtain the presheaf $\mathbf{G}/l^\nu[\beta^{-1}](\quad)$. (See [T-5], Chapter 5 for more details.) In addition, one may consider the localization of $\mathbf{G}(\quad)$ at $\mathbb{Q}$ in the sense of [B-K] or [T-5]: this will be denoted $\mathbf{G}(\quad) \otimes \mathbb{Q}$.

PROPOSITION 5.9. *All of the above presheaves are continuous in the following sense. Let* $\{S_\alpha \xleftarrow{f_{\alpha,\beta}} S_\beta | \alpha, \beta \in I\}$ *denote a filtered direct system of Noetherian algebraic stacks*

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where each map $f_{\alpha,\beta}$ is representable and affine. Let $\mathcal{S} = \lim_{\alpha} \mathcal{S}_{\alpha}$. Now colim$\Gamma_{\alpha}$ $(S_{\alpha}, P) \simeq \Gamma(S, P)$ *if P denotes any of the above presheaves.*

Proof. The existence of the inverse limit stack is shown in Proposition 4.5. Observe that a coherent sheaf F on an algebraic stack $\mathcal{S}$ is given by a coherent sheaf F_0 on an atlas $x: X \to \mathcal{S}$ along with descent data. In the above situation, if $x_{\alpha}: X_{\alpha} \to \mathcal{S}_{\alpha}$ are atlases, a coherent sheaf F on $X = \lim_{\alpha} X_{\alpha}$ along with descent data correspond to a compatible collection of coherent sheaves $\{F_{\alpha}$ *on* X_{α} *along with descent data*$|\alpha\}$. It follows that the presheaf $\mathbf{G}$ is continuous. One may prove similarly that the presheaf $\mathbf{K}$ is also continuous. Since localizations of spectra as well as smashing with a fixed spectrum commute with filtered colimits the remaining presheaves in (5.1.5) are also continuous. □

THEOREM 5.10. (i) *Let $\mathcal{S}$ denote an algebraic stack finitely presented over the base scheme S. Then the presheaves of spectra $\mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)}$, $G/l^{\nu}[\beta^{-1}](\quad)$ as well as* $\mathbf{G}(\quad) \otimes \mathbb{Q}$ *have cohomological descent on the isovariant étale site of $\mathcal{S}$. i.e. the obvious augmentations*

$$G_K(\mathcal{S}) \otimes \mathbb{Z}_{(J)} \xrightarrow{\simeq} \mathbb{H}_{iso.et}(\mathcal{S}, \mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)}),$$
$$G(\mathcal{S})/l^{\nu}[\beta^{-1}] \xrightarrow{\simeq} \mathbb{H}_{iso.et}(\mathcal{S}, \mathbf{G}/l^{\nu}[\beta^{-1}](\quad))$$

and

$$G(\mathcal{S}) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathbb{H}_{iso.et}(\mathcal{S}, \mathbf{G}(\quad, G) \otimes \mathbb{Q})$$

are weak equivalences.

(ii) *Assume in addition to the hypotheses in* (i) *that the stack $\mathcal{S}$ is provided with the action by a smooth group scheme G. Let $p: \mathcal{S} \to X$ denote a G-equivariant map to an algebraic space provided with a G-action. Then the augmentations*

$$G_K([\mathcal{S}/G]) \otimes \mathbb{Z}_{(J)} \xrightarrow{\simeq} \mathbb{H}_{iso.et}([X/G], p_{\#}\mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)}),$$
$$G([\mathcal{S}/G])/l^{\nu}[\beta^{-1}] \xrightarrow{\simeq} \mathbb{H}_{iso.et}([X/G], p_{\#}\mathbf{G}/l^{\nu}[\beta^{-1}](\quad))$$

and

$$G([\mathcal{S}/G]) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathbb{H}_{iso.et}([X/G], p_{\#}\mathbf{G}(\quad) \otimes \mathbb{Q})$$

are weak equivalences.

Proof. (i) Since the proofs of the last two quasi-isomorphisms are entirely similar to the first, we will explicitly consider only the first. Since hyper-cohomology on any site sends fibration sequences of presheaves (of spectra) to fibration sequences of spectra, and in view (5.1.2), it should be clear that both sides define localization sequences, i.e. if $\mathcal{S}_0 \to \mathcal{S}$ is a closed immersion with open complement $\mathcal{S}_1 \to \mathcal{S}$, one obtains a commutative diagram whose rows are fibration sequences:

$$\begin{array}{ccccc}
G(\mathcal{S}_0)_K \otimes \mathbb{Z}_{(J)} & \longrightarrow & G(\mathcal{S})_K \otimes \mathbb{Z}_{(J)} & \longrightarrow & G(\mathcal{S}_1)_K \otimes \mathbb{Z}_{(J)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}_{iso.et}(\mathcal{S}_0, \mathbf{G}_K(\ , G) \otimes \mathbb{Z}_{(J)}) & \longrightarrow & \mathbb{H}_{iso.et}(\mathcal{S}, \mathbf{G}_K(\ , G) \otimes \mathbb{Z}_{(J)}) & \longrightarrow & \mathbb{H}_{iso.et}(\mathcal{S}_1, \mathbf{G}_K(\ , G) \otimes \mathbb{Z}_{(J)})
\end{array}$$

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Therefore, (see Theorem 3.27) it suffices to consider the situation when the stack $\mathcal{S}$ is a gerbe over its coarse moduli space $\mathfrak{M}$. In this case, the equivalence of sites as in Theorem 3.13, shows that one may identify $\mathbb{H}_{iso.et}(\mathcal{S}, \mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})$ with $\mathbb{H}_{\mathrm{et}}(\mathfrak{M}, p_{\#}(\mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)}))$. Here $p_{\#}(\mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})$ is the presheaf on $\mathfrak{M}_{\mathrm{et}}$ defined by

$$\Gamma(U, p_{\#}(\mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})) = \Gamma(U \underset{\mathfrak{M}}{\times} \mathcal{S}), \mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)} = G_K(U \underset{\mathfrak{M}}{\times} \mathcal{S}) \otimes \mathbb{Z}_{(J)}.$$

Therefore, $\Gamma(\mathfrak{M}, p_{\#}(\mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})) = \Gamma(\mathcal{S}, \mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})$. i.e. It suffices to show that we have cohomological descent on the étale site of the algebraic space $\mathfrak{M}$ for the presheaf of spectra $p_{\#}(\mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})$. The continuity property and the localization sequence (5.1.2) reduce to establishing cohomological descent for the case $\mathfrak{M}$ is replaced by an Artin local ring–see [T-5], Section 2. Moreover, observe that the map $G(\mathcal{S}_0) \to G(\mathcal{S})$ induces a weak equivalence for any closed immersion $\mathcal{S}_0 \to \mathcal{S}$ of algebraic stacks defined by a nilpotent ideal. As in [T-5] Lemma (2.10), this shows it suffices to establish cohomological descent for the presheaf $p_{\#}(\mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})$ on the étale site of fields.

At this point, one needs to show that the presheaf of spectra $p_{\#}(\mathbf{G}_K(\ \ ) \otimes \mathbb{Z}_{(J)})$ has hyper-transfer in the sense of Thomason, [T-5] (2.25). (Given such a hypertransfer, Lemma (3.10) of [T-3] applies to complete the proof.) For this we will use the following arguments as outlined in [T-5] Example (2.30). First we begin with the definition of the hyper-transfer as in [T-5], Section (2.21). Let G denote a discrete group acting on a spectrum F. We consider G as a category with one object and whose morphisms are the elements of G. Now F may be viewed as a functor from this category to the category of spectra. One defines the *hyper-homology of G with respect to F* to be given by

$$\mathbb{H}_{\bullet}(G, F) = \underset{G}{\operatorname{hocolim}} F. \tag{5.1.6}$$

This functor preserves weak-homotopy equivalences and homotopy (co-)fibre sequences in F and moreover the homotopy colimit is characterized by a universal mapping property as shown in [T-5] (5.15). The group hyper-homology considered above has several properties of which the most important is the following:

Induction weak-equivalence (see [T-5] Lemma (2.22).). Let G denote a group with sub-group H and let F denote a spectrum on which H acts. Let $\sqcup_{G/H} F$ denote the wedge (= the co-product in the category of spectra) indexed by G/H. Now the inclusion of $H \to G$ and the map $F \to \sqcup_{G/H} F$ induce a weak-equivalence:

$$\mathbb{H}_{\bullet}(H, F) \xrightarrow{\simeq} \mathbb{H}_{\bullet}(G, \underset{G/F}{\sqcup} F). \tag{5.1.7}$$

DEFINITION 5.11. Let F denote a presheaf of spectra on the étale site of the spectrum of a field L. F is said to have a *hyper-transfer* if for all finite Galois extensions L'/L and all algebras A over L, there is a map of spectra $T : \mathbb{H}_{\bullet}(\mathrm{Gal}(L'/L); F(A \underset{L}{\otimes} L')) \to F(A)$ satisfying the following conditions:

- the transfer map T must be natural in A
- whenever $A = L''$ is a separable extension of L containing L', there is a homotopy commutative diagram formed

$$\begin{tikzcd}
\mathbb{H}_\bullet(Gal(L'/L), F(L''\underset{L}{\otimes} L')) \arrow[d, "\simeq"] \arrow[dr, "T"] & \\
\mathbb{H}_\bullet(Gal(L'/L), \prod_{Gal(L'/L)} F(L'')) & F(L'') \arrow[dl, "\simeq"] \\
\mathbb{H}_\bullet(Gal(L'/L), \sqcup_{Gal(L'/L)} F(L'')) \arrow[u, "\simeq"'] &
\end{tikzcd}$$

from the hyper-transfer T and the maps in the induction weak equivalence above.

- finally the following diagram homotopy commutes:

$$\begin{tikzcd}
H_\bullet(Gal(L_1/L); H_\bullet(Gal(L_2/L); F(A\underset{L}{\otimes}L_1\underset{L}{\otimes}L_2))) \arrow[r, "H_\bullet(L_1/L;T)"] \arrow[dd, "\simeq"] & H_\bullet(Gal(L_1/L); F(A\underset{L}{\otimes}L_1)) \arrow[d, "T"] \\
& F(A) \\
H_\bullet(Gal(L_2/L); H_\bullet(Gal(L_1/L); F(A\underset{L}{\otimes}L_1\underset{L}{\otimes}L_2))) \arrow[r, "H_\bullet(L_2/L;T)"] & H_\bullet(Gal(L_2/L); F(A\underset{L}{\otimes}L_2)) \arrow[u, "T"']
\end{tikzcd}$$

Given this, one needs to check the hypotheses in [T-3] Lemma (3.10) hold for the presheaf $F = p_\#(\mathbf{G}_K(\quad, G)\otimes\mathbb{Z}_{(J)})$. (Recall that these are the following:

(i) The presheaf F above is a presheaf of module spectra over the presheaf of K-theory spectra. (i.e. $F(A)$ is a module spectrum over $K(A)$ where $K(A)$ is the algebraic K-theory spectrum of A for all A as above and this structure is compatible with the structure of presheaves on the étale site of the field L).

(ii) The hypertransfer in Definition 5.11 is a map of $K(A)$-module spectra (i.e. the projection formula holds).

(iii) The hypertransfer in Definition 5.11 is compatible (in the sense of [T-3] (3.13)) with the hypertransfer for the presheaf of K-theory spectra).

The arguments as in [T-5] Example (2.30) and [T-4] (3.20) through (3.22) (see also Example (2.30) in [T-5]) apply to define a hyper-transfer for the presheaf $F = p_\#(\mathbf{G}(\quad))$ defined on the étale site of $\mathfrak{M}$ by $\Gamma(U, p_\#(\mathbf{G}(\quad))) = G(U\times_{\mathfrak{M}}\mathcal{S})$: one may readily verify the above hypotheses. Now $p_\#(\mathbf{G}_K(\quad)\otimes\mathbb{Z}_{(J)})$ inherits this hyper-transfer.

This completes the proof of (i) for the two presheaves $\mathbf{G}_K(\quad)\otimes\mathbb{Z}_{(J)}$ and $\mathbf{G}(\quad)/l^\nu[\beta^{-1}]$. The proof for the presheaf $\mathbf{G}(\quad)\otimes\mathbb{Q}$ is simpler since one has a strict transfer or what is called a *weak-transfer* in [T-5] and [J-1]: the same proof as above using the hyper-transfer works as well.

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Now we consider the proof of (ii). We will first consider the case when the group G is trivial and the stack is Deligne–Mumford. In this case, the localization sequence in (5.1.3) enables one to reduce to the case when the stack $\mathcal{S}$ is the quotient stack associated to a finite group action on a scheme: this case now follows from [T-3] Theorem (3.8). In case the map $p: \mathcal{S} \to X$ is the identity (i.e. $\mathcal{S}$ itself) is an algebraic space, (ii) also follows from [T-3] Theorem (3.8). Next we consider the general case. The localization sequence (5.1.3) enables one to reduce to the case when the quotient stack $[X/G]$ is a gerbe over its coarse moduli space which can be assumed to be a scheme: the coarse moduli space is a scheme-theoretic quotient in the sense of [T-3] Definition (2.3), which is also a geometric quotient. Therefore, we will denote this by X/G. If $\bar{p}: [X/G] \to X/G$ is the obvious map, now it suffices to establish cohomological descent for the presheaf $\bar{p}_{\#}(p_{\#}(\mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)}))$ on $(X/G)_{\mathrm{et}}$. The continuity property of the presheaf $\mathbf{G}_K(\quad)$ enables one to reduce to the case where X/G has been replaced by the spectrum of a local ring and the localization property as in (5.1.2) enables one to reduce to the case when X/G has been replaced by the spectrum of an Artin local ring. (See [T-5], Section 2 for details.) Moreover, as in the proof of (i), we may reduce to the case of fields. Now it suffices to show that a hypertransfer exists for the presheaf $\bar{p}_{\#}(p_{\#}(\mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)}))$ when X/G has been replaced by the spectrum of a field. The rest of the proof is entirely similar. □

COROLLARY 5.12 (Atiyah–Hirzebruch spectral sequences). (i) *Assume the hypotheses of Theorem* 5.10 (i). *Then there exists a strongly-convergent spectral sequence:*

$$E_2^{s,t} = H^s_{iso.et}(\mathcal{S}, \pi_t(\mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)})) \Rightarrow \pi_{-s+t}(\mathbf{G}_K(\mathcal{S}) \otimes \mathbb{Z}_{(J)}).$$

(ii) *Assume the hypotheses of Theorem* 5.10 (ii). *Then there exists a strongly-convergent spectral sequence:*

$$E_2^{s,t} = H^s_{iso.et}([X/G], \pi_t(p_{\#}\mathbf{G}_K(\quad) \otimes \mathbb{Z}_{(J)})) \Rightarrow \pi_{-s+t}(\mathbf{G}_K([\mathcal{S}/G]) \otimes \mathbb{Z}_{(J)}).$$

The corresponding statements also hold with the presheaf $\mathbf{G}_K$ *replaced by* $\mathbf{G}/l^{\nu}[\beta^{-1}]$ *and* $\mathbf{G}_{\mathbb{Q}}$.

Proof. (i) This spectral sequence is provided by (4.2.5) with $\phi = \mathbb{H}_{iso.et}(\mathcal{S}, \quad)$. The strong convergence follows from the observation that the hypotheses imply $E_2^{s,t} = 0$ for $s \gg 0$. The proofs of (ii) and the last assertion are similar. □

Remark 5.13. Observe that Theorem 5.10 and Corollary 5.12 extend the results of [T-3]: if the stacks are assumed to be algebraic spaces, we recover these results. Moreover, taking the group G to be trivial in the statements (ii) of Theorem 5.10 and the corollary 5.12, we see that the presheaves $p_{\#}\mathbf{G}_K$ and $p_{\#}\mathbf{G}_{\mathbb{Q}}$ have descent on the étale site of the moduli space $\mathfrak{M}$ provided it exists as an algebraic space with a proper map $p: \mathcal{S} \to \mathfrak{M}$.

Remark 5.14. In order to be able to use Theorem 5.10, one needs to be able to identify the stalks of the presheaf $\mathbf{G}_K \otimes \mathbb{Z}_{(J)}$ on the isovariant étale site. The following result shows that it is possible to do this generically, in general in characteristic 0, and globally for Deligne–Mumford stacks, which suffices for the applications. It suffices to do this for the nonequivariant case.

PROPOSITION 5.15. *Let $\mathcal{S}$ denote an algebraic stack as before so that it is a gerbe over its coarse moduli space $\mathfrak{M}$. Let $p: \mathcal{S} \to \mathfrak{M}$ denote the obvious map, let $\bar{x}: \mathrm{Spec}\quad \Omega \to \mathfrak{M}$ denote a fixed geometric point and let $R(\bar{x})$ denote the corresponding strict Henselization of $\mathcal{O}_{\mathfrak{M}}$ at $\bar{x}$. Then one obtains the identification of the stalk of the presheaf $p_{\#}(\mathbf{G}_K \otimes \mathbb{Z}_{(J)})$ at $\bar{x}$*

$$p_{\#}(\mathbf{G}_K \otimes \mathbb{Z}_{(J)})_{\bar{x}} \simeq \mathbf{G}_K(\mathcal{S}_{R(\bar{x})}) \otimes \mathbb{Z}_{(J)}$$

where $\mathcal{S}_{R(\bar{x})} = (\mathrm{Spec}\quad R(\bar{x})) \times_{\mathfrak{M}} \mathcal{S}$. (If the stack $\mathcal{S}$ is smooth and $\mathbf{K}$ denotes the K-theory of perfect complexes, one obtains a weak-equivalence $\mathbf{G}_K(\mathcal{S}_{R(\bar{x})}) \simeq \mathbf{K}_K(\mathcal{S}_{R(\bar{x})})$.)

(i) *If, in addition, $\mathcal{S}$ is smooth over its coarse moduli space (or more generally, if $\mathcal{S}_{R(\bar{x})}$ is smooth over* Spec $\quad R(\bar{x})$*), $\mathcal{S}_{R(\bar{x})}$ is neutral gerbe over* Spec $\quad R(\bar{x})$. *(In particular this holds generically if the map $p: \mathcal{S} \to \mathfrak{M}$ is smooth generically and the base scheme S is the spectrum of a field or more generally is an excellent scheme.) Moreover, in this case $\mathbf{K}_K(\mathcal{S}_{R(\bar{x})}) \otimes \mathbb{Z}_{(J)} \simeq \mathbf{K}_K(\mathrm{Spec}\quad R(\bar{x}), G_{\bar{x}})$ where $G_{\bar{x}}$ is the stabilizer at $R(\bar{x})$ in the stack $\mathcal{S}_{R(\bar{x})}$.*

(ii) *Moreover, if $\bar{x}$ corresponds to a regular point of $\mathfrak{M}$ (or if $\mathbf{K}$ denotes the K-theory of perfect complexes) and the stack $\mathcal{S}_{R(\bar{x})}$ is smooth over* Spec $\quad R(\bar{x})$*, the stalk $\mathbf{G}(\mathcal{S}_{R(\bar{x})})/l^{\nu}[\beta^{-1}] \simeq \mathbf{K}(\mathcal{S}_{R(\bar{x})})/l^{\nu}[\beta^{-1}] \simeq \mathbf{K}(\mathcal{S}_{k(\bar{x})})/l^{\nu}[\beta^{-1}]$.*

(iii) *If the stack is Deligne–Mumford, $\mathcal{S}_{R(\bar{x})}$ is the quotient stack associated to a finite group-scheme action for all geometric points $\bar{x}$ of $\mathfrak{M}$.*

Proof. The continuity property of the presheaf $p_{\#}(\mathbf{G}_K \otimes \mathbb{Z}_{(J)})$ provides the first weak-equivalence. Let $x: X \to \mathcal{S}$ denote an atlas for the stack. If the stack is smooth over Spec $\quad R(\bar{x})$, one may find a lifting of $\mathrm{id}_R(\bar{x})$ to a map Spec $\quad R(\bar{x}) \to X_{R(\bar{x})}$ over Spec $\quad R(\bar{x})$. Now the structure map of the stack $\mathcal{S}_{R(\bar{x})} \to$ Spec $\quad R(\bar{x})$ has a section which shows $\mathcal{S}_{\mathcal{R}(\bar{x})}$ is a neutral gerbe. This proves (i). In view of (i), the first weak equivalence in (ii) follows from the weak equivalence between the equivariant G-theory and equivariant K-theory of regular schemes: see [T-1]. The last weak equivalence in (ii) follows by the rigidity theorem for mod-l^{ν} topological K-theory of regular schemes. In order to prove (iii), observe that if the stack is Deligne–Mumford, one may localize on the étale topology of the moduli-space and assume the stack is a quotient stack. □

PROPOSITION 5.16 (See [Toe-1]) (Poincaré duality for smooth Deligne–Mumford stacks). *Assume that $\mathcal{S}$ is a Deligne–Mumford stack which is regular. Now the obvious map $G_K(\mathcal{S}) \otimes \mathbb{Z}_{(J)} \simeq \mathbb{H}_{\mathrm{et}}(\mathfrak{M}, p_{\#}G_K(\quad) \otimes \mathbb{Z}_{(J)}) \leftarrow \mathbb{H}_{\mathrm{et}}(\mathfrak{M}, p_{\#}K_K(\quad) \otimes \mathbb{Z}_{(J)})$ is a weak equivalence.*

Proof. It suffices to show the map is a weak equivalence locally on the étale topology of the moduli space. Therefore, we reduce to the case when the stack is a

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quotient stack for the action of a finite group. In this case the above weak-equivalence follows from [T-1]. □

5.1.8. We end this section with the following criterion for cohomological descent on the étale site of an algebraic space for presheaves of spectra that come up often in this paper. Let G denote a fixed smooth group scheme over the base scheme S and let $(alg.stacks/S, G)$ denote the category with objects all algebraic stacks over the base scheme S that are Noetherian and provided with a G-action. The morphisms are all G-equivariant maps of algebraic stacks. Let P denote a presheaf of spectra on this category having the following properties:

(i) There exists a Gysin map $i_*\colon P(\mathcal{S}_0) \to P(\mathcal{S})$ for any G-equivariant closed immersion $i\colon \mathcal{S}_0 \to \mathcal{S}$ (which is a weak-equivalence if the closed immersion is given by a nilpotent sheaf of ideals). The Gysin map is functorial in i.
(ii) Given a G-equivariant closed immersion as in (i) with open complement $j\colon \mathcal{S}_1 = \mathcal{S} - \mathcal{S}_0 \to \mathcal{S}$, there exists a fibration sequence $i_\# i^* P \to P \to j_\# j^* P$ of presheaves where $i_\#, j_\#$ (i^*, j^*) are the obvious direct image functors (inverse image functors, respectively) (as in Section 5). Moreover, the map $i_\# i^* P \to P$ is given by the Gysin map in (i).
(iii) The presheaf P has the following continuity property: let $\{\mathcal{S}_\alpha | \alpha\}$ denote an inverse system of algebraic stacks with G-action and where the structure maps of the inverse system are affine. Now the obvious map $\mathrm{colim}_\alpha \Gamma(\mathcal{S}_\alpha, P) \to \Gamma(\mathcal{S}, P)$ is a weak-equivalence.

Let $\mathcal{S}$ denote a given algebraic stack, finitely presented over the base scheme S, with a G-action, X a given algebraic space (with trivial action by G) and $p\colon \mathcal{S} \to X$ a G-equivariant map. We define a presheaf $p_\#(P)$ on X_{et} by $\Gamma(U, p_\# P) = \Gamma(U \times_X \mathcal{S}, P)$.

PROPOSITION 5.17. *Assume the above situation.*

(i) *Then the presheaf $p_\#(P)$ has cohomological descent on the étale site of X if for every L = a field that is étale over a residue field of X, one has cohomological descent for the restriction of $p_\#(P)$ to the étale site of L*
(ii) *If the presheaf of homotopy groups $\pi_n(p_\# P)$ are all $\mathbb{Q}$-vector spaces, the conclusion of* (i) *holds if the presheaf $p_\#(P)$ restricted to the étale site of every field L as in* (i) *has the weak-transfer property* (*as in [T-5]* (2.12) *or see Remark 5.18 below*)

Proof. This is essentially in [T-5], Section 2. □

Remark 5.18. The weak-transfer property for a presheaf F on the étale site of a field L means that for every finite étale map $\lambda\colon \mathrm{Spec}\ L' \to \mathrm{Spec}\ L$, there is given a transfer map $\lambda_*\colon F(\mathrm{Spec}\ L') \to F(\mathrm{Spec}\ L)$ satisfying the hypotheses in [T-5] Definition (2.12). The existence of a weak transfer suffices to obtain étale cohomological descent for presheaves of spectra satisfying the hypotheses in 5.18 whose homotopy groups are all $\mathbb{Q}$-vector spaces. The hypertransfer is a variant of the transfer that also provides étale cohomological descent for presheaves of spectra whose homotopy groups are not necessarily $\mathbb{Q}$-vector spaces.

6. Riemann–Roch Theorems for Algebraic vs. Topological G-Theory

In this section we obtain a general Riemann–Roch theorem relating algebraic and topological G-theories for algebraic stacks.

DEFINITION 6.1. Let $\mathcal{S}$ denote an algebraic stack as before provided with the action of a smooth group scheme G. Let J denote a set of primes in $\mathbb{Z}$. We define the G-equivariant topological G-theory of $\mathcal{S}$ to be $G(\mathcal{S}, G)_K \otimes \mathbb{Z}_{(J)}$. We will also often call $G(\mathcal{S}, G)/l^{\nu}[\beta^{-1}]$ G-equivariant topological G-theory of $\mathcal{S}$. Either of these will be denoted $G^{\mathrm{top}}(\mathcal{S}, G)$ in this section.

Remark 6.2. Theorem 5.10 shows that the G-equivariant topological G-theory has descent on the isovariant étale site of the stack $[\mathcal{S}/G]$ and therefore justifies being called topological G-theory. The following Riemann–Roch theorem might seem like a tautology: however the fact that topological G-theory has descent on the isovariant étale site makes the right-hand side computable by means of the spectral sequence in Corollary 5.12, whereas there is no such spectral sequence for the left-hand side.

THEOREM 6.3 (Riemann–Roch). *Let $f : \mathcal{S}' \to \mathcal{S}$ denote a proper map between algebraic stacks that are finitely presented over S. Assume that a smooth group scheme G acts on both the stacks making f G-equivariant and that f has finite cohomological dimension. Then the following square*

$$\begin{array}{ccc} G(\mathcal{S}', G) & \longrightarrow & G^{top}(\mathcal{S}', G) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ G(\mathcal{S}, G) & \longrightarrow & G^{top}(\mathcal{S}, G) \end{array}$$

commutes.

Proof. This is clear since the right-hand side is simply the localization of the left-hand side. □

We consider group actions on algebraic stacks and the resulting Lefschetz–Riemann–Roch in the rest of this section. We begin first by defining actions by group schemes on algebraic stacks and their fixed point stacks.

6.0.9. Throughout the rest of this section, the base scheme S is assumed to be the spectrum of an algebraically closed field k. We will further restrict to actions of algebraic groups G on smooth Deligne–Mumford stacks $\mathcal{S}$, all over k. It seems, to us, that the more systematic definition of the fixed point stack $\mathcal{S}$ would make it not a closed algebraic substack of $\mathcal{S}$, but one that is unramified over $\mathcal{S}$. However, the following approximation to the fixed point stack seems sufficient for the situation considered above.

DEFINITION 6.4. Let $\mathfrak{M}^G$ denote the fixed point algebraic subspace of the coarse moduli space. We let $\mathcal{S}^G = \mathfrak{M}^G \times_{\mathfrak{M}} \mathcal{S}$. We adopt this as the definition of the fixed point stack.

Remark 6.5. Since the coarse moduli space of $\mathcal{S}^G$ has the same points as $\mathfrak{M}^G$, it follows that the points of the group G act trivially on the points of the stack $\mathcal{S}^G$. However, the group G may not act trivially on residual gerbes at each point. Since the base scheme is assumed to be the spectrum of an algebraically closed field k, these residual gerbes may be identified with quotient stacks for the action of finite groups. The following result will, however, show that, we may find a finite étale cover of the group G that acts trivially on the stack $\mathcal{S}^G$, provided we work over an algebraically closed field k and the group G is a torus.

PROPOSITION 6.6. *Assume the base scheme is the spectrum of an algebraically closed field k and the group $G = T$ is a torus, that the stack $\mathcal{S}$ is smooth and the coarse moduli space $\mathfrak{M}$ is an algebraic space of finite type over k. Then there exists a finite étale cover $T \cong \tilde{T} \to T$ so that the torus $\tilde{T}$ (with the obvious induced action) acts trivially on the stack $\mathcal{S}^T$. We may now identify $\mathcal{S}^T$ with $\mathcal{S}^{\tilde{T}}$.*

Proof. We begin with the Cartesian square

$$\begin{array}{ccc} I^T_{\mathcal{S}^T} & \xrightarrow{\delta} & T \times \mathcal{S}^T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (\mu, pr_2)} \\ \mathcal{S}^T & \xrightarrow[\Delta]{} & \mathcal{S}^T \times \mathcal{S}^T \end{array} \tag{6.0.10}$$

defining $I^T_{\mathcal{S}^T}$. Let $X \to \mathcal{S}^T$ denote an atlas for the stack $\mathcal{S}^T$. The right column is the map defined by the two maps $\mu: T \times \mathcal{S}^T \to \mathcal{S}^T$ and the projection $\mathrm{pr}_2: T \times \mathcal{S}^T \to \mathcal{S}^T$. Clearly $I^T_{\mathcal{S}^T} \times_{\mathcal{S}^T} X$ is a *group scheme* over X. Moreover, the obvious map $\delta': I^T_{\mathcal{S}^T} \times_{\mathcal{S}^T} X \to T \times X$ induced by δ is *unramified and surjective*. (To see δ is surjective, one may take points of the diagram (6.0.10). Observe from [L-MB] Proposition (5.3.1) that the induced map $|I^T_{\mathcal{S}^T}| \to |\mathcal{S}^T| \times_{\mathcal{S}^T \times \mathcal{S}^T} |T \times \mathcal{S}^T|$ is surjective. The definition of the fixed point stacks above shows the last term is isomorphic to

$$|\mathfrak{M}^T| \underset{|\mathfrak{M}^T| \times |\mathfrak{M}^T|}{\times} |T| \times |\mathfrak{M}^T| \cong |\mathfrak{M}^T \underset{\mathfrak{M}^T \times \mathfrak{M}^T}{\times} T \times \mathfrak{M}^T|.$$

The latter is the set of points of the inertia stack associated to the trivial action of T on the moduli space $\mathfrak{M}^T$. Therefore, it maps surjectively to $|T \times \mathfrak{M}^T|$ and hence δ itself is surjective.) Since X is generically integral (recall the base scheme is a field), it follows that the map δ' is generically flat and, hence, finite étale. One may now

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stratify X by locally closed subschemes, U_i, which are the atlases of locally closed substacks of $\mathcal{S}^T$, so that over $U_i - U_{i-1}$ the map δ' is finite étale of degree n_i. Let $\mathcal{S}_i^T$ denote the algebraic substack corresponding to $U_i - U_{i-1}$.

If $\bar{x} : \mathrm{Spec}\quad k \to U_i - U_{i-1}$ is any geometric point of $U_i - U_{i-1}$, $I^T_{\mathcal{S}^T} \times_{\mathcal{S}^T} X \times_X \bar{x} = T_{\bar{x}}$ is a torus isomorphic to T, but the map $T_{\bar{x}} \to T$ induced by δ is finite étale of degree $= n_i$. If $\mathrm{Spec}\quad R(\bar{x})$ denotes the strict Henselization of X at $\bar{x}$, the corresponding induced map $T_{\mathrm{Spec}\quad R(\bar{x})} \to T \times \mathrm{Spec}\quad R(\bar{x})$ induced by δ will also be finite étale of the same degree. Therefore, we may find an étale covering $V \to U_i - U_{i-1}$ so that $I^T_{\mathcal{S}^T} \times_{\mathcal{S}^T} V \cong T \times V$ with the induced map to $T \times \mathcal{S}^T \times_{\mathcal{S}^T} V (= T \times V)$ is finite étale of the same degree. We will denote the torus T appearing in the former by T'_i: this is a finite étale cover of the original torus T. The algebraic groupoid $T'_i \times V \times_{\mathcal{S}^T} V \rightrightarrows T'_i \times V$ defines the algebraic stack $I^T_{(\mathcal{S}^T)_i} = I^T_{\mathcal{S}^T} \times_{\mathcal{S}^T} (\mathcal{S}^T)_i$. Therefore, we obtain the diagram with both squares cartesian:

$$\begin{array}{ccc}
I^{T'_i}_{\mathcal{S}_i^{T'_i}} & \longrightarrow & T'_i \times (\mathcal{S}^T)_i \\
\downarrow & & \downarrow \\
(I^T_{\mathcal{S}^T})_i \cong T'_i \times (\mathcal{S}^T)_i & \xrightarrow{\delta} & T \times (\mathcal{S}^T)_i \\
\downarrow{\scriptstyle p} & & \downarrow \\
(\mathcal{S}^T)_i & \xrightarrow[\Delta]{} & (\mathcal{S}^T)_i \times (\mathcal{S}^T)_i
\end{array} \tag{6.0.11}$$

Clearly the top row has a splitting. Therefore, if we consider the étale cover of T, $T'_i \to T$ of degree n_i, and we let T'_i act through the action of T, it will act trivially on the locally closed substack $(\mathcal{S}^T)_i$: see (5.1.1). Therefore, let $\tilde{T} \to T$ denote an étale cover of sufficiently large degree ($> n_i$, for all i) and let it act on the stack through the action of T and the homomorphism $\tilde{T} \to T$. Then the action of $\tilde{T}$ on $\mathcal{S}^T$ will be trivial. Since the homomorphism $\tilde{T} \to T$ is surjective, we see that $\mathcal{S}^T = \mathcal{S}^{\tilde{T}}$. $\square$

LEMMA 6.7. *Assume the above hypotheses. Then one may find a finite subgroup scheme F of T of order prime to the characteristic of k so that $\mathfrak{M}^T = \mathfrak{M}^F$.*

Proof. Observe that the elements of T of finite order different from the characteristic p are dense in T. If T_f denotes the subgroup generated by these elements, one may observe that $\mathfrak{M}^T = \mathfrak{M}^{T_f}$. On the other hand, T_f can be written as the union $\cup_i T_f^{(n_i)}$ where $T_f^{(n_i)}$ denotes the elements of order n_i in T_f, for a sequence of integers n_i different from the characteristic. Therefore $\mathfrak{M}^T = \cap_i \mathfrak{M}^{T_f^{(n_i)}}$. Since $\mathfrak{M}$ is Noetherian by hypothesis, it follows that $\mathfrak{M}^T = \mathfrak{M}^{T_f^{(n_i)}}$ for some i. (We thank Michel Brion for supplying this lemma.) $\square$

PROPOSITION 6.8. *Assume the above situation. Then $\mathcal{S}^T_{red}$ is smooth.*

Proof. Let $\bar{x}$: Spec $k \to \mathfrak{M}^T$ denote a fixed *geometric point* of $\mathfrak{M}^T$, let x: Spec $k \to \mathcal{S}^T$ denote lifting of $\bar{x}$ and let $\mathcal{G}_x$ denote the corresponding residual gerbe. One may observe that $\mathcal{G}_x$ is the neutral gerbe associated to a finite group scheme since the stack is assumed to be Deligne–Mumford. Moreover, we may therefore assume that x represents an atlas for this residual gerbe.

Let I denote the sheaf of ideals defining $\mathcal{S}^T$ as a closed sub-stack of $\mathcal{S}$: we will show that I_x is defined by a *regular sequence* in $\mathcal{O}_{\mathcal{S},x}$ which is strict Henselization of $\mathcal{O}_{\mathcal{S}}$ at x. If m_x is the maximal ideal of $\mathcal{O}_{\mathcal{S},x}$, let $\bar{x}_1, \dots, \bar{x}_k, \bar{x}_{k+1}, \dots, \bar{x}_n$ denote a basis for $k(x) = m_x/m_x^2$. Lift these basis vectors to $x_1, \dots, x_n$ in m_x: now they form a regular sequence in m_x.

By the preceding lemma, we may now find a finite subgroup scheme F of $\tilde{T}$ of order prime to the characteristic p, so that $\mathfrak{M}^F = \mathfrak{M}^T$. Observe that, by our definition, $\mathcal{S}^F = \mathcal{S}^T$. We may further find a map $\tilde{Y} \to \mathcal{S}$ and a lift $\tilde{y}$ of the point x so that $\tilde{Y}$ is an affine scheme, with an action of F, $\tilde{y}$ is fixed by F and with $y_1, \dots, y_n$ in its co-ordinate ring so that each x_i maps to the image of y_i in the strict Henselization of the local ring at $\tilde{y}$. Moreover, the action of F on $\tilde{Y}$ is compatible with the action of $\tilde{T}$ on $\mathcal{S}$. We may define $\tilde{Y}$ as follows. We may first find an affine smooth scheme with an étale map α: $\bar{Y} \to \mathcal{S}$ provided with a lift $\bar{y}$ of the point x and $\bar{y}_i$ in its co-ordinate ring so that each x_i maps to the image of $\bar{y}_i$ in the strict Henselization of the local ring at $\bar{y}$. $\bar{Y}$ may not have an action by F. Next replace $\bar{Y}$ by the iterated fibered product of $f(\bar{Y})$, (over $\mathcal{S}$), $f \in F$. $f(\bar{Y})$ is the fibered product of $\bar{Y}$ over $\mathcal{S}$ and the map f^{-1}: $\mathcal{S} \to \mathcal{S}$ with the map from $f(\bar{Y}) \to \mathcal{S}$ being the induced map $(= f \circ \alpha \circ f_{\bar{Y}}^{-1}$, where $f_{\bar{Y}}^{-1}$: $f(\bar{Y}) \to \bar{Y}$ is the map induced by f^{-1}: $\mathcal{S} \to \mathcal{S}$). This is a smooth separated scheme provided with an (obvious) action by F (which we denote by $\hat{Y}$), with an étale map to $\mathcal{S}$, a lift of the point x fixed by F (which we denote by $\hat{y}$) and $y_1, \dots, y_n$ in the stalk of its structure sheaf at the chosen point $\hat{y}$ so that x_i maps to the image of y_i in the strict Henselization of the local ring at $\hat{y}$. Observe that the action by F on $\hat{Y}$ is compatible with the action of $\tilde{T}$ on $\mathcal{S}$. However, $\hat{Y}$ is not necessarily affine. Now take an affine open neighborhood $N_{\hat{y}}$ of $\hat{y}$ in $\hat{Y}$: since $\hat{Y}$ is separated, $\tilde{Y} = \cap_{f \in F} f.N_{\hat{y}}$ is an affine open neighborhood of $\hat{y}$ stable by F. This also shows that such neighborhoods are co-final in the system of affine neighborhoods of $\hat{y}$ in $\hat{Y}$, so that we may lift the y_i to one such neighborhood. Since the group F is linearly reductive, we may also assume that F acts on y_j with nontrivial character χ_j, $j = k+1, \dots, n$ and trivially on $y_1, \dots y_k$. Observe that $Y = \mathcal{S}^{\tilde{T}} \times_{\mathcal{S}} \tilde{Y}$ is a closed subscheme of $\tilde{Y}$ defined by a sheaf of ideals I. Moreover, $Y \to \mathcal{S}^T$ is an atlas for $\mathcal{S}^T$.

We will next show that F acts trivially on Y and that the map $Y \to \mathcal{S}^T$ is fixed by every element of F. To see this recall F acts on $\mathcal{S}^T$ through the action of $\tilde{T}$ on $\mathcal{S}$: $\tilde{T}$ acts trivially on $\mathcal{S}^T$ and therefore the action of F on $\mathcal{S}^T$ is trivial. Now recall the definition of $\hat{Y}$ as the iterated fibered product of $f(\bar{Y})$, $f \in F$. Here $f(\bar{Y})$ is the fibered product of $\bar{Y} \xrightarrow{\alpha} \mathcal{S}$ and f^{-1}: $\mathcal{S} \to \mathcal{S}$. Since F acts trivially on $\mathcal{S}^T$, the composition $\mathcal{S}^T \to \mathcal{S} \xrightarrow{f^{-1}} \mathcal{S}$ is simply the original closed immersion $\mathcal{S}^T \to \mathcal{S}$; therefore $\mathcal{S}^T \times_{\mathcal{S}} f \circ \alpha \circ f_{\bar{Y}}^{-1}$: $\mathcal{S}^T \times_{\mathcal{S}} f(\bar{Y}) \to \mathcal{S}^T$ identifies with $\mathcal{S}^T \times_{\mathcal{S}} \alpha$. It follows that F leaves every point of Y fixed and fixes the map $Y \to \mathcal{S}^T$.

Let $j > k$ denote a fixed integer and let χ_j denote the corresponding character by which F acts on y_j. Recall $\tilde{Y}$ is affine; therefore the $\{y_j | j\}$ are elements of the co-ordinate ring of $\tilde{Y}$. Let $y' \in Y$ denote an arbitrary (closed) point. If y_j does not vanish at the point $y' \in Y$, the stabilizer $F_{y'}$ must be contained in $\ker(\chi_j)$. Since $\ker(\chi_j)$ is properly contained in F (otherwise F would act trivially on y_j contrary to the choice of y_j), it follows that y' would not be a fixed point for F. (Recall F acts trivially on every point y' of Y). Therefore, it follows that y_j vanishes at every point of Y: i.e. $(y_{k+1}, \ldots, y_n) \subseteq I$. Since $\tilde{Y}^F$ is defined by the ideal $(y_{k+1}, \ldots, y_n)$, it follows that $Y \subseteq \tilde{Y}^F$. Now observe that $\mathcal{O}_{\mathcal{S},x} \cong \mathcal{O}^{sh}_{\tilde{Y},\tilde{y}}$ = the strict Henselization of $\mathcal{O}_{\tilde{Y},\tilde{y}}$ at $\tilde{y}$ and that $I_x \cong I_{\tilde{y}} \otimes_{\mathcal{O}_{\tilde{Y},\tilde{y}}} \mathcal{O}^{sh}_{\tilde{Y},\tilde{y}}$ when $I_{\tilde{y}} = I \otimes_{\Gamma(\tilde{Y},\mathcal{O}_{\tilde{Y},\tilde{y}})} \mathcal{O}_{\tilde{Y},\tilde{y}}$. Therefore, it follows that $(x_{k+1}, \ldots, x_n) \subseteq I_x$.

Conversely we will show $I_x \subseteq (x_{k+1}, \ldots, x_n)$. For this, it suffices to show that $\tilde{Y}^F \subseteq Y$ which implies that $I \subseteq (y_{k+1}, \ldots, y_n)$. Since $Y = \mathcal{S}^T \times_{\mathcal{S}} \tilde{Y}$, the closed immersion $\tilde{Y}^F \to \tilde{Y}$ factors through Y. Now an argument as in the last paragraph shows $I_x = (x_{k+1}, \ldots, x_n)$ which is a regular sequence in $\mathcal{O}_{\mathcal{S},x}$. Therefore, $\mathcal{O}_{\mathcal{S}^T,x}$ is a regular local ring for every closed point x of $\mathcal{S}^T$. This proves $\mathcal{S}^T_{\text{red}}$ is smooth. □

6.0.12. *Proof of Theorem* 1.3. It follows from the above proposition that the closed immersion $i: \mathcal{S}^T_{red} \to \mathcal{S}$ is a regular immersion. Let N denote the conormal sheaf associated to this closed immersion. □

PROPOSITION 6.9. *Assume that the above situation is correct*. (i) *Then the class* $\lambda_{-1}(N)\varepsilon\pi_0(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}'))_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \mathbb{Q}$ *is a unit*. (ii) *Moreover the Gysin map* $i_*: \pi_*(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}'))_{(\mathfrak{p})} \to \pi_*(G(\mathcal{S}, \tilde{T}'))_{(\mathfrak{p})}$ *is an isomorphism*. (iii) *The inverse to this isomorphism tensored with* $\mathbb{Q}$ *is provided by* $i^*(\quad) \cap \lambda_{-1}(N)^{-1}$.

Proof. Throughout the proof, we will identify the G-theory of an algebraic stack with the G-theory of the associated reduced stack. We will first prove the first statement. We first recall the canonical isomorphism (see: 1.0.3):

$$\pi_*(G(\mathcal{S}, \tilde{T}')) \cong \mathbb{Z}[M'] \underset{\mathbb{Z}}{\otimes} \pi_*(G(\mathcal{S})) \tag{6.0.13}$$

where M' is the character group of $\tilde{T}'$ and $\tilde{T}'$ acts trivially on the stack $\mathcal{S}$. Throughout the proof we will let $\mathbf{G}(\quad, \tilde{T}')_{\mathcal{S}^{\tilde{T}'}}$ ($\mathbf{K}(\quad, \tilde{T}')_{\mathcal{S}^{\tilde{T}'}}$) denote the presheaf of spectra associated to the category of $\tilde{T}'$-equivariant coherent (coherent and locally free, respectively) sheaves on the stack $\mathcal{S}^{\tilde{T}'}$.

Let $p: \mathcal{S} \to \mathfrak{M}$ and $p^{\tilde{T}'}: \mathcal{S}^{\tilde{T}'} \to \mathfrak{M}^T$ denote the obvious maps. Using the notation as in 5.10, the presheaf $p^{\tilde{T}'}_{\#}(\mathbf{G}(\quad, \tilde{T}')_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q}$ satisfies the hypotheses in Proposition 5.17. Therefore, we obtain a spectral sequence:

$$E_2^{s,t} = H^s_{\text{et}}(\mathfrak{M}^{\tilde{T}'}; \pi_t(p^{\tilde{T}'}_{\#}(\mathbf{G}(\quad, \tilde{T}'))_{\mathcal{S}^{\tilde{T}'}} \otimes \mathbb{Q})) \to \pi_{t-s}(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}') \otimes \mathbb{Q}) \tag{6.0.14}$$

In view of the hypotheses, this spectral sequence converges strongly. One may localize this spectral sequence at the prime ideal $\mathfrak{p}$ in $R(T)$ corresponding to the subtorus $\tilde{T}'$: clearly the resulting spectral sequence also converges strongly. Therefore, the kernel of the edge map

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$$e\colon \pi_0(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}') \otimes \mathbb{Q})_{(\mathfrak{p})} \to E_\infty^{0,0} \to E_2^{0,0}$$
$$= H^0_{\mathrm{et}}(\mathfrak{M}^T, \pi_0(p_\#^{\tilde{T}'}(\mathbf{G}(\quad, \tilde{T}')_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q})_{(\mathfrak{p})})$$

is nilpotent and it suffices to show that $e(\lambda_{-1}(N))$ is a unit. Next observe that $e(\lambda_{-1}(N))$ is in the image of the natural map

$$H^0_{\mathrm{et}}(\mathfrak{M}^{\tilde{T}'}, \pi_0(p_\#^{\tilde{T}'}(\mathbf{K}(\quad, \tilde{T}')_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q})_{(\mathfrak{p})})$$
$$\xrightarrow{\epsilon} H^0_{\mathrm{et}}(\mathfrak{M}^T, \pi_0(p_\#^{\tilde{T}'}(\mathbf{G}(\quad, \tilde{T}')_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q})_{(\mathfrak{p})}) \tag{6.0.15}$$

Therefore, we will denote by $e(\lambda_{-1}(N))$ the corresponding class on the left-hand side of the previous equation. The isomorphism in (6.0.13) for K-theory (and the observation that $\tilde{T}'$ indeed acts trivially on the stack $\mathcal{S}^{\tilde{T}'}$) enables one to obtain the isomorphism:

$$H^0_{\mathrm{et}}(\mathfrak{M}^{\tilde{T}'}, \pi_0(p_\#^{\tilde{T}'}(\mathbf{K}(\quad, \tilde{T}')_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q})_{(\mathfrak{p})})$$
$$\cong \mathbb{Z}[M']_{(\mathfrak{p})} \underset{\mathbb{Z}}{\otimes} H^0_{\mathrm{et}}(\mathfrak{M}^{\tilde{T}'}, \pi_0(p_\#^{\tilde{T}'}(\mathbf{K}(\quad)_{\mathcal{S}^{\tilde{T}'}}) \otimes \mathbb{Q})). \tag{6.0.16}$$

Next, in order to show $e(\lambda_{-1}(N))$ is a unit, it suffices to show that it maps to a unit at each of the stalks (taken at the geometric points of the moduli space $\mathfrak{M}^{\tilde{T}'}$) of the presheaf $\mathbb{Z}[M]'_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \pi_0(p_\#^{\tilde{T}'}(\mathbf{K}(\quad)_{\mathcal{S}^{\tilde{T}'}} \otimes \mathbb{Q}))$. Since the stack is Deligne–Mumford, one may localize on the moduli space $\mathfrak{M}^{\tilde{T}'}$ and assume the stack is a quotient stack associated to a finite group action. Therefore, we reduce to showing that the class $\lambda_{-1}(N)$ is a unit in $\mathbb{Z}[M']_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \pi_0(K(\mathcal{S}^{\tilde{T}'})) \otimes_{\mathbb{Z}} \mathbb{Q}$ when the stack $\mathcal{S}^{\tilde{T}'}$ is a quotient stack associated to the action of a finite group on a scheme of finite type over k.

At this point, one observes that the γ-filtration on the Grothendieck group of equivariant vector bundles on a scheme of finite type over k, equivariant with respect to the action of a finite group is *nilpotent modulo torsion*. (See [A].) Therefore, it suffices to show that the image of $\lambda_{-1}(N)$ in $\mathbb{Z}[M']_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a unit. In fact it suffices to do this for the image of $\lambda_{-1}(N)$ in $\mathbb{Z}[M']_{(\mathfrak{p})}$. The stalk $N_{\bar{x}}$ is the sum of nontrivial one-dimensional representations $\Sigma_i m_i$: therefore the image of $(\lambda_{-1}(N)) = \prod_i (1 - m_i)$. One may readily show $m_i \neq 1 (\mathrm{mod}) \mathfrak{p}$. It follows that $(1 - m_i)$ is a unit in $\mathbb{Z}[M']_{(\mathfrak{p})}$ for all i: i.e. $(\lambda_{-1}(N))$ maps to a unit in the given stalk.

This completes the proof of the first statement of the proposition.

Next we will show the Gysin map

$$i_* : \pi_*(G(\mathcal{S}^{\tilde{T}'}, \tilde{T}'))_{(\mathfrak{p})} \to \pi_*(G(\mathcal{S}, \tilde{T}'))_{(\mathfrak{p})} \tag{6.0.17}$$

is an isomorphism. By the localization sequence in G-theory and induction on the dimension of the stack, it suffices to prove that on any sufficiently small open substack $\mathcal{S}_V$ of $\mathcal{S} - \mathcal{S}^{\tilde{T}'}$, $\pi_*(G(\mathcal{S}_V, \tilde{T}'))_{(\mathfrak{p})} \simeq 0$. Let $V \subseteq \mathfrak{M} - \mathfrak{M}^{\tilde{T}'}$ denote any open non-empty $\tilde{T}'$-stable and *smooth* subalgebraic space. We may in fact assume that it is a scheme. Let $\mathcal{S}_V = \mathcal{S} \times_{\mathfrak{M}} V$. Observe that $G(V, \tilde{T}') \simeq K(V, \tilde{T}')$ since V is regular and that $G(\mathcal{S}_V, \tilde{T}')$ is a module over $K(V, \tilde{T}')$. The latter is trivial on localization at the prime ideal $\mathfrak{p}$ by [T-2]. Therefore, $\pi_*(G(\mathcal{S}_V, \tilde{T}'))_{(\mathfrak{p})} \simeq 0$ and hence $\pi_*(G^{top}(\mathcal{S}_V, \tilde{T}'))_{(\mathfrak{p})} \simeq 0$. This shows that the Gysin map in (6.0.17) is an isomorphism and completes the proof of the second statement of the proposition. The last assertion follows readily since the composition of i_* and i^* corresponds to multiplying by $\lambda_{-1}(N)$. This completes the proof of the proposition. □

6.0.18. *Proof of Theorem* 1.3. This is clear in view of the previous proposition. The key observation is that the *map* Ψ commutes with proper push-forward by [Toe-1], Lemme (4.12). In fact, in [Toe-1] Lemme (4.12) is stated in a restrictive form with the hypothesis that every coherent sheaf on the stacks $\mathcal{S}^{\tilde{T}'}$ and $\mathcal{S}'^{\tilde{T}'}$ is a quotient of a locally free coherent sheaf. This is a very restrictive hypotheses which, fortunately may be removed as follows. Let $\mathcal{S}$ denote either of the above stacks: recall these are both smooth. It suffices to show that there exists a Chow envelope $\tilde{\mathcal{S}} \to \mathcal{S}$ which is strongly projective, i.e. factors through a closed immersion into Proj($\mathcal{E}$) where $\mathcal{E}$ is a locally free coherent sheaf on $\mathcal{S}$ followed by the obvious projection to $\mathcal{S}$. Since the stack $\mathcal{S}$ is smooth and defined over a field k, one may find such a $\tilde{\mathcal{S}}$ as follows. Since $\mathcal{S}$ is smooth, it is well known that $\mathcal{S}$ is a gerbe over $\mathcal{S}_0$ where the latter is another smooth Deligne–Mumford stack which is generically a scheme. By Theorem 2.18, [EHKV], $\mathcal{S}_0$ is a quotient stack and therefore every coherent sheaf on $\mathcal{S}_0$ is the quotient of a locally free coherent sheaf. Therefore every morphism $\mathcal{S}' \to \mathcal{S}_0$ that factors as the composition of a closed immersion into Proj(E), with E a coherent sheaf on $\mathcal{S}_0$ and the obvious projection is in fact strongly projective in the sense above, i.e. one may assume without loss of generality that E is in fact locally free. In particular, if $\mathcal{S}'_0 \to \mathcal{S}_0$ is a Chow envelope, it is strongly projective. Now one takes the pull-back $\tilde{\mathcal{S}} = \mathcal{S}'_0 \times_{\mathcal{S}_0} \mathcal{S} \to \mathcal{S}$. This is strongly projective and is a Chow envelope, since $\mathcal{S} \to \mathcal{S}_0$ is a gerbe. Therefore one may apply Lemme (4.12) of Toen without further restrictions on the stacks. (We thank Bertrand Toen for supplying the above argument.) The map $i^*(\quad) \cap \lambda_{-1}(N)^{-1}$ being inverse to i_* also commutes with proper push-forward for equivariant maps. □

6.0.19 *Proof of Corollary* 1.4. This is also clear in view of the previous results. Étale cohomological descent for the presheaf $\mathbf{K}(\quad)) \otimes \mathbb{Q}$ provides the isomorphism $\pi_0(K(X, \tilde{T}'))_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty) \cong \pi_0(K_{\mathrm{et}}(X, \tilde{T}'))_{(\mathfrak{p})} \otimes_{\mathbb{Z}} \mathbb{Q}(\mu_\infty)$.

7. Appendix: Quotient Stacks of Algebraic Stacks

In this section we will briefly show that the quotient of an algebraic stack by the action of a smooth group scheme exists as an algebraic stack. This seems well known, though nothing appears in the literature.

7.1. Let $\mathcal{S}$ denote an algebraic stack with an action by a smooth group scheme G. We define the category, $[\mathcal{S}/G]$ fibered in groupoids over schemes as follows. For a

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given scheme T, the objects of the category $[\mathcal{S}/G](T)$ are given by diagrams of the following form:

$$\begin{tikzcd} \psi' \arrow[d, "s"] & \\ \psi \arrow[r, "g"] & T \end{tikzcd} \tag{7.1.1}$$

where g is a principal G-bundle over T, s corresponds to an object in the stack $\mathcal{S}$ over ψ, the object ψ is provided with an action by G, so that if μ, $pr_2 \colon G \times \psi \to \psi$ are the group action and the projection, then there is given an isomorphism $\phi \colon \mu^*(s) \cong pr_2^*(s)$ satisfying an obvious co-cycle condition on further pull-back to $G \times G \times \psi$ by the obvious maps and so that the pull-back to ψ by the identity section $e \colon \psi \to G \times \psi$ is the identity. A morphism between two such objects in the category $[\mathcal{S}/G](T)$ is an isomorphism preserving all the structure.

PROPOSITION 7.1. *Assume the above situation. Then* $[\mathcal{S}/G]$ *is an algebraic stack so that there exists a representable smooth map* $\mathcal{S} \to [\mathcal{S}/G]$ *of algebraic stacks. If* $x \colon X \to \mathcal{S}$ *is an atlas for the stack* $\mathcal{S}$*, the composition* $X \to \mathcal{S} \to [\mathcal{S}/G]$ *defines an atlas for the stack* $[\mathcal{S}/G]$.

Proof. We skip the verification that $[\mathcal{S}/G]$ is a stack. The *map* $\mathcal{S} \to [\mathcal{S}/G]$ is given by sending an object η' in $\mathcal{S}(T)$ to the diagram

$$\begin{tikzcd} \psi' = G \times \eta' \arrow[d] & \\ G \times T \arrow[r, "pr_2"] & T \end{tikzcd} \tag{7.1.2}$$

One may verify that the map $\mathcal{S} \to [\mathcal{S}/G]$ is representable. Finally, to show that the stack $[\mathcal{S}/G]$ is algebraic, one may proceed as follows. First let $x_0 \colon X_0 \to \mathcal{S}$ denote an atlas for the stack $\mathcal{S}$ with X_0 a separated scheme. If $\mu \colon G \times \mathcal{S} \to \mathcal{S}$ and $\mathrm{pr}_2 \colon G \times \mathcal{S} \to \mathcal{S}$ are the projections, one obtains an isomorphism (not necessarily satisfying any co-cycle conditions) between the two pull-backs $\mu^*(X_0)$ and $pr_2^*(X_0)$. We will denote $pr_2^*(X_0)$ by X_1. Making use of this isomorphism, one obtains the commutative square:

$$\begin{tikzcd} G \times \mathcal{S} \arrow[r, shift left, "\mu"] \arrow[r, shift right, "pr_2"'] & \mathcal{S} \\ X_1 \arrow[u] \arrow[r, shift left, "\mu'"] \arrow[r, shift right, "pr_2'"'] & X_0 \arrow[u, "x_0"'] \end{tikzcd}$$

where the two maps in the bottom row are the obvious ones induced by the ones in the top row. Now the square is Cartesian with the maps μ and μ' (pr_2 and pr_2', respectively). Moreover, all the maps are smooth and the schemes are all separated (and quasi-compact). We may now extend this to the diagram:

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$$\begin{tikzcd}
G \times \mathcal{S} \arrow[rr, shift left, "\mu"] \arrow[rr, shift right, "pr_2"'] & & \mathcal{S} \\
X_1 \arrow[u] \arrow[rr, shift left, "\mu'"] \arrow[rr, shift right, "pr_2'"'] & & X_0 \arrow[u, "x_0"'] \\
X_1 \underset{G\times\mathcal{S}}{\times} X_1 \arrow[u, shift left] \arrow[u, shift right] \arrow[rr, shift left, "\mu''"] \arrow[rr, shift right, "pr_2''"'] & & X_0 \underset{\mathcal{S}}{\times} X_0 \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd}$$

Once again all the maps are smooth and the schemes are all separated (and quasi-compact). Therefore, the diagonal of the above diagram:

$$X_1 \underset{G\times\mathcal{S}}{\times} X_1 \rightrightarrows X_0$$

defines an algebraic groupoid. (Observe that $X_1 \cong G \times X_0$. Therefore one obtains a composition $X_1 \times_{X_0} X_1 \to X_1$ induced by the group-law $G \times G \to G$. Next observe that, since $G \times \mathcal{S}$ is an algebraic stack, $X_1 \times_{G\times\mathcal{S}} X_1 \rightrightarrows X_1$ is an algebraic groupoid. Therefore one has a composition

$$X_1 \underset{G\times\mathcal{S}}{\times} X_1 \underset{X_1}{\times} X_1 \underset{G\times\mathcal{S}}{\times} X_1 \to X_1 \underset{G\times\mathcal{S}}{\times} X_1.$$

Combining these two compositions, one obtains a composition

$$X_1 \underset{G\times\mathcal{S}}{\times} X_1 \underset{X_0}{\times} X_1 \underset{G\times\mathcal{S}}{\times} X_1 \to X_1 \underset{G\times\mathcal{S}}{\times} X_1$$

that defines the *groupoid law*. Now one needs to verify that the required identities hold.) The associated stack may be identified with $[\mathcal{S}/G]$. $\square$

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References

[Ab-W] Abramovich, D. and Wang, J.: Equivariant resolution of singularities in characteristic 0, *Math. Res. Lett.* **4**(3) (1997), 481–494.

[Ar] Artin, M.: Versal deformations and algebraic stacks, *Invent. Math.* **27** (1974), 165–189.

[A] Atiyah, M.: Characters and the cohomology of finite groups, *Publ. Math. IHES*, (1961).

[A.B] Atiyah, M. and Bott, R.: A Lefschetz fixed point theorem for elliptic operators, *Ann. Math*. **86** (1967), 374–407; **87** (1968), 451–491.

[A.S1] Atiyah, M. and Segal, G.: The index of elliptic operators II, *Ann. Math*. **87** (1968), 531–545.

[A.S2] Atiyah, M. and Segal, G.: Equivariant K-theory and completion, *J. Differential Geom*. **3** (1969), 1–18.

[B-F-M] Baum, P., Fulton, W. and Macpherson, R.: Riemann–Roch and topological K-theory, for singular varieties, *Acta Math*. **143** (1979), 155–191.

[B-F-Q] Baum, P., Fulton, W. and Quart, G.: Lefschetz–Riemann–Roch for singular varieties, *Acta Math*. **143**(3–4) (1979), 193–211.

[B-R] Bardsley, P. and Richardson, R. W.: Étale slices for algebraic transformation groups in characteristic p, *Proc. London Math. Soc*. **51** (1985), 295–317.

[Bl1] Bloch, S.: Algebraic cycles and higher K-theory, *Adv. Math*. **61** (1986), 267–304.

[Bl2] Bloch, S.: The moving lemma for higher Chow groups, *J. Algebraic Geom*. **3** (1994), 537–568.

[Bo] Borel, A. *et al*.: *Seminar on Transformation Groups*, Ann. Math. Study. 46, Princeton Univ. Press, 1960.

[B-K] Bousfield, A. K. and Kan, D. M.: *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer, New York, 1974.

[Bous] Bousfield, A. K.: Localizations of spaces with respect to homology theories, *Topology* **14** (1975), 133–150.

[Br] Bredon, G.: *Equivariant Cohomology Theories*, Lecture Notes in Math. 34, Springer, New York, 1967.

[EHKV] Edidin, D., Hasett, B., Kresch, A. and Vistoli, A.: Brauer groups and quotient stacks, *Amer J. Math* **123** (2001), 761–777.

[FL] Fulton, W. and Lang, S.: *Riemann–Roch Algebra*, Grundlehren Math. Wiss. 277, Springer, New York, 1983.

[Fr] Friedlander, E.: *Etale Homotopy of Simplicial Schemes*, Ann. Math Study, Princeton, 1983.

[Hak] Hakim, M.: *Topos anneles et schémas relatifs*, Lecture Notes in Math. 64, Springer, New York, 1972.

[Hs] Hsiang, W. Y.: *Cohomology Theory of Topological Transformation Groups*, Ergeb. Math. Grenzgeb., Springer, Berlin.

[Iv] Iversen, B.: A fixed point formula for actions of tori on algebraic varieties, *Invent. Math*. **16** (1972), 229–236.

[J-1] Joshua, R.: Equivariant Riemann–Roch for G-quasi-projective varieties-I, *K-theory*, **17** (1999), 1–35.

[J-2] Joshua, R.: Intersection theory for algebraic stacks-I, *K-Theory*, **27** (2002), 133–195.

[J-3] Joshua, R.: Intersection theory for algebraic stacks-II, *K-Theory*, **27** (2002), 197–244.

[KS] Kelly, G. M. and Street, R.: *Review of the Elements of 2-Categories, Category Seminar*, Lecture Notes in Math. 420, Springer, New York, 1972/73, pp. 75–103.

[Kn] Knutson, D.: *Algebraic Spaces*, Lecture Notes in Math. 203, Springer, New York, 1971.

[L-MB] Laumon, G. Moret-Bailly: Champs algébriques, Prepublication de Université de Paris, Orsay, 1992.

[LMS] Lewis, G., May, J. P. and Steinberger, M.: *Equivariant Stable Homotopy Theory*, Lecture Notes in Math. 1213, Springer, New York, 1986.

[Mil] Milne, J.: *Étale Cohomology*, Princeton Univ. Press, 1980.

[Mur] Murre, J.: Lectures on an introduction to Grothendieck's theory of the fundamental group, *Tata Institute Lecture Notes* 40, 1967.

[Mm2] Mumford, D., Fogarty, J. and Kirwan, F.: *Geometric Invariant Theory*, 3rd edn, Springer, New York.

[Qu] Quillen, D.: Higher algebraic *K*-theory I, In: *Higher K-theories*, Lecture Notes in Math. 341, Springer, New York, 1973, pp. 85–147.

[SGA3] Demazure, M. and Grothendieck, A.: *Schémas en groupes*, Lecture Notes in Math. 151, 152, 153, Springer, New York, 1970.

[SGA4] Artin, M., Grothendieck, A. and Verdier, J. L.: *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Math. 269, 270, 305, Springer, New York, 1971.

[SGA6] Berthelot, P., Grothendieck, A. and Illusie, L.: *Théorie des intersections et théorèm de Riemann–Roch*, Lecture Notes in Math. 225, Springer, New York, 1971.

[Sum] Sumihiro, H.: Equivariant completion II, *J. Math Kyoto Univ*. **15** (1975), 573–605.

[T-1] Thomason, R. W.: Algebraic *K*-theory of group scheme actions, In: W. Browder (ed.), *Algebraic Topology and Algebraic K-theory*, Ann. Math. Stud. 113, Princeton Univ. Press, 1987, pp. 539–563.

[T-2] Thomason, R. W.: Lefschetz–Riemann–Roch and coherent trace formula, *Invent. Math*. **85** (1986), 515–543.

[T-3] Thomason, R. W.: Equivariant algebraic vs. topological *K*-homology Atiyah–Segal-style, *Duke Math. J*. **56**(3) 1988, 589–636.

[T-4] Thomason, R. W.: First quadrant spectral sequences in algebraic *K*-theory via homotopy colimits, *Comm. Algebra* **10**(15) (1982), 1589–1668.

[T-5] Thomason, R. W.: Algebraic *K*-theory and étale cohomology, *Ann. Sci. Ecol. Norm. Sup*. **18** (1985), 437–552.

[Toe-1] Toen, B.: Théorèmes de Riemann–Roch pour les champs de Deligne–Mumford, *K-theory*, **18** (1999), 33–76.

[Toe-2] Toen, B.: PhD thesis, Université Paul Sabatier, Toulouse, 1998.

[Vi] Vistoli, A.: Higher algebraic *K*-theory of finite group actions, *Duke Math. J*. **63** (1991), 399–419.