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Closeness to singularities of manipulators based on geometric average normalized volume spanned by weighted screws

Wanghui Bu*

School of Mechanical Engineering, Tongji University, Shanghai 200092, P. R. China

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SUMMARY
In order to prevent robot manipulators from reaching singularities, the “distance” from the current configuration to a singular configuration should be measured. This paper presents a novel metric based on geometric average normalized volume spanned by weighted screws to measure closeness to singularities for both serial and parallel manipulators. The weighted screws are proposed to reinterpret the physical meaning of twists and wrenches, so the problem of inconsistent dimensions in the dot product of screws has been eliminated. Compared with other existing methods, the proposed metric can obtain an identical result for similar manipulators with different sizes. Furthermore, the metric is independent of the selection of base screws, which is very suitable for the overconstrained or lower mobility parallel manipulator whose base screws are not uniquely definite. Besides, the geometrical meaning of the metric is related to the dimensionless volume of a high dimensional polyhedron, and hence the metric is insensitive to screw magnitude.

KEYWORDS: Robot manipulators, Singularity, Metric, Screw theory, Overconstrained mechanisms

1. Introduction

Singularities have a great influence on the performance of robot manipulators.[1−12] Singularities can be categorized into many ways according to different conceptions such as the classification of the singularities as kinematic singularities and static singularities.[13,14] Recently, a novel concept of a topological singularity was also presented.[15] The kinematic singularities emerge in the case that joint twists in a serial kinematic chain become linearly dependent, which may occur in serial manipulators or in a limb of parallel manipulators. The static singularities arise when constraint wrenches of limbs become linearly dependent, which may occur in parallel manipulators. A manipulator far away from singularities has good manipulability,[16−18] whereas if it is near the kinematic singularities, its joint velocities cannot be solved for a given velocity of its end-effector or moving platform; if it is near the static singularities, its limbs cannot provide necessary constraints to the moving platform. Hence, manipulators should be prevented from reaching singular configurations, and the singularity margin that indicates the “distance” between the current configuration and a singular configuration should be measured effectively.

Some well-known indicators, such as the determinant of the Jacobian, the smallest singular value, and the condition number,[16,17] may not be the best choice to measure closeness to singularities, since these indicators lack physical or geometrical meaning,[18−20] and their values are not invariant with respect to origin selection or scaling of the coordinate frame.[21] In order to explore an eligible metric of closeness to singularities, Hubert and Merlet[19] employed thresholds of robot joint forces or torques to determine the closeness to singularities. Actually these forces and torques not only depended on the robot configuration, but also on the action lines of external loads. Voglewede and Ebert-Uphoff[22] adopted constrained optimization in both velocity and force domains to evaluate closeness

* Corresponding author. E-mail: buwanghui@tongji.edu.cn

to singularities in terms of power, stiffness, kinetic energy, potential energy, and natural frequency. They declared that there was no invariant metric on the group of rigid body displacements, and any metric necessitated a choice to weight the translational and rotational portions together. Hartley and Kerr[23,24] investigated closeness to singularities of abstract screw systems. For a system of six screws, they adopted the reciprocal screw theory to find a translation invariant metric.[23] However, for a system containing less than six screws, the invariance of their proposed metric was hard to prove.[24] Liu *et al.*[25,26] studied the reciprocal product between transmission wrenches and input or output twists, and their metric of closeness to singularities had a physical meaning. For a limb with less than six DOF (degrees of freedom); however, their method to determine the transmission wrench employed the orthogonal operation between the transmission wrench and limb constraint wrenches, where the orthogonality of screws was not translation invariant.[19,27] Huang *et al.*[28] analyzed the force and motion transmissibility between robot joint space and operation space, and presented an overall transmission index that was suitable for measuring closeness to singularities for six DOF parallel robots. Bu[29] proposed an approach based on characteristic angles to measure closeness to singularities, and the proposed metric could get an identical result for similar mechanisms with different sizes, yet the metric met some trouble if the screws were not uniquely definite, such as the limb constraint wrenches of an overconstrained mechanism. Park[35] constructed the left-invariant and right-invariant distance metrics depending on the choice of length scale, and declared there was no bi-invariant metric on the group of rigid body displacements in the three dimensional space.

One of the shortfalls in most of the existing methods is that measuring closeness to singularities of *similar manipulators* at the same posture cannot get an identical result. Here, the similar manipulators are defined as the serial or parallel manipulators with the same architecture and the same length ratio of corresponding links, but with different sizes. There are the following merits to achieve the identical result of the closeness to singularities for similar mechanisms. (1) In many engineering cases, a prototype is needed to check the performance, such as singularities and the closeness to singularities, before the real machine has been fabricated. The prototype has smaller size yet similar structure compared with the real machine. This requires the results of measuring the closeness to singularities should be identical for the real machine and the prototype. (2) The metric that gets the identical result for similar mechanisms helps in writing a unified control law, where two or more similar manipulators with different sizes are being used.

Another shortfall in the existing methods is that for a screw system whose base screws are not uniquely definite, such as limb constraint wrenches of an overconstrained or lower mobility parallel manipulator, the result of measuring closeness to singularities varies as different base screws are selected.

Hence, a novel approach based on geometric average normalized volume spanned by weighted screws is presented in this paper to measure the closeness to singularities. One of the merits of this approach lies in the acquisition of an identical measurement for similar manipulators. Furthermore, the measurement is independent of the selection of base screws, which is very suitable for measuring closeness to static singularities of an overconstrained or lower mobility parallel manipulator whose base wrench screws cannot be uniquely determined. Besides, the proposed metric is insensitive to screw magnitude, and has a geometrical meaning.

The remainder of this paper is organized as follows. A novel concept of weighted screws is proposed in Section 2, where the physical meaning of twists and wrenches are reinterpreted. In Section 3, the normalized volume spanned by weighted screws is presented. The invariance with respect to both the selection of base screws and the similar manipulators is studied in Sections 4 and 5 respectively. In Section 6, the relation between the geometric average normalized volume and the characteristic angle is discussed. The 3-UPU parallel manipulator is taken as an example in Section 7. Finally, conclusions are given in Section 8.

2. Dot Product of Weighted Screws

It is well-known that $\det(\mathbf{M}^{\mathrm{T}}\mathbf{M})$ can be used to detect singularities of a screw system containing N screws, $\$_i$ $(i = 1, 2, \ldots, N)$, where $\mathbf{M}$ denotes the $6 \times N$ screw matrix as follows.

$$\mathbf{M} = \begin{bmatrix} \$_1 & \$_2 & \cdots & \$_N \end{bmatrix}.$$

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However, this indicator refers to the dot product of two screws, which falls into two problems: (1) the problem of origin selection, since this indicator varies as the origin of the coordinate frame changes; (2) the problem of inconsistent dimensions, since the real and dual parts of a screw have different units.

For the first problem, it is obvious that the magnitudes of linear velocity and external force moment acting on the end-effector or moving platform may change if the point of interest alters. Hence, the metric value of closeness to singularities should be related to the location of the point of interest on the end-effector or moving platform. In fact, it has been proved that the dot product of two screws is imposed by the translation of the origin of the coordinate frame, but is invariant with respect to the rotation of the coordinate frame.[3,29] In order to conveniently measure closeness to singularities, the origin of the coordinate frame is demanded to be settled at the point of interest on the end-effector or moving platform. In this way, the coordinate frame coincides with the robotic tool frame.

For the second problem, a screw is generally expressed as $\$ = (\mathbf{S} \quad \mathbf{S}_0)^{\mathrm{T}}$, where $\mathbf{S}$ and $\mathbf{S}_0$ denote the real and dual parts respectively, and these two parts have different units. To solve this problem, the screw should be redefined as the weighted screw $\$_l = \mathbf{R}_{6\times6}\mathbf{L}_{6\times6}(\,\mathbf{S}\ \mathbf{S}_0\,)^{\mathrm{T}}$, where $\mathbf{L}$ denotes a six order diagonal matrix whose upper left three components l_x, l_y, and l_z have the dimension of length,[29–31] and $\mathbf{R}$ denotes a six order square matrix whose upper left 3×3 submatrix is a rotation matrix as follows.

$$\mathbf{L} = \begin{bmatrix} l_x & & & & & \\ & l_y & & & \mathbf{0}_{3\times3} & \\ & & l_z & & & \\ & & & 1 & & \\ & \mathbf{0}_{3\times3} & & & 1 & \\ & & & & & 1 \end{bmatrix}_{6\times6},$$

$$\mathbf{R} = \begin{bmatrix} n_x & o_x & a_x & & & \\ n_y & o_y & a_y & \mathbf{0}_{3\times3} & & \\ n_z & o_z & a_z & & & \\ & & & 1 & & \\ & \mathbf{0}_{3\times3} & & & 1 & \\ & & & & & 1 \end{bmatrix}_{6\times6}.$$

The three components l_x, l_y, and l_z can be artificially chosen, such as the width, depth, and height of the manipulator; or the three components can be chosen as the same value $l, l_x = l_y = l_z = l$. The rotation submatrix in $\mathbf{R}$ is also artificially chosen, such as the posture matrix of the robotic tool frame described in the base frame, or just an identity matrix. The selection of the matrices $\mathbf{L}$ and $\mathbf{R}$ depends on various applications, and the matrix $\mathbf{R}\ \mathbf{L}$ may become asymmetric or even anisotropic.

In this manner, an alternative representation of rigid-body kinematics and statics is interpreted as follows. For a twist that is usually expressed as $\$_t = (\,\omega\ \mathbf{v}\,)^{\mathrm{T}}$, although it contains both linear velocity $\mathbf{v}$ and angular velocity ω, the angular velocity can be numerically expressed as the linear velocity at a point l meters away from the twist axis, that is $\$_{lt} = (\,l\cdot\omega\ \mathbf{v}\,)^{\mathrm{T}}$; alternatively, the three components of the angular velocity can also be represented as other linear velocities, such as $\$_{lt} = \mathbf{L}_{6\times6}(\,\omega\ \mathbf{v}\,)^{\mathrm{T}}$ or even $\$_{lt} = \mathbf{R}_{6\times6}\mathbf{L}_{6\times6}(\,\omega\ \mathbf{v}\,)^{\mathrm{T}}$. Similarly, for a wrench that is usually expressed as $\$_w = (\,\mathbf{f}\ \mathbf{m}\,)^{\mathrm{T}}$, although it contains both force $\mathbf{f}$ and moment $\mathbf{m}$, the force can be numerically described by the moment at a point l meters away from the wrench axis, that is $\$_{lw} = (\,l\cdot\mathbf{f}\ \mathbf{m}\,)^{\mathrm{T}}$; alternatively, the three components of the force can also be represented as other moments, such as $\$_{lw} = \mathbf{L}_{6\times6}(\,\mathbf{f}\ \mathbf{m}\,)^{\mathrm{T}}$ or even $\$_{lw} = \mathbf{R}_{6\times6}\mathbf{L}_{6\times6}(\,\mathbf{f}\ \mathbf{m}\,)^{\mathrm{T}}$.

By this means, the six components of the twist $\$_{lt}$ or of the wrench $\$_{lw}$ have the consistent dimension. Therefore, the dot product of two twists or two wrenches has the dimension of the square of velocity or the square of moment. Since $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}_{6\times6}$, the dot product of weighted screws, $\$_{lt}^{\mathrm{T}}\$_{lt}$ or

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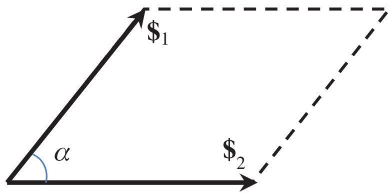

Fig. 1. A parallelogram spanned by two weighted screws.

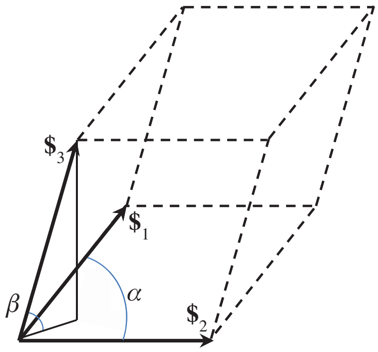

Fig. 2. A parallelepiped spanned by three weighted screws.

$\$_{lw}^{\mathrm{T}}\$_{lw}$, is equivalent to $\$_t^{\mathrm{T}}\mathbf{Q}\$_t$ or $\$_w^{\mathrm{T}}\mathbf{Q}\$_w$, where $\mathbf{Q}$ is the following weight matrix.[18]

$$\mathbf{Q} = \begin{bmatrix} l_x^2 & & & & & \\ & l_y^2 & & & \mathbf{0}_{3\times3} & \\ & & l_z^2 & & & \\ & & & 1 & & \\ & \mathbf{0}_{3\times3} & & & 1 & \\ & & & & & 1 \end{bmatrix}_{6\times6}.$$

The matrix $\mathbf{Q}$ is positive definite, and therefore the following equation holds.

$$\mathrm{Rank}\left(\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\right) = \mathrm{Rank}\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) = \mathrm{Rank}\left(\mathbf{M}\right) \tag{1}$$

Hence, the weighted twists or weighted wrenches can be adopted in the screw matrix to detect singularities of a screw system. In the remainder of this paper, screws used in dot product or determinant are all represented as the weighted screws.

3. Normalized Volume Spanned by Weighted Screws

Weighted screws in an N order system span an N dimensional polyhedron whose edges are the N screw vectors, such as a parallelogram for a two order system in Fig. 1 and a parallelepiped for a three order system in Fig. 2. It can be proved that the volume of the N dimensional polyhedron, V_{pol}, can be expressed as follows.

$$V_{pol} = \sqrt{\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)}, \tag{2}$$

where $\mathbf{M}$ denotes the weighted screw matrix.

Another N dimensional rectangular polyhedron is established with its edges along coordinate axes, and the length of each edge is expressed as $\|\$_i\|$, $i = 1, 2, \ldots, N$. The volume of this rectangular

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polyhedron is

$$V_{rec} = \prod_{i=1}^{N} \|\$_i\|. \tag{3}$$

The normalized volume spanned by weighted screws is defined as the following ratio.

$$\chi = \frac{V_{pol}}{V_{rec}} = \frac{\sqrt{\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)}}{\prod_{i=1}^{N} \|\$_i\|}. \tag{4}$$

In some cases, the base screws of a screw system cannot be uniquely determined, especially for the constraint wrenches of an overconstrained parallel manipulator, or for the wrenches of a lower mobility parallel manipulator when its actuators are locked. For instance, there are three constraint wrenches in a four DOF limb where the actuator is locked, and although the screw submanifold spanned by the three wrenches is definite, the three base wrenches are not unique. Suppose there are J screw subsystems with indefinite base screws, and the dimension and screw matrix of the jth ($j = 1, 2, \ldots, J$) screw subsystem are N_j and $\mathbf{M}_j$ respectively. Then the amount of definite screws in the N dimensional screw manifold is

$$I = N - \sum_{j=1}^{J} N_j. \tag{5}$$

These definite screws can establish an I dimensional rectangular sub-polyhedron. Other indefinite base screws can form J nonrectangular sub-polyhedrons, where the dimension of the jth ($j = 1, 2, \ldots, J$) sub-polyhedron is N_j. Hence, the volume of the polyhedron consisting of these $1{+}J$ sub-polyhedrons is

$$V_{1+J} = \prod_{i=1}^{I} \|\$_i\| \cdot \prod_{j=1}^{J} \sqrt{\det\left(\mathbf{M}_j^{\mathrm{T}}\mathbf{M}_j\right)}.$$

Thus, the normalized volume spanned by the N weighted screws can be defined as follows.

$$\chi = \frac{\sqrt{\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)}}{\prod_{i=1}^{I} \|\$_i\| \cdot \prod_{j=1}^{J} \sqrt{\det\left(\mathbf{M}_j^{\mathrm{T}}\mathbf{M}_j\right)}}. \tag{6}$$

There are some merits for Eq. (6) as the metric of closeness to singularities. First, the geometric meaning of the metric is clarified by interpreting the square root of the determinant as the volume of a high dimensional polyhedron. Second, this metric is insensitive to screw magnitude, since Eq. (6) is dimensionless. Furthermore, this metric is invariant with respect to the selection of base screws for the overconstrained or lower mobility parallel mechanism, and is also invariant with respect to similar mechanisms, as presented in the following sections.

4. Invariance With Respect to Selection of Base Screws

The constraint wrenches of an overconstrained parallel manipulator are not uniquely definite, nor are the wrenches of a lower mobility parallel manipulator when its actuators are locked. In these cases, the base wrenches may be artificially selected.[32,33] It is reasonable to expect that the closeness to singularities is independent of the selection of base screws; however, all existing metrics get different results when these base screws change. In ref. [35], Park announced the non-existence of any bi-invariant metric on the group of rigid body displacements in the three dimensional space.

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For the jth ($j = 1, 2, \ldots, J$) screw subsystem whose N_j base screws are not uniquely determined, these base screws are artificially chosen as $\$_{10}, \$_{20}, \ldots, \$_{Nj0}$, and the screw matrix is denoted by $\mathbf{M}_{j0} = [\$_{10}, \$_{20}, \ldots, \$_{Nj0}]$. Another set of base screws can be expressed as the linear combination of these selected base screws as follows.

$$\$_p = \sum_{k=1}^{N_j} x_{pk} \$_{k0}, \tag{7}$$

where $p = 1, 2, \ldots, N_j$, and x_{pk} ($k = 1, 2, \ldots, N_j$) is the weight coefficient of the pth screw in the N_j order subsystem. Let $\mathbf{M}_j$ be the screw matrix of these new base screws. It can be proved that

$$\det\left(\mathbf{M}_j^{\mathrm{T}}\mathbf{M}_j\right) = \det^2\left(\mathbf{X}_j\right)\det\left(\mathbf{M}_{j0}^{\mathrm{T}}\mathbf{M}_{j0}\right), \tag{8}$$

where $\mathbf{X}_j$ denotes the $N_j \times N_j$ coefficient matrix

$$\mathbf{X}_j = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N_j} \\ x_{21} & x_{22} & \cdots & x_{2N_j} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N_j1} & x_{N_j2} & \cdots & x_{N_jN_j} \end{bmatrix}.$$

Let $\mathbf{M}_0$ be the screw matrix of the N order system whose all indefinite base screws are artificially selected, and $\mathbf{M}$ the screw matrix of the N order system with another set of base screws. It can be proved that

$$\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) = \prod_{j=1}^{J}\det^2\left(\mathbf{X}_j\right)\det\left(\mathbf{M}_0^{\mathrm{T}}\mathbf{M}_0\right). \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (6) yields

$$\chi = \frac{\sqrt{\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)}}{\prod_{i=1}^{I}\|\$_i\| \cdot \prod_{j=1}^{J}\sqrt{\det\left(\mathbf{M}_j^{\mathrm{T}}\mathbf{M}_j\right)}} = \frac{\sqrt{\det\left(\mathbf{M}_0^{\mathrm{T}}\mathbf{M}_0\right)}}{\prod_{i=1}^{I}\|\$_i\| \cdot \prod_{j=1}^{J}\sqrt{\det\left(\mathbf{M}_{j0}^{\mathrm{T}}\mathbf{M}_{j0}\right)}}. \tag{10}$$

The above equation indicates that the proposed metric of closeness to singularities is invariant with respect to the selection of base screws for the overconstrained or lower mobility parallel manipulator with indefinite base screws.

5. Invariance With Respect to Similar Manipulators

The similar manipulators are defined as the mechanisms with the same architecture and the same length ratio of corresponding links, but with different sizes, as shown in Fig. 3. When two similar manipulators are at the same posture, it is reasonable to expect that the closeness to singularities of these two manipulators should be the same; however, almost neither existing metrics can obtain the identical result, except the method of characteristic angles proposed by Bu.[29]

The weight factor l proposed in Section 2 can be regarded as a property of the manipulator, which may be defined as the mean or maximum value of distances from the origin of the coordinate frame to axes of all the revolute joints. Under this definition, this weight factor varies as the manipulator moves to different configurations. Alternatively, the weight factor l may be defined as a constant, for example, the length of some link in the manipulator, or the height of the manipulator at its initial configuration.

There are two similar manipulators A and B in Fig. 3, and the length ratio of the corresponding links in two manipulators is λ. Suppose $\$_{Ai}$ and $\$_{Bi}$ ($i = 1, 2, \ldots, N$) denote the corresponding screws of two similar manipulators respectively, and for each manipulator there are p ($0 \leq p \leq N$)

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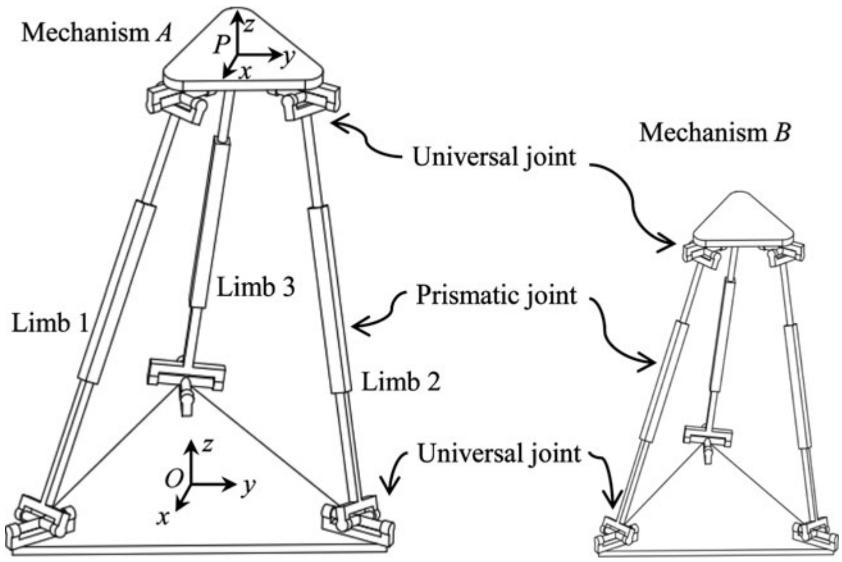

Fig. 3. Two similar 3-UPU manipulators with different sizes.

screws whose real parts are **0**. The p screws with zero real parts are expressed as $\$_{Ai} = (\mathbf{0} \quad \mathbf{S}_{0Ai})^{\mathrm{T}}$ and $\$_{Bi} = (\mathbf{0}\ \mathbf{S}_{0Bi})^{\mathrm{T}}$ $(i = 1, 2, \ldots, p)$ for the two similar manipulators respectively. Here, $\mathbf{S}_{0Ai}$ and $\mathbf{S}_{0Bi}$ denote the dual parts, and it is required $\mathbf{S}_{0Ai} = \eta_i \mathbf{S}_{0Bi}$ for similarity, where η_i is a scalar coefficient. Hence, $\$_{Ai} = \eta_i \$_{Bi}$.

The remaining N-p screws with non-zero real parts are expressed as $\$_{Aj} = (l_A \cdot \mathbf{S}_{Aj}\ \mathbf{S}_{0Aj})^{\mathrm{T}}$ and $\$_{Bj} = (l_B \cdot \mathbf{S}_{Bj}\ \mathbf{S}_{0Bj})^{\mathrm{T}}$ $(j = p{+}1, p{+}2, \ldots, N)$ for the two similar manipulators respectively, where l_A and l_B denote the weight factors, and $l_A/l_B = \lambda$; $\mathbf{S}_{Ai}$ and $\mathbf{S}_{Bi}$ denote the real parts without weight factors, and it is required $\mathbf{S}_{Aj} = \eta_j \mathbf{S}_{Bj}$ for similarity, where η_j is a scalar coefficient; $\mathbf{S}_{0Aj}$ and $\mathbf{S}_{0Bj}$ denote the dual parts. Since $\mathbf{S}_{0Aj} = \mathbf{r}_{Aj} \times \mathbf{S}_{Aj} + h_{Aj}\mathbf{S}_{Aj}$ and $\mathbf{S}_{0Bj} = \mathbf{r}_{Bj} \times \mathbf{S}_{Bj} + h_{Bj}\mathbf{S}_{Bj}$, where $\mathbf{r}_{Aj}$ and $\mathbf{r}_{Bj}$ denote distances from the origins of two coordinate frames to the screw axes, $\mathbf{r}_{Aj} = \lambda \mathbf{r}_{Bj}$, and h_{Aj} and h_{Bj} denote pitches of the two screws, $h_{Aj} = \lambda h_{Bj}$, therefore $\mathbf{S}_{0Aj} = \lambda \eta_j \mathbf{S}_{0Bj}$, and furthermore, $\$_{Aj} = \lambda \eta_j \$_{Bj}$.

Let $\mathbf{M}_A$ and $\mathbf{M}_B$ be the N order screw matrices of the two similar manipulators, and then

$$\det\left(\mathbf{M}_A^{\mathrm{T}}\mathbf{M}_A\right) = \lambda^{2(N-p)} \prod_{i=1}^{N} \eta_i^2 \det\left(\mathbf{M}_B^{\mathrm{T}}\mathbf{M}_B\right). \tag{11}$$

Let $\mathbf{M}_{Aj}$ and $\mathbf{M}_{Bj}$ be the N_j $(j = 1, 2, \ldots, J)$ order screw matrices whose base screws are not uniquely definite, and then

$$\prod_{i=1}^{I} \$_{Ai}^{\mathrm{T}}\$_{Ai} \cdot \prod_{j=1}^{J} \det\left(\mathbf{M}_{Aj}^{\mathrm{T}}\mathbf{M}_{Aj}\right) = \lambda^{2(N-p)} \prod_{i=1}^{N} \eta_i^2 \cdot \prod_{i=1}^{I} \$_{Bi}^{\mathrm{T}}\$_{Bi} \cdot \prod_{j=1}^{J} \det\left(\mathbf{M}_{Bj}^{\mathrm{T}}\mathbf{M}_{Bj}\right). \tag{12}$$

Taking Eqs. (11) and (12) into Eq. (6) yields

$$\chi = \frac{\sqrt{\det\left(\mathbf{M}_A^{\mathrm{T}}\mathbf{M}_A\right)}}{\prod_{i=1}^{I} \|\$_{Ai}\| \cdot \prod_{j=1}^{J} \sqrt{\det\left(\mathbf{M}_{Aj}^{\mathrm{T}}\mathbf{M}_{Aj}\right)}} = \frac{\sqrt{\det\left(\mathbf{M}_B^{\mathrm{T}}\mathbf{M}_B\right)}}{\prod_{i=1}^{I} \|\$_{Bi}\| \cdot \prod_{j=1}^{J} \sqrt{\det\left(\mathbf{M}_{Bj}^{\mathrm{T}}\mathbf{M}_{Bj}\right)}}. \tag{13}$$

The above equation indicates that the proposed metric of closeness to singularities is invariant with respect to similar manipulators at the same posture with different sizes.

6. Geometric Average Normalized Volume and Characteristic Angle

As the order of the screw system becomes higher, the normalized volume spanned by the weighted screws becomes smaller if these screws are not perpendicular to each other, which can be seen from the comparison between Figs. 1 and 2. In order to compare the closeness to singularities of screw systems with different orders, the geometric average of the normalized volume should be adopted as the metric, which is defined as follows.

$$\xi = \sqrt[(I+J-1)]{\chi} = \left(\frac{\sqrt{\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)}}{\prod_{i=1}^{I} \|\$_i\| \cdot \prod_{j=1}^{J} \sqrt{\det\left(\mathbf{M}_j^{\mathrm{T}}\mathbf{M}_j\right)}} \right)^{\frac{1}{I+J-1}}. \tag{14}$$

The symbols in Eq. (14) are coincident with the symbols in Eqs. (6), and (14) is the $(I + J\text{-}1)$th root of Eq. (6). Note that the power in Eq. (14) is $1/(I+J\text{-}1)$, not $1/(I+J)$, since this geometric average value can be related to the average characteristic angle between screws. In Fig. 1, there is an angle α for two order screw system; in Fig. 2, there are two angles α and β for three order screw system. For the manifold spanned by I definite screws, there are I-1 characteristic angles between these screws, as shown in Figs. 1 and 2. For the manifold spanned by J submanifolds whose base screws are indefinite, there are J-1 characteristic angles between these submanifolds, but there are no characteristic angles within each submanifold since its base screws are indefinite. Hence, for the entire manifold consisting of I definite screws and J submanifolds, there are $I + J$-1 characteristic angles between these screws and submanifolds.

Since the possible value of the geometric average normalized volume ranges from 0 to 1, the average characteristic angle can be defined as follows.

$$\overline{\varphi} = \arcsin(\xi). \tag{15}$$

7. An Example

Two similar 3-UPU parallel manipulators are shown in Fig. 3. The moving platform of each manipulator is connected to the base with three limbs. Each limb consists of two universal joints and a prismatic joint equipped with an actuator. The structural parameters of the larger manipulator are as follows. The three universal joints on the base form an equilateral triangle with the side length 0.5 m, and the origin O of the base frame is at the center of this triangle. Similarly, the three universal joints on the moving platform form an equilateral triangle with the side length 0.2m, and the origin P of the tool frame is at the center of this triangle. Suppose the moving platform goes from point (0, 0, 1m) to location (0, 0.4m, 1m) keeping with the constant horizontal posture, and the closeness to singularities is measured during its motion. Note that the coordinate frame of the screw system coincides with the robotic tool frame; the expressions of all the screws should be transformed into the tool frame.

In this case, the weighted screws are constructed as follows. The three components in the diagonal matrix are picked as the same value l, $l_x = l_y = l_z = l$. The rotation submatrix is chosen as an identity matrix.

The three curves of closeness to singularities for three limbs are plotted by the geometric average normalized volume using Eq. (14) in Fig. 4, where the red solid curve, green dash dotted curve, and blue dashed curve stand for limbs one, two and three respectively, and the three curves start at the same initial value (about 0.838) since the 3-UPU parallel manipulator has a symmetrical structure at the initial location. The curve of closeness to singularities for the moving platform is plotted using Eq. (14) in Fig. 5, where the value at the start point is 0 which indicates the manipulator is in a singular configuration at the initial location. Actually, singularities of the 3-UPU parallel manipulator are related to the arrangement of its universal joints.[34] Alternatively, the closeness to singularities for three limbs and the moving platform can also be estimated by average characteristic angles using Eq. (15), as shown in Figs. 6 and 7. It can be found the average characteristic angle here is complementary to the characteristic angle presented in ref. [29].

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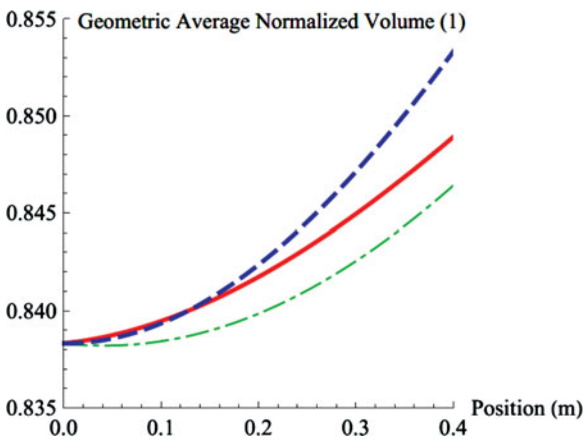

Fig. 4. Geometric average normalized volume for three limbs.

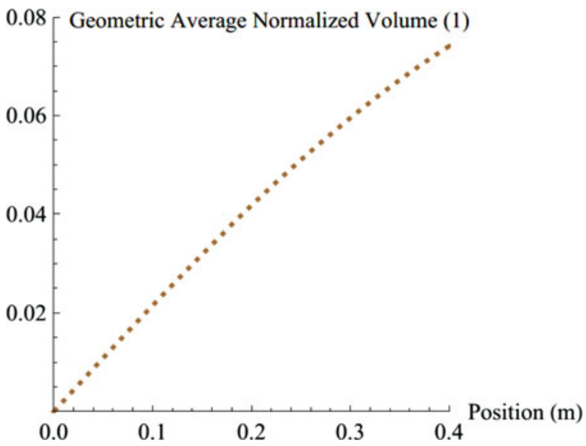

Fig. 5. Geometric average normalized volume for the moving platform.

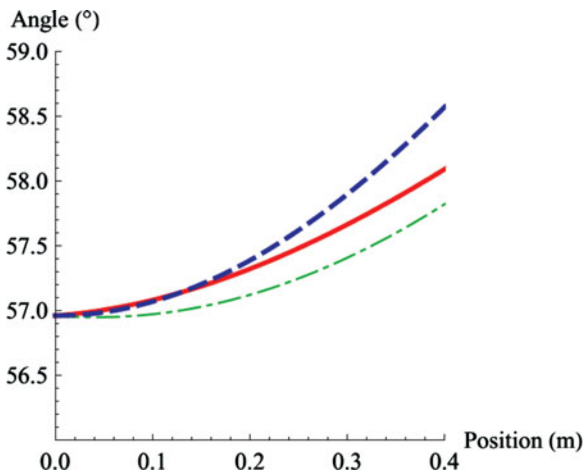

Fig. 6. Average characteristic angles for three limbs.

Note that the 3-UPU parallel manipulator is a lower mobility mechanism, which indicates the limb constraint wrenches are indefinite when its actuators are locked. Nevertheless, the proposed metric of the geometric average normalized volume or the average characteristic angle can solve this problem. Furthermore, for the smaller 3-UPU manipulator in Fig. 3, the proposed metric can obtain the same result when the smaller manipulator moves along the similar trajectory.

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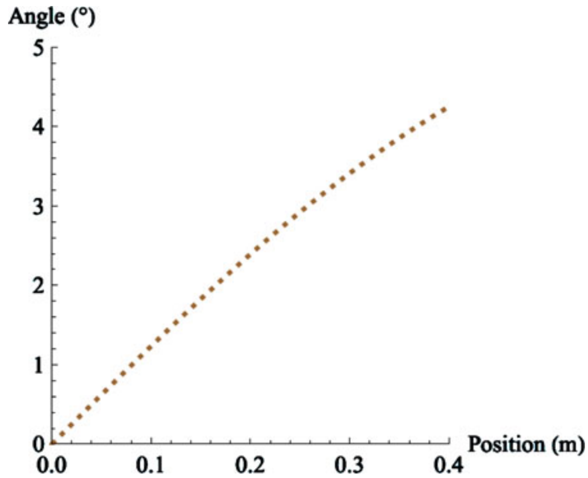

Fig. 7. Average characteristic angle for the moving platform.

8. Conclusions

Manipulators should be prevented from reaching singular configurations, and the singularity margin that indicates the "distance" between the current configuration and a singular configuration should be measured. This paper presents a novel approach based on geometric average normalized volume spanned by weighted screws to measure closeness to singularities for both serial and parallel manipulators. Compared with other existing methods, the proposed metric can obtain an identical result for similar manipulators with different sizes. Furthermore, the result is independent of the selection of base screws, which is very suitable for the overconstrained or lower mobility parallel manipulator whose base screws are not uniquely definite. Besides, the proposed metric is insensitive to screw magnitude, and has a geometrical meaning.

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