Proceedings of the Edinburgh Mathematical Society (2017) **60**, 251–272
DOI:10.1017/S0013091516000109

FISCHER DECOMPOSITION AND CAUCHY–KOVALEVSKAYA EXTENSION IN FRACTIONAL CLIFFORD ANALYSIS: THE RIEMANN–LIOUVILLE CASE

N. VIEIRA

CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal (nloureirovieira@gmail.com)

(Received 11 April 2014)

Abstract In this paper we present the basic tools of a fractional function theory in higher dimensions by means of a fractional correspondence to the Weyl relations via fractional Riemann–Liouville derivatives. A Fischer decomposition, Almansi decomposition, fractional Euler and Gamma operators, monogenic projection, and basic fractional homogeneous powers are constructed. Moreover, we establish the fractional Cauchy–Kovalevskaya extension (FCK extension) theorem for fractional monogenic functions defined on $\mathbb{R}^d$. Based on this extension principle, fractional Fueter polynomials, forming a basis of the space of fractional spherical monogenics, i.e. fractional homogeneous polynomials, are introduced. We study the connection between the FCK extension of functions of the form $\boldsymbol{x}P_l$ and the classical Gegenbauer polynomials. Finally, we present an example of an FCK extension.

Keywords: fractional monogenic polynomials; Fischer decomposition; Almansi decomposition; Cauchy–Kovalevskaya extension theorem; fractional Clifford analysis; fractional Dirac operator

2010 *Mathematics subject classification:* Primary 30G35; 26A33; 35A10; 30A05; 31B05; 30G20

1. Introduction

In recent decades, interest in fractional calculus has increased substantially. This is due, on the one hand, to the fact that different problems can be considered in the framework of fractional derivatives like, for example, those in optics and quantum mechanics, and, on the other hand, to the fact that fractional calculus provides us with a new degree of freedom that can be used for more complete characterization of an object or as an additional encoding parameter.

The study of the fractional Dirac operator is motivated by its physical and geometrical interpretations. Physically, this fractional differential operator is related to some aspects of fractional quantum mechanics such as the derivation of the fractal Schrödinger-type wave equation, the resolution of the gauge hierarchy problem, and the study of supersymmetries. Geometrically, the classical part of this operator may be identified with the scalar curvature in Riemannian geometry.

There are some disadvantages in the implementation of a fractional approach. For example, complicated problems arise during the mathematical manipulations due to the inexistence of a simple Leibnitz rule for the product. Restricting ourselves to the purposes of this work, the main drawback is that in spite of similarities in the formulation of some fractional results with the corresponding classical ones, the proofs are very different in nature because we cannot apply polar or spherical coordinates when we are dealing with fractional derivatives. An explicit and complete derivation of fractional operators in polar or spherical coordinates is still an open task, despite several attempts in the past (see [**18**, **24**, **29**, **31**]). A possible route to overcoming this problem is to adapt the approach presented for the discrete case [**10**].

Clifford analysis is a generalization of classical complex analysis in the plane to the case of an arbitrary dimension $d \in \mathbb{Z}$ (in the case of negative dimensions, one is dealing with the so-called super Clifford analysis). At the heart of the theory lies the Dirac operator D on $\mathbb{R}^d$, a conformally invariant first-order differential operator that plays the same role in classical Clifford analysis as the Cauchy–Riemann operator $\partial_{\bar{z}}$ does in complex analysis. Over the last decades Sommen and his collaborators developed a method for establishing a higher dimension function theory based on the so-called Weyl relations [**7**, **11**, **14**]. In more restrictive settings it is nowadays called the Howe dual pair technique (see [**27**]). There are two focal points: the construction of an operator algebra (classically $\mathfrak{osp}(1|2)$), and the establishment of a Fischer decomposition.

The traditional Fischer decomposition in harmonic analysis yields an orthogonal decomposition of the space of homogeneous polynomials of given homogeneity in terms of spaces of harmonic homogeneous polynomials. In classical continuous Clifford analysis one obtains a refinement yielding an orthogonal decomposition with respect to the so-called Fischer inner product of homogeneous polynomials in terms of spaces of monogenic polynomials, i.e. null solutions of the Dirac operator (see [**14**]). Generalizations of the Fischer decomposition in other frameworks can be found, for example, in [**2**, **7**, **11**, **15**, **21**, **25**, **27**].

Another well-known result in Clifford analysis is the Cauchy–Kovalevskaya extension theorem, which we will denote simply as the CK extension (see [**6**, **23**]). It corresponds to a direct generalization to higher dimension of the complex plane case, and can be found in [**14**]. Other generalizations of the CK extension can be found, for instance, in [**5**, **9**, **10**, **12**, **13**, **26**, **32**].

The aim of this paper is to present an analogue of the results in [**21**, **32**] for a fractional Dirac operator defined via fractional Riemann–Liouville derivatives. The author would like to reinforce that this is not a direct generalization because, contrarily to what happens in the classical and Caputo cases, the constant functions are not null solutions of the Euclidian/fractional Dirac operator.

The structure of the paper is as follows. In the preliminaries we recall some basic facts about Clifford analysis and fractional calculus. In § 3 we introduce the corresponding Weyl relations for this fractional setting and we introduce the notion of a fractional homogeneous polynomial. Here, we also present the fractional correspondence to the Fischer decomposition and its extension to a fractional Almansi decomposition. At the end of this section we construct the projection of a given fractional homogeneous polynomial into the space of fractional homogeneous monogenic polynomials. We also calculate the dimension of the space of fractional homogeneous monogenic polynomials. In § 4 we establish a fractional Cauchy–Kovalevskaya extension (FCK extension) theorem for fractional monogenic functions. Based on this extension principle, we introduce fractional Fueter polynomials, which form a basis of the space of fractional spherical monogenics, i.e. of fractional homogeneous monogenic polynomials. Moreover, we go into detail about the connection between the FCK extension of functions of the form $\boldsymbol{x}P_l$ and the classical Gegenbauer polynomials. We end this paper with an example of an FCK extension.

2. Preliminaries

It is well known that the treatment of the two-dimensional vector space $\mathbb{R}^2$ in terms of complex numbers has the advantage of providing an additional multiplication operator on $\mathbb{R}^2$. Appropriate higher-dimensional associative analogues of the complex numbers are real Clifford algebras. For details about Clifford algebras and basic concepts of its associated function theory, we refer the interested reader to, for example, [**3**, **14**, **19**].

Let $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\}$ be the standard basis of the Euclidian vector space in $\mathbb{R}^d$. The associated Clifford algebra $\mathbb{R}_{0,d}$ is the free algebra generated by $\mathbb{R}^d$ modulo $x^2 = -\|x\|^2\boldsymbol{e}_0$, where $x \in \mathbb{R}^d$ and $\boldsymbol{e}_0$ is the neutral element with respect to the multiplication operation in the Clifford algebra $\mathbb{R}_{0,d}$. The defining relation induces the multiplication rule $\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = -2\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. In particular, $\boldsymbol{e}_i^2 = -1$ for all $i = 1, \dots, d$. The standard basis vectors thus operate as imaginary units.

A vector space basis for $\mathbb{R}_{0,d}$ is given by the set $\{\boldsymbol{e}_A \colon A \subseteq \{1, \dots, d\}\}$ with $\boldsymbol{e}_A = \boldsymbol{e}_{l_1}\boldsymbol{e}_{l_2}\cdots\boldsymbol{e}_{l_r}$, where $1 \leqslant l_1 < \cdots < l_r \leqslant d$, $\boldsymbol{e}_\emptyset := \boldsymbol{e}_0 := 1$. Each $a \in \mathbb{R}_{0,d}$ can be written in the form $a = \sum_A a_A\boldsymbol{e}_A$, with $a_A \in \mathbb{R}$. The conjugation in the Clifford algebra $\mathbb{R}_{0,d}$ is defined by $\bar{a} = \sum_A a_A\bar{e}_A$, where $\bar{e}_A = \bar{e}_{l_r}\bar{e}_{l_{r-1}}\cdots\bar{e}_{l_1}$, and $\bar{e}_j = -\boldsymbol{e}_j$ for $j = 1, \dots, d$, $\bar{e}_0 = \boldsymbol{e}_0 = 1$. An important property of the Clifford algebra $\mathbb{R}_{0,d}$ is that each non-zero vector $a \in \mathbb{R}_1^d$ has a multiplicative inverse given by $\bar{a}/\|a\|^2$. An important subspace of the real Clifford algebra $\mathbb{R}_{0,d}$ is the so-called space of paravectors $\mathbb{R}_1^d = \mathbb{R} \oplus \mathbb{R}^d$, that being the sum of scalars and vectors. An element $\underline{a} = (a_0, a_1, \dots, a_d)$ of $\mathbb{R}^d$ will be identified by $\underline{a} = a_0 + a$, with $a = \sum_{i=1}^d \boldsymbol{e}_i a_i$.

We now introduce the complexified Clifford algebra $\mathbb{C}_d$ as the tensor product

$$\mathbb{C} \otimes \mathbb{R}_{0,d} = \left\{ w = \sum_A w_A\boldsymbol{e}_A,\ w_A \in \mathbb{C},\ A \subset M \right\},$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e. $\mathrm{i}\boldsymbol{e}_j = \boldsymbol{e}_j\mathrm{i}$ for all $j = 1, \dots, d$. To avoid ambiguities with the Clifford conjugation, we denote the complex conjugation mapping a complex scalar $w_A = a_A + \mathrm{i}b_A$, with real components a_A and b_A, onto $\bar{w}_A = a_A - \mathrm{i}b_A$ by $\sharp$. The complex conjugation leaves the elements $\boldsymbol{e}_j$ invariant, i.e. $\boldsymbol{e}_j^\sharp = \boldsymbol{e}_j$ for all $j = 1, \dots, d$. We also have a pseudo-norm on $\mathbb{C}$, namely, $|w| := \sum_A |w_A|$, where $w = \sum_A w_A\boldsymbol{e}_A$, as usual. Notice also that for $a, b \in \mathbb{C}_d$ we only have $|ab| \leqslant 2^d|a|\,|b|$. The other norm criteria are fulfilled.

Clifford analysis can be regarded as a higher-dimensional generalization of complex function theory in the sense of the Riemann approach. A $\mathbb{C}_d$-valued function f over $\Omega \subset \mathbb{R}_1^d$ has representation $f = \sum_A \boldsymbol{e}_A f_A$, with components $f_A \colon \Omega \to \mathbb{C}$. Properties such as continuity will be understood component-wise. Next, we recall the Euclidean Dirac operator $\mathrm{D} = \sum_{j=1}^d \boldsymbol{e}_j \partial_{x_j}$, which factorizes the d-dimensional Euclidean Laplacian, i.e.

$$\mathrm{D}^2 = -\Delta = -\sum_{j=1}^{d} \partial x_j^2.$$

A $\mathbb{C}_d$-valued function f is called *left monogenic* if it satisfies $\mathrm{D}u = 0$ on Ω (respectively, *right monogenic* if it satisfies $u\mathrm{D} = 0$ on Ω).

The most widely known definition of the fractional derivative is the so-called Riemann–Liouville definition. This definition appears as a result of the unification of the notions of integer-order integration and differentiation, and is expressed as follows:

$$ {}_a\mathrm{D}_t^p f(t) = \frac{1}{\Gamma(m+1-p)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)\,\mathrm{d}\tau, \quad m \leqslant p < m+1. \tag{2.1}$$

The previous definition requires that the function $f(t)$ must be $m+1$ times continuously differentiable, which corresponds, in some sense, to a narrow class of functions; however, this class of functions is very important for applications because the character of the majority of dynamical processes is smooth enough and does not allow discontinuities. Understanding this fact is important for proper use of the methods of fractional calculus in applications, especially because of the fact that the Riemann–Liouville definition provides an excellent opportunity to weaken the conditions of the function $f(t)$; then integral (2.1) exists for $t > a$ and can be differentiated $m+1$ times. The weak condition on the function $f(t)$ in (2.1) is necessary, for example, for obtaining the solution of the Abel equation. For more details about fractional calculus and applications we refer the reader to [**22**, **28**, **30**].

In [**16**] the fractional derivative (2.1) was successfully applied in the definition of the fractional equivalent of the Dirac operator in the context of Clifford analysis. In fact, the fractional Dirac operator corresponds to

$$\mathrm{D}^\alpha = \sum_{j=1}^{d} \boldsymbol{e}_j \mathrm{D}_j^\alpha = \sum_{j=1}^{d} \boldsymbol{e}_j (\mathrm{D}_j + Y_j),$$

where D_j is the classical derivative with respect to x_j, and $Y_j = (1-\alpha)/(\xi_j - x_j)$, with $0 < \alpha < 1$, and $\xi = (\xi_1, \dots, \xi_d)$ is the observer time vector. A $\mathbb{C}_n$-valued function f is called *fractional left monogenic* if it satisfies $\mathrm{D}^\alpha u = 0$ on Ω (respectively, *fractional right monogenic* if it satisfies $u\mathrm{D}^\alpha = 0$ on Ω). We observe that due to the definition of the fractional Dirac operator we have that

$$\mathrm{D}^\alpha \left(\prod_{i=1}^{d} (\xi_i - x_i)^{1-\alpha} \right) = 0, \tag{2.2}$$

i.e. $\prod_{i=1}^d (\xi_i - x_i)^{1-\alpha}$ is a fractional monogenic function.

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

Remark 2.1. In (2.2) the fractional power $(\xi_i - x_i)^{1-\alpha}$ should be understood in the following way:

$$(\xi_j - x_j)^{1-\alpha} = \begin{cases} \exp((1-\alpha)\ln|\xi_j - x_j|), & \xi_j > x_j, \\ 0, & \xi_j = x_j, \\ \exp((1-\alpha)\ln|\xi_j - x_j| + \mathrm{i}\alpha\pi), & \xi_j < x_j, \end{cases}$$

with $0 < \alpha < 1$ and $j = 0, 1, \ldots, d$.

Hereafter we will consider paravectors of the form $\underline{\boldsymbol{x}}^{\alpha} = \boldsymbol{x}_0 + \boldsymbol{x}$, where $\boldsymbol{x}_j = \alpha(\xi_j - x_j)$.

Remark 2.2. During the paper we will restrict ourselves to the case in which the fractional parameter α belongs to the interval $]0,1[$. Cases in which α is outside this range can be reduced to the considered one. In fact, for $\alpha \in \mathbb{R}$ we have that $\alpha = [\alpha] + \tilde{\alpha}$, with $[\alpha]$ the integer part of α and $\tilde{\alpha} \in]0,1[$.

3. Weyl relations and Fractional Fischer decomposition

The aim of this section is to provide the basic tools for a function theory for the fractional Dirac operator defined via fractional Riemann–Liouville derivatives.

3.1. Fractional Weyl relations

Now we introduce the fractional correspondence of the classical Euler and Gamma operators. Moreover, we will show that the two natural operators D^{α} and $\boldsymbol{x}$, considered as odd elements, generate a finite-dimensional Lie superalgebra in the algebra of endomorphisms generated by the partial fractional Riemann–Liouville derivatives, the basic *vector variables* $\boldsymbol{x}_j$ (seen as multiplication operators), and the basis of the Clifford algebra $\boldsymbol{e}_j$. Before we proceed, we recall the definition of a Lie superalgebra (for more details about Lie superalgebras and their connections with Clifford analysis, see [**4**, **20**]).

Definition 3.1. A Lie superalgebra $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$) is a $\mathbb{Z}_2$-graded vector space, i.e., a direct sum of two vector spaces $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a graded bracket $[[\cdot,\cdot]]$, satisfying

- the $\mathbb{Z}_2$-grading: $[[a_i, a_j]] \in \mathfrak{g}_{i+j \mod 2}$, with $a_i \in \mathfrak{g}_i$ and $a_j \in \mathfrak{g}_j$;
- the graded antisymmetry: $[[a_i, a_j]] = (-1)^{ij}[[a_j, a_i]]$, with $a_i \in \mathfrak{g}_i$ and $a_j \in \mathfrak{g}_j$;
- the generalized Jacobi identity:

$$(-1)^{ik}[[a_i, [[a_j, a_k]]]] + (-1)^{ji}[[a_j, [[a_k, a_i]]]] + (-1)^{kj}[[a_k, [[a_i, a_j]]]] = 0,$$

  with $a_i \in \mathfrak{g}_i$, $a_j \in \mathfrak{g}_j$, $a_k \in \mathfrak{g}_k$.

In order to obtain our results, we will use a standard technique in higher dimensions, namely, we study the commutator and the anti-commutator between $\boldsymbol{x}$ and D^{α}. We start by proposing the following fractional Weyl relations:

$$[\mathrm{D}_i^{\alpha}, \boldsymbol{x}_i] = \mathrm{D}_i^{\alpha}\boldsymbol{x}_i - \boldsymbol{x}_i\mathrm{D}_i^{\alpha} = -\alpha, \tag{3.1}$$

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

with $i = 1, \dots, d$, $0 < \alpha < 1$. This leads to the following relations for $\boldsymbol{x}$ and D^α:

$$\{\mathrm{D}^\alpha, \boldsymbol{x}\} = \mathrm{D}^\alpha \boldsymbol{x} + \boldsymbol{x}\mathrm{D}^\alpha = -2\mathbb{E}^\alpha + \alpha d, \tag{3.2}$$

$$[\boldsymbol{x}, \mathrm{D}^\alpha] = \boldsymbol{x}\mathrm{D}^\alpha - \mathrm{D}^\alpha \boldsymbol{x} = -2\Gamma^\alpha - \alpha d, \tag{3.3}$$

where $\mathbb{E}^\alpha$, Γ^α are, respectively, the fractional Euler and Gamma operators and are expressed as

$$\mathbb{E}^\alpha = \sum_{i=1}^{d} \boldsymbol{x}_i \mathrm{D}_j^\alpha, \qquad \Gamma^\alpha = \sum_{i<j} \boldsymbol{e}_i \boldsymbol{e}_j (\boldsymbol{x}_i \mathrm{D}_j^\alpha - \mathrm{D}_j^\alpha \boldsymbol{x}_i). \tag{3.4}$$

From (3.4) we derive, via straightforward calculations, the following relations:

$$\left.\begin{array}{ccc} \mathbb{E}^\alpha + \Gamma^\alpha = -\boldsymbol{x}\mathrm{D}^\alpha, & [\mathbb{E}^\alpha, \Gamma^\alpha] = 0, & [\boldsymbol{x}, \mathbb{E}^\alpha] = \alpha \boldsymbol{x}, \\ [\mathrm{D}^\alpha, \mathbb{E}^\alpha] = -\alpha \mathrm{D}^\alpha, & \{\boldsymbol{x}, \boldsymbol{x}\} = -2|\boldsymbol{x}|^2, & \{\mathrm{D}^\alpha, \mathrm{D}^\alpha\} = 2\Delta^{2\alpha}, \\ [\boldsymbol{x}, |\boldsymbol{x}|^2] = 0, & [\boldsymbol{x}, \Delta^{2\alpha}] = 2\alpha \mathrm{D}^\alpha, & [\mathrm{D}^\alpha, |\boldsymbol{x}|^2] = -2\alpha \boldsymbol{x}, \\ [\mathrm{D}^\alpha, \Delta^{2\alpha}] = 0, & [\mathbb{E}^\alpha, \Delta^{2\alpha}] = 2\alpha \Delta^{2\alpha}, & [\mathbb{E}^\alpha, |\boldsymbol{x}|^2] = -2\alpha |\boldsymbol{x}|^2, \\ & [|\boldsymbol{x}|^2, \Delta^{2\alpha}] = 4\alpha\left(\mathbb{E}^\alpha - \dfrac{\alpha d}{2}\right), & \end{array}\right\} \tag{3.5}$$

where $\Delta^{2\alpha} = -\mathrm{D}^\alpha \mathrm{D}^\alpha$. Relations (3.5) show that we have a finite-dimensional Lie superalgebra, in the sense of Definition 3.1, generated by $\boldsymbol{x}$ and D^α, isomorphic to $\mathfrak{osp}(1|2)$. At the end of this section we give a brief description (with references) of the $\mathfrak{osp}(1|2)$ algebra and its realizations. The normalization

$$H^\alpha = \frac{1}{2}\left(\mathbb{E}^\alpha - \frac{\alpha d}{2}\right),$$

$$(E^\alpha)^- = \frac{-\alpha}{2}|\boldsymbol{x}|^2, \qquad (E^\alpha)^+ = \frac{\alpha}{2}\Delta^{2\alpha}, \qquad (F^\alpha)^- = \frac{-\mathrm{i}\alpha}{2\sqrt{2}}\boldsymbol{x}, \qquad (F^\alpha)^+ = \frac{-\mathrm{i}\alpha}{2\sqrt{2}}\mathrm{D}^\alpha,$$

leads to the standard commutation relations for $\mathfrak{osp}(1|2)$ (see [**17**]):

$$\begin{array}{cc} [H^\alpha, (E^\alpha)^\pm] = \pm\alpha (E^\alpha)^\pm, & [(E^\alpha)^+, (E^\alpha)^-] = -2\alpha^3 H^\alpha, \\ [H^\alpha, (F^\alpha)^\pm] = \pm\dfrac{\alpha}{2}(F^\alpha)^\pm, & \{(F^\alpha)^+, (F^\alpha)^-\} = \dfrac{\alpha^2}{2} H^\alpha, \\ [(E^\alpha)^\pm, (F^\alpha)^\mp] = -\alpha^2 (F^\alpha)^\pm, & \{(F^\alpha)^\pm, (F^\alpha)^\pm\} = \pm\dfrac{\alpha}{2}(E^\alpha)^\pm. \end{array}$$

We now introduce the definition of fractional homogeneity of a polynomial by means of the fractional Euler operator.

Definition 3.2. A polynomial P_l is called fractional homogeneous of degree $l \in \mathbb{N}_0$ if and only if $\mathbb{E}^\alpha P_l = -\alpha l P_l$.

We observe that from the previous definition the basic fractional homogeneous powers are given by $\prod_{j=1}^{d} \boldsymbol{x}_j^{\beta_j}$, with $l = |\beta| = \beta_1 + \cdots + \beta_d$. In combination with the third relation in (3.5),

$$[\boldsymbol{x}, \mathbb{E}^\alpha] = \alpha \boldsymbol{x},$$

this definition also implies that:

- the multiplication of a fractional homogeneous polynomial of degree l by $\boldsymbol{x}$ will result in a fractional homogeneous polynomial of degree $l + 1$, and thus may be seen as a raising operator;
- for a fractional homogeneous polynomial P_l of degree l, $\mathrm{D}^\alpha P_l$ is a fractional homogeneous polynomial of degree $l - 1$;
- the fractional Weyl relations (3.1) will now enable us to construct fractional homogeneous polynomials recursively.

The variables $\boldsymbol{x}_j$, $\boldsymbol{x}_j^k$ are the basic fractional homogeneous polynomials of degree k. In the following result their fundamental properties are listed.

Theorem 3.3. *For all $k \in \mathbb{N}$ and $i, j = 1, \ldots, d$ we have*

$$\left.\begin{aligned} \mathrm{D}_j^\alpha \boldsymbol{x}_j^k &= -k \boldsymbol{x}_j^{k-1}, & \\ \mathrm{D}_i^\alpha \boldsymbol{x}_j^k &= 0, & i \neq j, \\ \mathrm{D}_j^\alpha \boldsymbol{x}_j^{k_1} \boldsymbol{x}_i^{k_2} &= -k_1 \boldsymbol{x}_j^{k_1 - 1} \boldsymbol{x}_i^{k_2}, & i \neq j. \end{aligned}\right\} \tag{3.6}$$

Moreover, for any two multi-index $\gamma = (\gamma_1, \ldots, \gamma_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ with $|\gamma| = |\beta|$, it holds that

$$\mathrm{D}_1^{\gamma_1} \cdots \mathrm{D}_1^{\gamma_d} (\boldsymbol{x}_1^{\beta_1} \cdots \boldsymbol{x}_d^{\beta_d}) \prod_{j=1}^{d} \boldsymbol{x}_j^{1-\alpha} = \begin{cases} (-1)^{\gamma!} \gamma! & \text{if } \gamma = \beta, \\ 0 & \text{if } \gamma \neq \beta, \end{cases}$$

where we have put $\gamma! = \gamma_1! \cdots \gamma_d!$.

The proof of this result is immediate and we therefore omit it from the text. Furthermore, from the previous theorem we conclude that a closed form for the fractional homogeneous polynomials is given by

$$(\boldsymbol{e}_j \boldsymbol{x}_j)^{2n+1} = (-1)^n \boldsymbol{e}_j (\boldsymbol{x}_j)^{2n+1}, \qquad (\boldsymbol{e}_j \boldsymbol{x}_j)^{2n} = (-1)^n (\boldsymbol{x}_j)^{2n} \tag{3.7}$$

for $n = 1, 2, \ldots$ and $j = 1, \ldots, d$. Moreover, from Theorem 3.3 we have the equalities

$$\mathrm{D}^\alpha \left(\prod_{j=1}^{d} \boldsymbol{x}_j^{1-\alpha} \right) = 0, \qquad \mathbb{E}^\alpha \left(\prod_{j=1}^{d} \boldsymbol{x}_j^{1-\alpha} \right) = 0, \qquad \mathbb{E}^\alpha \left(\boldsymbol{x}^\beta \prod_{j=1}^{d} \boldsymbol{x}_j^{1-\alpha} \right) = -|\beta| \boldsymbol{x}^\beta \prod_{j=1}^{d} \boldsymbol{x}_j^{1-\alpha},$$

and therefore in the fractional Fischer decomposition that we will present in the next section we will have that

$$P_0 = \operatorname{span} \left\{ \prod_{j=1}^{d} \boldsymbol{x}_j^{1-\alpha} \right\} \quad \text{and} \quad P_l = \operatorname{span} \left\{ \boldsymbol{x}^\beta \prod_{j=1}^{d} \boldsymbol{x}_j^{1-\alpha}, \ |\beta| = l \right\},$$

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

where $\beta = (\beta_1, \ldots, \beta_d)$ with $|\beta| = \beta_1 + \cdots + \beta_d$. Furthermore, we have the following remark.

Remark 3.4. We would like to point out the main difference between this approach and the correspondent one presented in previous work (even when considering a fractional Dirac operator defined via fractional Caputo derivatives): due to the fact that $\mathrm{D}^\alpha c \neq 0$, with c a constant and D^α the fractional Dirac operator defined via Riemann–Liouville derivatives, we cannot assume that $P_0 = \mathrm{span}\{1\}$.

3.2. Fractional Fischer decomposition

A fractional Fischer inner product of two fractional homogeneous polynomials P and Q has the form

$$\langle P(\boldsymbol{x}), Q(\boldsymbol{x})\rangle = \mathrm{Sc}[\overline{P(\partial_{\boldsymbol{x}})}Q(\boldsymbol{x})], \tag{3.8}$$

where $\partial_{\boldsymbol{x}}$ represents D_j^α, and $P(\partial_{\boldsymbol{x}})$ is a differential operator obtained by replacing in the polynomial P each variable $\boldsymbol{x}_j$ by the corresponding fractional derivative, i.e. $\mathrm{D}_j + Y_j$. From (3.8) we have that for any polynomial P_{l-1} of homogeneity $l-1$ and any polynomial Q_l of homogeneity l,

$$\langle \boldsymbol{x}P_{l-1}, Q_l\rangle = \langle P_{l-1}, \mathrm{D}^\alpha Q_l\rangle. \tag{3.9}$$

This fact allows us to prove the following result.

Theorem 3.5. *For each $l \in \mathbb{N}_0$ we have $\Pi_l = \mathcal{M}_l + \boldsymbol{x}\Pi_{l-1}$, where Π_l denotes the space of fractional homogeneous polynomials of degree l, and $\mathcal{M}_l$ denotes the space of fractional monogenic homogeneous polynomials of degree l. Moreover, the subspaces $\mathcal{M}_k$ and $\boldsymbol{x}\Pi_{l-1}$ are orthogonal with respect to the Fischer inner product (3.8).*

Proof. Since $\Pi_l = \boldsymbol{x}\Pi_{l-1} + (\boldsymbol{x}\Pi_{l-1})^\perp$, it suffices to prove that $(\boldsymbol{x}\Pi_{l-1})^\perp = \mathcal{M}_{l-1}$. To this end, assume that for some $P_l \in \Pi_l$ we have $\langle \boldsymbol{x}P_{l-1}, P_l\rangle = 0$ for all $P_{l-1} \in \Pi_{l-1}$. From (3.9) we then have that $\langle P_{l-1}, \mathrm{D}^\alpha P_l\rangle = 0$ for all $P_{l-1} \in \Pi_{l-1}$. As $\mathrm{D}^\alpha P_l \in \Pi_{l-1}$, we obtain that $\mathrm{D}^\alpha P_l = 0$, or that $P_l \in \mathcal{M}_l$. This means that $(\boldsymbol{x}\Pi_{l-1})^\perp \subset \mathcal{M}_{l-1}$. Conversely, take $P_l \in \mathcal{M}_l$. Then we have, for any $P_{l-1} \in \Pi_{l-1}$, that

$$\langle \boldsymbol{x}P_{l-1}, P_l\rangle = \langle P_{l-1}, \mathrm{D}^\alpha P_l\rangle = \langle P_{l-1}, 0\rangle = 0, \tag{3.10}$$

from which it follows that $\mathcal{M}_{l-1} \subset (\boldsymbol{x}\Pi_{l-1})^\perp$, and therefore $\mathcal{M}_{l-1} = (\boldsymbol{x}\Pi_{l-1})^\perp$. □

As a result we obtain the fractional Fischer decomposition with respect to the fractional Dirac operator D^α.

Theorem 3.6. *Let P_l be a fractional homogeneous polynomial of degree l. Then*

$$P_l = M_l + \boldsymbol{x}M_{l-1} + \boldsymbol{x}^2 M_{l-2} + \cdots + \boldsymbol{x}^l M_0, \tag{3.11}$$

where each M_j denotes the fractional homogeneous monogenic polynomial of degree j. More specifically,

$$M_0 = P_0 \quad \text{and} \quad M_l = \{u \in P_l \colon \mathrm{D}^\alpha u = 0\}.$$

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

The spaces represented in (3.11) are orthogonal to each other with respect to the Fischer inner product (3.8). This a consequence of the construction of the fractional Euler operator $\mathbb{E}^\alpha$, and in particular of (3.2) and (3.3). Moreover, the space $\mathcal{P}(S)$ of all Clifford-valued polynomials decomposes in a multiplicity free by means of our fractional Weyl relations, and the decomposition has the form of an infinite triangle

$$\begin{array}{ccccccccc}
\mathcal{P}_0(S) & & \mathcal{P}_1(S) & & \mathcal{P}_2(S) & & \mathcal{P}_3(S) & \\
\mathcal{M}_0 & \xleftarrow{\mathrm{D}^\alpha} & \boldsymbol{x}\mathcal{M}_0 & \xleftarrow{\mathrm{D}^\alpha} & |\boldsymbol{x}|^2\mathcal{M}_0 & \xleftarrow{\mathrm{D}^\alpha} & \boldsymbol{x}^3\mathcal{M}_0 & \cdots \\
 & & \oplus & & \oplus & & \oplus & \\
 & & \mathcal{M}_1 & \xleftarrow{\mathrm{D}^\alpha} & \boldsymbol{x}\mathcal{M}_1 & \xleftarrow{\mathrm{D}^\alpha} & |\boldsymbol{x}|^2\mathcal{M}_1 & \cdots \\
 & & & & \oplus & & \oplus & \\
 & & & & \mathcal{M}_2 & \xleftarrow{\mathrm{D}^\alpha} & \boldsymbol{x}\mathcal{M}_2 & \cdots \\
 & & & & & & \oplus & \\
 & & & & & & \mathcal{M}_3 & \cdots
\end{array}$$

While in the classic case all the summands in the same row are isomorphic to $\mathrm{Pin}(d)$-modules, and each row is an irreducible module for the Howe dual pair $\mathrm{Pin}(d) \times \mathfrak{osp}(1|2)$ (see [**8**] for more details), the same cannot be said in the fractional case yet. The reason is that while the classical Laplacian and the Fischer dual are invariant under $O(d)$ ($\mathrm{Pin}(d)$ is the double cover of $O(d)$), we can easily show that this is not true in the fractional case.

Nevertheless, we have that all the summands in the same row are modules. The author's conjecture is that they can be invariant under a certain 'fractional' Pin group that does not coincide with the classical one; this subject will be studied in forthcoming work.

The Dirac operator shifts all spaces in the same row to the left, the multiplication by $\boldsymbol{x}$ shifts them to the right, and both of these actions are isomorphisms of modules. Moreover, any spinor-valued polynomial can be written (in a unique way) as a sum of monogenic polynomials and a product of powers of $\boldsymbol{x}$ with other monogenic polynomials.

From Theorem 3.6 we have the following direct extension to the fractional case of the Almansi decomposition.

Theorem 3.7. *For any fractional polyharmonic polynomial P_l of degree $l \in \mathbb{N}_0$ in a star-like domain D in $\mathbb{R}^d$ with respect to* 0*, i.e. $\Delta^{2\alpha} P_l = 0$, there exist uniquely fractional harmonic functions $P_0, P_1, \ldots, P_{l-1}$ such that*

$$P_l = P_0 + |\boldsymbol{x}|^2 P_1 + \cdots + |\boldsymbol{x}|^{2(l-1)} P_{l-1} \quad \forall \boldsymbol{x} \in D.$$

3.3. Explicit formulae

Here we obtain an explicit formula for the projection $\pi_{\mathcal{M}}(P_l)$ of a given fractional homogeneous polynomial P_l into the space of fractional homogeneous monogenic polynomials. We start with the following auxiliary result.

Theorem 3.8. *For any fractional homogeneous polynomial P_l and any positive integer s, we have*

$$\mathrm{D}^\alpha \boldsymbol{x}^s P_l = g_{s,l} \boldsymbol{x}^{s-1} P_l + (-1)^s \boldsymbol{x}^s \mathrm{D}^\alpha P_l,$$

where $g_{2k,l} = 2k\alpha$ and $g_{2k+1,l} = 2(\alpha k + \alpha l) + \alpha d$.

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

Proof. The proof follows, by induction and straightforward calculations, from the commutation between D^α and $\boldsymbol{x}^s$ using the relations

$$\mathrm{D}^\alpha \boldsymbol{x} = -2\mathbb{E}^\alpha + \alpha d - \boldsymbol{x}\mathrm{D}^\alpha, \qquad \mathbb{E}^\alpha \boldsymbol{x} = \boldsymbol{x}\mathbb{E}^\alpha - \alpha \boldsymbol{x}.$$

□

Let us now compute an explicit form of the projection $\pi_{\mathcal{M}}(P_l)$.

Theorem 3.9. *Consider the constants $c_{j,l}$ defined by*

$$c_{0,l} = 1, \qquad c_{j,l} = \frac{(-1)^j}{(2[j/2])!! \prod_{i=0}^{[j/2]} g_{2i+1,l-(2i+1)}},$$

where $j = 1, \dots, l$ and $[\cdot]$ represents the integer part. Then the map $\pi_{\mathcal{M}}$ given by

$$\pi_{\mathcal{M}}(P_l) := P_l + c_{1,l}\boldsymbol{x}\mathrm{D}^\alpha P_l + c_{2,l}\boldsymbol{x}^2(\mathrm{D}^\alpha)^2 P_l + \cdots + c_{l,l}\boldsymbol{x}^l(\mathrm{D}^\alpha)^l P_l$$

is the projection of the fractional homogeneous polynomial P_l into the space of fractional homogeneous monogenic polynomials.

Proof. Let us consider the linear combination

$$r = a_0 P_l + a_1\boldsymbol{x}\mathrm{D}^\alpha P_l + a_2\boldsymbol{x}^2(\mathrm{D}^\alpha)^2 P_l + \cdots + a_k\boldsymbol{x}^l(\mathrm{D}^\alpha)^l P_l$$

with $a_0 = 1$. If there are constants a_j, $j = 1, \dots, l$, such that $r \in \mathcal{M}_l$, then r is equal to $\pi_{\mathcal{M}}(P_l)$. Indeed, we know that $P_l = \mathcal{M}_l \oplus \boldsymbol{x}P_{l-1}$ and

$$r = P_l + Q_{l-1} \quad \text{with} \quad Q_{l-1} = \sum_{i=1}^{l} a_i\boldsymbol{x}^i(\mathrm{D}^\alpha)^i P_l.$$

Applying Theorem 3.8, we get

$$\begin{aligned}
0 &= \mathrm{D}^\alpha(\pi_{\mathcal{M}}(P_l)) \\
&= \mathrm{D}^\alpha P_l + a_1\mathrm{D}^\alpha\boldsymbol{x}\mathrm{D}^\alpha P_l + a_2\mathrm{D}^\alpha\boldsymbol{x}^2(\mathrm{D}^\alpha)^2 P_l + \cdots + a_l\mathrm{D}^\alpha\boldsymbol{x}^l(\mathrm{D}^\alpha)^l P_l \\
&= (1 + a_1 g_{1,l-1})\mathrm{D}^\alpha P_l + (-a_1 + a_2 g_{2,l-2})\boldsymbol{x}(\mathrm{D}^\alpha)^2 P_l + (a_2 + a_3 g_{3,l-3})\boldsymbol{x}^2(\mathrm{D}^\alpha)^3 P_l \\
&\qquad + \cdots + ((-1)^{l-1}a_{l-1} + a_l g_{l,0})\boldsymbol{x}^{l-1}(\mathrm{D}^\alpha)^l P_l.
\end{aligned}$$

Hence, if the relation $(-1)^{j-1}a_{j-1} + a_i g_{j,l-j} = 0$ holds for each $j = 1, \dots, l$, then the function r is fractional monogenic. By induction we get

$$a_j = \frac{(-1)^j}{(2[j/2])!! \prod_{i=0}^{[j/2]} g_{2i+1,l-(2i+1)}}.$$

□

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

Theorem 3.10. *Each polynomial P_l can be written in a unique way as*

$$P_l = \sum_{j=0}^{l} \boldsymbol{x}^j M_{l-j}(P_l),$$

where

$$M_{l-j}(P_l) = c_j' \sum_{n=0}^{j} c_{j,l-n} \boldsymbol{x}^n (\mathrm{D}^\alpha)^n (\mathrm{D}^\alpha)^{l-j} P_l, \quad j = 0, \dots, l,$$

and the coefficients c_j' are defined by

$$c_j' = \frac{(-1)^j}{(2[j/2])!! \prod_{i=0}^{[j/2]} g_{2i+1,l-(2i+1)}}.$$

Proof. We know that for any P_l there is a unique decomposition

$$P_l = \boldsymbol{x}^l M_0 + \boldsymbol{x}^{l-1} M_1 + \cdots + \boldsymbol{x} M_{l-1} + M_l,$$

where $M_{l-j} \in \mathcal{M}_{l-j}$ with $j = 0, \dots, l$. To compute a component M_{l-j} explicitly, we apply $(\mathrm{D}^\alpha)^j$ to both sides of the previous equality:

$$(\mathrm{D}^\alpha)^j P_l = (\mathrm{D}^\alpha)^j \boldsymbol{x}^l M_0 + (\mathrm{D}^\alpha)^j \boldsymbol{x}^{l-1} M_1 + \cdots + (\mathrm{D}^\alpha)^j \boldsymbol{x}^{j+1} M_{l-j-1} + (\mathrm{D}^\alpha)^j \boldsymbol{x}^j M_{l-j}.$$

The summands on the right-hand side belong, in turn, to the spaces

$$\boldsymbol{x}^{l-j}\mathcal{M}_0, \quad \boldsymbol{x}^{l-j-1}\mathcal{M}_1, \quad \dots, \quad \boldsymbol{x}\mathcal{M}_{l-j-1}, \quad \mathcal{M}_{l-j}.$$

Hence, $(\mathrm{D}^\alpha)^j \boldsymbol{x}^j M_{l-j}$ is equal to $\pi_{\mathcal{M}}((\mathrm{D}^\alpha)^j P_l)$. We can now use the expression for the harmonic projection proved above. So, to get our result it is sufficient to show that

$$(\mathrm{D}^\alpha)^j \boldsymbol{x}^j M_{l-j} = \frac{1}{c_j} M_j.$$

By Theorem 3.8, we get by induction that $c_j' = a_j$, and therefore the proof is finished. □

Under these conditions it is possible to calculate the dimension of the space of fractional homogeneous monogenic polynomials of degree l. From the Fischer decomposition (3.11) we get $\dim(\mathcal{M}_l) = \dim(\Pi_l) - \dim(\Pi_{l-1})$, with the dimension of the space of fractional homogeneous polynomials of degree l given by $\dim(\Pi_l) = (k+d-1)!/k!(d-1)!$. This leads to the following theorem.

Theorem 3.11. *The space of fractional homogeneous monogenic polynomials of degree l has dimension*

$$\dim(\mathcal{M}_k) = \frac{(l+d-1)! - l(l+d-2)!}{l!(d-1)!} = \frac{(l+d-2)!}{l!(d-2)!}.$$

4. Fractional Cauchy–Kovalevskaya extension

Taking into account the formal similarities with the classical setting, we propose the following form for the FCK extension:

$$F(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_d) = \sum_{k=0}^{\infty} \frac{(\boldsymbol{e}_1 \boldsymbol{x}_1)^k \boldsymbol{x}_1^{1-\alpha}}{k!} f_k(\boldsymbol{x}_2, \dots, \boldsymbol{x}_d),$$

with $f_0 = f$. From (3.7) we conclude that the function F takes the correct values and satisfies $F|_{x_1=0} = f$. For F to be fractional monogenic it must vanish under the action of the fractional Dirac operator D^α, which can be rewritten as

$$\mathrm{D}^\alpha = \mathrm{D}_1^\alpha + \sum_{j=2}^{d} \boldsymbol{e}_j \mathrm{D}_j^\alpha = \mathrm{D}_1^\alpha + \mathrm{D}_*^\alpha.$$

In order to determine the coefficient functions f_k, $k = 1, 2, \dots, d$, such that $\mathrm{D}^\alpha F = 0$, we proceed by direct calculation. From the action of D_j^α over $\boldsymbol{x}_1$ (see Theorem 3.3), and taking into account that D_1^α only acts on $\boldsymbol{x}_1^k$ and D_*^α anticommutes with $\boldsymbol{x}_1$, we obtain

$$\begin{aligned} 0 = \mathrm{D}^\alpha F &= (\mathrm{D}_1^\alpha + \mathrm{D}_*^\alpha) \left(\sum_{k=0}^{\infty} \frac{(\boldsymbol{e}_1 \boldsymbol{x}_1)^k \boldsymbol{x}_1^{1-\alpha}}{k!} f_k \right) \\ &= \sum_{k=0}^{\infty} \frac{(\boldsymbol{e}_1 \boldsymbol{x}_1)^k \boldsymbol{x}_1^{1-\alpha}}{k!} f_{k+1} + \sum_{k=0}^{\infty} (-1)^k \frac{(\boldsymbol{e}_1 \boldsymbol{x}_1)^k \boldsymbol{x}_1^{1-\alpha}}{k!} \mathrm{D}_*^\alpha f_k, \end{aligned}$$

which results in the recurrence relation

$$f_{k+1} = (-1)^{k+1} \mathrm{D}_*^\alpha f_k.$$

Hence, we obtain the following definition for the FCK extension.

Definition 4.1. The FCK extension of a function $f = f(\boldsymbol{x}_2, \dots, \boldsymbol{x}_d)$ is the fractional monogenic function

$$\mathrm{FCK}[f](\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_d) = \sum_{k=0}^{\infty} \frac{(\boldsymbol{e}_1 \boldsymbol{x}_1)^k \boldsymbol{x}_1^{1-\alpha}}{k!} f_k(\boldsymbol{x}_2, \dots, \boldsymbol{x}_d), \tag{4.1}$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} \mathrm{D}_*^\alpha f_k$.

Let us observe that the previous definition does not impose any conditions on the original function f. From (3.7) follow

$$\begin{aligned} (\boldsymbol{e}_1 \boldsymbol{x}_1)^{2n+1} \boldsymbol{x}_1^{1-\alpha} &= 0 \quad \text{for } n \leqslant |\boldsymbol{x}_1|, \\ (\boldsymbol{e}_1 \boldsymbol{x}_1)^{2n} \boldsymbol{x}_1^{1-\alpha} &= 0 \quad \text{for } n \leqslant |\boldsymbol{x}_1| + 1, \end{aligned}$$

which imply that for every point $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_d) \in \mathbb{R}^d$ there exists $N \in \mathbb{N}$ such that all but the first N terms in the series (4.1) vanish, and therefore the series reduces to a finite sum at every point of $\mathbb{R}^d$. This fact implies that the FCK extension of the function $f(\boldsymbol{x}_2, \dots, \boldsymbol{x}_d)$ is well defined on $\mathbb{R}^d$. The uniqueness of the extension is a corollary of the following result.

Theorem 4.2. *Let F be a fractional monogenic function defined on $\mathbb{R}^d$ with $F|_{\boldsymbol{x}_1=0} \equiv 0$. Then F is the null function.*

Proof. The fractional monogenicity of F explicitly reads as $(\mathrm{D}_1^\alpha + \mathrm{D}_*^\alpha)F = 0$. Now take $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_d) \in \mathbb{R}^d$ with $\boldsymbol{x}_1 = 0$. Since $F|_{\boldsymbol{x}_1=0} \equiv 0$, the above expression reduces to $\mathrm{D}_i^\alpha F = 0$. Furthermore, $-\Delta^{2\alpha} F = \mathrm{D}^\alpha \mathrm{D}^\alpha F = 0$, from which we obtain, for $(0, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_d) \in \mathbb{R}^d$ with $\boldsymbol{x}_1 = 0$, that $F \equiv 0$. Repeating this procedure, we find that $F \equiv 0$ on $\mathbb{R}^d$. □

Corollary 4.3 (uniqueness of the FCK extension). *Let F_1 and F_2 be two fractional monogenic functions such that $F_1|_{\boldsymbol{x}_1=0} = f$ and $F_2|_{\boldsymbol{x}_1=0} = f$. Then F_1 and F_2 coincide.*

4.1. Fractional Fueter polynomials

The fractional FCK extension procedure establishes a homomorphism between the space $\mathit{\Pi}_l^{(d-1)}$ of fractional homogeneous polynomials of degree l in $d-1$ variables and the space $\mathcal{M}_l^{(d)}$ of spherical fractional monogenics of degree l in d variables. Theorem 4.2 and Corollary 4.3 imply that this homomorphism is injective. Moreover, a basis for the space $\mathit{\Pi}_l^{(d-1)}$ is given by the fractional homogeneous polynomials $\boldsymbol{x}_2^{\beta_2} \cdots \boldsymbol{x}_d^{\beta_d}$, with $\beta_2 + \cdots + \beta_d = l$, and its dimension is

$$\dim(\mathit{\Pi}_l^{(d-1)}) = \frac{(l-d)!}{l!(d-2)!},$$

which corresponds to the dimension of $\mathcal{M}_l^{(d)}$ (see Theorem 3.11 with $d = d-1$), whence the homomorphism is also surjective. The FCK extension procedure thus establishes an isomorphism between $\mathit{\Pi}_l^{(d-1)}$ and $\mathcal{M}_l^{(d)}$, allowing us to determine a basis for the space $\mathcal{M}_l^{(d)}$.

Definition 4.4. Let $\beta = (\beta_2, \ldots, \beta_d) \in \mathbb{N}^{d-1}$ with $\beta_2 + \cdots + \beta_d = l$. Then the fractional spherical monogenics

$$V_\beta = \mathrm{FCK}\left[\boldsymbol{x}_2^{\beta_2} \cdots \boldsymbol{x}_d^{\beta_d} \prod_{j=2}^{d} \boldsymbol{x}_j^{1-\alpha}\right]$$

are called the fractional Fueter polynomials of degree l.

Theorem 4.5. *The set $\{V_\beta \mid \beta_2 + \cdots + \beta_d = l\}$ is a basis for $\mathcal{M}_l^{(d)}$.*

Proof. The FCK extension procedure is an isomorphism between both spaces. □

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

Example 4.6. The space $\mathcal{M}_2^{(3)}$ has dimension $d=3$. A basis for it is given by the elements

$$\begin{aligned} V_{2,0} &= \mathrm{FCK}[\boldsymbol{x}_2^2\boldsymbol{x}_2^{1-\alpha}] \\ &= \boldsymbol{x}_2^2\boldsymbol{x}_2^{1-\alpha} - 2\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_1^{1-\alpha}\boldsymbol{x}_2^{1-\alpha} - \boldsymbol{x}_1^2\boldsymbol{x}_1^{1-\alpha}, \\ V_{1,1} &= \mathrm{FCK}[\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_2^{1-\alpha}\boldsymbol{x}_3^{1-\alpha}] \\ &= \boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_2^{1-\alpha}\boldsymbol{x}_3^{1-\alpha} - \boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_1^{1-\alpha}\boldsymbol{x}_3^{1-\alpha} + \boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_1^{1-\alpha}\boldsymbol{x}_2^{1-\alpha} + \boldsymbol{x}_1^2\boldsymbol{x}_1^{1-\alpha}, \\ V_{0,2} &= \mathrm{FCK}[\boldsymbol{x}_3^2\boldsymbol{x}_3^{1-\alpha}] \\ &= \boldsymbol{x}_3^2\boldsymbol{x}_3^{1-\alpha} - 2\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_1^{1-\alpha}\boldsymbol{x}_3^{1-\alpha} - \boldsymbol{x}_1^2\boldsymbol{x}_1^{1-\alpha}, \end{aligned}$$

from which it can be checked directly that they are fractional monogenic, of homogeneity degree 2 in $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, and are linearly independent.

4.2. Fractional Cauchy–Kovalevskaya extension of $\boldsymbol{x}^s M_l$

In the Euclidian setting functions of the form $x^s P_l(x)$ are building blocks of homogeneous polynomials in $\mathbb{R}^d$ and whence, in order to characterize spaces of inner spherical monogenics in $\mathbb{R}^{d+1}$, it suffices to determine the CK extension of polynomials of the form $x^s P_l(x)$ (see [**14**] for more details), which was formulated in the following theorem.

Theorem 4.7 (cf. Delanghe *et al.* [14]). *Let $s \in \mathbb{N}$ and let $P_l \in M^+(l; d; C)$. Then the CK extension of $x^s P_l(x)$ has the form $X_l^s(x_0, x)P_l(x)$, where*

$$X_l^s(x_0,x) = \lambda_l^s r^s \left[C_s^{(d-1)/2+l}\left(\frac{x_0}{r}\right) + \frac{2l+d-1}{s+2l+d-1} C_{s-1}^{(d+1)/2+l}\left(\frac{x_0}{r}\right)\frac{x_0}{r}\right].$$

*In this equation, $r^2 = x_0^2 - x^2$ and the polynomials $C_n^{\lambda}(x)$ are the standard Gegenbauer polynomials [**1**] given by*

$$C_n^{\lambda}(x) = \sum_{j=0}^{[n/2]} \frac{(-1)^j(\lambda)_{n-j}}{j!(n-2j)!}(2x)^{n-2j}, \tag{4.2}$$

where the Pochhammer symbol $(a)_n$ denotes $a(a+1)\cdots(a+n-1)$. Furthermore, the coefficients λ_k^s are

$$\lambda_l^{2k} = (-1)^k (C_{2k}^{(d-1)/2+l}(0))^{-1}, \qquad \lambda_l^{2k+1} = (-1)^k \frac{2k+2l+d}{2l+d-1}(C_{2k}^{(d+1)/2+l}(0))^{-1},$$

and explicitly

$$\lambda_l^{2k} = \frac{k!\Gamma(l+\frac{1}{2}(d-1))}{\Gamma(k+l+\frac{1}{2}(d-1))}, \qquad \lambda_l^{2k+1} = \frac{2k+2l+d}{2l+d-1}\frac{k!\Gamma(l+\frac{1}{2}(d+1))}{\Gamma(k+l+\frac{1}{2}(d+1))}. \tag{4.3}$$

We now consider the fractional version of the previous theorem. We will consider P_l to be a fractional homogeneous monogenic function in d variables $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ and will determine the FCK extension of $\boldsymbol{x}^s P_l$. The result is a fractional monogenic in $d+1$ variables $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ such that

$$\mathrm{FCK}[f] = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{x}_1 f_k, \qquad f_0 = f, \qquad f_{k+1} = (-1)^{k+1} \mathrm{D}^\alpha f_k,$$

where D^α is the fractional Dirac operator in d variables. The operators D^α and $\boldsymbol{x}$ satisfy Theorem 3.8. Denote by R the fractional vector variable in $d+1$ dimensions, i.e.

$$R = \boldsymbol{x}_0 - \sum_{j=1}^{d} \boldsymbol{e}_j \boldsymbol{x}_j = \boldsymbol{x}_0 - \boldsymbol{x},$$

with $R^2 = \boldsymbol{x}_0^2 + \boldsymbol{x}^2$.

Remark 4.8. As was done in the Euclidian case (see our Theorem 4.7 from [**14**]), in this section we will use the formal notation $\boldsymbol{x}_0/R$ and $\boldsymbol{x}/R$ as arguments in the Gegenbauer polynomials, by which we mean that we first of all expand the Gegenbauer polynomials using $(\boldsymbol{x}_0/R)^k = \boldsymbol{x}_0^k/R^k$, and then cancel out all appearances of R in the denominators, after which no ambiguity is left.

4.2.1. *Auxiliary results*

In this section we present some necessary results for the proof, in the next section, of the main theorem. Taking into account Theorem 3.3, we present the following auxiliary lemmas.

Lemma 4.9. *Let $k \in \mathbb{N}$ and let P_l be a fractional spherical monogenic of degree l in the variables $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$. Then*

$$\mathrm{D}_0^\alpha R^{2k} P_l = -2\alpha k \boldsymbol{x}_0 R^{2k-2} P_l, \tag{4.4}$$

$$\mathrm{D}^\alpha R^{2k} P_l = 2\alpha k \boldsymbol{x} R^{2k-2} P_l, \tag{4.5}$$

$$\mathbb{E}^\alpha R^{2k} P_l = (-l R^2 - \alpha k \boldsymbol{x}^2) R^{2k-2} P_l. \tag{4.6}$$

Proof. We start by expanding R^{2k} in the following way:

$$R^{2k} = \sum_{s=0}^{k} \binom{k}{s} \boldsymbol{x}_0^{2k-2s} \boldsymbol{x}^{2s}.$$

Taking into account that $\boldsymbol{x}$ and P_l do not depend on $\boldsymbol{x}_0$, we get

$$\mathrm{D}_0^\alpha R^{2k} P_l = -2\alpha k \sum_{s=0}^{k-1} \frac{(k-1)!}{s!(k-s-1)!} \boldsymbol{x}_0^{2k-2s-1} \boldsymbol{x}^{2s} P_l = -2\alpha k \boldsymbol{x}_0 R^{2k-2} P_l.$$

The proof of the second statement uses the relations presented in Theorem 3.8 and the fact that P_l is a fractional spherical monogenic in the variables $\boldsymbol{x}_1,\dots,\boldsymbol{x}_d$ (thus, $\mathrm{D}^\alpha P_l = 0$):

$$\begin{aligned}\mathrm{D}^\alpha R^{2k}P_l &= 2s\alpha \sum_{s=0}^{k} \frac{k!}{(s-1)!(k-s)!}\boldsymbol{x}_0^{2k-2s}\boldsymbol{x}^{2s-1}P_l\\ &= 2s\alpha k \sum_{s=1}^{k} \frac{k!}{(s-1)!(k-s)!}\boldsymbol{x}_0^{2k-2s}\boldsymbol{x}^{2s-1}P_l\\ &= 2\alpha k\boldsymbol{x}R^{2k-2}P_l.\end{aligned}$$

For the final relation we use the commutation relation

$$\boldsymbol{x}\mathbb{E}^\alpha - \mathbb{E}^\alpha\boldsymbol{x} = \alpha\boldsymbol{x} \iff \mathbb{E}^\alpha\boldsymbol{x} = \boldsymbol{x}\mathbb{E}^\alpha - \alpha\boldsymbol{x},$$

which implies that

$$\mathbb{E}^\alpha\boldsymbol{x}^{2s} = \boldsymbol{x}^{2s}\mathbb{E}^\alpha - 2s\alpha\boldsymbol{x}^{2s},$$

to show that

$$\begin{aligned}\mathbb{E}^\alpha R^{2k}P_l &= -lR^{2k}P_l - 2\alpha k\sum_{p=0}^{k-1}\binom{k-1}{p}\boldsymbol{x}_0^{2(k-1)-2p}\boldsymbol{x}^{2p+2}P_l\\ &= (-lR^2 - 2\alpha k\boldsymbol{x}^2)R^{2k-2}P_l.\end{aligned}$$

□

Lemma 4.10. *For a parameter λ and $k \geqslant 1$ we have*

$$(\mathrm{D}_0^\alpha + \mathrm{D}^\alpha)\left[C_{2k}^{\lambda}\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k}P_l\right] = -2\alpha\lambda\left[C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right) - C_{2k-2}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\right]R^{2k-1}P_l.$$

Proof. Taking into account the series expansion (4.2) for the Gegenbauer polynomials and the relations presented in Lemma 4.9, after straightforward calculations we have

$$\begin{aligned}&(\mathrm{D}_0^\alpha + \mathrm{D}^\alpha)\left[C_{2k}^{\lambda}\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k}P_l\right]\\ &\quad= \sum_{j=0}^{k}\frac{(-1)^j(\lambda)_{2k-j}}{j!(2k-2j)!}2^{2k-2j}[-\alpha(2k-2j)\boldsymbol{x}_0^{2k-2j-1}R^{2j} + \boldsymbol{x}_0^{2k-2j}\mathrm{D}_0^\alpha R^{2j}\\ &\qquad\qquad + (-1)^{2k-2j}\boldsymbol{x}_0^{2k-2j}\mathrm{D}^\alpha R^{2j}]P_l\\ &\quad= -2\alpha\lambda\left[C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right) - C_{2k-2}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\right]R^{2k-1}P_l.\end{aligned}$$

□

We remark that in Lemma 4.10, on the right-hand side, there is no ambiguity about whether the Rs should be left or right since the first thing one has to do is eliminate the powers of R in the denominator, which leaves only even powers of R (in the denominator), which commute with $\boldsymbol{x}_0$ and $\boldsymbol{x}$. We now continue presenting auxiliary lemmas.

Lemma 4.11. *For a parameter λ and $k \geqslant 1$ we have*

$$(\mathrm{D}_0^\alpha + \mathrm{D}^\alpha)\left[C_{2k-1}^\lambda\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k}P_l\right] = \bigg[-2\alpha\lambda C_{2k-2}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\boldsymbol{x}R^{2k-2} + \alpha(d+2l)C_{2k-1}^\lambda\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k-1} - 2\alpha\lambda C_{2k-3}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k-1}\bigg]P_l.$$

Proof. Taking into account the series expansion (4.2) for the Gegenbauer polynomials and the relations presented in Lemma 4.9, after straightforward calculations we have

$$\begin{aligned}
&(\mathrm{D}_0^\alpha + \mathrm{D}^\alpha)\left[C_{2k-1}^\lambda\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k}P_l\right] \\
&= -\alpha\sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-2)!}2^{2k-2j-1}\boldsymbol{x}_0^{2k-2j-2}\boldsymbol{x}R^{2j}P_l \\
&\quad - \sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-1)!}(2\boldsymbol{x}_0)^{2k-2j-1}[\boldsymbol{x}\mathrm{D}_0^\alpha R^{2j} - (2\mathbb{E}^\alpha - \alpha d + \boldsymbol{x}\mathrm{D}^\alpha)R^{2j}]P_l \\
&= -2\alpha\lambda C_{2k-2}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\boldsymbol{x}R^{2k-2}P_l + \alpha(d+2l)C_{2k-1}^\lambda\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k-1}P_l \\
&\quad - 2\alpha\lambda\sum_{p=0}^{k-2}\frac{(-1)^p(\lambda)_{2k-p-3}}{p!(2k-2p-3)!}\left(\frac{2\boldsymbol{x}_0}{R}\right)^{2k-2p-3}\frac{\boldsymbol{x}}{R}R^{2k-1}P_l \\
&= \bigg[-2\alpha\lambda C_{2k-2}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\boldsymbol{x}R^{2k-2} + \alpha(d+2l)C_{2k-1}^\lambda\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k-1} \\
&\qquad - 2\alpha\lambda C_{2k-3}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k-1}\bigg]P_l.
\end{aligned}$$

□

We remark that in Lemma 4.11, after elimination of the powers of R in the denominator, there is no ambiguity in the first and last terms of the right-hand side. For the second term, however, we must clarify how the elimination should be made. For example, letting $d = 2$ we have

$$C_1^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^3 = 2(\lambda+1)\frac{\boldsymbol{x}_0}{R}\frac{\boldsymbol{x}}{R}R^3 = 2(\lambda+1)\boldsymbol{x}_0\boldsymbol{x}R,$$

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

which is not the same as

$$C_1^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)R^3\frac{\boldsymbol{x}}{R} = 2(\lambda+1)\boldsymbol{x}_0 R\boldsymbol{x}$$

or

$$R^3 C_1^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R} = 2(\lambda+1)R\boldsymbol{x}_0\boldsymbol{x}.$$

For the second term on the right-hand side we thus put as a convention that (after elimination of R in the denominator) the remaining (odd) powers of R are written on the far right of both $\boldsymbol{x}_0$ and $\boldsymbol{x}$.

In a very similar way to in Lemma 4.11, we can prove the following results.

Lemma 4.12. *For a parameter λ and $k \geqslant 1$ we have*

$$(\mathrm{D}_0^\alpha + \mathrm{D}^\alpha)\left[C_{2k+1}^{\lambda}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k}P_l\right] = -2\alpha\lambda\left[C_{2k}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right) - C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\left(\frac{\boldsymbol{x}_0}{R} - \frac{\boldsymbol{x}}{R}\right)\right]R^{2k}P_l.$$

Lemma 4.13. *For a parameter λ and $k \geqslant 1$ we have*

$$(\mathrm{D}_0^\alpha + \mathrm{D}^\alpha)\left[C_{2k}^{\lambda}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k+1}P_l\right] = \left[\alpha(d+2l)C_{2k}^{\lambda}\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k} - 2\alpha\lambda C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k} - 2\alpha\lambda C_{2k-2}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}\boldsymbol{x}R^{2k-1}\right]P_l.$$

4.2.2. *Main result*

We now present the main result of this section.

Theorem 4.14. *For a fractional spherical monogenic P_l of degree l in the fractional variables $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ and for $s \in \mathbb{N}$, the fractional FCK extension of $\boldsymbol{x}^s P_l$ is the fractional monogenic polynomial in $d+1$ variables $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ given by*

$$\mathrm{FCK}[\boldsymbol{x}^{2k}P_l] = (-1)^k\lambda_l^{2k}R^{2k}\left[C_{2k}^{(d+1)/2+k}\left(\frac{\boldsymbol{x}_0}{R}\right) + \frac{2l+d-1}{2l+d-1+2k}C_{2k-1}^{(d-1)/2+k}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}\right]P_l, \quad (4.7)$$

$$\mathrm{FCK}[\boldsymbol{x}^{2k+1}P_l] = (-1)^k\lambda_l^{2k+1}R^{2k+1}\left[-C_{2k+1}^{(d-1)/2+k}\left(\frac{\boldsymbol{x}_0}{R}\right) + \frac{2l+d-1}{2l+d-1+2k}C_{2k}^{(d+1)/2+k}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}\right]P_l. \quad (4.8)$$

In this formula $r^2 = \boldsymbol{x}_0^2 - \boldsymbol{x}^2$, $C_n^\lambda(x)$ are the standard Gegenbauer polynomials given by (4.2), and the coefficients λ_k^s are given by (4.3).

Before we present the proof we give the following remark concerning notation, which is similar to the one presented in [**10**].

Remark 4.15. Please note that in the main theorem terms like $C_k^\lambda(\boldsymbol{x}_0/R)$ appear, while in fact $\boldsymbol{x}_0/R$ is not well defined. Since $R\boldsymbol{x}_0 \neq \boldsymbol{x}_0 R$ and $R\boldsymbol{x} \neq \boldsymbol{x}R$, the notation has to be understood in the following way: because $C_k^\lambda(x)$ contains only powers of x of degree at most k, we first multiply it by R^k, after which there is no ambiguity.

Let us now consider the proof.

Proof. We start with expression (4.7). The proof has two parts. In the first we show that the restriction of

$$F := (-1)^k \lambda_l^{2k} n R^{2k}\left[C_{2k}^{(d-1)/2+k}\left(\frac{\boldsymbol{x}_0}{R}\right) + \frac{2l+d-1}{2l+d-1+2k} C_{2k-1}^{(d+1)/2+k}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}_0}{R}\right] P_l$$

to the hyperplane $\boldsymbol{x}_0 = 0$ is exactly $\boldsymbol{x}^{2k} P_l$. In fact, $R^{2j}|_{\boldsymbol{x}_0=0} = \boldsymbol{x}^{2j}$ and

$$C_{2k}^\lambda\left(\frac{\boldsymbol{x}_0}{R}\right)\bigg|_{\boldsymbol{x}_0=0} = \sum_{j=0}^{k} \frac{(-1)^j(\lambda)_{2k-j}}{j!(2k-2j)!}\left(\frac{2\boldsymbol{x}_0}{R}\right)^{2k-2j}\bigg|_{\boldsymbol{x}_0=0} = \frac{(-1)^k(\lambda)_k}{k!},$$

$$C_{2k-1}^\lambda\left(\frac{\boldsymbol{x}_0}{R}\right)\bigg|_{\boldsymbol{x}_0=0} = \sum_{j=0}^{k-1} \frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-1)!}\left(\frac{2\boldsymbol{x}_0}{R}\right)^{2k-2j-1}\bigg|_{\boldsymbol{x}_0=0} = 0,$$

which implies that

$$F|_{\boldsymbol{x}_0=0} = \frac{k!\Gamma(l+(d-1)/2)}{\Gamma(l+(d-1)/2+k)}\frac{1}{k!}\frac{\Gamma(l+(d-1)/2+k)}{\Gamma(l+(d-1)/2)}\boldsymbol{x}^{2k}P_l = \boldsymbol{x}^{2k}P_l.$$

In the second part of the proof we show that F is fractional monogenic in the $d+1$ variables $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$. By the uniqueness of the FCK extension, we know that F must be exactly $\mathrm{FCK}[\boldsymbol{x}^{2k}P_l]$. Since F consists of two terms, we will first consider $\mathrm{D}_0^\alpha + \mathrm{D}^\alpha$ acting on each term separately via Lemmas 4.10 and 4.11 with $\lambda = l + (d-1)/2$. We will then continue by combining the obtained results. We have

$$\begin{aligned}
(\mathrm{D}_0^\alpha + \mathrm{D}^\alpha)F
&= (-1)^{k+1}\lambda_l^{2k} 2\alpha\lambda\left[C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right) - C_{2k-2}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)\right]R^{2k-1}P_l \\
&\quad + (-1)^k\lambda_l^{2k}\frac{\lambda}{\lambda+k}\bigg[-2\alpha(\lambda+1)C_{2k-2}^{\lambda+2}\left(\frac{\boldsymbol{x}_0}{R}\right)\boldsymbol{x}R^{2k-2} \\
&\qquad + \alpha(2\lambda-1)C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k-1} \\
&\qquad - 2\alpha(\lambda+1)C_{2k-3}^{\lambda+2}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k-1}\bigg]P_l \\
&= (-1)^{k+1}\lambda_l^{2k}2\alpha\lambda\bigg[\frac{2k+1}{2\lambda+2k}C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k-1} - C_{2k-1}^{\lambda+1}\left(\frac{\boldsymbol{x}_0}{R}\right)R^{2k-1} \\
&\qquad + \frac{\lambda+1}{\lambda+k}C_{2k-2}^{\lambda+2}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k-1} + \frac{\lambda+1}{\lambda+k}C_{2k-3}^{\lambda+2}\left(\frac{\boldsymbol{x}_0}{R}\right)\frac{\boldsymbol{x}}{R}R^{2k-1}\bigg]P_l.
\end{aligned}$$

https://doi.org/10.1017/S0013091516000109 Published online by Cambridge University Press

Taking into account the series expansion (4.2) for the Gegenbauer polynomials, the previous expression becomes

$$\begin{aligned}
&(\mathrm{D}_0^\alpha+\mathrm{D}^\alpha)F\\
&\quad=(-1)^{k+1}\lambda_l^{2k}\alpha\biggl[\frac{\lambda(2k+1)}{\lambda+k}\sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j}}{j!(2k-2j-1)!}2^{2k-2j-1}\boldsymbol{x}_0^{2k-2j-1}R^{2j}\\
&\qquad\qquad-\sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-2)!}2^{2k-2j-1}\boldsymbol{x}_0^{2k-2j-2}R^{2j+1}\\
&\qquad\qquad+\frac{1}{\lambda+k}\sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j}}{j!(2k-2j-2)!}2^{2k-2j-1}\boldsymbol{x}_0^{2k-2j-2}\boldsymbol{x}R^{2j}\\
&\qquad\qquad+\frac{1}{\lambda+k}\sum_{j=0}^{k-2}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-3)!}2^{2k-2j-2}\boldsymbol{x}_0^{2k-2j-3}\boldsymbol{x}R^{2j+1}\biggr]P_l. \quad (4.9)
\end{aligned}$$

From the series expansions of R^{2j} and R^{2j+1},

$$R^{2j}=\sum_{s=0}^{j}\binom{j}{s}\boldsymbol{x}_0^{2j-2s}\boldsymbol{x}^{2s},\qquad R^{2j+1}=\sum_{s=0}^{j}\binom{j}{s}(\boldsymbol{x}_0^{2j-2s+1}\boldsymbol{x}^{2s}+\boldsymbol{x}_0^{2j-2s}\boldsymbol{x}^{2s+1}),$$

there are two possible combinations (with respect to the powers of $\boldsymbol{x}_0$ and $\boldsymbol{x}$): either an odd power of $\boldsymbol{x}_0$ combined with an even power of $\boldsymbol{x}$ or vice-versa. We will look at both possibilities separately and show that both must be zero. We first consider the terms of (4.9) containing a combination of an even power of $\boldsymbol{x}_0$ and an odd power of $\boldsymbol{x}$, which we call the 'even part' and denote by EP:

$$\begin{aligned}
\mathrm{EP}&=(-1)^{k+1}\lambda_l^{2k}\alpha\biggl[-\sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-2)!}2^{2k-2j-1}\boldsymbol{x}_0^{2k-2j-2}\boldsymbol{x}R^{2j}\\
&\qquad+\sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j}}{j!(2k-2j-2)!}\frac{2^{2k-2j-1}}{\lambda+k}\boldsymbol{x}_0^{2k-2j-2}\boldsymbol{x}R^{2j}\\
&\qquad+\frac{1}{\lambda+k}\sum_{j=0}^{k-2}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-3)!}2^{2k-2j-2}\boldsymbol{x}_0^{2k-2j-3}\boldsymbol{x}_0\boldsymbol{x}R^{2j}\biggr]P_l\\
&=(-1)^{k+1}\lambda_l^{2k}\alpha\biggl[\frac{1}{\lambda+k}\sum_{j=0}^{k-1}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-2)!}(k-j-1)2^{2k-2j-1}\boldsymbol{x}_0^{2k-2j-2}\boldsymbol{x}R^{2j}\\
&\qquad\qquad+\frac{1}{\lambda+k}\sum_{j=0}^{k-2}\frac{(-1)^j(\lambda)_{2k-j-1}}{j!(2k-2j-3)!}2^{2k-2j-2}\boldsymbol{x}_0^{2k-2j-2}\boldsymbol{x}R^{2j}\biggr]P_l\\
&=0.
\end{aligned}$$

For the 'odd part' we proceed in a very similar way.

One checks for different values of j for which the coefficients of R^{2j} will be zero, and hence the total sum will be zero. Regarding (4.8), we proceed in a similar way and consider Lemmas 4.12 and 4.13. □

4.3. Example

To end the paper we present an example of an FCK extension.

Example 4.16. The FCK extension of $\boldsymbol{x}^2 P_l$ is given by

$$\mathrm{FCK}[\boldsymbol{x}^2 P_l] = \sum_{k=0}^{\infty} \frac{\boldsymbol{x}_0^k}{k!} f_k,$$

where the functions f_k are $f_0 = \boldsymbol{x}^2 P_l$, $f_1 = -2\alpha \boldsymbol{x} P_l$, $f_2 = 2\alpha(-2l - \alpha d)P_l$, whence explicitly

$$\mathrm{FCK}[\boldsymbol{x}^2 P_l] = [\boldsymbol{x}^2 - 2\alpha\boldsymbol{x} + 2\alpha(-2l - \alpha d)]P_l.$$

Acknowledgements. This work was supported through the Centro de Investigação e Desenvolvimento em Matemática e Aplicações (CIDMA), and the Fundação para a Ciência e a Tecnologia (FCT) within Project UID/MAT/0416/2013. The author was also supported by the FCT via the FCT Researcher Program 2014 (Grant IF/00271/2014).

The author wishes to express his gratitude to the anonymous referee for his corrections and suggestions, which greatly improved the quality and presentation of the manuscript.

References

1. G. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and Its Applications, Volume 71 (Cambridge University Press, 2000).
2. G. Bernardes, P. Cerejeiras and U. Kähler, Fischer decomposition and Cauchy kernel for Dunkl–Dirac operators, *Adv. Appl. Clifford Alg.* **19**(2) (2009), 163–171.
3. F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis*, Research Notes in Mathematics, Volume 76 (Pitman, Boston, MA, 1982).
4. F. Brackx, H. De Schepper, D. Eelbode and V. Souček, The Howe dual pair in Hermitean Clifford analysis, *Rev. Mat. Ibero.* **26**(2) (2010), 449–479.
5. F. Brackx, H. De Schepper, R. Lávička and V. Souček, The Cauchy–Kovalevskaya extension theorem in Hermitian Clifford analysis, *J. Math. Analysis Applic.* **381**(2) (2011), 649–660.
6. A. Cauchy, *Oeuvres complètes*, Série 1, Tome VII, pp. 17–58 (Gauthier-Villars, Paris, 1892).
7. H. De Bie and F. Sommen, Fischer decompositions in superspace, in *Function spaces in complex and Clifford analysis*, pp. 170–188 (National University Publishers, Hanoi, 2008).
8. H. De Bie, P. Somberg and V. Souček, The metaplectic Howe duality and polynomial solutions for the symplectic Dirac operator, *J. Geom. Phys.* **75** (2014), 120–128.
9. B. De Knock, D. Peña-Peña and F. Sommen, On the CK-extension for a special overdetermined system in complex Clifford analysis, in *Hypercomplex analysis* (ed. I. Sabadini, M. Shapiro and F. Sommen), Trends in Mathematics, pp. 115–124 (Birkhäuser, 2009).
10. H. De Ridder, Discrete Clifford analysis: function theory and integral transforms, PhD thesis, University of Gent (2013).
11. H. De Ridder, H. De Schepper, U. Kähler and F. Sommen, Discrete function theory based on skew Weyl relations, *Proc. Am. Math. Soc.* **138**(9) (2010), 3241–3256.

12. H. De Ridder, H. De Schepper and F. Sommen, The Cauchy–Kovalevskaya extension theorem in discrete Clifford analysis, *Commun. Pure Appl. Analysis* **10**(4) (2011), 1097–1109.
13. N. De Schepper and F. Sommen, Cauchy–Kowalevski extensions and monogenic plane waves using spherical monogenics, *Bull. Braz. Soc.* **44**(2) (2013), 321–350.
14. R. Delanghe, F. Sommen and V. Souček, *Clifford algebras and spinor-valued functions: a function theory for the Dirac operator*, Mathematics and Its Applications, Volume 53 (Kluwer Academic, Dordrecht, 1992).
15. D. Eelbode, Stirling numbers and spin-Euler polynomials, *Exp. Math.* **16**(1) (2007), 55–66.
16. R. A. El-Nabulsi, Fractional Dirac operators and deformed field theory on Clifford algebra, *Chaos Solitons Fractals* **42**(5) (2009), 2614–2622.
17. L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie algebras and superalgebras* (Academic Press, 2000).
18. E. Goldfain, Complexity in quantum field theory and physics beyond the standard model, *Chaos Solitons Fractals* **28**(4) (2006), 913–922.
19. K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford calculus for physicists and engineers*, Mathematical Methods in Practice (Wiley, 1997).
20. V. G. Kac, Lie superalgebras, *Adv. Math.* **26** (1977), 8–96.
21. U. Kähler and N. Vieira, Fractional Clifford analysis, in *Hypercomplex Analysis: new perspectives and applications* (ed. S. Bernstein, U. Kähler, I. Sabadini and F. Sommen), Trends in Mathematics, pp. 191–201 (Birkhäuser, 2014).
22. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Volume 204 (Elsevier, 2006).
23. S. Kowalevsky, Zur Theorie der partiellen Differentialgleichungen, *J. Reine Angew. Math.* **80** (1875), 1–32.
24. M. F. Li, J. R. Ren and T. Zhu, Fractional vector calculus and fractional special function, Preprint (arXiv:1001.2889v1; 2010).
25. H. R. Malonek and D. Peña-Peña, Fischer decomposition by inframonogenic functions, *Cubo* **12**(2) (2010), 189–197.
26. H. Malonek, D. Peña-Peña and F. Sommen, A Cauchy–Kowalevski theorem for inframonogenic functions, *Math. J. Okayama Univ.* **53** (2011), 167–172.
27. B. Ørsted, P. Somberg and V. Souček, The Howe duality for the Dunkl version of the Dirac operator, *Adv. Appl. Clifford Alg.* **19**(2) (2009), 403–415.
28. I. Podlubny, *Fractional differential equations: an introdunction to fractional derivatives, fractional diffrential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, Volume 198 (Academic Press, 1999).
29. M. D. Roberts, Fractional derivative cosmology, Preprint (arXiv:0909.1171v1; 2009).
30. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications* (Gordon and Breach, New York, 1993).
31. V. E. Tarasov, Fractional vector calculus and fractional Maxwell's equations, *Annals Phys.* **323**(11) (2008), 2756–2778.
32. N. Vieira, Cauchy–Kovalevskaya extension theorem in fractional Clifford analysis, *Complex Analysis Operat. Theory* **9**(5) (2015), 1089–1109.