106.42 Another proof of Rolle's Theorem

There are many different proofs of Rolle's Theorem. Here we will propose another one, which uses simple properties of continuous functions.

Theorem 1: ([1], Rolle) Let f be a real valued function defined on a closed interval $[a, b] \subset \mathbb{R}$. Assume that f has a derivative at each point of the open interval (a, b) and that f is continuous at both endpoints a and b. If $f(a) = f(b)$, then there is at least one point c in (a, b) such that the derivative of f at c is 0, that is $f'(c) = 0$.

The main strategy of our proof is: We will construct a nested sequence of closed and bounded intervals $[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$ such that $f(a_n) = f(b_n)$ for $n = 1, 2, 3, \dots$ and $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains precisely one point c (say), which is in (a, b). Then we will show that $f'(c) = 0$.

Proof: Let us denote the interval $[a, b]$ by the indexed interval $[a_0, b_0]$. Set

$$g(x) = f\left(\frac{b_0 - a_0}{2} + x\right) - f(x), \qquad x \in \left[a_0, \frac{a_0 + b_0}{2}\right].$$

Then $g(a_0) = -g\left(\frac{1}{2}(a_0 + b_0)\right)$; and hence by the Intermediate Value Theorem (see [1]), there exists $x_0 \in \left[a_0, \frac{1}{2}(a_0 + b_0)\right]$ such that $g(x_0) = 0$, that is $f\left(\frac{1}{2}(b_0 - a_0) + x_0\right) = f(x_0)$. If $x_0 \in \left(a_0, \frac{1}{2}(b_0 + a_0)\right)$, then we denote the interval $\left[x_0, \frac{1}{2}(b_0 - a_0) + x_0\right]$ by the indexed interval $[a_1, b_1]$. If $x_0 = a_0$ or $x_0 = \frac{1}{2}(b_0 + a_0)$, then observe that $f(a_0) = f\left(\frac{1}{2}(b_0 + a_0)\right) = f(b_0)$; and in this case, we apply the same process for the function f on $\left[a_0, \frac{1}{2}(a_0 + b_0)\right]$ as we did for $[a_o, b_0]$. For, if, we consider the interval $\left[a_0, \frac{1}{2}(a_0 + b_0)\right]$ or $\left[\frac{1}{2}(a_0 + b_0), b_0\right]$ as $[a_1, b_1]$ and apply the same process on $[a_1, b_1]$, we obtain another interval $[a_2, b_2]$ such that $f(a_2) = f(b_2)$ and $b_2 - a_2 = \frac{1}{4}(b_0 - a_0)$. Now, if we continue this process we get a sequence of closed and bounded intervals $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$ such that $b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$ for $n = 0, 1, 2, \dots$. Now, by the Cantor Intersection Theorem (see [1]), $\bigcap_{n=0}^{\infty} [a_n, b_n]$ is closed and bounded and contains exactly one point c (say) as $\lim_{n\to\infty} (b_n - a_n) = 0$. But there is a possibility that $c \notin (a_0, b_0)$, in other words $c = a_0$ or $c = b_0$. To avoid this possibility we apply the same process for f on $\left[a_0, \frac{1}{2}(a_0 + b_0)\right]$ as we did for $[a_0, b_0]$ and we get an interval $[y, z] \subset \left[a_0, \frac{1}{2}(a_0 + b_0)\right]$ such that $f(y) = f(z)$ and $z - y = \frac{1}{4}(b_0 - a_0)$. Now, if $y > a_0$, then we denote $[y, z]$ by $[a_1, b_1]$ and if $y = a_0$, we denote the interval $\left[z, \frac{1}{2}(a_0 + b_0)\right]$ by $[a_1, b_1]$. Hence, we obtain an interval $[a_1, b_1] \subset (a_0, b_0)$ on which we have $f(a_1) = f(b_1)$ and $0 < b_1 - a_1 \leqslant \frac{1}{2}(b_0 - a_0)$.

Now, applying the same process for the function f on $[a_1, b_1]$ as above, we get an interval $[a_2, b_2] \subset (a_1, b_1)$ such that $f(a_2) = f(b_2)$ and $0 < b_2 - a_2 \leqslant \frac{1}{4}(b_0 - a_0)$. If we continue this process, we obtain a countable collection of intervals $[a_0, b_0], [a_1, b_1], [a_2, b_2], \dots$, where on

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the $(n + 1)$th interval $[a_n, b_n]$ we have $f(a_n) = f(b_n)$. Also $[a_{n+1}, b_{n+1}] \subset (a_n + b_n)$ and $0 < b_n - a_n \leqslant \frac{1}{2^n}(b_0 - a_0)$ for $n = 0, 1, 2, \ldots$. Now, $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots$. Hence, by the Cantor Intersection Theorem, $\bigcap_{n=0}^{\infty} [a_n, b_n]$ is closed and nonempty. Also, $\lim_{n \to \infty} (b_n - a_n) = 0$. Hence, $\bigcap_{n=0}^{\infty} [a_n, b_n]$ consists of exactly one point and let us assume that the point is c. Also observe that $c \in (a_0, b_0)$. Since $\{a_n\}$ is an increasing sequence of points in $[a_0, c)$ converging to c and $\{b_n\}$ is a decreasing sequence of points in $(c, b_0]$ converging to c, by the sequential criterion of derivative we have $\lim_{n \to \infty} \frac{f(c) - f(a_n)}{c - a_n} = f'(c)$ and $\lim_{n \to \infty} \frac{f(b_n) - f(c)}{b_n - c} = f'(c)$. Also, $a_n < c < b_n$ for $n = 0, 1, 2, \ldots$, hence the sequences $\left\{\frac{c - a_n}{b_n - a_n}\right\}$ and $\left\{\frac{b_n - c}{b_n - a_n}\right\}$ are bounded. Hence,

$$\lim_{n \to \infty} \frac{c - a_n}{b_n - a_n}\left(\frac{f(c) - f(a_n)}{c - a_n} - f'(c)\right) = 0$$

$$\text{and} \quad \lim_{n \to \infty} \frac{b_n - c}{b_n - a_n}\left(\frac{f(b_n) - f(c)}{b - c} - f'(c)\right) = 0.$$

Now, from the identity

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'(c) = \frac{b_n - c}{b_n - a_n}\left(\frac{f(b_n) - f(c)}{b_n - c} - f'(c)\right) + \frac{c - a_n}{b_n - a_n}\left(\frac{f(c) - f(a_n)}{c - a_n} - f'(c)\right)$$

we have

$$-f'(c) = \frac{b_n - c}{b_n - a_n}\left(\frac{f(b_n) - f(c)}{b_n - c} - f'(c)\right) + \frac{c - a_n}{b_n - a_n}\left(\frac{f(c) - f(a_n)}{c - a_n} - f'(c)\right)$$

as $f(b_n) = f(a_n)$ for $n = 0, 1, 2, \ldots$. So, letting $n \to \infty$ we have

$$f'(c) = -\lim_{n \to \infty} \frac{b_n - c}{b_n - a_n}\left(\frac{f(b_n) - f(c)}{b_n - c} - f'(c)\right) - \lim_{n \to \infty} \frac{c - a_n}{b_n - a_n}\left(\frac{f(c) - f(a_n)}{c - a_n} - f'(c)\right) = 0.$$

Hence, there is a point $c \in (a, b)$ such that $f'(c) = 0$.

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Reference

1. T. M. Apostol, *Mathematical Analysis*, Addison-Wesley (1974).

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