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EXACT PAIRS FOR THE IDEAL OF THE K-TRIVIAL SEQUENCES IN THE TURING DEGREES

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Abstract. The K-trivial sets form an ideal in the Turing degrees, which is generated by its computably enumerable (c.e.) members and has an exact pair below the degree of the halting problem. The question of whether it has an exact pair in the c.e. degrees was first raised in [22, Question 4.2] and later in [25, Problem 5.5.8].

We give a negative answer to this question. In fact, we show the following stronger statement in the c.e. degrees. There exists a K-trivial degree $\mathbf{d}$ such that for all degrees $\mathbf{a}, \mathbf{b}$ which are not K-trivial and $\mathbf{a} > \mathbf{d}, \mathbf{b} > \mathbf{d}$ there exists a degree $\mathbf{v}$ which is not K-trivial and $\mathbf{a} > \mathbf{v}, \mathbf{b} > \mathbf{v}$. This work sheds light to the question of the definability of the K-trivial degrees in the c.e. degrees.

§1. Introduction. The algebraic study of the Turing degrees has been a topic of considerable research in computability theory, ever since the establishment of degree theory as a research area in [14]. In this study, the ideals of this uppersemilattice are of particular interest. These are downward closed sets of degrees that also closed under the join operator. The recent study of algorithmic information theory by people in computability theory has brought forward a wealth of interactions between the two areas, including the discovery of a new ideal in the Turing degrees: the degrees of sequences with trivial initial segment complexity, the so-called K-trivial sequences. Since this discovery in [11, 24], the study of the K-trivial sequences and degrees has been established as a major area of research in the interface between computability theory and algorithmic information theory.

Issues of definability have been of special interest in the study of ideals in the Turing degrees. Such issues were already present in [14], where the notion of exact pairs of ideals was introduced. Two degrees $\mathbf{a}, \mathbf{b}$ form an *exact pair* of an ideal $\mathcal{C}$ in the Turing degrees if they are both upper bounds for the degrees in $\mathcal{C}$ and any degree below both $\mathbf{a}$ and $\mathbf{b}$ is in $\mathcal{C}$. By [14, 30] every ideal in the Turing degrees has an exact pair. By [24] every K-trivial degree is bounded by a computably enumerable (c.e. for short) K-trivial degree. Hence for the purpose of finding exact pairs for this ideal it suffices to consider its restriction to the c.e. degrees. This turns out to be a Σ^0_3 ideal, in the sense that the index set of its members is Σ^0_3. Moreover, by [7] it has a c.e.

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upper bound that is strictly below the degree $\mathbf{0}'$ of the halting problem.[1] By [27], such ideals have an exact pair strictly below $\mathbf{0}'$. However, it is well known that such an ideal may or may not have an exact pair in the c.e. degrees (this follows from the existence of branching and nonbranching degrees that was established in [17, 32]). Hence whether or not such an ideal has an exact pair in the c.e. degrees depends on the specific properties of it. The following question has come into focus.

Problem (Question 4.2 in [22] and Problem 5.5.8 in [25]). *Is there an exact pair for the ideal of the K-trivial sequences in the c.e. degrees*?

The purpose of this paper is to give a negative answer to this question. In fact, our main result can be seen as a very strong negative answer to this question.

Theorem 1.1. *There exists a K-trivial c.e. degree $\mathbf{d}$ with the following property. For each pair of c.e. degrees $\mathbf{a}, \mathbf{b}$ which are not K-trivial, there exists a c.e. degree $\mathbf{v}$ which is not K-trivial and $\mathbf{v} < \mathbf{a} \cup \mathbf{d}, \mathbf{v} < \mathbf{b} \cup \mathbf{d}$.*

Here $\mathbf{a} \cup \mathbf{d}$ denotes the join (i.e., supremum) of the degrees $\mathbf{a}, \mathbf{d}$.

This theorem provides new and interesting information about the K-trivial sequences and their computational power. Moreover, as we elaborate in Section 2, it rests upon deeper information-theoretic properties that are specific to the K-trivial sequences, rather than some general property that this ideal happened to have. In contrast, the existence of a low bound of this ideal (another question from [22]) was obtained in [16] by observing that it satisfied a certain domination property, and proving that all ideals which share this property have a low bound.

We may obtain a negative answer to our problem from Theorem 1.1 by using some known properties of the K-trivial sequences.

Corollary 1.2. *The ideal of the K-trivial sequences does not have an exact pair of c.e. degrees.*

Proof. By [23] there is no low c.e. upper bound for the K-trivial degrees. By [24] every K-trivial degree is low. Therefore, if two c.e. degrees are an exact pair for the K-trivial degrees, then both of them are not K-trivial. The corollary now follows directly from Theorem 1.1. $\dashv$

Note that the proof of Corollary 1.2 rests on the following weak (and nonuniform) version of Theorem 1.1: "given a pair $\mathbf{a}, \mathbf{b}$ of c.e. degrees which are not K-trivial, there exists a K-trivial c.e. degree $\mathbf{d}$ and c.e. degree $\mathbf{v}$ which is not K-trivial such that $(\mathbf{d} \le \mathbf{a} \wedge \mathbf{d} \le \mathbf{b}) \rightarrow (\mathbf{v} \le \mathbf{a} \wedge \mathbf{v} \le \mathbf{b})$".

The following fact is a direct consequence of the splitting theorem from [2, Section 5] and [31, Chapter 2]. It shows that by replacing $\mathbf{v} < \mathbf{a} \cup \mathbf{d}, \mathbf{v} < \mathbf{b} \cup \mathbf{d}$ with $\mathbf{v} \le \mathbf{a} \cup \mathbf{d}, \mathbf{v} \le \mathbf{b} \cup \mathbf{d}$ in Theorem 1.1 we obtain an equivalent statement.

Proposition 1.3. *If $\mathbf{c}$ is a c.e. degree which is not K-trivial then there exist c.e. degrees $\mathbf{a} < \mathbf{c}$ and $\mathbf{b} < \mathbf{c}$ which are not K-trivial and $\mathbf{c} = \mathbf{a} \cup \mathbf{b}$.*

Since there exists a Δ^0_2 exact pair for the K-trivial degrees, the phenomenon described in Theorem 1.1 is specific to c.e. sets. The following observation contrasts Proposition 1.3 and confirms this intuition from a different angle.

[1] By [16] it also has a low upper bound $\mathbf{b}$, which means that the halting problem relativized to $\mathbf{b}$ has degree $\mathbf{0}'$; however the latter bound cannot be c.e.

PROPOSITION 1.4. *There exists a degree* $\mathbf{x} < \mathbf{0}'$ *which is not* K*-trivial and for every* K*-trivial degree* $\mathbf{d}$, *the only c.e. degrees that are computable from* $\mathbf{x} \cup \mathbf{d}$ *are also computable from* $\mathbf{d}$.

PROOF. A degree that is 1-generic relative to every K-trivial degree has the desired properties, but is not necessarily below $\mathbf{0}'$. Moreover, no 1-generic set has degree $\mathbf{0}'$. Hence it suffices to show that there exists a degree that is 1-generic relative to every K-trivial degree and computable from the halting problem. This follows from the fact (see [16]) that there exists a function that is computable from the halting problem and dominates all partial computable functions relative to any K-trivial set. ⊣

The proof of Theorem 1.1 rests on a few facts about K-trivial sequences and initial segment Kolmogorov complexity. We present these, along with their use in the proof, in Section 2. Some background on Kolmogorov complexity and K-trivial sequences that is directly relevant to our result is given in Section 2.1. For background material on computability theory we refer to [26]. The main property of Kolmogorov complexity that is used in the proof of Theorem 1.1 is discussed in Section 2.3. It is a result from [3] which roughly says that any two c.e. sets of non-trivial initial segment complexity must have common lengths in their characteristic sequences where their complexity rises *simultaneously*. Our proof is essentially a derivation of Theorem 1.1 from this result. This route reduces the complexity of the main construction and results in a transparent presentation.

Two more tools from Kolmogorov complexity are used in order to reduce the calculations further and avoid the dynamic construction of machines in the main construction. The first is the use of Solovay functions to express K-triviality, which is based on [4, 6]. The second one is the standard computable invariance property that is intrinsic to most notions in Kolmogorov complexity. Both of these tools are discussed in Section 2.2. Section 2.5 provides the exact form of the result from [3] that will be used in the main argument, which is given in Section 3. These few preparatory steps (including the formulation of a sufficient set of requirements in Section 3.1) reduce the main argument to the simple construction and verification of Sections 3.3 and 3.4.

§2. Preliminary facts.

In this section we provide a number of notions and results that are needed for the proof of Theorem 1.1. Some of these facts are known, while others are original.

2.1. Background on Kolmogorov complexity and K-trivial sequences.

A standard measure of the complexity of a finite string was introduced by Kolmogorov in [13] (an equivalent approach was due to Solomonoff [28]). The basic idea behind this approach is that simple strings have short descriptions relative to their length while complex or random strings are hard to describe concisely. Kolmogorov (and Solomonoff) formalized this idea using the theory of computation. In this context, Turing machines play the role of our idealized computing devices, and we assume that there are Turing machines capable of simulating any mechanical process which proceeds in a precisely defined and algorithmic manner. Programs can be identified with binary strings.

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A string τ is said to be a description of a string σ with respect to a Turing machine M if this machine halts when given program τ and outputs σ. Then the Kolmogorov complexity of σ with respect to M (denoted by $K_M(\sigma)$) is the length of its shortest description with respect to M. It can be shown that there exists an *optimal* machine V, that is, a machine which gives optimal complexity for all strings, up to a certain constant number of bits. This means that for each Turing machine M there exists a constant c such that $K_V(\sigma) < K_M(\sigma) + c$ for all finite strings σ. Hence the choice of the underlying optimal machine does not change the complexity distribution significantly and the theory of Kolmogorov complexity can be developed without loss of generality, based on a fixed underlying optimal machine U.

When we come to consider the initial segment complexity of infinite strings, it becomes important to consider machines whose domain satisfies a certain condition; the machine M is called *prefix-free* if it has prefix-free domain (which means that no program for which the machine halts and gives output is an initial segment of another). Prefix-free complexity was introduced by Levin [18] and Chaitin [8]. Similarly to the case of ordinary Turing machines, there exists an *optimal* prefix-free machine U so that for each prefix-free machine M the complexity of any string with respect to U is up to a constant number of bits larger than the complexity of it with respect to M. We let K denote the prefix-free complexity with respect to a fixed optimal prefix-free machine. Order the binary strings first by length and then lexicographically. This standard ordering of the strings induces a computable bijections between $\mathbb{N}$ and the binary strings. Under this bijection we may identify numbers and strings. In this sense we may talk about the complexity $K(n)$ of a number n as being the complexity of the string that is represented by n.

The original motivation behind Kolmogorov complexity was a mathematical definition of random infinite sequences. Kolmogorov's idea was that these should be infinite sequences with very complex initial segments. Based on this intuition, Levin [18] and Chaitin [8] gave a robust definition of randomness for infinite binary sequences, which coincided with Martin-Löf randomness (already defined in [21]). They called X random if $\exists c \forall n, K(X \restriction_n) \geq n - c$. In other words, X is random if its initial segments cannot be "compressed" (i.e., be described more concisely) by more than a constant number of bits.

In this paper we are concerned with the other end of the spectrum: sequences with trivial initial segment complexity. These are sequences whose initial segments are very highly compressible, in the sense that they have very short descriptions.

Definition 2.1 (K-trivial sequences). An infinite binary sequence X is called K-trivial if $\exists c \forall n,\ K(X \restriction_n) \leq K(n) + c$.

Here $K(n)$ denotes the complexity of the number n. It follows from the basic properties of Kolmogorov complexity that $K(n)$ and $K(0^n)$ are equal up to an additive constant. Hence the first n bits of a K-trivial sequence have the same complexity as the sequence 0^n. By identifying subsets of $\mathbb{N}$ with their characteristic sequence we can also talk about K-trivial sets of numbers. Chaitin drew some attention to K-trivial sets by noticing that they are computable from the halting problem and by asking whether they are all computable. Solovay [29] produced the first example of a noncomputable K-trivial set. The work in [11] signaled a renewed interest on this notion and initiated a deeper study of K-triviality which revealed surprising connections between initial segment complexity and classical computability. For example, Hirschfeldt and Nies showed in [24] that K-triviality is downward closed under Turing computation. Moreover, the K-trivial sets form an ideal in the Turing degrees, which is generated by its c.e. members (in the sense that every K-trivial set is computable by a c.e. K-trivial set).

2.2. Solovay functions and computable invariance. Building on work from [29], the following characterisation of K-trivial sets was given in [4]. There exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that

$$X \textit{ is } K\textit{-trivial} \iff \exists c \forall n \, (K(X \upharpoonright_n) \leq g(n) + c) \tag{2.1}$$

for all sets X and also $\sum_n 2^{-g(n)}$ is a random real. Here by a random real we mean a real number in $(0, 1)$ whose binary expansion is a random sequence. Later it was demonstrated in [6] that the functions g of (2.1) are exactly the *computable tight upper bounds* of the Kolmogorov function $K(n)$, in the sense for some constant c we have $K(n) \leq g(n) + c$ for all n and $g(t) \leq K(t) + c$ for infinitely many t. These functions were called *Solovay functions*. By [4, 12] a computable function g is a Solovay function if and only if $\sum_n 2^{-g(n)}$ is a random real.

Note that (2.1) replaces a noncomputable component in the definition of K-triviality (namely $K(n)$) with a computable function. In certain situations this allows for a simplification of the calculations involved in arguments about the K-trivial sets. This is the case with the proof of Theorem 1.1.

Before we fix a Solovay function for use in the proof of Theorem 1.1, let us discuss a few basic facts about Solovay functions. We start with a certain computable invariance, which appears in [25, Exercise 5.2.9].

LEMMA 2.2. *Let m_i be a computable increasing sequence. Then a set X is K-trivial if and only if $K(X \upharpoonright_{m_i}) \leq K(m_i) + c$ for some constant c and all i.*

The following observation is a direct consequence of the fact that $\sum_t 2^{-f(t)}$ is noncomputable when f is a Solovay function.

LEMMA 2.3 (Accumulation of weight in Solovay functions). *If (m_i) is a computable increasing sequence and f is a Solovay function then for every k there exist infinitely many n such that $\sum_{t > m_n} 2^{-f(t)} > \frac{1}{n-k-1}$.*

In the following sections, we fix a computable function g as in (2.1) and use (2.1) as a characterisation of K-triviality. A c.e. real is a real that is the limit of a computable nondecreasing sequence of rationals. Define

$$\Omega = \sum_i 2^{-g(i)} \quad \text{and} \quad \Omega_n = \sum_{i<n} 2^{-g(i)}. \tag{2.2}$$

The letter Ω is often used to denote the halting probability of a universal prefix-free machine. Since these numbers coincide with the random c.e. reals (e.g. see [10, Section 9.2] or the original references [9, 15]) we may use it in order to denote $\sum_n 2^{-g(n)}$. Without loss of generality we may assume that $\Omega < 1/4$. All facts that we are going to prove in the following about Ω hold independently of the choice of the Solovay function g. Moreover, let

$$\Omega^* = \sum_i 2^{-K(i)} \quad \text{and} \quad \Omega^*_s = \sum_{i<s} 2^{-K(i)}[s].$$

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A set X is called *low for* Ω if Ω is random relative to X. Here are some facts about this class of sets that we are going to use in this article (for more information on this topic we refer to [25, Section 8.1]). The low for Ω sets form a proper superclass of the K-trivial sets. Inside the Δ^0_2 sets, the low for Ω sets coincide with the K-trivial sets.

In this and the previous background section we focused on aspects of K-triviality that are directly relevant to the proof of Theorem 1.1. For a more thorough presentation of the research area algorithmic randomness and complexity-theoretic weakness we refer to the monographs [10, 25], while [19] is a standard reference for the more general theory of Kolmogorov complexity.

2.3. Common complexity in pairs of c.e. sets of nontrivial complexity. Much of the excitement about the K-trivial sequences comes from the fact that they provide an ideal platform for the study of the interaction between the information that can be coded into an infinite binary sequence and the complexity of its initial segments. The latter has been a primary focus of research in the interface between computability theory and Kolmogorov complexity. The fact that there are noncomputable K-trivial sequences showed that one can code nontrivial information into a sequence without increasing the complexity of its initial segments. A limitation to this phenomenon was revealed in [11] where it was shown that K-trivial sequences cannot compute the halting problem (in other words, they are not Turing complete). In contrast, there are Turing complete sequences of arbitrarily low nontrivial prefix-free initial segment complexity. More precisely, in [3] it was shown that for every c.e. set A which is not K-trivial, there exists a Turing complete c.e. set V of lower complexity, that is, such that $\exists c \forall n,\ K(V\restriction_n) \le K(A\restriction_n) + c$. This was also generalized for the case of any finite collection $A_i, i < k$ of c.e. sets which are not K-trivial, producing a Turing complete c.e. set V such that $\exists c \forall n \forall i < k,\ K(V\restriction_n) \le K(A_i\restriction_n) + c$. A consequence this fact (see [3, Corollary 1.7]) is that given an infinite computable set I,

$$\text{if } A, B \text{ are c.e. sets which are not } K\text{-trivial, then for each } c \text{ there exists } n \in I \text{ such that } \min\{K(A\restriction_n), K(B\restriction_n)\} > K(n) + c. \tag{2.3}$$

This fact is the crux of the proof of Theorem 1.1. It says that any pair of c.e. sets of nontrivial initial segment prefix-free complexity exhibit common lengths of nontrivial prefix-free complexity. It is just one of a series of results which indicate that any two c.e. sets of nontrivial initial segment complexity have some kind of common complexity, or even information. In view of the existence of minimal pairs in the c.e. Turing degrees (a classic result from [17]), such information is not common in terms of Turing reducibility but in terms of weaker measures of relative complexity. See [3, Theorem 1.2] and [1, Theorem 1.3].

Instead of a direct proof, we have chosen to derive Theorem 1.1 as a consequence of (2.3). This route reduces the bulk of the proof to the rather simple construction and verification of Section 3. We note that even the proof of (2.3) from [3] is not direct (strictly speaking) in the sense that it rests on the nontrivial result from [11] that Turing complete sets are not K-trivial.

2.4. Construction of prefix-free machines. A (rather simple) direct construction of a prefix-free machine will be used in Section 2.5. There are certain notions and tools associated with such constructions, which are standard in the arguments employed in algorithmic randomness and also relate to the main argument of Section 3. We briefly discuss them. The *weight* of a prefix-free set S of strings, denoted $\mathtt{wgt}(S)$, is defined to be the sum $\sum_{\sigma\in S} 2^{-|\sigma|}$. The *weight* of a prefix-free machine M is defined to be the weight of its domain and is denoted $\mathtt{wgt}(M)$. Prefix-free machines are most often built in terms of *request sets*. A request set L is a set of pairs $\langle \rho, \ell\rangle$ where ρ is a string and ℓ is a positive integer. A "request" $\langle \rho, \ell\rangle$ represents the intention of describing ρ with a string of length ℓ. We define the *weight of the request* $\langle \rho, \ell\rangle$ to be $2^{-\ell}$. We say that L is a *bounded request set* if the sum of the weights of the requests in L is less than 1. This sum is the *weight of the request set* L and is denoted by $\mathtt{wgt}(L)$.

The Kraft-Chaitin theorem (see e.g. [10, Section 2.6]) says that for every bounded request set L which is c.e., there exists a prefix-free machine M with the property that for each $\langle \rho, \ell\rangle \in L$ there exists a string τ of length ℓ such that $M(\tau) = \rho$. Hence the dynamic construction of a prefix-free machine can be reduced to a mere description of a corresponding c.e. bounded request set.

A function is called right-c.e. if it has a computable nonincreasing approximation. Recall that a c.e. real is a real that is the limit of a computable nondecreasing sequence of rationals. Note that c.e. sets are c.e. reals but the converse does not hold. The Kraft-Chaitin theorem also implies that the definition of a prefix-free machine N may be reduced (as far as the function $n \mapsto K_N(n)$ is concerned) to a definition of a right-c.e. function h such that $\sum_n 2^{-h(n)} < 1$. Indeed, given such a function h we may define $K_N = h$. Then the Kraft-Chaitin theorem guarantees that such a machine N exists. This useful method of defining prefix-free machines (when we are only concerned with the corresponding complexity function) will be used in several proofs in this paper, starting with a proof in the following section. The prefix relation amongst finite or infinite strings is denoted by $\prec$.

2.5. Modulus functions of c.e. sets and K-triviality. Let us fix a computable bijection $(m, n) \mapsto \langle m, n\rangle$ between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$. We use the following notion of "modulus of convergence" which is associated with the enumeration of a set or the monotone approximation to a real.

Definition 2.4 (Modulus functions of c.e. sets). Let A be a c.e. set (or real) with a computable enumeration $(A[s])$. The modulus function $n \mapsto a(n)$ of A maps each n to $\langle n, s\rangle$ where s is the least stage such that $A[s] \upharpoonright_n \prec A$ and $s > n$.

Note that the modulus function of a c.e. set A always refers to a particular computable enumeration $(A[s])$ of it. In this paper all c.e. sets will be given via a certain computable enumeration of them. Hence we may talk about *the* modulus function of a given c.e. set (suppressing the corresponding computable enumeration) without causing confusion. This also means that the modulus function of a c.e. sets comes automatically with a computable monotone approximation. Moreover, according to Definition 2.4, the image $a(n)$ of the modulus function of a c.e. set encodes n. This property will be used in the proof of Lemma 2.9.

Modulus functions and K-triviality are related, as we show in this section. We start with a result which says that functions that are computed from K-trivial sets do not speed up the canonical computable approximation to Ω. Without extra effort,

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we prove this for the larger class of low for Ω sets. This result is not needed for the proof of Theorem 3 but it gives a pleasing characterization of the c.e. K-trivial sets which we present in Corollary 2.8.

LEMMA 2.5 (Low for Ω functions are slow growing). *If A is low for Ω and $f \le_T A$ then there exists a constant c such that $\Omega - \Omega_n < 2^c \cdot (\Omega - \Omega_{f(n)})$ for all n.*

PROOF. Without loss of generality we may assume that f is increasing. Indeed, otherwise we may consider $f'(n) = \max_{i \le n} f(n) + 1$ and since $\Omega - \Omega_{f(n)} > \Omega - \Omega_{f'(n)}$ the lemma about f' implies the lemma about f.

Define a Martin-Löf test (V_n) relative to A as follows. At stage $s+1$, do the following for each $n < s$. If $\Omega_{f(s)} \in V_n[s]$ do nothing. Otherwise let $t_n[s]$ be the last stage since we put something into V_n (and $t_n[s] = 0$ if such a stage does not exist) and put the interval $(\Omega_{f(s)}, \Omega_{f(s)} + 2^{-n} \cdot (\Omega_s - \Omega_{t_n[s]}))$ into V_n.

Clearly $\mu(V_n) \le 2^{-n} \cdot \Omega$. Moreover, (V_n) is uniformly c.e. in A. Hence (V_n) is a Martin-Löf test relative to A. Since A is low for Ω, there exists n such that $\Omega \notin V_n$. Let (s_i) be an increasing enumeration of the stages where an enumeration occurred in V_n and note that there are infinitely many such stages. Then $\Omega_{f(s_{i+1})} - \Omega_{f(s_i)} > 2^{-n} \cdot (\Omega_{s_i} - \Omega_{s_{i-1}})$ for all $i > 0$. Hence for each $i > 0$ we have $\Omega - \Omega_{s_{i-1}} < 2^n \cdot (\Omega - \Omega_{f(s_i)})$. Hence for each $i > 0$ and each $t \in [s_{i-1}, s_i)$ we have

$$\Omega - \Omega_t \le \Omega - \Omega_{s_{i-1}} < 2^n \cdot (\Omega - \Omega_{f(s_i)}) \le 2^n \cdot (\Omega - \Omega_{f(t)}).$$

Hence for each t we have $\Omega - \Omega_t \le 2^n \cdot (\Omega - \Omega_{f(t)})$ as required. ⊣

We do not know if the converse of Lemma 2.5 holds, thereby giving a characterization of the low for Ω sets. The interested reader may consult two other characterizations of this class that were obtained in [20] and [5], respectively. We are able to prove that this equivalence holds inside the class of Δ^0_2 sets. We present this result later in this section, in the form of Corollary 2.8.

The following notion of movable markers is implicit in many recursion-theoretic constructions. We isolate it since it can be used in order to elegantly describe the enumeration of a c.e. sets and prove Proposition 2.10, which is the ultimate goal of this section.

DEFINITION 2.6 (Movable markers). A function $(n, s) \mapsto m_n[s]$ is a system of movable markers if it is nondecreasing in n, s and if $m_n[s] \ne m_n[s+1]$ then $m_n[s+1] > s$.

According to the above definition, a movable marker need not be convergent. Recall that g is the Solovay function that we fixed in Section 2.2.

LEMMA 2.7 (Complexity of movable markers). *Let $m_n[s]$ be a computable system of movable markers such that $\Omega - \Omega_n < 2^c \cdot (\Omega - \Omega_{m_n})$ for some constant c and all n, where $m_n = \lim_s m_n[s]$. Then there exists d such that $K(m_n) \le K(n) + d$ for all n. The same holds if "for all n" is replaced by "for all $n \in I$" where I is an infinite computable set.*

PROOF. For simplicity we give the argument for $I = \mathbb{N}$. The more general case is a simple translation of the proof below. Let g be the Solovay function from (2.2). Since Ω is c.e. and random, by the hypothesis, there is some constant c such that $\Omega^* - \Omega^*_n < 2^c \cdot (\Omega - \Omega_{m_n})$ for all n. Then for all n the limit of m_n is finite. It suffices to define a prefix-free machine N such that $K_N(m_n) \le K(n) + c$ for all n.

Let k_s be the largest number such that $K_N(m_n)[s] \leq K(n)[s] + c$ for all $n < k_s$. Then k_s is defined for all s and by the properties of the following construction, $k_s \leq s$ for all s. The following construction defines N, along with a partition (I_s) of the set of stages which will be used in order to count the weight of N in the verification. Let $I_0 = \emptyset$.

At each stage $s + 1$ we first check if $m_i[s] \neq m_i[s + 1]$ for some $i < k_s$. If not, then we enumerate an N-description of $m_{k_s}[s]$ of length $K(k_s)[s + 1] + c$ and for each $i < k_s$ such that $K(i)[s + 1] < K(i)[s]$ enumerate an N-description of $m_i[s]$ of length $K(i)[s + 1]$; we say that these enumerations are *primary*. In this case we also define $I_{s+1} = \emptyset$. Otherwise we search for a stage $p > s + 1$ such that

$$\Omega^*_{m_r[p]} - \Omega^*_r < 2^c \cdot (\Omega_p - \Omega_{m_r[p]}), \tag{2.4}$$

where r is the least such that m_r moved during the stages in $(s, p]$ (note that $r < k_s$). By the hypothesis, such a stage p exists. For each $i \in (s, p)$ let $I_i = \emptyset$. Define $I_p = [m_r[p], p]$. Note that by Definition 2.6 we have $m_r[p] \geq s + 1$, so $m_r[p]$ is greater than all numbers in the intervals I_i for $i < p$. By (2.4) we have

$$\sum_{r \leq i < k_s} 2^{-K(i)[s+1]} < 2^c \cdot \sum_{i \in I_p} 2^{-g(i)}. \tag{2.5}$$

For each $i \in [r, k_s)$ enumerate an N-description of m_i of length $K(i)[s+1]+c$. We say that this enumeration of N-descriptions is *secondary*. Go to stage $p + 1$.

By the construction, the intervals I_i are pairwise disjoint. Moreover, the weight of the N-descriptions that correspond to primary enumerations amount to weight at most the weight of the universal machine, because (k_s) is nondecreasing on the stages where primary enumerations occur. At each stage p where a secondary enumeration of N-descriptions takes place, by (2.5) the weight of these descriptions is bounded by $\sum_{i \in I_p} 2^{-g(i)}$. Hence the total weight of the N-descriptions that correspond to secondary enumerations is bounded by

$$\sum_p \sum_{i \in I_p} 2^{-g(i)} = \Omega.$$

Hence the total weight of N is bounded by 1. So N is a prefix-free machine. By the construction, $k_s \to \infty$ as $s \to \infty$ and (through the secondary enumerations) we also have $K_N(m_n) \leq K(n) + c$ for all n. $\dashv$

The following characterization is not needed for the proof of Theorem 1.1 but it is worth mentioning. It says that a c.e. set (or real) is K-trivial if and only if the monotone approximations to Ω that it can provide are no better than those that can be provided by a computable function.

Corollary 2.8 (Characterisation of K-trivial c.e. reals). *Given a c.e. set (or real) A the following are equivalent:*

(a) *A is K-trivial;*
(b) *for all functions $f \leq_T A$ we have $\exists c \forall n\, \Omega - \Omega_n < 2^c \cdot (\Omega - \Omega_{f(n)})$;*
(c) $\exists c \forall n\, \Omega - \Omega_n < 2^c \cdot (\Omega - \Omega_{a(n)})$.

where $(a(n))$ is the modulus of its computable enumeration (or monotone approximation, in the case of c.e. reals). Moreover, if $I \subseteq \mathbb{N}$ is computable and infinite, then the above equivalence holds with '$\forall n$' replaced by '$\forall n \in I$'.

Proof. The implication (a)⇒(b) follows from Lemma 2.5 and the fact that K-trivial sets are low for Ω. The implication (b)⇒(c) is trivial. For the remaining implication (c)⇒(a), assume that there exists a constant c such that $\Omega - \Omega_n < 2^c \cdot (\Omega - \Omega_{a(n)})$ for all $n \in I$. Let m_n be a computable increasing enumeration of I. Note that $(a(m_n)[s])$ is a computable system of movable markers according to Definition 2.6. Hence by Lemma 2.7 there exists a constant c_0 such that $K(a(m_n)) \leq K(m_n) + c_0$ for all n. On the other hand there exists a constant c_1 such that $K(A \upharpoonright_n) \leq K(a(n)) + c_1$ for each n. Hence $K(A \upharpoonright_{m_n}) \leq K(m_n) + c_0 + c_1$ for all n, which means that A is K-trivial by Lemma 2.2. ⊣

The following observation is the last step that we need in order to derive the main result of this section which is Proposition 2.10; it connects the complexity of certain system of movable markers with the initial segment complexities of a pair of c.e. sets.

Lemma 2.9. *Let A, B be c.e. sets (or reals), let $(a(n)), (b(n))$ be their modulus functions and let $d(n) = \min\{a(n), b(n)\}$. If $\forall n\ K(d(n)) \leq K(n) + p$ for some constant p then there exists c such that $\forall n\ \min\{K(A \upharpoonright_n), K(B \upharpoonright_n)\} < K(n) + c$. Moreover, if I is an infinite computable set then the this implication holds with '$\forall n$' replaced by '$\forall n \in I$'.*

Proof. It suffices to show that there exist prefix-free machines M_a, M_b such that $\min\{K_{M_a}(A \upharpoonright_n), K_{M_b}(B \upharpoonright_n)\} \leq K(d(n))$ for all n. Consider M_a which, on input σ operates as follows. First it waits until $U(\sigma) \downarrow = \langle n, t \rangle$ for some t, n (where U is the optimal machine). Then it defines $M_a(\sigma) = A_t \upharpoonright_n$. Similarly, on input σ machine M_b waits until $U(\sigma) \downarrow = \langle n, t \rangle$ for some t, n and then defines $M_b(\sigma) = B_t \upharpoonright_n$. Clearly M_a, M_b are prefix free (as they have the same domain as U). Moreover, for each n let t_n be such that $d_n = \langle n, t_n \rangle$. Since $d(n) = a(n)$ or $d(n) = b(n)$ for each n, we have $A_{t_n} \upharpoonright_n = A \upharpoonright_n$ or $B_{t_n} \upharpoonright_n = B \upharpoonright_n$ for each n. Hence for each n we have $K_{M_a}(A \upharpoonright_n) \leq K(d(n))$ or $K_{M_b}(B \upharpoonright_n) \leq K(d(n))$, which completes the proof. ⊣

Proposition 2.10 (Tool for the main construction). *Let A, B be c.e. sets (or reals), let $(a(n)), (b(n))$ be their modulus functions and let $d(n) = \min\{a(n), b(n)\}$. If A, B are not K-trivial and I is an infinite computable set then for each c there exists $n \in I$ such that $\Omega - \Omega_{d(n)} < 2^{-c} \cdot (\Omega - \Omega_n)$.*

Proof. Note that $(d(n))$ is a computable system of movable markers. Hence the proposition is a direct consequence of Lemma 2.7, Lemma 2.9 and (2.3). ⊣

Note that the conclusion $\Omega - \Omega_{d(n)} < 2^{-c} \cdot (\Omega - \Omega_n)$ can be written as

$$\sum_{i>d(n)} 2^{-g(i)} < 2^{-c} \cdot \sum_{i>n} 2^{-g(i)}$$

by (2.2). Hence if (n_t) is a computable increasing sequence of numbers then (under the same assumptions about A, B) we have that for each c there exists t such that

$$\sum_{i>d(n_t)} 2^{-g(i)} < 2^{-c} \cdot \sum_{i>n_t} 2^{-g(i)}.$$

This is the form that we are going to use when we formulate the requirements for the proof of Theorem 1.1 in Section 3.1.

§**3. Proof of Theorem 1.1.** We wish to construct a c.e. set D, whose Turing degree $\mathbf{d}$ meets the conditions of Theorem 1.1. We formulate a sufficient set of requirements for D in Section 3.1 and give the specifics of the construction in Section 3.2. We conclude with the formal construction in Section 3.3 and the verification of the requirements in Section 3.4.

3.1. Requirements for the construction of D. Let U be the universal prefix-free machine which underlies the prefix-free Kolmogorov complexity function, that is, such that $K = K_U$. We may assume that $\mathtt{wgt}(U) < 2^{-4}$. Also let (A_e, B_e) be an effective list of all pairs of c.e. sets. Note that the sets A_e, B_e are given via specific computable enumerations that are provided by a fixed universal Turing machine. The sets A_e, B_e correspond to guesses about representatives of the degrees $\mathbf{a}, \mathbf{b}$ of Theorem 1.1. For each pair (A_e, B_e) let a_e, b_e denote the corresponding modulus functions. Moreover, let $a_e[s], b_e[s]$ denote their approximations at stage s. In particular, $a_e(n)[s]$ is n if $s \le n$ and the least stage $t > n$ with $t \le s$ such that $A[t] \restriction_n \prec A[s]$ otherwise; similarly for $b_e(n)[s]$. Let $(i, j) \mapsto \langle i, j\rangle$ be a standard computable increasing (in both arguments) pairing function and define $\mathbb{N}^{[k]} = \{\langle k, n\rangle \mid n \in \mathbb{N}\}$.

We define a version of the parameter $\min\{a_e(n), b_e(n)\}$ which can be treated dynamically (at any stage of the construction) as a number that is eligible for enumeration into the set D that will be constructed. Define $d_e(n)[s]$ to be the least number in $\mathbb{N}^{[\langle e,n\rangle]} - D[s]$ which is larger than $\min\{a_e(n)[s], b_e(n)[s]\}$. Moreover, let $d_e(n) = \lim_s d_e(n)[s]$. The parameters $a_e(n)[s], b_e(n)[s], d_e(n)[s]$ can be seen as movable markers on $\mathbb{N}$. Moreover, a direct consequence of their definition is that they always move monotonically, that is, $a_e(n)[s] \le a_e(n)[s+1]$ and similarly for $b_e(n)[s], d_e(n)[s]$.

We will define a K-trivial c.e. set D and a sequence of c.e. sets (V_e) such that the following conditions are met.

$$R_e : V_e \le_T A_e \oplus D \ \wedge\ V_e \le_T B_e \oplus D.$$

We will also ensure the following condition on V_e.

$$P_e : \text{ If } A_e, B_e \text{ are not } K\text{-trivial then } V_e \text{ is not } K\text{-trivial.}$$

These conditions on $D, (V_e)$ are sufficient for the proof of Theorem 1.1. Let g be a fixed Solovay function, that is, a function satisfying (2.1), for the duration of this proof. Without loss of generality we may assume that $\sum_i 2^{-g(i)} < 2^{-4}$. We may split each condition P_e into more elementary conditions P^*_{ekt}. Let $(k, i) \mapsto n_k(i)$ be a computable function such that $n_k(i) < n_k(i+1)$ and $n_k(i) \in \mathbb{N}^{[k]}$. In Section 3.2 we will define a specific such function, but at this point we may express P^*_{ekt} in terms of any fixed such choice. We may write n_{kt} to denote $n_k(t)$ in the interest of space.

$$P^*_{ekt} : \Big(\sum_{i > d_e(n_{kt})} 2^{-g(i)} < 2^{-e-k} \cdot \sum_{i > n_{kt}} 2^{-g(i)}\Big) \Rightarrow \exists i\ K(V_e \restriction_i) > g(i) + k.$$

We let P^*_{ek} denote the conjunction of all P^*_{ekt} , $t \in \mathbb{N}$. We verify that the satisfaction of P_e may be reduced to the satisfaction of P^*_{ek}, $k \in \mathbb{N}$. Fix e. Assume that A_e, B_e are not K-trivial and P^*_{ekt} are met for all k, t. Then by Proposition 2.10, for each k there are infinitely many t such that the left-hand-side of the implication

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in P^*_{ekt} holds. Since each P^*_{ekt} is met, it follows that for each k there are infinitely many t such that $K(V_e \upharpoonright_{n_{kt}}) > g(n_{kt}) + k$. Since g is a Solovay function, this means that V_e is not K-trivial with constant k. Hence $(\forall k\ P^*_{ek})$ implies P_e.

The requirement that D is K-trivial can be expressed as

$$\exists c \forall n,\ K(D \upharpoonright_n) \leq g(n) + c. \tag{3.1}$$

The cost associated with the enumeration of a number n in D at stage $s+1$ of the construction in view of (3.1) is given by

$$c(n,s) = \sum_{n \leq i \leq s} 2^{-g(i)}. \tag{3.2}$$

The satisfaction of (3.1) will be achieved by ensuring that the total cost of the enumerations into D is bounded, in other words

$$\sum_{(n,s)\in I_D} c(n,s) < 1 \text{ where } I_D = \big\{(n,s) \mid n = \min\{x \mid x \in D[s+1] - D[s]\}\big\}. \tag{3.3}$$

The fact that (3.3) implies (3.1) was established in [11] when g is replaced by the Kolmogorov function $K(n)$ (also see [10, Section 11.1] and [25, Section 5.3] for elaborate presentations of this method). The same argument shows that this implication holds when $K(n)$ is replaced by any right-c.e function f such that $\sum_i 2^{-f(i)} < 1$.

We close this section by providing a condition which implies R_e and shows explicitly the required Turing reductions. By the definition of $a_e[s]$ it follows that $a_e(n)$ (the final position of $a(n)[s]$) is computable from A_e. Similarly, $b_e(n)$ is computable from B_e. Hence $A_e \oplus D$ computes an upper bound of $n \mapsto d_e(n)$ (provided that $\mathbb{N}^{[\langle e,n\rangle]} \cap D$ is finite) and the same is true of $B_e \oplus D$. The following condition expresses a *weak coding of V_e into D*.

$$R^*_e: \quad \left(\begin{array}{l} \text{For all } k, t, s \text{ and all } n \in [n_k(t-1), n_k(t)) \cap \mathbb{N}^{[k]} \\ n \in V_e[s+1] - V_e[s] \Rightarrow d_e(n_k(t))[s] \in D[s+1] - D[s] \end{array} \right)$$

Condition R^*_e implies condition R_e. Indeed, suppose that R^*_e holds. Then to determine if $n \in V_e$ we can first find k such that $n \in \mathbb{N}^{[k]}$ and then find t such that $n \in [n_k(t-1), n_k(t))$. Assuming that $\mathbb{N}^{[\langle e,n_k(t)\rangle]} \cap D$ is finite, $d_e(n_k(t))[s]$ reaches a limit as $s \to \infty$. Moreover, by the definition of $d_e(n_k(t))[s]$ it follows that it changes value only when one of the following holds:

- the minimum of $a_e(n_k(t))[s]$ and $b_e(n_k(t))[s]$ changes value;
- the current value of $d_e(n_k(t))[s]$ is enumerated into D.

If the first case above holds but not the second, and

$$\min\{a_e(n_k(t)), b_e(n_k(t))\}[s] \text{ changes but } \max\{a_e(n_k(t)), b_e(n_k(t))\}[s] \text{ does not}$$

then $d_e(n_k(t))$ will move to the least number in $\mathbb{N}^{[e,n_k(t)]} - D[s]$; this in a number that can be computed at stage s. Since $a_e(n_k(t))$ is computable from A_e, it follows that we can use $A_e \oplus D$ in order to compute a stage where the approximation to $D(d_e(n_k(t)))$ has reached a limit. By R^*_e, at that stage the approximation to $V_e(n)$ has also reached a limit. So we have computed $V_e(n)$. The same procedure can be

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performed via $B_e \oplus D$-computations, by first computing $b_e(n_k(t))$. Hence $V_e(n)$ is also computable from $B_e \oplus D$.

We have established that a construction of $D, (V_e)$ which meets conditions (3.3) and R_e^*, P_{ek} for $e, k \in \mathbb{N}$ (and any choice of a computable function $(k,i) \mapsto n_k(i)$ which is increasing on i and such that $n_k(i) \in \mathbb{N}^{[k]}$) is sufficient for the proof of Theorem 1.1. An underlying assumption is that for each e, n the set $\mathbb{N}^{[\langle e,n\rangle]} \cap D$ is finite, so that $d_e(n)[s]$ reaches a limit. The latter will be an immediate feature of the construction.

3.2. Strategy and witnesses for conditions P_{ek}^*. Recall that P_{ek}^* denotes the conjunction of the conditions P_{ekt}^* of Section 3.1 (which depend on the choice of $(k,i) \mapsto n_k(i)$). The construction of Section 3.3 is driven by actions (enumerations into D, V_e) for the satisfaction of P_{ek}^*. Here we define some parameters that are used in these actions. For each k we define an increasing sequence $(n_k(i))$ of numbers. Recall the definitions of $(i,j) \mapsto \langle i,j\rangle$ and $\mathbb{N}^{[k]}$ from Section 3.1. Define

$$J(\langle k,x\rangle) = \Big\{\langle k,m\rangle \mid m > x+1 \ \wedge \sum_{t>\langle k,m\rangle} 2^{-g(t)} > \frac{1}{m-x-1}\Big\}.$$

The sets $J(i)$ are uniformly c.e. and by Lemma 2.3 they are all infinite. Hence we may choose a uniformly computable family of sets $J^*(i)$ such that $J^*(i) \subseteq J(i)$ for each i. Define $(n_k(i))$ recursively as follows.

$$n_k(-1) = \min \mathbb{N}^{[k]}$$
$$n_k(i) = \min J^*\big(n_k(i-1)\big).$$

Note that the function $(k,i) \mapsto n_k(i)$ is computable. Moreover, there exists some number x such that

$$\textstyle\sum_{n_k(t)<i<x} 2^{-g(i)} > 1/\big|\big(n_k(t-1), n_k(t)\big) \cap \mathbb{N}^{[k]}\big|. \tag{3.4}$$

From this point on, P_{ekt}^* refers to this choice of $(k,i) \mapsto n_k(i)$. We say that P_{ek}^* *requires attention at stage* $s+1$ if there is some $t < s$ such that

$$\sum_{d_e(n_k(t))[s]<i\le s} 2^{-g(i)} < 2^{-e-k} \cdot \sum_{n_k(t)<i\le s} 2^{-g(i)} \tag{3.5}$$

and

$$\forall i \le p_{ek}[s],\ K(V_e \restriction_i)[s] \le g(i) + k, \tag{3.6}$$

where $p_{ek}[s]$ is the maximum of the following two numbers:

(a) the largest stage $\le s$ where P_{ek}^* required attention (or 0 if such a stage does not exist);
(b) the least number x satisfying (3.4).

In this case we say that P_{ek}^* *requires attention for* t at stage $s+1$.

The intuition for the main action of the construction is that if (3.5) holds, by enumerating $d_e(n_k(t))[s]$ into D and changing the approximation to $V_e \restriction_{n_k(t)}$ *the cost of the opponent for maintaining* (3.6) *is a large multiple of our cost for maintaining* (3.3). Our choice of the sequence $(n_k(i))$ ensures that such attacks are sufficient in order to drive the opponent out of the available descriptions that are needed for maintaining (3.6). Moreover, recall that by the analysis of Section 3.1 (which was

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based on Proposition 2.10) property (3.5) has to hold for infinitely many t, if A_e, B_e are indeed not K-trivial.

3.3. Construction of the sets D, V_e**.** At stage $s+1$ check if there is some $\langle e,k\rangle < s$ such that P^*_{ek} requires attention. If there is such a number, let $\langle e,k\rangle$ be the least one and let t be the least number such that (3.5) and (3.6) hold. Enumerate $d_e(n_k(t))[s]$ into D and enumerate the largest number of

$$\mathbb{N}^{[k]} \cap \big(n_k(t-1), n_k(t)\big) - V_e[s] \tag{3.7}$$

into V_e.

3.4. Verification of the requirements. At every stage $s+1$ where P^*_{ek} requires attention for t and $\langle e,k\rangle < s$, a change in $V_e \restriction_{n_k(t)}$ is caused by an enumeration of a number of the set in (3.7) into V_e (provided that the set in (3.7) is nonempty). There are

$$\big|\big(n_k(t-1), n_k(t)\big) \cap \mathbb{N}^{[k]}\big|$$

many such enumerations that can be performed. Because of (3.4) and (3.6), each time that P^*_{ek} requires attention after such an enumeration, we can count an additional weight of

$$1/\big|\big(n_k(t-1), n_k(t)\big) \cap \mathbb{N}^{[k]}\big|$$

in the underlying universal prefix-free machine U. Since $\mathtt{wgt}(U) < 2^{-2}$,

$$P^*_{ek} \text{ requires attention less than } \big|\big(n_k(t-1), n_k(t)\big) \cap \mathbb{N}^{[k]}\big| \text{ times for } t. \tag{3.8}$$

Hence whenever P^*_{ek} requires attention in the construction, an enumeration into V_e will occur. Moreover, (the current value of) $d_e(n_k(t))[s]$ will only be enumerated into D finitely many times. Marker $d_e(i)$ moves at stage $s+1$ only if one of the following events occur:

(a) $A \restriction_i [s] \not\prec A[s+1]$ or $B \restriction_i [s] \not\prec B[s+1]$;
(b) $d_e(i)[s] \in D_e[s+1] - D_e[s]$.

Clearly (a) can only occur at most finitely many times. Moreover, (b) only occurs if $i = n_k(t)$ for some t such that P^*_{ek} requires attention for t at stage $s+1$. By (3.8), case (b) only occurs at most finitely many times. Consequently,

$$\lim_s d_e(i)[s] \text{ exists for each } e. \tag{3.9}$$

In other words $\mathbb{N}^{[\langle e,i\rangle]} \cap D$ is finite, which was an underlying assumption for the requirements of Section 3.1.

Lemma 3.1. *For each e, condition R_e is met.*

Proof. Fix e. The construction clearly meets condition R^*_e. By (3.9) and the analysis in Section 3 it follows that R_e is met. ⊣

Lemma 3.2. *For each e, condition P_e is met.*

Proof. By the discussion of Section 3.1, it suffices to show that P^*_{ekt} is met for each k, t. Fix k, t and assume that the left hand side of the implication in P^*_{ekt} holds. Then according to the construction, (3.8) implies that $\exists i\ K(V_e \restriction_i) > g(i) + k$. ⊣

Lemma 3.3. *The set D is K-trivial.*

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Proof. By the analysis in Section 3.1 it suffices to show (3.3). Let

$$I_D(e,k) = \Big\{ \big(d_e(n_{kt})[s], s\big) \in I_D \mid s, t > 0 \Big\}.$$

Note that $I_D(e,k)$ contains the pairs in I_D that correspond to actions for P^*_{ek}. In particular, $I_D = \bigcup_{e,k} I_D(e,k)$ and it suffices to show that

$$\sum_{(n,s)\in I_D(e,k)} c(n,s) < 2^{-e-k-3} \tag{3.10}$$

for each e, k. Fix e, k and let (x_i, s_i) be a monotone enumeration of $I_D(e,k)$, in the sense that $s_i < s_{i+1}$ for each i. Let us say that at stage s_{i+1} the ith cycle of P^*_{ek} is completed. Note that the sequence (x_i, s_i) is possibly infinite. However, upon the completion of the ith cycle of P_{ek} we may count an additional set of descriptions of the universal machine U (describing current values of V_e) of weight at least $2^{e+k} \cdot c(x_i, s_i)$. This is a consequence of (3.5) and (3.6). For the case that (x_i, s_i) is finite (so the last cycle is never completed) note that $c(x_i, s_i) < 2^{-e-k-4}$ for all i due to (3.5). Since $\mathtt{wgt}(U) < 2^{-4}$ we obtain $\sum_i c(x_i, s_i) < 2^{-e-k-3}$, that is (3.10). ⊣

According to the analysis of Section 3.1, this concludes the proof of Theorem 1.1.

§4. **Conclusion.** The class of K-trivial sequences and their Turing degrees is far from trivial and, in fact, has very rich structure. There are several ways one can reveal the complexities of this class. One of these is the study of the quotient structure of the c.e. Turing degrees modulo the K-trivial degrees. Intuitively, this structure gives information about the degrees of unsolvability of c.e. sets when K-trivial information is available "for free". The following is a direct consequence of Theorem 1.1.

Corollary 4.1. *The quotient upper semi-lattice of the c.e. Turing degrees modulo the K-trivial degrees has no minimal pairs.*

We do not know much more about this structure; for example, the following basic question is open.

Is the quotient upper semi-lattice of the c.e. Turing degrees modulo the K-trivial degrees dense?

Our result shows that a certain simple definition with parameters of the ideal of the K-trivial degrees is not possible in the c.e. degrees. In particular, the K-trivial degrees cannot be defined as the intersection of two lower cones in the c.e. Turing degrees. The question of parameter definability of this ideal in the c.e. degrees (briefly discussed in the end of [24]) remains open.

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